Solving Polynomial Systems using Numerical Homotopies

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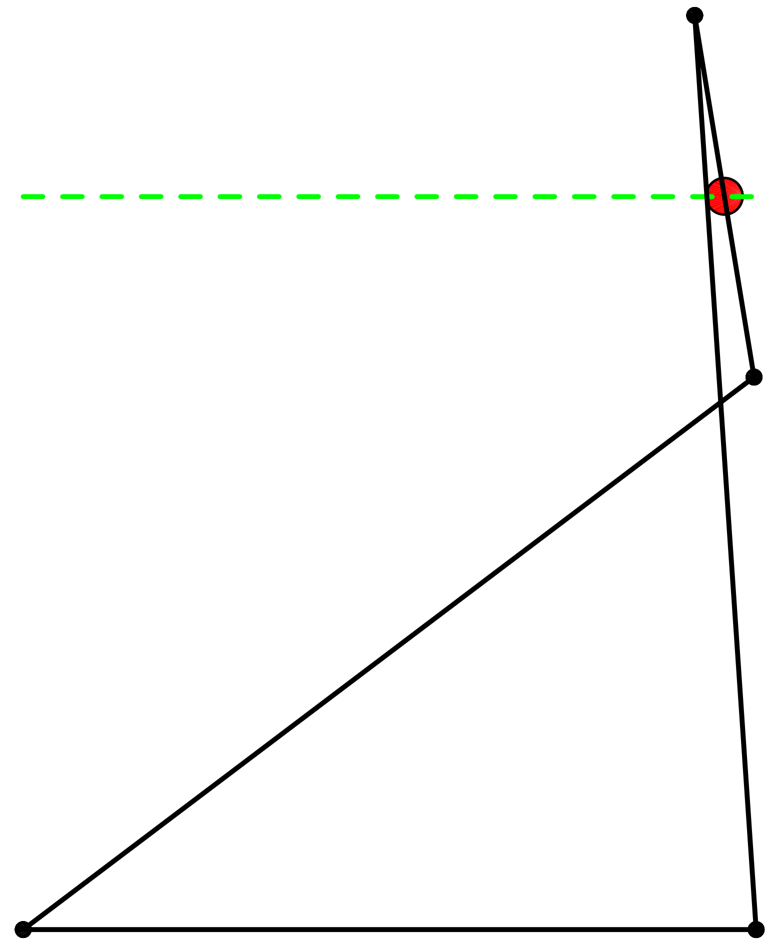
Outline

- Polynomial systems arise in science and engineering
  — for this talk: mainly mechanism design.

- Numerical homotopy continuation algorithms
  are pleasingly parallel.

- Newton with deflation to recondition isolated singularities.

Chebyshev's straight line mechanism (1875)



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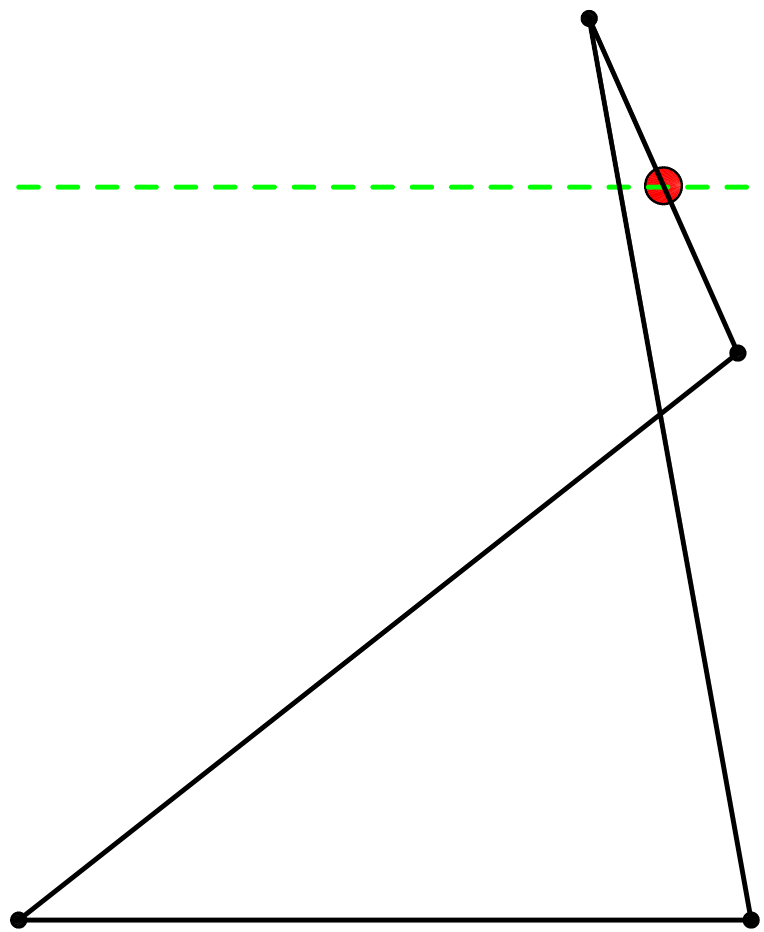

