Overdetermined Polynomial Homotopies

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polynomial homotopy continuation

$f(\mathbf{x}) = \mathbf{0}$ is a polynomial system we want to solve,
$g(\mathbf{x}) = \mathbf{0}$ is a start system (g is similar to f) with known solutions.

A homotopy $h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t\,f(\mathbf{x}) = \mathbf{0}$, $t \in [0, 1]$,
to solve $f(\mathbf{x}) = \mathbf{0}$ defines solution paths $\mathbf{x}(t)$: $h(\mathbf{x}(t), t) \equiv \mathbf{0}$.

Numerical continuation methods track the paths $\mathbf{x}(t)$, from $t = 0$ to 1.

Predictor-corrector method operate in two stages:

1. The predictor sets the new value for t and predicts $\mathbf{x}(t)$.
2. The corrector applies Newton's method to $h(\mathbf{x}, t) = \mathbf{0}$.

problem statement

Current predictor methods apply higher-order extrapolation,

- which may cause path crossing: the predicted point lies so close to another path that it gets corrected to that other path;
- which may not be sufficient to reach convergence in the corrector.

Our solution: apply Newton's method on truncated power series.

numerical analysis and symbolic computation

- E. L. Allgower and K. Georg: Introduction to Numerical Continuation Methods. Volume 45 of *Classics in Applied Mathematics*, SIAM, 2003.
- A. Morgan: Solving polynomial systems using continuation for engineering and scientific problems. Volume 57 of *Classics in Applied Mathematics*, SIAM, 2009.

Newton-Hensel iteration is discussed in the following:

- J. Heintz, T. Krick, S. Puddu, J. Sabia, and A. Waissbein: Deformation techniques for efficient polynomial equation solving. *Journal of Complexity* 16(1):70-109, 2000.
- D. Castro, L.M. Pardo, K. Hägele, and J.E. Morais, Kronecker's and Newton's Approaches to Solving: A First Comparison. *Journal of Complexity* 17(1):212-303 2001.
- A. Bompadre, G. Matera, R. Wachenchauzer, and A. Waissbein: Polynomial equation solving by lifting procedures for ramified fibers. *Theoretical Computer Science* 315(2-3):335-369, 2004.

truncated power series

A series $s(t)$ in t with coefficients $s_k \in \mathbb{C}$:

$$s(t) = s_0 + s_1 t + s_2 t^2 + \cdots + s_n t^n + O(t^{n+1}),$$

is truncated to a polynomial of degree n, after dropping $O(t^{n+1})$.

The inverse $x(t)$ of $s(t)$ is defined via $x(t) \times s(t) = 1 + O(t^{n+1})$.

The coefficients x_k of the inverse $x(t)$ are computed as

$$\begin{array}{rcl} x_0 & = & 1/s_0 \\ x_1 & = & -(s_1 x_0)/s_0 \\ x_2 & = & -(s_1 x_1 + s_2 x_0)/s_0 \\ & \vdots & \\ x_n & = & -(s_1 x_{n-1} + s_2 x_{n-2} + \cdots + s_n x_0)/s_0 \end{array}$$

Newton's method on truncated power series

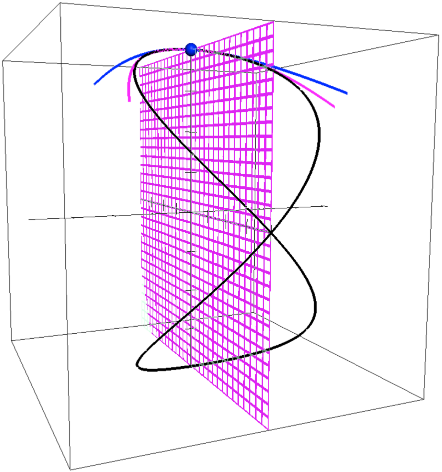
Given $c = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$, compute $\sqrt{c}$.

Apply Newton's method on the equation $x^2 - c = 0$,
doubling the degrees of the truncated power series in each step:

$$
\begin{array}{l}
x := \sqrt{c_0} + x_1 t \\
c := c_0 + c_1 t \\
k := 1 \\
\text{while } (k \leq n) \text{ do} \\
\quad \Delta x := (x^2 - c)/(2x) \\
\quad x := x - \Delta x \\
\quad x := x + x_{k+1} t^{k+1} + \cdots + x_{2k} t^{2k} \\
\quad c := x + c_{k+1} t^{k+1} + \cdots + c_{2k} t^{2k} \\
\quad k := 2 \times k
\end{array}
$$

Quadratic convergence: the order of Δx doubles in each step.

the Viviani curve



computing a power series solution

$$h(x(t), y(t), z(t), t) = \left\{ \begin{array}{r} (1-t)y + t(y-1) = 0 \\ x^2 + y^2 + z^2 - 4 = 0 \\ (x-1)^2 + y^2 - 1 = 0 \end{array} \right.$$

After 3 steps with Newton's method:

$$\begin{array}{rcl} y & = & t \\ x & = & 0.5t^2 \\ z & = & 2 - 0.25t^2 \end{array}$$

