Sampling Algebraic Sets in Local Intrinsic Coordinates

Yun Guan   Jan Verschelde

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
http://www.math.uic.edu/˜guan   http://www.math.uic.edu/˜jan
guan@math.uic.edu   jan@math.uic.edu

FoCM 2011 – Real Number Complexity Workshop
Budapest, Hungary, 4-6 July 2011

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates

2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots

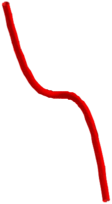
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

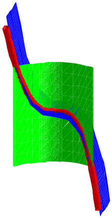
Outline

Representing a Space Curve

Consider the twisted cubic:

$$\begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \end{cases}$$





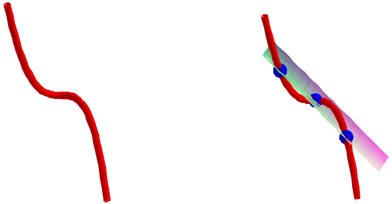
Important attributes are dimension and degree:

- dimension: cut with one random plane,
- degree: #points on the curve and in the plane.

Witness Set for a Space Curve

Consider the twisted cubic:

$$\begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \end{cases} \qquad \begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \\ c_0 + c_1 x + c_2 y + c_3 z = 0 \end{cases}$$



Intersect with a random plane $c_0 + c_1 x + c_2 y + c_3 z = 0$
$\rightarrow$ find three generic points on the curve.

Generic Points on Algebraic Sets

A polynomial system $f(\mathbf{x}) = \mathbf{0}$ defines an algebraic set $f^{-1}(\mathbf{0}) \subset \mathbb{C}^n$.

We assume

1. $f^{-1}(\mathbf{0})$ is pure dimensional, k is codimension; and moreover
2. $f(\mathbf{x}) = \mathbf{0}$ is a complete intersection, $k = $ #polynomials in f.

For example, consider all adjacent minors of a general 2-by-3 matrix:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \quad f(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \end{cases}$$

$n = 6$, $k = 2$: $\dim(f^{-1}(\mathbf{0})) = n - k = 4$.

To compute $\deg(f^{-1}(\mathbf{0}))$, add $n - k$ general linear equations $L(\mathbf{x}) = \mathbf{0}$ to $f(\mathbf{x}) = \mathbf{0}$ and solve $\{f(\mathbf{x}) = \mathbf{0}, L(\mathbf{x}) = \mathbf{0}\}$.

$\rightarrow$ 4 generic points for all adjacent minors of a general 2-by-3 matrix.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Intrinsic Coordinates save Work

Generic points for all adjacent minors of a general 2-by-3 matrix satisfy (for random coefficients $c_{ij} \in \mathbb{C}$):

$$\left. \begin{array}{r} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ c_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{21} + c_{15}x_{22} + c_{16}x_{23} = 0 \\ c_{20} + c_{21}x_{11} + c_{22}x_{12} + c_{23}x_{13} + c_{24}x_{21} + c_{25}x_{22} + c_{26}x_{23} = 0 \\ c_{30} + c_{31}x_{11} + c_{32}x_{12} + c_{33}x_{13} + c_{34}x_{21} + c_{35}x_{22} + c_{36}x_{23} = 0 \\ c_{40} + c_{41}x_{11} + c_{42}x_{12} + c_{43}x_{13} + c_{44}x_{21} + c_{45}x_{22} + c_{46}x_{23} = 0 \end{array} \right\}$$

$L^{-1}(\mathbf{0})$ is a 2-plane in $\mathbb{C}^6$, spanned by

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} + \xi_1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \end{bmatrix} + \xi_2 \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \\ v_{25} \\ v_{26} \end{bmatrix}$$

$\mathbf{b}$ is offset point
$\mathbf{v}_1, \mathbf{v}_2$ orthonormal basis

(ξ_1, ξ_2) intrinsic coordinates

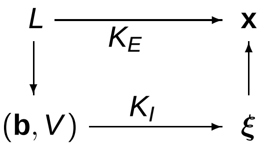
A Commutative Diagram

- $f(\mathbf{x}) = 0$ a system of k polynomials in n variables $\mathbf{x}$,
- $L(\mathbf{x}) = 0$ a system of $n - k$ general linear equations in $\mathbf{x}$,
- $\mathbf{b} \in \mathbb{C}^n$ is offset point, $V = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_k]$, $V^H V = I_k$.