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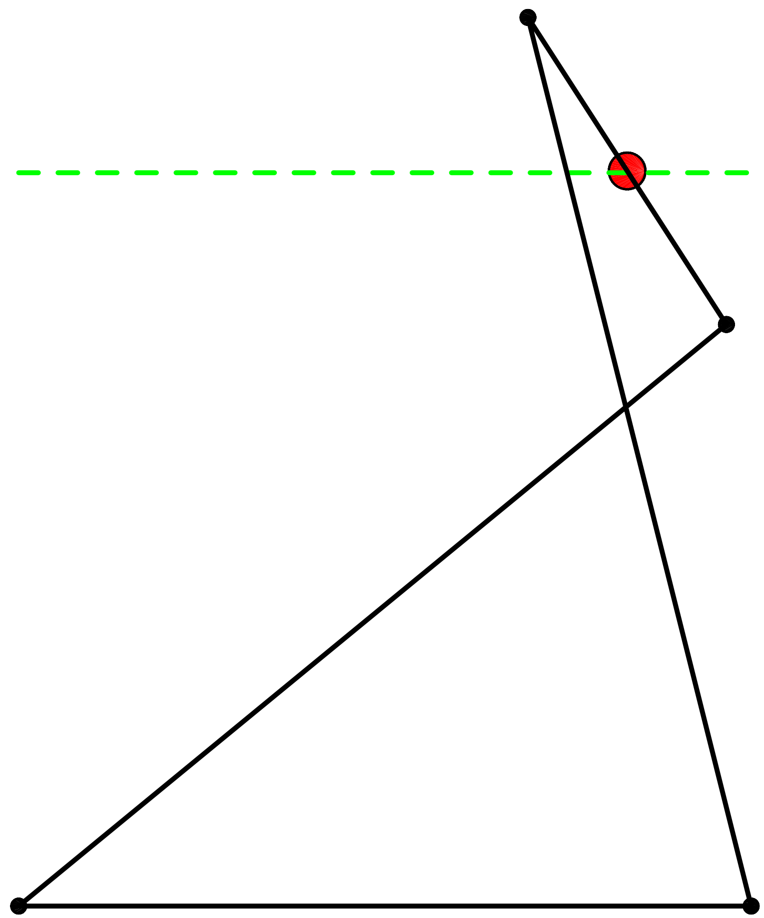

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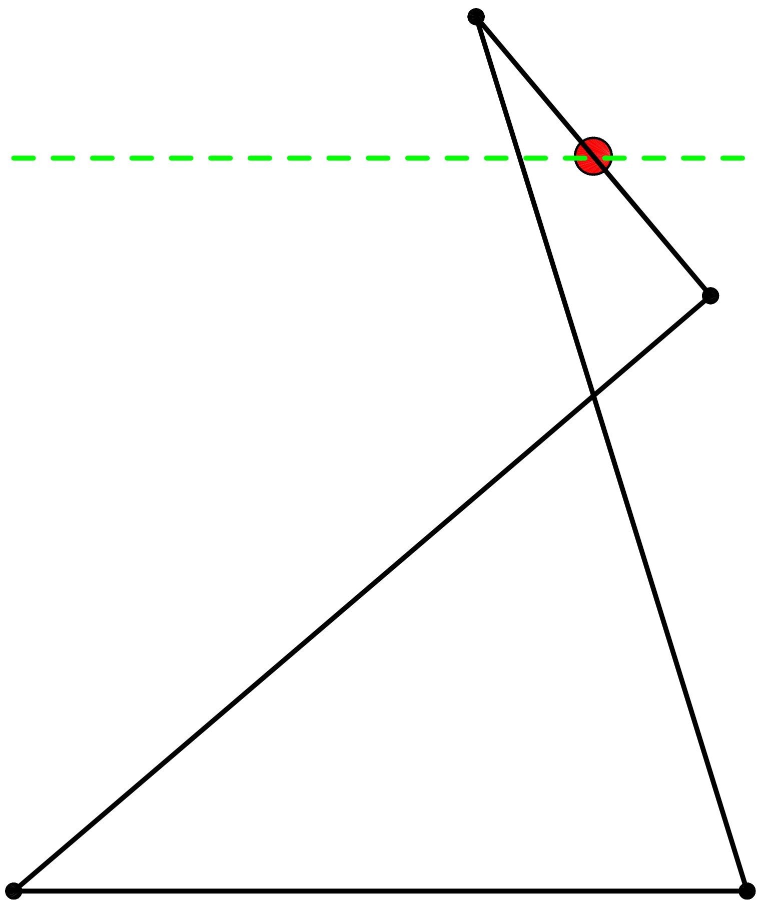

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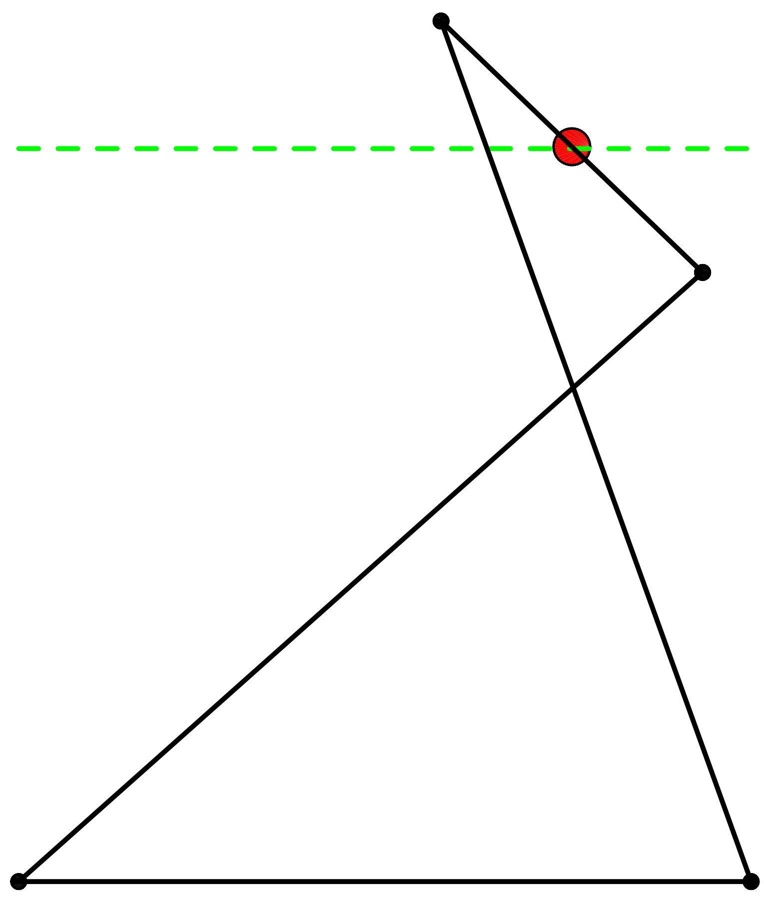

