Newton's Method with Deflation for Isolated Singularities*

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Newton's method slows down when approaching a singular root and convergence may even be lost if the working precision is not at least as high as the multiplicity of the root multiplied with the distance of the current approximation to the root. To restore the quadratic convergence of Newton's method, T. Ojika, S. Watanabe, and T. Mitsui ([1], see also [2]) developed a deflation algorithm.

We studied the methods proposed in [1] and [2] and – for the sake of numerical stability – implemented and experimented with two modifications to their deflation algorithm:

1. Instead of replacing equations of the original system by conditions from derivatives of the system, we propose to add equations, introducing random constants for uniform treatment.

2. Instead of using Gaussian Elimination, we propose to apply Singular Value Decomposition to determine the numerical rank of the Jacobian matrix and to solve linear systems.

In particular, if the numerical rank of the Jacobian matrix at the current approximation equals R, we introduce $R+1$ additional variables which serve as multipliers to selections of random combinations of columns of the Jacobian matrix. These selections determine new equations added to the original system, in order to cut out the isolated singular root more precisely. By construction of our deflation method, the isolated singular root of the original system remains isolated and the corank of the Jacobian matrix can only decrease after deflation.

Our modifications to the methods of [1] and [2] were developed in Maple, exploiting its facilities for polynomial manipulations and convenient multi-precision arithmetic. We tested our ideas on more than a dozen of examples and our experiments all confirm the conjecture (proven in [1] for a special class of systems): if m is the multiplicity of the isolated root, then no more than $m-1$ successive deflations are needed to restore quadratic convergence. The method is currently being implemented and tested to become part of the next public release of PHCpack [3].

Key words and phrases. Newton's method, deflation, numerical homotopy algorithms, symbolic-numeric computations.

References

[1] T. Ojika, S. Watanabe, and T. Mitsui. Deflation algorithm for the multiple roots of a system of nonlinear equations. *J. Math. Anal. Appl.* 96, 463–479, 1983.

[2] T. Ojika. Modified deflation algorithm for the solution of singular problems. I. A system of nonlinear algebraic equations. *J. Math. Anal. Appl.* 123, 199–221, 1987.

[3] J. Verschelde. Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Transactions on Mathematical Software* 25(2): 251–276, 1999. Software available at `http://www.math.uic.edu/~jan`.

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