Intrinsic coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_k)$ for $\mathbf{x}$:

$$\mathbf{x} = \mathbf{b} + \xi_1 \mathbf{v}_1 + \xi_2 \mathbf{v}_2 + \cdots + \xi_k \mathbf{v}_k = \mathbf{b} + V\boldsymbol{\xi}.$$

Use $f(\mathbf{x} = \mathbf{b} + V\boldsymbol{\xi}) = \mathbf{0}$ to compute generic points:



$$\frac{||\Delta \mathbf{x}||}{||\mathbf{x}||} \leq K_E \frac{||\Delta L||}{||L||}$$

$$\frac{||\Delta \boldsymbol{\xi}||}{||\boldsymbol{\xi}||} \leq K_I \frac{||\Delta(\mathbf{b}, V)||}{||(\mathbf{b}, V)||}$$

We observe worsening of the numerical conditioning: $K_I \gg K_E$.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Sampling in Intrinsic Coordinates

Represent L via $(\mathbf{b}, V)$ and use intrinsic coordinates $\boldsymbol{\xi} \in \mathbb{C}^k$.

Moving from $(\mathbf{b}, V)$ to $(\mathbf{c}, W)$, as t goes from 0 to 1, homotopy:

$$f\left(\mathbf{x} = \underbrace{(1-t)\mathbf{b} + t\mathbf{c}}_{\text{moving offset point}} + \underbrace{((1-t)V + tW)}_{\text{moving basis vectors}} \boldsymbol{\xi} \right) = \mathbf{0}.$$

Track paths $\boldsymbol{\xi}(t)$ via predictor-corrector methods.

Binomial expansion destroys sparse monomial structure of f.
For example, evaluate $x_1^{a_1} x_2^{a_2}$ at $x_1 = b_1 + \xi_1 v_1$ and $x_2 = b_2 + \xi_2 v_2$:

$$\left(\sum_{i=0}^{a_1} \binom{a_1}{i} b_1^i (\xi_1 v_1)^{a_1 - i} \right) \left(\sum_{j=0}^{a_2} \binom{a_2}{j} b_2^j (\xi_2 v_2)^{a_2 - j} \right).$$

In general: $f(\mathbf{b} + V(\boldsymbol{\xi} + \Delta\boldsymbol{\xi})) = f(\mathbf{b} + V\boldsymbol{\xi}) + \Delta f$, with very large $||\Delta f||$.

Local Intrinsic Coordinates

What if we could keep $||\boldsymbol{\xi}||$ small?

$$
\begin{aligned}
&(b_1 + \xi_1 v_1)^{a_1} (b_2 + \xi_2 v_2)^{a_2} \\
&= \left(b_1^{a_1} + a_1 b_1^{a_1 - 1} \xi_1 v_1 + O(\xi_1^2)\right) \left(b_2^{a_2} + a_2 b_2^{a_2 - 1} \xi_2 v_2 + O(\xi_2^2)\right) \\
&= b_1^{a_1} b_2^{a_2} + a_1 b_1^{a_1 - 1} b_2^{a_2} \xi_1 v_1 + a_2 b_1^{a_1} b_2^{a_2 - 1} \xi_2 v_2 + O(\xi_1^2, \xi_1 \xi_2, \xi_2^2)
\end{aligned}
$$

Now we have: $f(\mathbf{b} + V\boldsymbol{\xi}) = f(\mathbf{b}) + \Delta f$,

where $||\Delta f||$ is $O(||V\boldsymbol{\xi}||) = O(||\boldsymbol{\xi}||)$ as V is orthonormal basis.

Use extrinsic coordinates of generic point as offset point for k-plane: for $d = \deg(f^{-1}(\mathbf{0}))$ and d generic points $\{\mathbf{z}_1, \mathbf{z}_1, \ldots, \mathbf{z}_d\}$:

$$\mathbf{x} = \mathbf{z}_\ell + V\boldsymbol{\xi}, \quad \ell = 1, 2, \ldots, d.$$

The local intrinsic coordinates are defined by $(\{\mathbf{z}_1, \mathbf{z}_1, \ldots, \mathbf{z}_d\}, V)$.

