Computing Puiseux Series for Algebraic Surfaces

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Puiseux Series for Algebraic Sets

problem statement

A polynomial in n variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ consists of a vector of nonzero complex coefficients with corresponding exponents in A:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

Solve $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$ with supports $(A_0, A_1, \ldots, A_{N-1})$.

- Systems are ***sparse***: few monomials have a nonzero coefficient.
- For $\mathbf{a} \in \mathbb{Z}^n$, we consider ***Laurent*** polynomials, $f \in \mathbb{C}[\mathbf{x}^{\pm 1}]$
  $\Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
- Many applications give rise to ***symmetric*** polynomial systems. The solution set is invariant under permutations of the variables.

the cyclic n-roots system

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ x_0x_1 + x_1x_2 + \cdots + x_{n-2}x_{n-1} + x_{n-1}x_0 = 0 \\ i = 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\ x_0x_1x_2 \cdots x_{n-1} - 1 = 0. \end{cases}$$

Lemma (Backelin)

If m^2 divides n, then the cyclic n-roots system has a solution set of dimension $m - 1$.

J. Backelin: *Square multiples n give infinitely many cyclic n-roots*. Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

Newton polytopes and Puiseux series

The sparse structure is modeled by its Newton polytope.

Definition

Consider the support $A \subset \mathbb{Z}^n$ of $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, $c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}$.

The ***Newton polytope of f*** is the convex hull of A.

Definition

Consider a curve C defined by $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.
A ***Puiseux series of the curve C*** has the form

$$\left\{ \begin{array}{rcl} x_0 & = & t^{v_0} \\ x_k & = & z_k t^{v_k}(1 + O(t)), \quad k = 1, 2, \ldots, n-1, \end{array} \right.$$

where $(z_1, \ldots, z_{n-1}) \in (\mathbb{C}^*)^{n-1}$.

The Newton-Puiseux algorithm is in Walker's *Algebraic Curves*, 1950.

initial forms and tropisms

Denote the inner product of vectors **u** and **v** as $\langle \mathbf{u}, \mathbf{v} \rangle$.

Definition

Let $\mathbf{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ be a direction vector. Consider $f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$.

The ***initial form of f in the direction v*** is

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad \text{where } m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}.$$

Definition

Let the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ define a curve. A ***tropism*** consists of the leading powers $(v_0, v_1, \ldots, v_{n-1})$ of a Puiseux series of the curve.

The leading coefficients of the Puiseux series satisfy $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \mathbf{0}$.

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relevant software

- Maple, of course...
- `cddlib` by Komei Fukuda and Alain Prodon implements the double description method to efficiently enumerate all extreme rays of a general polyhedral cone.
- `Gfan` by Anders Jensen to compute Gröbner fans and tropical varieties uses `cddlib`.
- The `Singular` library `tropical.lib` by Anders Jensen, Hannah Markwig and Thomas Markwig for computations in tropical geometry.
- `Macaulay2` interfaces to Gfan.
- `Sage` interfaces to Gfan.
- `PHCpack` (published as Algorithm 795 ACM TOMS) provides our numerical blackbox solver.

Puiseux Series for Algebraic Sets

binomial systems

Definition

A ***binomial system*** has exactly two monomials with nonzero coefficient in every equation.

The binomial equation $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} - c_{\mathbf{b}}\mathbf{x}^{\mathbf{b}} = 0$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, $c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{C} \setminus \{0\}$, has normal representation $\mathbf{x}^{\mathbf{a}-\mathbf{b}} = c_{\mathbf{b}}/c_{\mathbf{a}}$.

A binomial system of N equations in n variables is then defined by an exponent matrix $A \in \mathbb{Z}^{N \times n}$ and a coefficient vector $\mathbf{c} \in (\mathbb{C}^*)^N$: $\mathbf{x}^A = \mathbf{c}$.

Solution sets of binomial systems are related to ***toric varieties***.

Solution sets of binomial systems can be represented exactly by the first term of their Puiseux series.

an example

Consider as an example for $\mathbf{x}^A = \mathbf{c}$ the system

$$\left\{ \begin{array}{r} x_0^2 x_1 x_2^4 x_3^3 - 1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right. \quad A = \left[\begin{array}{cccc} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{array} \right]^T \quad \mathbf{c} = \left[\begin{array}{c} 1 \\ 1 \end{array} \right].$$

As basis of the null space of A we can for example take $\mathbf{u} = (-3, 2, 1, 0)$ and $\mathbf{v} = (-2, 1, 0, 1)$.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are tropisms for a two dimensional algebraic set.