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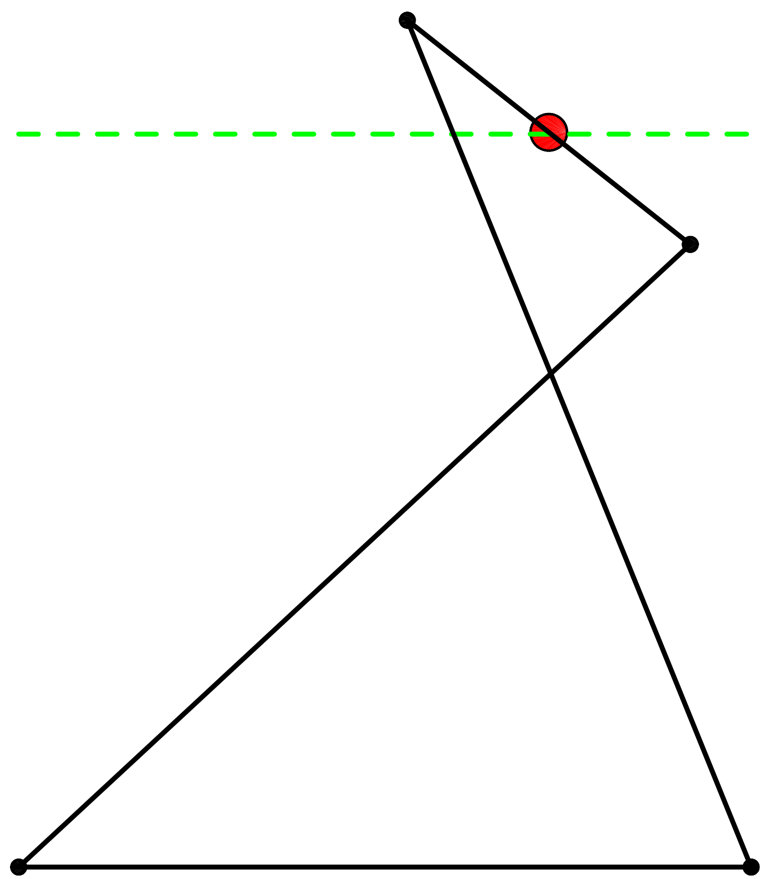

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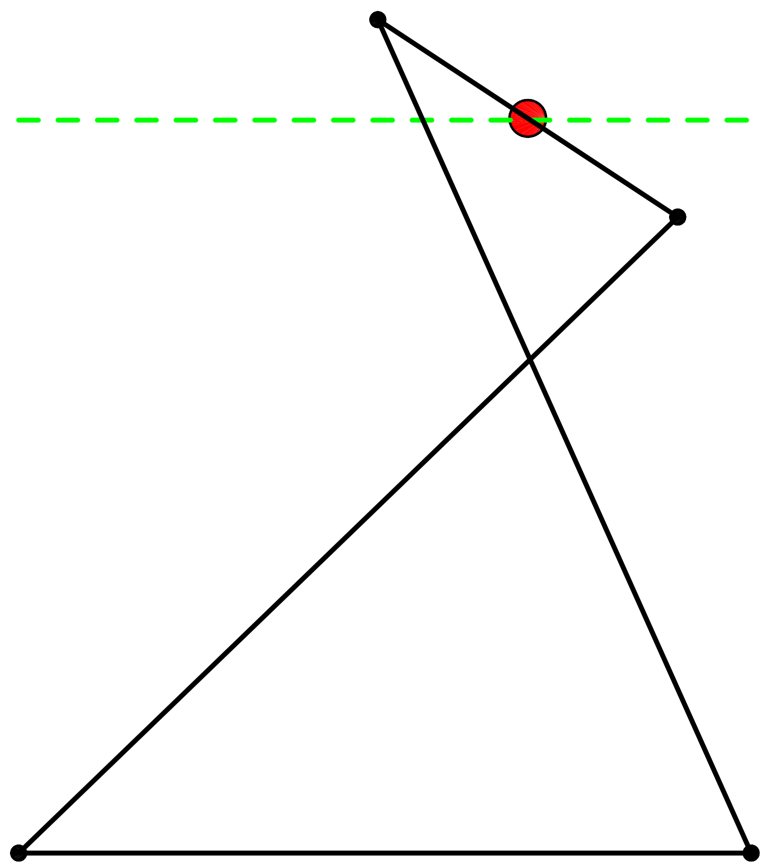