Origin and Assumptions

The problem has its origin in the implementation of
an intrinsic homotopy for intersecting algebraic varieties
J. Complexity 21(4):593-608, 2005 (with Sommese & Wampler).

Intrinsic coordinates were introduced to mitigate the doubling of the number of variables in the diagonal homotopy.

Assumptions:

- *no* rewriting of the equations for $f(\mathbf{x}) = \mathbf{0}$;
- the algebraic set we sample is reduced;
- coefficients and solutions are well scaled; and
- our working precision remains fixed.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates

2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots

3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

a Maple experiment

Via the companion matrix of a polynomial,
we relate the numerical conditioning of a root to that of an eigenvalue.

We use `LinearAlgebra[EigenConditionNumbers]` of Maple 12, with default settings of the `balance` parameter, and `UseHardwareFloats` set to `true`.

We consider one sparse polynomial f (5 terms) in $n = 10$ variables, of increasing degrees d, with coefficients on the complex unit circle.

```
[> n := 10: d := 10: t := 5:
[> c := () -> exp(I*stats[random,uniform[0,2*Pi]](1)):
[> X := [seq(x[i],i=1..n)]:
[> f := X[1]^d + randpoly(X,coeffs=c,degree=d-1,terms=5)
       + sum(c()*x[i],i=1..n);
```

Influence of Offset Point

Consider intrinsic coordinates once with and once without offset $\mathbf{b}$:

$$\mathbf{x} = \mathbf{b} + \mathbf{v}\xi \quad \text{and} \quad \mathbf{x} = \mathbf{v}\xi, \quad \mathbf{b}, \mathbf{v} \in \mathbb{C}^n,$$

With $f(\mathbf{v}\xi) = 0$ all coefficients are on the complex unit circle.
With $f(\mathbf{b} + \mathbf{v}\xi) = 0$, the offset $\mathbf{b}$ causes the variation in the condition numbers. The table displays *inverse* condition numbers:

d	$f(\mathbf{b} + \mathbf{v}\xi) = 0$ largest	smallest	$f(\mathbf{v}\xi) = 0$ largest	smallest	ratios of smallest	ratios of largest
10	5.9e-01	9.0e-02	8.8e-01	4.0e-01	6.6e+00	2.2e+00
20	2.8e-01	1.8e-03	8.9e-01	3.3e-01	1.6e+02	2.7e+00
30	2.8e-01	6.2e-05	9.5e-01	7.3e-02	4.5e+03	1.3e+01
40	4.5e-01	7.1e-06	9.7e-01	1.9e-01	6.3e+04	5.8e+00

The conditioning for $f(\mathbf{b} + \mathbf{v}\xi) = 0$ worsens for increasing degree d, whereas for $f(\mathbf{v}\xi) = 0$, all roots of $f(\mathbf{v}\xi) = 0$ are well conditioned.

Global versus Local

To compare the conditioning of global intrinsic with local intrinsic coordinates, we first solve $f(\mathbf{b} + \mathbf{v}\xi) = 0$ and take one root, say $\xi = z$.

Then let $\mathbf{b}_z = \mathbf{b} + \mathbf{v}z$ so $f(\mathbf{b}_z + \mathbf{v}\xi) = 0$ has one solution $\xi = 0$ corresonding to z.

d	$f(\mathbf{b} + \mathbf{v}\xi) = 0$			$f(\mathbf{b}_z + \mathbf{v}\xi) = 0$		
	largest	2nd largest	smallest	largest	2nd largest	smallest
10	5.9e-01	4.7e-01	6.2e-02	1.0e+00	2.8e-03	2.0e-06
20	4.0e-01	3.3e-01	6.7e-03	1.0e+00	9.9e-06	7.0e-11
30	2.5e-01	1.1e-01	8.1e-04	1.0e+00	4.0e-08	3.4e-11
40	5.6e-01	2.4e-01	1.4e-04	1.0e+00	1.5e-08	3.9e-11