Placing $\mathbf{u}$ and $\mathbf{v}$ in the first two rows of a matrix M, extended so $\det(M) = 1$, we obtain a coordinate transformation, $\mathbf{x} = \mathbf{y}^M$:

$$M = \left[\begin{array}{rrrr} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{l} x_0 = y_0^{-3} y_1^{-2} y_2 \\ x_1 = y_0^2 y_1 y_3 \\ x_2 = y_0 \\ x_3 = y_1. \end{array} \right.$$

monomial transformations

By construction, as $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$:

$$MA = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = B.$$

The corresponding monomial transformation $\mathbf{x} = \mathbf{y}^M$ performed on $\mathbf{x}^A = \mathbf{c}$ yields $\mathbf{y}^{MA} = \mathbf{y}^B = \mathbf{c}$, eliminating the first two variables:

$$\left\{ \begin{array}{l} y_2^2 y_3 - 1 = 0 \\ y_2 y_3 - 1 = 0. \end{array} \right.$$

Solving this reduced system gives values z_2 and z_3 for y_2 and y_3. Leaving y_0 and y_1 as parameters t_0 and t_1 we find as solution

$$(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1).$$

unimodular coordinate transformations

Definition

A ***unimodular coordinate transformation*** $\mathbf{x} = \mathbf{y}^M$ is determined by an invertible matrix $M \in \mathbb{Z}^{n \times n}$: $\det(M) = \pm 1$.

For a d-dimensional solution set of a binomial system:

1. The null space of A gives d tropisms, stored in the rows of a d-by-n matrix B.
2. Compute the Smith normal form S of B: $UBV = S$.
3. There are three cases:
   1. $U = I \Rightarrow M = V^{-1}$
   2. If $U \neq I$ and S has ones on its diagonal, then extend U^{-1} with an identity matrix to form M.
   3. Compute the Hermite normal form H of B and let D be the diagonal elements of H, then $M = \begin{bmatrix} D^{-1}B \\ \mathbf{0} \quad I \end{bmatrix}$.

cyclic 4-roots and binomial systems

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 = 0 \\ x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_3 x_0 x_1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right.$$

Looking for a special solution,
we apply the binomial system solver to

$$\left\{ \begin{array}{r} x_0 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_0 x_1 + x_1 x_2 = 0 \\ x_2 x_3 + x_3 x_0 = 0 \\ x_0 x_1 x_2 + x_2 x_3 x_0 = 0 \\ x_1 x_2 x_3 + x_3 x_0 x_1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right. \quad \ldots$$

the output of phc -b

```
4 5
t1 - t1 +
x0 - t1^1;
x2 - (-1 - 1.22464679914735E-16*i)*t1^1;
x1 - (-1)*t1^-1;
x3 - (1 - 1.22464679914735E-16*i)*t1^-1;
4 5
t1 - t1 +
x0 - t1^1;
x2 - (-1 - 1.22464679914735E-16*i)*t1^1;
x1 - (1 - 1.22464679914735E-16*i)*t1^-1;
x3 - (-1 + 2.44929359829471E-16*i)*t1^-1;
```

This output corresponds to the two solutions
$(x_0 = t, x_1 = -t^{-1}, x_2 = -t, x_3 = t^{-1})$ and
$(x_0 = t, x_1 = t^{-1}, x_2 = -t, x_3 = -t^{-1})$ of the original problem.

Puiseux Series for Algebraic Sets

the Cayley embedding – an example

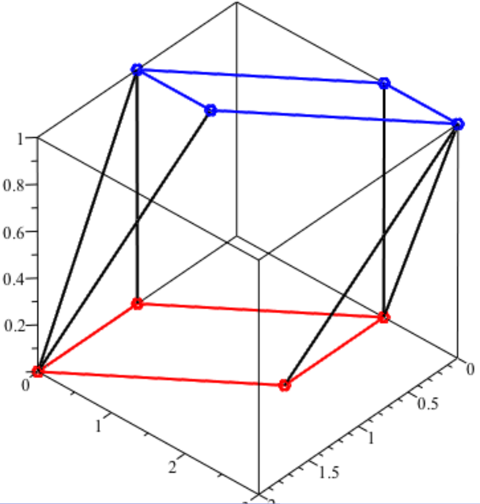
$$\begin{cases} p = (x_0 - x_1^2)(x_0 + 1) = x_0^2 + x_0 - x_1^2 x_0 - x_1^2 = 0 \\ q = (x_0 - x_1^2)(x_1 + 1) = x_0 x_1 + x_0 - x_1^3 - x_1^2 = 0 \end{cases}$$