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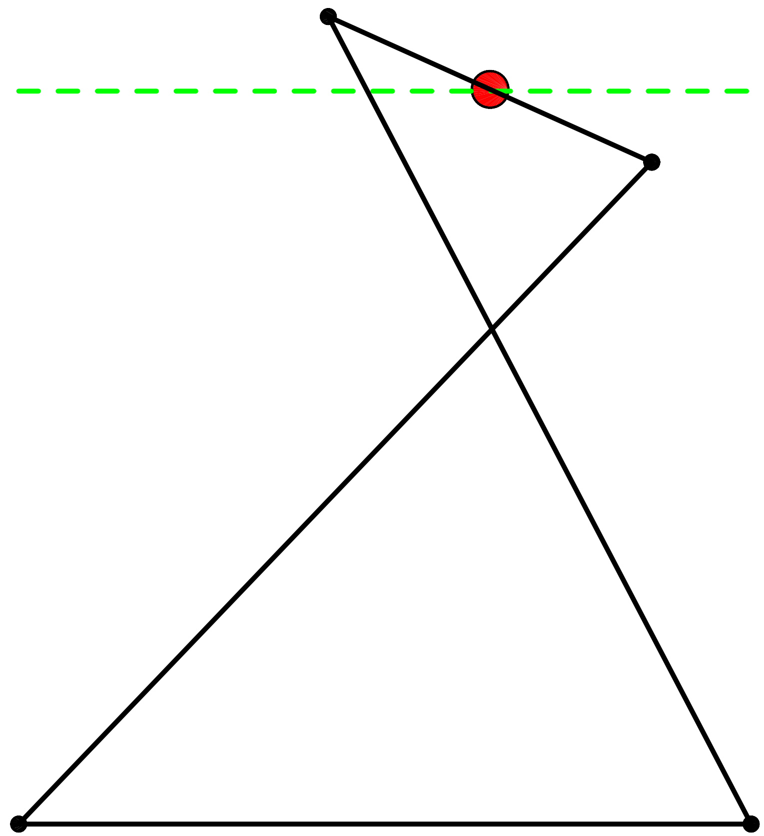

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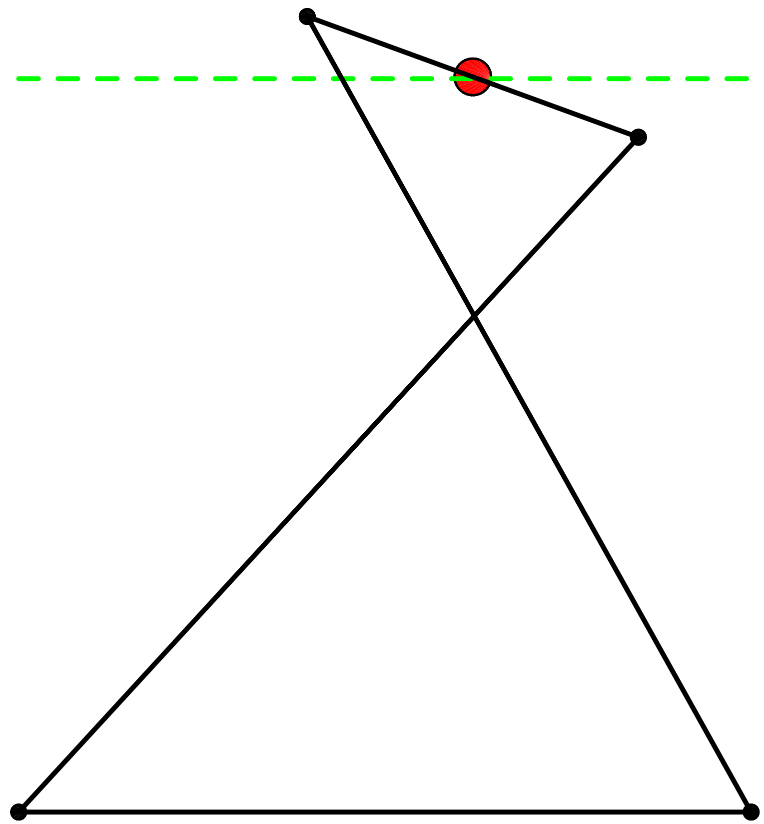

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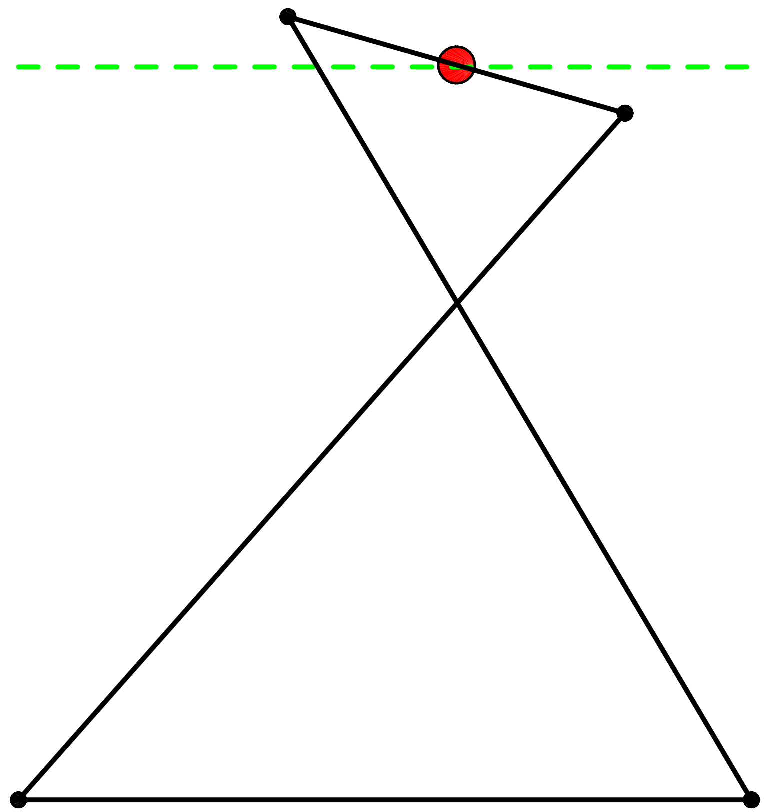