For growing degree d, the condition of z of $f(\mathbf{b}_z + \mathbf{v}\xi) = 0$ is `1.0e+00`, while the condition of other roots worsens.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Numerical Polynomial Evaluation

Definition (Demmel 1997, Applied Linear Algebra)

The *relative condition number* to evaluate a polynomial p of degree d in one variable x with complex coefficients is

$$\mathrm{cond}(p,x) = \frac{\displaystyle\sum_{i=0}^{d} |c_i x^i|}{|p(x)|} \quad \text{for} \quad p(x) = \sum_{i=0}^{d} c_i x^i \quad \text{with} \quad c_i \in \mathbb{C}.$$

Observe:

- At $p(x) = 0$: $\mathrm{cond}(p,x) = \infty$, an ill-posed problem.
- For bounded $\mathrm{cond}(p,x)$, we evaluate at x: $|x| \approx 1$.

Global versus Local

We compare evaluating a polynomial p

1. at $x = b + v\xi$, for random $b, v \in \mathbb{C}$, $|b| = 1$, $|v| = 1$; and
2. at $x = z + vh$, with $v \in \mathbb{C}$ as above and h: $0 < |h| \ll 1$.

With $0 < |h| \ll 1$, we neglect $O(h^2)$ terms.
The equation $b + v\xi = z + vh$ defines the relation between ξ and z.

Lemma (monomial evaluation)

For $d > 1$, $|b| = 1$, $|v| = 1$, $|z| = 1$, and $0 < |h| \ll 1$, the ratio

$$\frac{\mathrm{cond}(x^d, x = b + v\xi)}{\mathrm{cond}(x^d, x = z + vh)} \leq \frac{3^d}{1 - O(h)}$$

compares the condition of evaluating x^d as a polynomial in ξ to x^d as a polynomial in h.

Proof Idea: apply binomial expansion.

Polynomials in one Variable

Proposition

Let $p = \sum_{i=0}^{d} c_i x^i$. *For* $|b| = 1$, $|v| = 1$, $|z| = 1$, $|p(z)| \gg |h|$,

and $0 < |h| \ll 1$: $\dfrac{\mathrm{cond}(p, x = b + v\xi)}{\mathrm{cond}(p, x = z + vh)} \leq \dfrac{\sum_{i=0}^{d} |c_i| 3^i}{|p(z)| - O(h)}$.

Proof Idea: apply triangle inequalities.

Corollary

For $|c_i| = 1$ *in* p, *the ratio* $\dfrac{\mathrm{cond}(p, x = b + v\xi)}{\mathrm{cond}(p, x = z + vh)} \leq \dfrac{1}{2} \dfrac{3^{d+1} - 1}{|p(z)| - O(h)}$

compares the condition of evaluating p *as a polynomial in* ξ *to* p *as a polynomial in* h.

Polynomials in several Variables

Definition

The *relative condition number* to evaluate a sparse polynomial f in n variables, with support set $A \in \mathbb{N}^n$, $\#A < \infty$, is

$$\mathrm{cond}(f, \mathbf{x}) = \frac{\displaystyle\sum_{\mathbf{a} \in A} |c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}|}{|f(\mathbf{x})|},$$

for

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \ \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The degree of f is

$$\deg(f) := \max_{\mathbf{a} \in A} (a_1 + a_2 + \cdots + a_n).$$

Global versus Local

We compare evaluating a sparse polynomial f at $\mathbf{x} = \mathbf{b} + \mathbf{v}\xi$ to evaluating f at $\mathbf{x} = \mathbf{z} + \mathbf{v}h$.

Theorem

Let $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$. *For* $|b_i| = 1$, $|v_i| = 1$, $|z_i| = 1$, $i = 1, 2, \ldots, n$, $|f(\mathbf{z})| \gg |h|$, *and* $0 < |h| \ll 1$:

$$\frac{\mathrm{cond}(f, \mathbf{x} = \mathbf{b} + \mathbf{v}\xi)}{\mathrm{cond}(f, \mathbf{x} = \mathbf{z} + \mathbf{v}h)} \leq \frac{\displaystyle\sum_{\mathbf{a} \in A} |c_{\mathbf{a}}| 3^{a_1 + a_2 + \cdots + a_n}}{|f(\mathbf{z})| - O(h)}.$$

Proof Idea: apply binomial expansion and triangle inequalities.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Roots as Eigenvalues

We define the condition number of roots of a polynomial in one variable via the condition numbers of the eigenvalues of the companion matrix.