The Cayley polytope is the convex hull of

$\{(2,0,0),(1,0,0),(1,2,0),(0,2,0)\}$

$\cup$

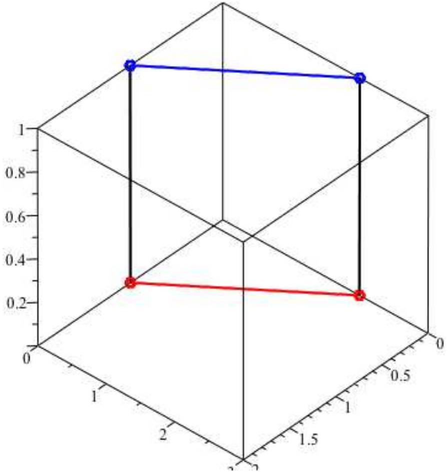
$\{(1,1,1),(1,0,1),(0,3,1),(0,2,1)\}$.



facet normals and initial forms

The Cayley polytope
has facets spanned by
one edge of the
Newton polygon of p
and
one edge of the
Newton polygon of q.

Consider $\mathbf{v} = (2, 1, 0)$.



$$\begin{cases} \mathrm{in}_{(2,1)}(p) = \mathrm{in}_{(2,1)}\left(x_0^2 + x_0 - x_1^2 x_0 - x_1^2\right) = x_0 - x_1^2 \\ \mathrm{in}_{(2,1)}(q) = \mathrm{in}_{(2,1)}\left(x_0 x_1 + x_0 - x_1^3 - x_1^2\right) = x_0 - x_1^2 \end{cases}$$

computing all pretropisms

Definition

A nonzero vector $\mathbf{v}$ is a ***pretropism*** for the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ if $\#\mathrm{in}_{\mathbf{v}}(f_k) \geq 2$ for all $k = 0, 1, \ldots, N-1$.

Application of the Cayley embedding to $(A_0, A_1, \ldots, A_{N-1})$:

$$E = \{ (\mathbf{a}, \mathbf{0}) \mid \mathbf{a} \in A_0 \} \cup \bigcup_{k=1}^{N-1} \{ (\mathbf{a}, \mathbf{e}_k) \mid \mathbf{a} \in A_k \} \subset \mathbb{Z}^{n+N-1},$$

where $\mathbf{0}, \mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, ..., $\mathbf{e}_{N-1} = (0, 0, \ldots, 1)$ span the standard unit simplex in $\mathbb{R}^{N-1}$.

The set of all facet normals to the convex hull of E contains all normals to facets spanned by at least two points of each support.

We used `cddlib` to compute all pretropisms of the cyclic n-roots system, up to $n = 12$ (148.5 hours on a 3.07GHz CPU with 4GB RAM).

cones of pretropisms

Definition

A ***cone of pretropism*** is a polyhedral cone spanned by pretropisms.

If we are looking for an algebraic set of dimension d and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $f(\mathbf{x}) = \mathbf{0}$ of dimension d,
  then the system $f(\mathbf{x}) = \mathbf{0}$ has no solution set of dimension d that intersects the first d coordinate planes properly; otherwise
- if a d-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For the cyclic 9-roots system,
we found a two dimensional cone of pretropisms.

the tropical prevariety of cyclic n-roots

All facets normals of the Cayley polytope computed with `cddlib` on a 3.07GHz Linux computer with 4Gb RAM:

n	#normals	#pretropisms	#generators	user cpu time
8	831	94	11	$<$ 1 sec
9	4,840	276	17	1 sec
12	907,923	38,229	290	148 hours 27 min

Tropical intersections with `Gfan` on a 2.26GHz MacBook:

n	#rays	f-vector	user cpu time
8	94	1 94 108 48	15 sec
9	276	1 276 222 54	1 min 11 sec
12	5,582	1 5582 37786 66382 42540 8712	21 hours 1 min

Note that `Gfan` can exploit permutation symmetry.

solving the cyclic 4-roots system

$$f(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{array} \right.$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \left\{ \begin{array}{c} x_1 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_1x_2x_3 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} x_0 = y_0^{+1} \\ x_1 = y_0^{-1}y_2 \\ x_2 = y_0^{+1}y_3 \\ x_3 = y_0^{-1}y_4 \end{array} \right.$$

The system $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{y}) = \mathbf{0}$ has two solutions.
We find two solution curves: $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$.