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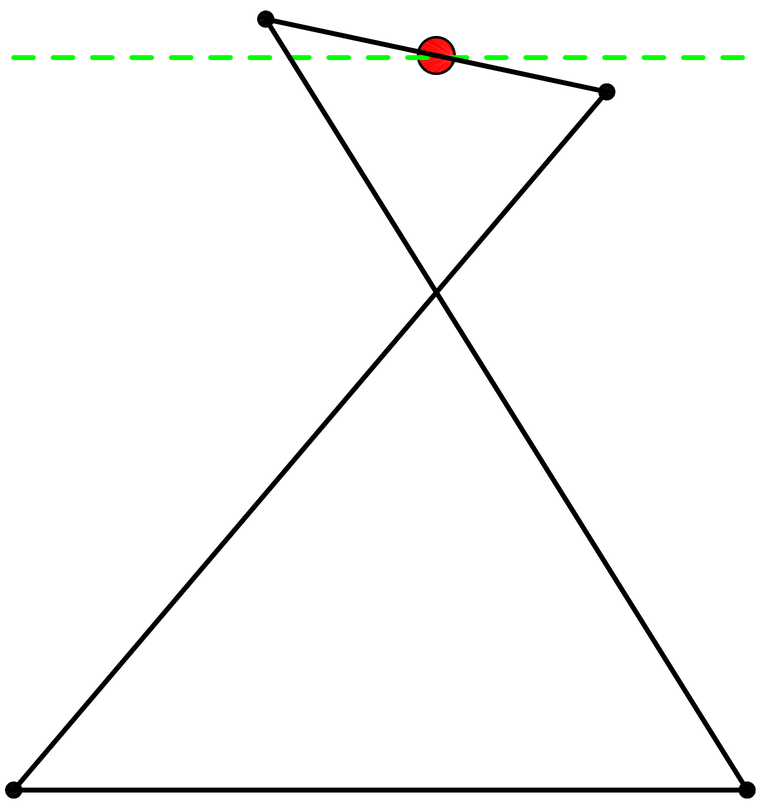

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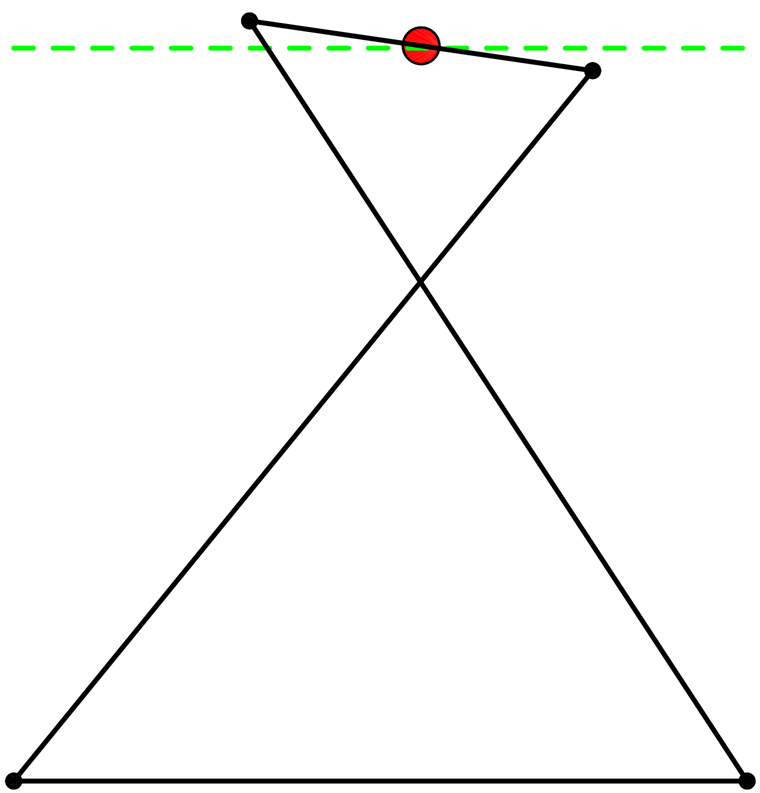

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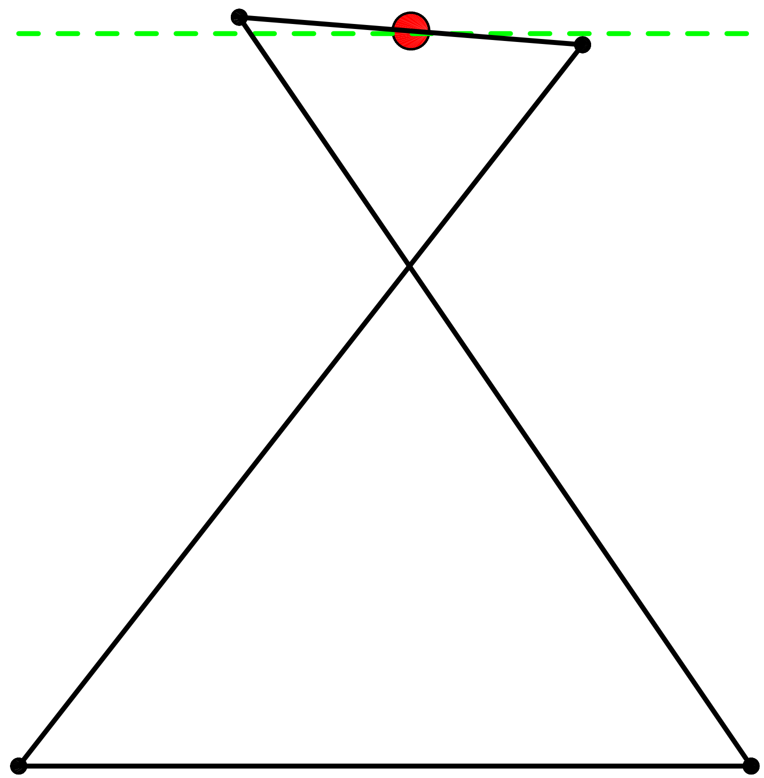