Definition (Tyrtyshnikov 1997, numerical analysis textbook)

Let C_p be the companion matrix of a polynomial p in one variable x and with complex coefficients.

Solutions to $p(x) = 0$ are eigenvalues denoted by z with corresponding right eigenvectors $\mathbf{r} \in \mathbb{C}^n$: $C_p \mathbf{r} = z\mathbf{r}$ and left eigenvectors $\mathbf{q} \in \mathbb{C}^n$: $\mathbf{q}^H C_p = \mathbf{q}^H z$.

The *condition number* $\kappa(p, z)$ *of a zero* z *of* p with corresponding left and right eigenvectors $\mathbf{q}_z$ and $\mathbf{r}_z$ is

$$\kappa(p, z) = \frac{||\mathbf{q}_z||_2 ||\mathbf{r}_z||_2}{|\mathbf{q}_z^H \mathbf{r}_z|}.$$

Roots of Unity

We consider polynomials with perfectly conditioned roots.

Lemma

Consider $p = x^d - 1$*. For all* z*,* $p(z) = 0$*, we have* $\kappa(p, z) = 1$*.*

Proof Idea: eigenvectors are powers of a root.

Notes:

- With eigenvalues we ignore the sparsity of p.
- Distances between the roots decrease as d increases.
- Sparse condition numbers are ϵ/d for perturbations ϵ.
  See Questions of numerical condition related to polynomials [Gautschi 1984].

Perturbed Roots of Unity

Lemma

Let $v \in \mathbb{C}$, $|v| = 1$, and h, $0 < |h| \ll 1$ consider $p = (x + vh)^d - 1$. For all z, $p(z) = 0$ we have $\kappa(p, z) = 1 + O(h)$.

Proof Idea: view companion matrix of p as $C_p(h) = C_p + C_1 h + O(h^2)$.

Consider $p(x) = (b + vx)^d - 1 = 0$
for constants b and v on the complex unit circle.

Our notion of numerical conditioning is algebraic, not geometric.

In the geometric point of view, the roots of p compared to those of $x^d - 1$ are merely translated.

As this translation preserves the distance between the roots one would not expect a worsening of the condition number.

Shifted Roots of Unity

Lemma

Let $b, v \in \mathbb{C}$, $|b| = 1$, $|v| = 1$, *and consider* $p = (b + xv)^d - 1$.
For all z, $p(z) = 0$ *we have* $\kappa(p, z) \leq d\sqrt{\frac{4^d\Gamma(d+1/2)}{\sqrt{\pi}\Gamma(d+1)}}$.

Proof Idea: apply the theorem of Bauer-Fike and Maple 12 to bound a spectral radius via $\displaystyle\sum_{i=0}^{d} \left(\begin{array}{c} d \\ i \end{array} \right)^2 = \frac{4^d\Gamma(d+1/2)}{\sqrt{\pi}\Gamma(d+1)}$.

Because $\log_2\left(\sqrt{\frac{4^d\Gamma(d+1/2)}{\sqrt{\pi}\Gamma(d+1)}}\right)$ increases fairly linearly and is bounded by $d - 1$, we replace $\sqrt{\frac{4^d\Gamma(d+1/2)}{\sqrt{\pi}\Gamma(d+1)}}$ by 2^{d-1}.

Comparing Condition Numbers

The lemmas imply

Theorem

Let $b, v \in \mathbb{C}$, $|b| = 1$, $|v| = 1$, and h such that $0 < |h| \ll 1$. Then, the ratio

$$\frac{\kappa\left((b + vx)^d - 1, z\right)}{\kappa\left((x + vh)^d - 1, z\right)} \leq 2^{d-1} - O(h)$$

compares the conditioning of the solutions of $(b + vx)^d - 1 = 0$ with the solutions of $(x + vh)^d - 1 = 0$.