After 4 steps with Newton's method:

$$\begin{array}{rcl} y & = & t \\ x & = & 0.5t^2 + 0.125t^4 + 0.0625t^6 + 0.03125t^8 \\ z & = & 2 - 0.25t^2 - 0.078125t^4 - 0.041015625t^6 - 0.020751953125t^8 \end{array}$$

Gauss-Newton on truncated power series

Orthogonality is defined via an inner product on vectors:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n, \quad ||\mathbf{u}||_2^2 = \langle \mathbf{u}, \mathbf{u} \rangle.$$

To make a vector $\mathbf{x}$ parallel to $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$:

$$\mathbf{v} = \mathbf{x} + ||\mathbf{x}||_2 \mathbf{e}_1, \quad H\mathbf{x} = \mathbf{x} - \frac{2\langle \mathbf{v}, \mathbf{x} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v},$$

where H is a Householder transformation.

To transform a matrix A into an upper triangular matrix R, apply a sequence of Householder transformations:

$$H_n H_{n-1} \cdots H_1 A = R, \quad Q = H_1 \cdots H_{n-1} H_n, \quad A = QR.$$

This is well defined for matrices of truncated power series.

biunimodular vectors and cyclic n-roots

$$
\left\{
\begin{array}{c}
x_0 + x_1 + \cdots + x_{n-1} = 0 \\
\displaystyle i = 2, 3, 4, \ldots, n-1 : \sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\
x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0.
\end{array}
\right.
$$

The system arises in the study of biunimodular vectors.
A vector $\mathbf{u} \in \mathbb{C}^n$ of a unitary matrix A is biunimodular if for $k = 1, 2, \ldots, n$: $|u_k| = 1$ and $|v_k| = 1$ for $\mathbf{v} = A\mathbf{u}$.

- J. Backelin: *Square multiples n give infinitely many cyclic n-roots.* Technical Report, 1989.
- H. Führ and Z. Rzeszotnik. On biunimodular vectors for unitary matrices. *Linear Algebra and its Applications* 484:86–129, 2015.

series developments for cyclic 8-roots

Cyclic 8-roots has solution curves not reported by Backelin.

With Danko Adrovic (ISSAC 2012, CASC 2013): a tropism is $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$, the leading exponents of the series.

The corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ is

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} x_0 &= z_0 \\ x_1 &= z_1 z_0^{-1} \\ x_2 &= z_2 \\ x_3 &= z_3 z_0 \\ x_4 &= z_4 \\ x_5 &= z_5 \\ x_6 &= z_6 z_0^{-1} \\ x_7 &= z_7. \end{aligned}$$

Solving $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ gives the leading term of the series.

version 2.4.21 of PHCpack and 0.5.0 of phcpy

The source code (GNU GPL License) is availale at `github`.

After 2 Newton steps with `phc -u`, the series for z_1:

```
(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 2.37676980513323E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

After 3 Newton steps with `phc -u`, the series for z_1:

```
( 7.12500000000000E+00 + 7.12500000000000E+00*i)*z0^4
+(-1.52745512076048E-16 - 4.25000000000000E+00*i)*z0^3
+(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 1.45255178343636E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

Gauss-Newton power series predictor

To correct a solution, apply Gauss-Newton in complex arithmetic, on vectors of complex numbers.

The predictor is symbolic-numeric:

- Gauss-Newton on truncated power series: $\mathbf{x}(t)$, where $\mathbf{x}(t)$ is a vector of series, each series is of degree n.
- Step control via evaluation of the series, $\mathbf{y}(t) = f(\mathbf{x}(t))$. Let k be the order of $\mathbf{y}(t)$, $k < n$.

Let $\epsilon > 0$ be the tolerance on the residual $||f(\mathbf{x}(t))||$.

To compute the step size τ, solve $\epsilon = |y_k|\tau^k$:

$$\tau = \left(\frac{\epsilon}{|y_k|} \right)^{1/k}.$$

one method to predict *and* correct

Polynomials in the homotopy, with support A, have the form

$$h(\mathbf{x}, t) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}}(t)\mathbf{x}^{\mathbf{a}}, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

where the coefficients $c_{\mathbf{a}}(t)$ are truncated power series.

Three stages in one step with the path tracker:

1. Given a tolerance $\epsilon > 0$, set the step size τ: $||h(\mathbf{x}(\tau), \tau)||$ is $O(\epsilon)$.
2. Shift the coefficient series $c_{\mathbf{a}}(t)$ into $c_{\mathbf{a}}(t - \tau)$ in the homotopy.
3. Correct the solution with Newton's method on series of degree 0. Continue with truncated power series of increasing degrees to compute solution series $\mathbf{x}(t)$, accurate up to a prescribed order.

conclusions

Solving polynomial systems with power series is inspired by tropical algebraic geometry. The leading exponents of series are *tropisms*.

Predicting the solution on a path defined by a homotopy with Gauss-Newton on truncated power series is promising.

One future research direction:

- shared memory parallel implementations,
- acceleration with Graphics Processing Units (GPUs),
- quality up: compensate extra cost with parallel computations.