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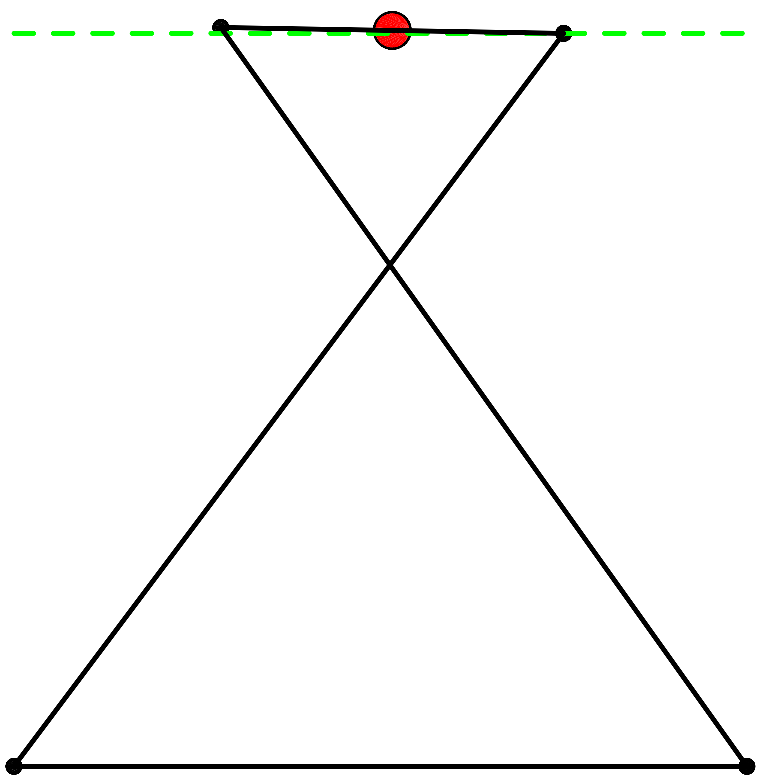

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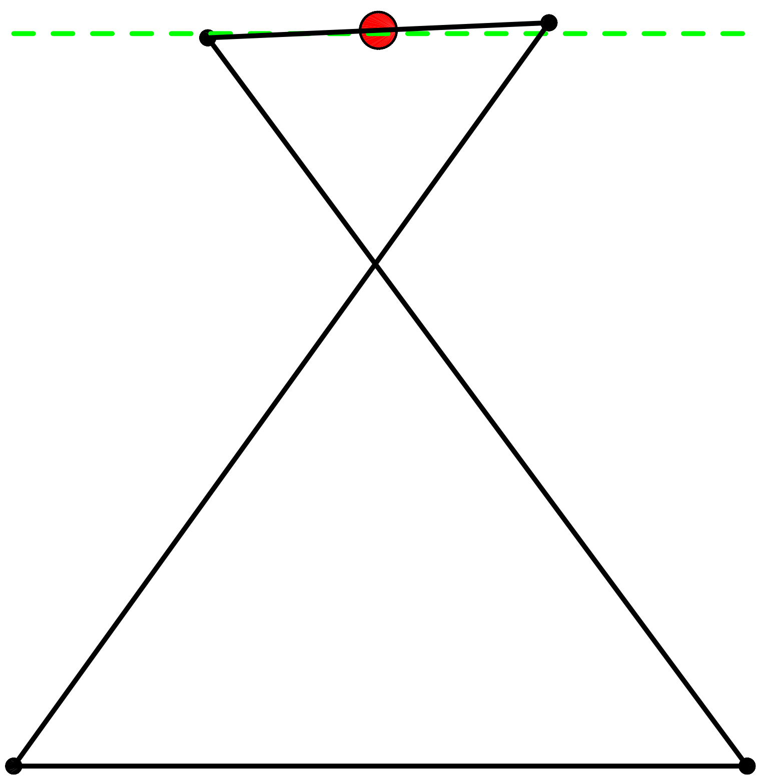

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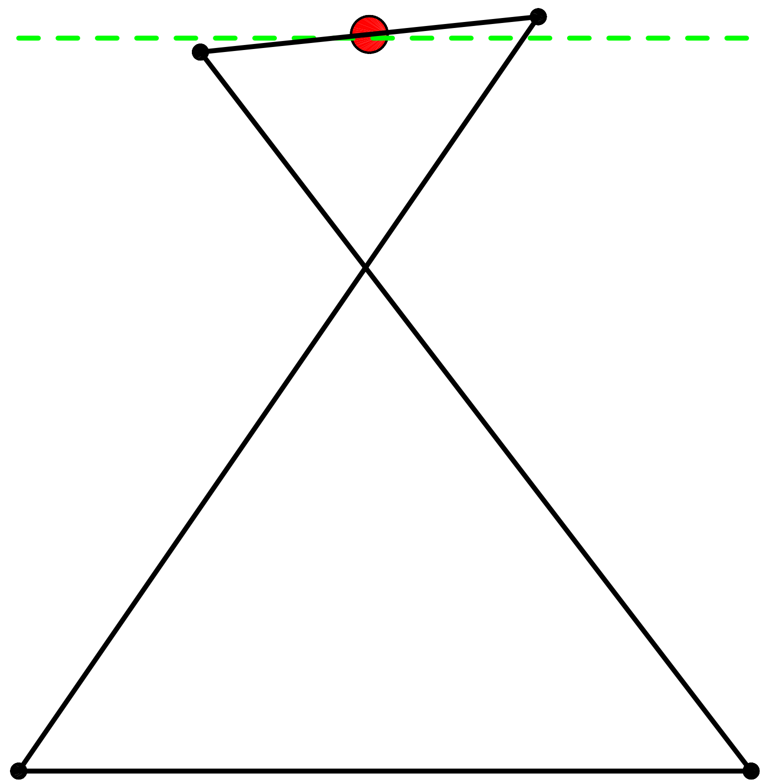