The upper bound of the theorem is attained for the case of $(-1 + x)^d - 1 = 0$ where 2 is a solution and powers of 2 appear in the companion matrix.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

an isolated Root of a Polynomial System

Definition (Rheinboldt, 1976)

Let $f(\mathbf{x}) = \mathbf{0}$ be a polynomial system of n equations in n unknowns. Denote the Jacobian matrix of f by J_f and let $\mathbf{z} \in \mathbb{C}^n$ be an isolated solution. Then, *the relative condition number of the zero* $\mathbf{z}$ *as a solution of* $f(\mathbf{x}) = \mathbf{0}$ is

$$\kappa(f, \mathbf{z}) = ||J_f(\mathbf{z})||_2 ||J_f^{-1}(\mathbf{z})||_2,$$

i.e.: $\kappa(f, \mathbf{z})$ is the condition number of the Jacobian matrix of the polynomials in the system evaluated at $\mathbf{z}$.

Notes:

- In Newton's method we solve $J_f(\mathbf{x})\Delta\mathbf{x} = -f(\mathbf{x})$.
- We have $||C||_2 = \sqrt{\rho(C^H C)}$ where $\rho(\cdot)$ is the spectral radius. For univariate f, we use the companion matrix for C.
- $\kappa(f, \mathbf{z})$ is local: for one solution $\mathbf{z}$ and particular: it depends on the coefficients of f, determined by a coordinate system.

Numerical Condition of generic Points

Definition (extrinsic, intrinsic, local condition number)

Let $\mathbf{z} \in \mathbb{C}^n$ be a generic point on an $(n-k)$-dimensional component of $f^{-1}(\mathbf{0})$, satisfying k linear equations $L(\mathbf{z}) = \mathbf{0}$. Then, *the relative extrinsic condition number of* $\mathbf{z}$*, as a generic point on* $f^{-1}(\mathbf{0}) \cap L$ is

$$\kappa_{\mathcal{E}}(f, L, \mathbf{z}) = \kappa(f = (f, L), \mathbf{z}).$$

Writing the solutions to the linear equations $L(\mathbf{x}) = \mathbf{0}$ as $\mathbf{x} = \mathbf{b} + V\xi$, for some offset point $\mathbf{b}$ and orthonormal matrix $V \in \mathbb{C}^{n \times k}$, we have $\mathbf{z} = \mathbf{b} + V\xi_z$, where ξ_z are the intrinsic coordinates of $\mathbf{z}$. Then, *the relative intrinsic condition number of* $\mathbf{z}$*, as a generic point of* $f^{-1}(\mathbf{0})$ is

$$\kappa_{\mathcal{I}}(f, \mathbf{b}, V, \mathbf{z}) = \kappa(f = f(\mathbf{b} + V\xi_{\mathbf{z}}), \xi_{\mathbf{z}}).$$

The relative local intrinsic condition number of $\mathbf{z}$ *as a generic point on* $f^{-1}(\mathbf{0})$ is

$$\kappa_{\mathcal{L}}(f, V, \mathbf{z}) = \kappa(f = f(\mathbf{z} + V\xi), \xi = \mathbf{0}).$$

the Test Equation in extrinsic Coordinates

Similar to $x^d - 1 = 0$, we consider the multivariable version as test equation $\mathbf{x}^\mathbf{a} - 1 = 0$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$.

Lemma

Let $f = \mathbf{x}^\mathbf{a} - 1 = \mathbf{0}$, $\mathbf{x}^\mathbf{a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, denote $d = a_1 + a_2 + \cdots + a_n$.

*There is a choice for the coefficients of L defining a generic point **z**, $f(\mathbf{z}) = 0$, $L(\mathbf{z}) = \mathbf{0}$ so $\kappa_{\mathcal{E}}(f, L, \mathbf{z}) \leq d^2$.*

Our proof considers the Jacobian matrix of $f(\mathbf{x}) = \mathbf{0}$ with the coefficients of $L(\mathbf{x}) = \mathbf{0}$ as indeterminates.

Note that **z** is *not* considered as given (and thus fixed), because otherwise we could still obtain a badly scaled Jacobian matrix.

the Test Equation in intrinsic Coordinates

We consider the condition of intrinsic coordinates of our test equation.