Puiseux series for algebraic sets

Proposition

If $f(\mathbf{x}) = \mathbf{0}$ is in Noether position and defines a d-dimensional solution set in $\mathbb{C}^n$, intersecting the first d coordinate planes in regular isolated points, then there are d linearly independent tropisms $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_{d-1} \in \mathbb{Q}^n$ so that the initial form system $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has a solution $\mathbf{c} \in (\mathbb{C} \setminus \{0\})^{n-d}$.

This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

$$
\begin{aligned}
x_0 &= t_0^{v_{0,0}} & x_d &= c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
x_1 &= t_0^{v_{0,1}} t_1^{v_{1,1}} & x_{d+1} &= c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
&\vdots & &\vdots \\
x_{d-1} &= t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} & x_n &= c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots
\end{aligned}
$$

our polyhedral approach

For every d-dimensional cone C of pretropisms:

1. We select d linearly independent generators to form the d-by-n matrix A and the unimodular transformation $\mathbf{x} = \mathbf{y}^M$.
2. If $\mathrm{in}_{\mathbf{v}_0}(\mathrm{in}_{\mathbf{v}_1}(\cdots \mathrm{in}_{\mathbf{v}_{d-1}}(f)\cdots))(\mathbf{x} = \mathbf{y}^M) = \mathbf{0}$ has no solution in $(\mathbb{C}^*)^{n-d}$, then return to step 1 with the next cone C, else continue.
3. If the leading term of the Puiseux series satisfies the entire system, then we report an explicit solution of the system and return to step 1 to process the next cone C.
   Otherwise, we take the current leading term to the next step.
4. If there is a second term in the Puiseux series,
   then we have computed an initial development for an algebraic set and report this development in the output.

Note: to ensure the solution of the initial form system is not isolated, it suffices to compute a series development *for a curve*.

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applied to the cyclic 8-roots system

Our approach applied to the cyclic 8-roots system:

- 831 facet normals (computed with `cddlib` in less than one second on one core of 3.07GHz Linux computer with 4GB RAM)
- 29 pretropism generators
- 5 lead to initial forms with solutions
  - $(1, -1, 0, 1, 0, 0, -1, 0)$
  - $(1, -1, 1, -1, 1, -1, 1, -1)$
  - $(1, 0, -1, 0, 0, 1, 0, -1)$
  - $(1, 0, -1, 1, 0, -1, 0, 0)$
  - $(1, 0, 0, -1, 0, 1, -1, 0)$

For the initial form solutions we used the blackbox solver of PHCpack.

Symbolic manipulations for the computation of the second term of the Puiseux series were done with Sage.

an initial form system

The pretropism $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ defines

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \begin{cases} x_1 + x_6 = 0 \\ x_1x_2 + x_5x_6 + x_6x_7 = 0 \\ x_4x_5x_6 + x_5x_6x_7 = 0 \\ x_0x_1x_6x_7 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_6x_7 + x_0x_1x_5x_6x_7 = 0 \\ x_0x_1x_2x_5x_6x_7 + x_0x_1x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases}$$

v defines the unimodular coordinate transformation: $x_0 = y_0$, $x_1 = y_1/y_0, x_2 = y_2, x_3 = y_0y_3, x_4 = y_4, x_5 = y_5, x_6 = y_6/y_0, x_7 = y_7$.
Using the new coordinates, we transform $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x})$.