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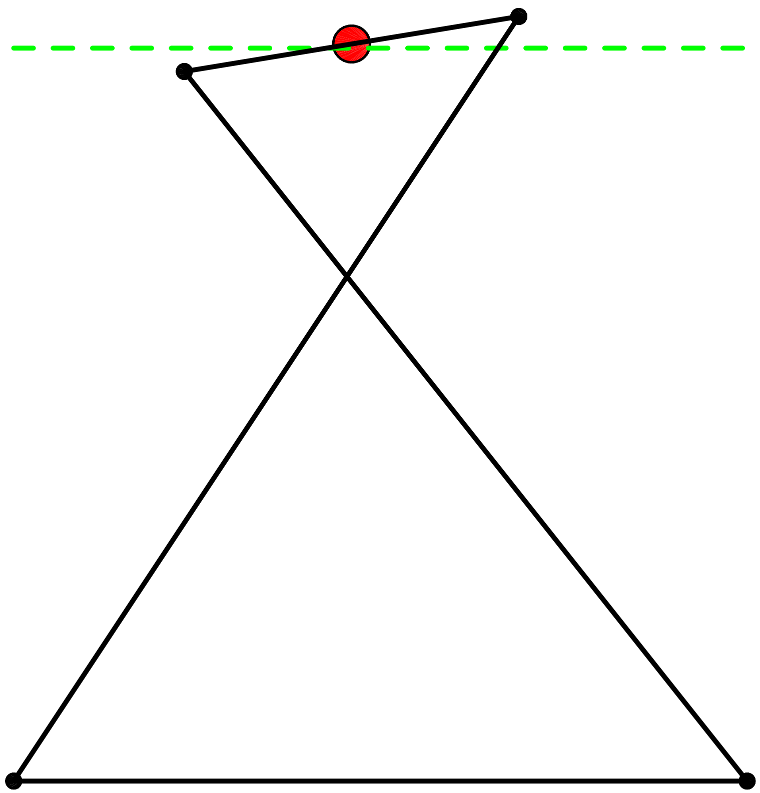

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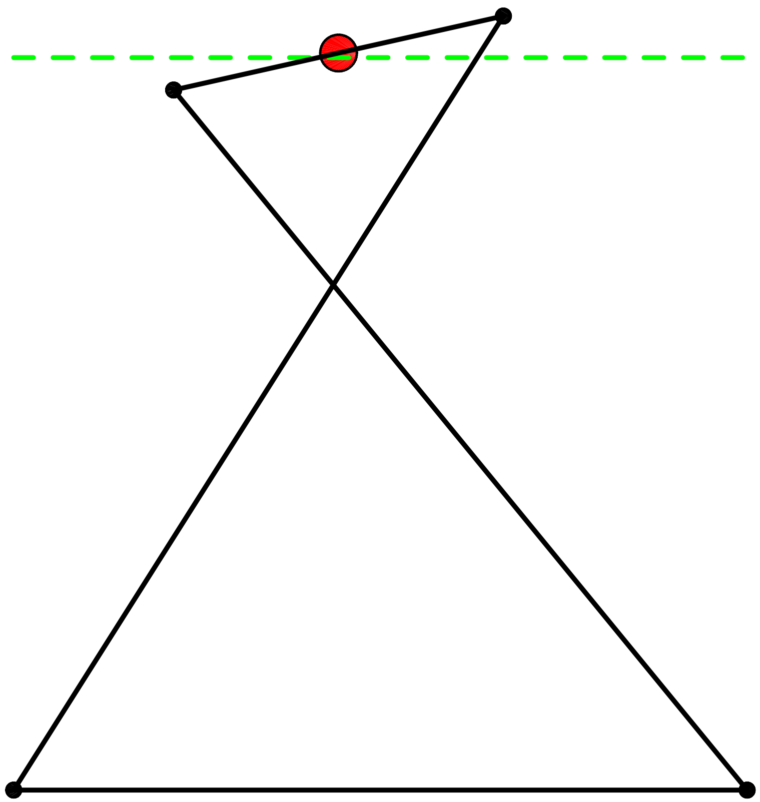