Lemma

Let $\mathbf{z} \in \mathbb{C}^n$ *be a generic point of* $f(\mathbf{x}) = \mathbf{x}^{\mathbf{a}} - 1 = 0$, $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, $d = a_1 + a_2 + \cdots + a_n$.
Let $\mathbf{z} = \mathbf{b} + \mathbf{v}\xi_{\mathbf{z}}$ *for some offset point* $\mathbf{b}$ *and a vector* $\mathbf{v}$.
Then $\kappa_{\mathcal{I}}(f, \mathbf{b}, \mathbf{v}, \xi_{\mathbf{z}}) \leq 2^{d-1}$.

Apply repeated substitution to reduced to the univariate case and use 2^{d-1} for the expression $\sqrt{\frac{4^d \Gamma(d+1/2)}{\sqrt{\pi}\Gamma(d+1)}}$.

The bound is pessimistic but is attained in bad cases.

the Test Equation in local intrinsic Coordinates

Lemma

Consider $\mathbf{x} = \mathbf{z} + \mathbf{v}\xi$ *for some vector* $\mathbf{v}$, $||\mathbf{v}||_2 = 1$, *and* $\mathbf{z} \in \mathbb{C}^n$ *a generic point for* $f(\mathbf{x}) = \mathbf{x}^{\mathbf{a}} - 1$. *Then,* $\kappa_{\mathcal{L}}(f, \mathbf{v}, \mathbf{z}) = 1$.

To summarize:

Theorem

For a generic point $\mathbf{z}$ *for the equation* $f(\mathbf{x}) = \mathbf{x}^{\mathbf{a}} - 1 = 0$, *with* $d = \deg(f)$, *we have:*

$$\kappa_{\mathcal{L}}(f, \mathbf{v}, \mathbf{z}) \leq \kappa_{\mathcal{E}}(f, L, \mathbf{z}) \leq \kappa_{\mathcal{I}}(f, \mathbf{b}, \mathbf{v}, \mathbf{z}) \leq 2^{d-1},$$

where $\mathbf{z}$ *lies on some generic line with offset* $\mathbf{b}$, *direction* $\mathbf{v}$, *and linear equations* $L(\mathbf{x}) = 0$.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Sampling in Local Intrinsic Coordinates

Generic points $\{\mathbf{z}_1, \mathbf{z}_1, \ldots, \mathbf{z}_d\}$ are offset points for k-plane L with directions in the orthonormal matrix V.

Moving from $(\mathbf{z}_\ell, V)$ to $(\mathbf{b}, W)$, as t goes from 0 to 1, homotopy:

$$f(\mathbf{x} = (1-t)\mathbf{z}_\ell + t\mathbf{b} + W\xi) = \mathbf{0}$$

$\rightarrow$ *only the offset point moves!*

Instead of moving to $\mathbf{b}$, let $\mathbf{c}$ be the orthogonal projection of $\mathbf{z}_\ell$ onto the k-plane L.

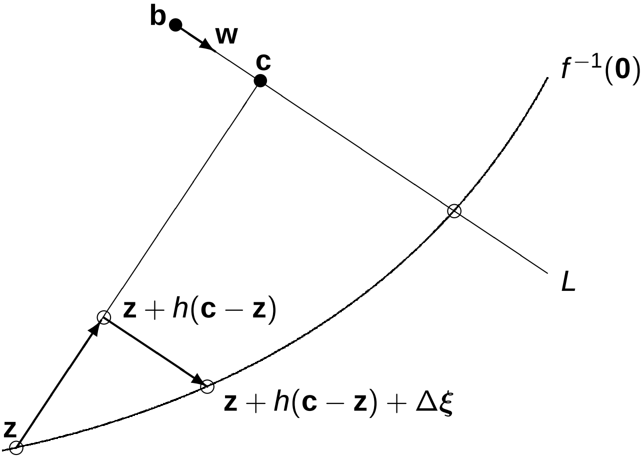
For some step size h, consider:

$$f(\mathbf{x} = \mathbf{z}_\ell + h(\mathbf{c} - \mathbf{z}_\ell) + W\xi) = \mathbf{0}$$

and apply Newton's method to find the correction $\Delta\xi$.