the transformed initial form system

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{y}) = \left\{ \begin{array}{r} y_1 + y_6 = 0 \\ y_1y_2 + y_5y_6 + y_6y_7 = 0 \\ y_4y_5y_6 + y_5y_6y_7 = 0 \\ y_4y_5y_6y_7 + y_1y_6y_7 = 0 \\ y_1y_2y_6y_7 + y_1y_5y_6y_7 = 0 \\ y_1y_2y_3y_4y_5y_6 + y_1y_2y_5y_6y_7 + y_1y_4y_5y_6y_7 = 0 \\ y_1y_2y_3y_4y_5y_6y_7 + y_1y_2y_4y_5y_6y_7 = 0 \\ y_1y_2y_3y_4y_5y_6y_7 - 1 = 0 \end{array} \right.$$

Solving $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{y})$, we obtain 8 solutions (all in the same orbit), e.g.:

$$y_0 = t, y_1 = -I, y_2 = \frac{-1}{2} - \frac{I}{2}, y_3 = -1, y_4 = 1 + I,$$
$$y_5 = \frac{1}{2} + \frac{I}{2}, y_6 = I, y_7 = -1 - I, I = \sqrt{-1}.$$

developing a series for the solution

Taking solution at infinity, we build a series of the form:

$$
\begin{aligned}
y_0 &= t \\
y_1 &= -I + c_1 t \\
y_2 &= \frac{-1}{2} - \frac{I}{2} + c_2 t \\
y_3 &= -1 + c_3 t \\
y_4 &= 1 + I + c_4 t \\
y_5 &= \frac{1}{2} + \frac{I}{2} + c_5 t \\
y_6 &= I + c_6 t \\
y_7 &= (-1 - I) + c_7 t
\end{aligned}
$$

Plugging series form into transformed system, collecting all coefficients of t^1, solving yields

$$
\begin{aligned}
c_1 &= -1 - I \\
c_2 &= \frac{1}{2} \\
c_3 &= 0 \\
c_4 &= -1 \\
c_5 &= \frac{-1}{2} \\
c_6 &= 1 + I \\
c_7 &= 1
\end{aligned}
$$

The second term in the series, still in the transformed coordinates:

$$
\begin{aligned}
y_0 &= t \\
y_1 &= -I + (-1 - I)t \\
y_2 &= \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t \\
y_3 &= -1 \\
y_4 &= 1 + I - t \\
y_5 &= \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t \\
y_6 &= I + (1 + I)t \\
y_7 &= (-1 - I) + t
\end{aligned}
$$

degree computations

Definition (Branch Degree)

Let $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ be a tropism and let R be the set of initial roots of the initial form system $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{y}) = \mathbf{0}$.

Then the degree of the branch is $\#R \times \left| \max_{i=0}^{n-1} v_i - \min_{i=0}^{n-1} v_i \right|$.

Tropisms, their cyclic permutations, and degrees:

$$
\begin{array}{rr}
(1,-1,1,-1,1,-1,1,-1) & 8 \times 2 = 16 \\
(1,-1,0,1,0,0,-1,0) \rightarrow (1,0,0,-1,0,1,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,0,0,1,0,-1) \rightarrow (1,0,-1,1,0,-1,0,0) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,-1,1,0,-1,0,0) \rightarrow (1,0,-1,0,0,1,0,-1) & 8 \times 2 + 8 \times 2 = 32 \\
(1,0,0,-1,0,1,-1,0) \rightarrow (1,-1,0,1,0,0,-1,0) & 8 \times 2 + 8 \times 2 = 32 \\
 & \text{TOTAL} = 144
\end{array}
$$

144 is the degree of the solution curve of the cyclic 8-root system.

an initial form of cyclic 9-roots

$\mathbf{v}_0 = (1,1,-2,1,1,-2,1,1,-2)$ and $\mathbf{v}_1 = (0,1,-1,0,1,-1,0,1,-1)$ define the initial form system

$$
\left\{
\begin{array}{rcl}
x_2 + x_5 + x_8 & = & 0 \\
x_0x_8 + x_2x_3 + x_5x_6 & = & 0 \\
x_0x_1x_2 + x_0x_1x_8 + x_0x_7x_8 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 & & \\
+ x_4x_5x_6 + x_5x_6x_7 + x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_8 + x_2x_3x_4x_5 + x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_8 + x_0x_5x_6x_7x_8 + x_2x_3x_4x_5x_6 & = & 0 \\
x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_8 + x_0x_1x_2x_3x_7x_8 & & \\
+ x_0x_1x_2x_6x_7x_8 + x_0x_1x_5x_6x_7x_8 + x_0x_4x_5x_6x_7x_8 + x_1x_2x_3x_4x_5x_6 & & \\
+ x_2x_3x_4x_5x_6x_7 + x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_8 + x_0x_1x_2x_5x_6x_7x_8 + x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_8 + x_0x_1x_2x_3x_5x_6x_7x_8 + x_0x_2x_3x_4x_5x_6x_7x_8 & = & 0 \\
x_0x_1x_2x_3x_4x_5x_6x_7x_8 - 1 & = & 0
\end{array}
\right.
$$