Schematic of the new Sampling Algorithm

one predictor-corrector step



pseudocode for one predictor-corrector step

Input: $\mathbf{b} \in \mathbb{C}^n$, $W = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_k] \in \mathbb{C}^{n \times k}$, $W^* W = I_k$
$\mathbf{z} \in \mathbb{C}^n$, $f(\mathbf{z}) = \mathbf{0}$, $K(\mathbf{z}) = \mathbf{0}$, $h > 0$, $\epsilon > 0$, some L.

Output: $\widehat{\mathbf{z}}$, $f(\widehat{\mathbf{z}}) = \mathbf{0}$: $\widehat{\mathbf{z}}$ closer to L.

$$\mathbf{v} := \mathbf{z} - \mathbf{b}; \qquad \mathbf{v} := \mathbf{v} - \sum_{i=1}^{k} (\overline{\mathbf{w}_i}^T \mathbf{v})\mathbf{w}_i; \qquad \mathbf{v} := \mathbf{v}/||\mathbf{v}||;$$

$$\widetilde{\mathbf{z}} := \mathbf{z} + h\,\mathbf{v}; \qquad \widehat{\mathbf{z}} := \widetilde{\mathbf{z}}; \qquad \xi := \mathbf{0};$$

while $||f(\widehat{\mathbf{z}} + W\xi)|| > \epsilon$ do
$\Delta\xi := f(\widehat{\mathbf{z}} + W\xi)/f'(\widehat{\mathbf{z}} + W\xi)$;
$\xi := \xi + \Delta\xi$.

Numerical Stability

For some step size h, we evaluate

$$f(\mathbf{x} = \mathbf{z}_\ell + h(\mathbf{c} - \mathbf{z}_\ell)) = f(\mathbf{z}_\ell) + O(h) = O(h).$$

If step size h is too large, then Newton is unlikely to converge.

If step size h is too large, then $f(\mathbf{x} = \mathbf{z}_\ell + h(\mathbf{c} - \mathbf{z}_\ell)) \gg h$.

If $f(\mathbf{x} = \mathbf{z}_\ell + h(\mathbf{c} - \mathbf{z}_\ell)) \gg h$, then reduce h immediately.

Do not wait for (costly) Newton corrector to fail.

We can control size of residual $||f(\xi)||$ *to be always* $O(h)$.

Outline

1. Generic Points on Algebraic Sets
   - numerical representation of an algebraic set
   - intrinsic coordinates save work
   - sampling in intrinsic coordinates
2. Evaluation and Root Finding
   - condition number estimates
   - the numerical condition of polynomial evaluation
   - the numerical condition of polynomial roots
3. Improving the Numerical Conditioning
   - extrinsic, intrinsic, and local condition numbers
   - a recentering algorithm and the numerical stability
   - computational results on benchmark systems

Implementation and Benchmark Systems

Available since version 2.3.53 of PHCpack

Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, 25(2):251–276, 1999.

`http://www.math.uic.edu/~jan/download.html`

Three classes, families of systems:

1. all adjacent minors of a general 2-by-n matrix, $n = 3, 4, \ldots, 13$
2. cyclic n-roots, $n = 4, 8, 9$ (an academic benchmark)
3. Griffis-Duffy platforms and other systems from mechanical design

Computational experimental setup:

- given one set of generic points, generate another random k-plane
- move the given set of generic points to the new random k-plane
- check results for accuracy, #predictor-corrector steps, timings

Computational Results

Characteristics of three families of polynomial systems:

	polynomial system	n	$n-k$	d
1	Griffis-Duffy platform	8	1	40
2	cyclic 8-roots system	8	1	144
3	all adjacent minors of 2-by-11 matrix	22	12	1,024

n: number of variables, k: codimension, d: degree

Sampling in global intrinsic/local intrinsic coordinates:

system	#iterations	timings
1	207/164	550/535 μsec
2	319/174	5.3/3.2 sec
3	285/219	44.6/40.3 sec

Done on a Mac OS X 3.2 Ghz Intel Xeon, using 1 core.

Conclusions

Advantages of using local intrinsic coordinates:

- only offset point moves during sampling
- keep sparse structure of the polynomials
- control step size by evaluation

Applications to numerical algebraic geometry:

- implicitization via interpolation
- monodromy breakup algorithm
- diagonal homotopies to intersect solution sets