the unimodular transformation $\mathbf{x} = \mathbf{y}^M$

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_0 &= y_0 \\ x_1 &= y_0 y_1 \\ x_2 &= y_0^{-2} y_1^{-1} y_2 \\ x_3 &= y_0 y_3 \\ x_4 &= y_0 y_1 y_4 \\ x_5 &= y_0^{-2} y_1^{-1} y_5 \\ x_6 &= y_0 y_6 \\ x_7 &= y_0 y_1 y_7 \\ x_8 &= y_0^{-2} y_1^{-1} y_8 \end{aligned}$$

We use the coordinate change to transform the initial form system and the original cyclic 9-roots system.

the transformed initial form system

$$\left\{\begin{array}{rcl}
y_2 + y_5 + y_8 & = & 0 \\
y_2 y_3 + y_5 y_6 + y_8 & = & 0 \\
y_2 y_3 y_4 + y_3 y_4 y_5 + y_4 y_5 y_6 + y_5 y_6 y_7 & & \\
+ y_6 y_7 y_8 + y_2 y_3 + y_7 y_8 + y_2 + y_8 & = & 0 \\
y_2 y_3 y_4 y_5 + y_5 y_6 y_7 y_8 + y_2 y_8 & = & 0 \\
y_2 y_3 y_4 y_5 y_6 + y_5 y_6 y_7 y_8 + y_2 y_3 y_8 & = & 0 \\
y_2 y_3 y_4 y_5 y_6 y_7 + y_3 y_4 y_5 y_6 y_7 y_8 + y_2 y_3 y_4 y_5 y_6 & & \\
+ y_4 y_5 y_6 y_7 y_8 + y_2 y_3 y_4 y_5 + y_2 y_3 y_4 y_8 & & \\
+ y_2 y_3 y_7 y_8 + y_2 y_6 y_7 y_8 + y_5 y_6 y_7 y_8 & = & 0 \\
y_3 y_4 y_6 y_7 + y_3 y_4 + y_6 y_7 & = & 0 \\
y_4 y_7 + y_4 + y_7 & = & 0 \\
y_2 y_3 y_4 y_5 y_6 y_7 y_8 - 1 & = & 0
\end{array}\right.$$

A solution is
$y_2 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$, $y_3 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$, $y_4 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$, $y_5 = 1$, $y_6 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$,
$y_7 = -\frac{1}{2} - \frac{\sqrt{3}I}{2}$, $y_8 = -\frac{1}{2} + \frac{\sqrt{3}I}{2}$, where $I = \sqrt{-1}$.

an exact representation of a two dimensional set

$$
\begin{aligned}
x_0 &= y_0 \\
x_1 &= y_0 y_1 \\
x_2 &= y_0^{-2} y_1^{-1} y_2 \\
x_3 &= y_0 y_3 \\
x_4 &= y_0 y_1 y_4 \\
x_5 &= y_0^{-2} y_1^{-1} y_5 \\
x_6 &= y_0 y_6 \\
x_7 &= y_0 y_1 y_7 \\
x_8 &= y_0^{-2} y_1^{-1} y_8
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= t_1 \\
y_1 &= t_2 \\
y_2 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
y_3 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2} \\
y_4 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2} \\
y_5 &= 1 \\
y_6 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
y_7 &= -\frac{1}{2} - \frac{\sqrt{3}I}{2} \\
y_8 &= -\frac{1}{2} + \frac{\sqrt{3}I}{2}
\end{aligned}
$$

$$
\begin{aligned}
x_0 &= t_1 \\
x_1 &= t_1 t_2 \\
x_2 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_3 &= t_1 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_4 &= t_1 t_2 \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right) \\
x_5 &= t_1^{-2} t_2^{-1} \\
x_6 &= t_1 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_7 &= t_1 t_2 \left(-\frac{1}{2} - \frac{\sqrt{3}I}{2}\right) \\
x_8 &= t_1^{-2} t_2^{-1} \left(-\frac{1}{2} + \frac{\sqrt{3}I}{2}\right)
\end{aligned}
$$

a tropical interpretation of Backelin's Lemma

Denoting by $u = e^{i2\pi/3}$ the primitive third root of unity, $u^3 - 1 = 0$:

$$\begin{array}{lll} x_0 = t_0 & x_3 = t_0 u & x_6 = t_0 u^2 \\ x_1 = t_0 t_1 & x_4 = t_0 t_1 u & x_7 = t_0 t_1 u^2 \\ x_2 = t_0^{-2} t_1^{-1} u^2 & x_5 = t_0^{-2} t_1^{-1} & x_8 = t_0^{-2} t_1^{-1} u. \end{array}$$

Introducing new variables $y_0 = t_0$, $y_1 = t_0 t_1$, and $y_2 = t_0^{-2} t_1^{-1} u^2$:

$$\begin{array}{lll} x_0 = y_0 & x_3 = y_0 u & x_6 = y_0 u^2 \\ x_1 = y_1 & x_4 = y_1 u & x_7 = y_1 u^2 \\ x_2 = y_2 & x_5 = y_2 u & x_8 = y_2 u^2 \end{array}$$

which modulo $y_0^3 y_1^3 y_2^3 u^9 - 1 = 0$ satisfies by plain substitution the cyclic 9-roots system, as in the proof of Backelin's Lemma, see J.C. Faugère. **Finding all the solutions of Cyclic 9 using Gröbner basis techniques.** In *Computer Mathematics - Proceedings of the Fifth Asian Symposium (ASCM 2001)*, pages 1–12. World Scientific, 2001.

degree computations

For $u^3 = 1$, our representation of the solution set is

$$\begin{array}{lll} x_0 = t_0 & x_3 = t_0 u & x_6 = t_0 u^2 \\ x_1 = t_0 t_1 & x_4 = t_0 t_1 u & x_7 = t_0 t_1 u^2 \\ x_2 = t_0^{-2} t_1^{-1} u^2 & x_5 = t_0^{-2} t_1^{-1} & x_8 = t_0^{-2} t_1^{-1} u. \end{array}$$

We compute the degree of the surface using two random hyperplanes:

$$\left\{ \begin{array}{rcl} \alpha_1 t_0 + \alpha_2 t_0 t_1 + \alpha_3 t_0^{-2} t_1^{-1} & = & 0 \\ \alpha_4 t_0 + \alpha_5 t_0 t_1 + \alpha_6 t_0^{-2} t_1^{-1} & = & 0, \quad \alpha_1, \alpha_2, \ldots, \alpha_6 \in \mathbb{C}. \end{array} \right.$$

Simplifying, the system becomes

$$\left\{ \begin{array}{rcl} t_0^{-2} t_1^{-1} - \beta_1 & = & 0 \\ t_1 - \beta_2 & = & 0, \quad \beta_1, \beta_2 \in \mathbb{C}. \end{array} \right.$$

There are 3 solutions, so we have a cubic surface of cyclic 9 roots.
Applying symmetry, we find an orbit of 6 cubic surfaces.

cyclic m^2-roots

Proposition

For $n = m^2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$\begin{aligned}
x_{km+0} &= u_k t_0 \\
x_{km+1} &= u_k t_0 t_1 \\
x_{km+2} &= u_k t_0 t_1 t_2 \\
&\vdots \\
x_{km+m-2} &= u_k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} &= u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{aligned}$$

for $k = 0, 1, 2, \ldots, m-1$ and $u_k = e^{i 2k\pi/m}$.

The degree of this solution set equals m.

conclusion

Promising results on the cyclic n-roots problem give a proof of concept for a new polyhedral method to compute algebraic sets.

Version 2.3.68 of PHCpack solves binomial systems with `phc -b`.

For the computation of pretropisms, we rely on

- `cddlib` on the Cayley embedding of the Newton polytopes, or
- `Gfan` for the tropical intersection.

To process the pretropisms, we

- use `Sage` to extract initial form systems
  and look for the second term in the Puiseux series;
- solve initial form systems with the blackbox solver of `PHCpack`.