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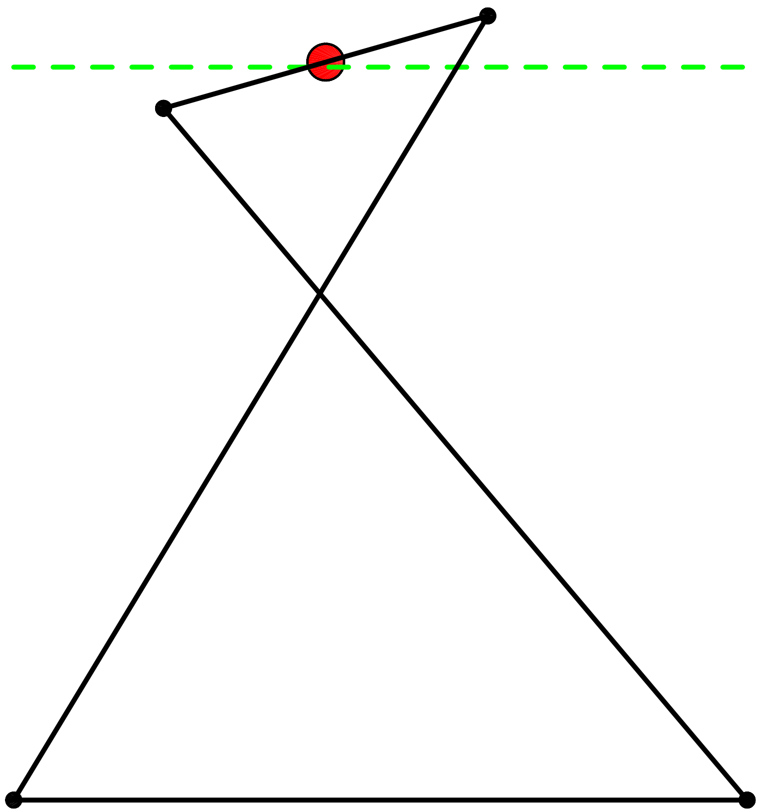

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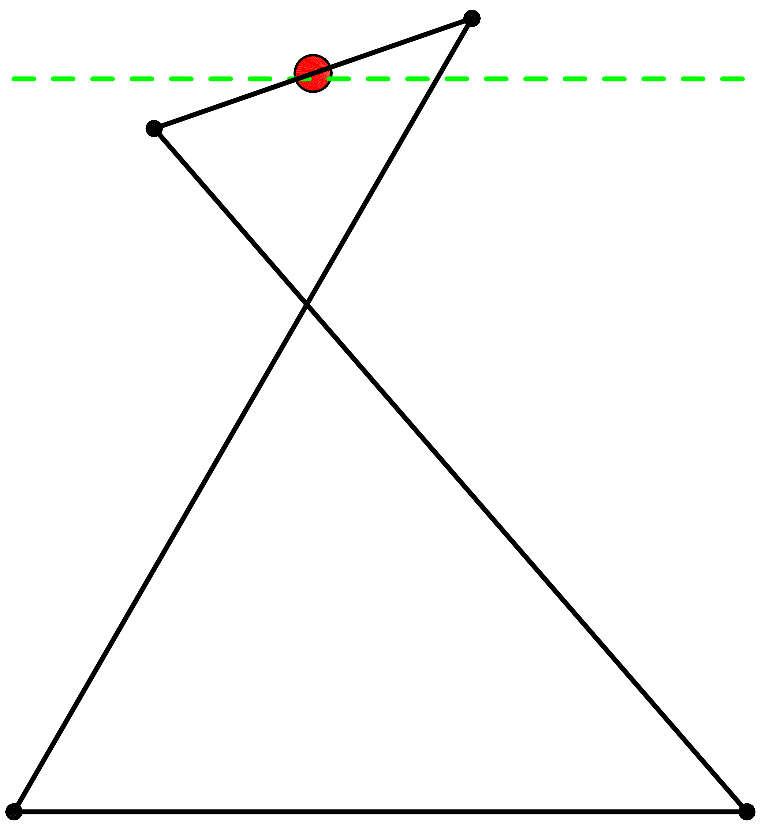

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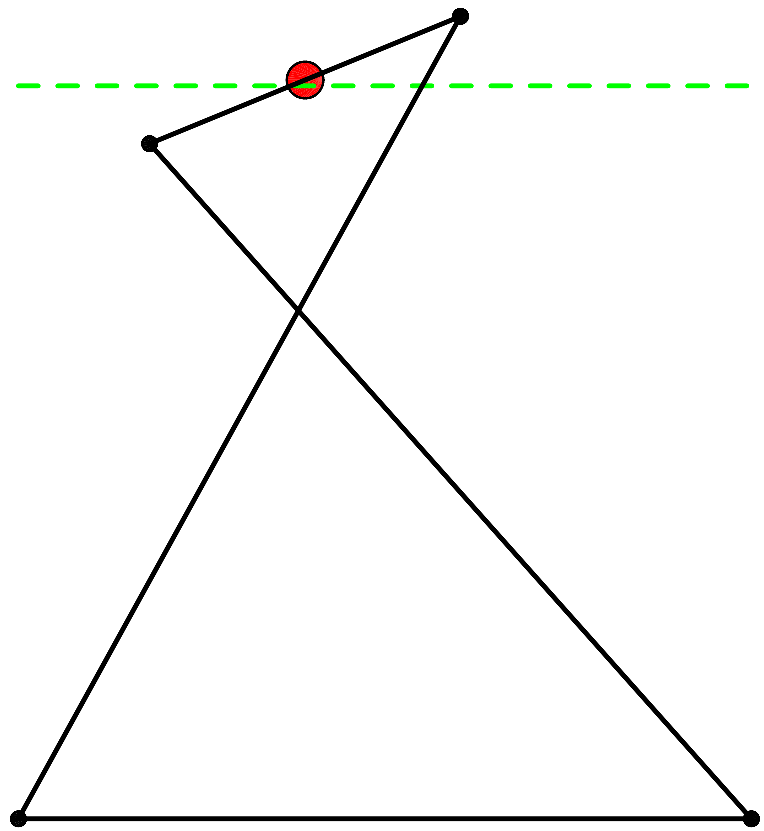

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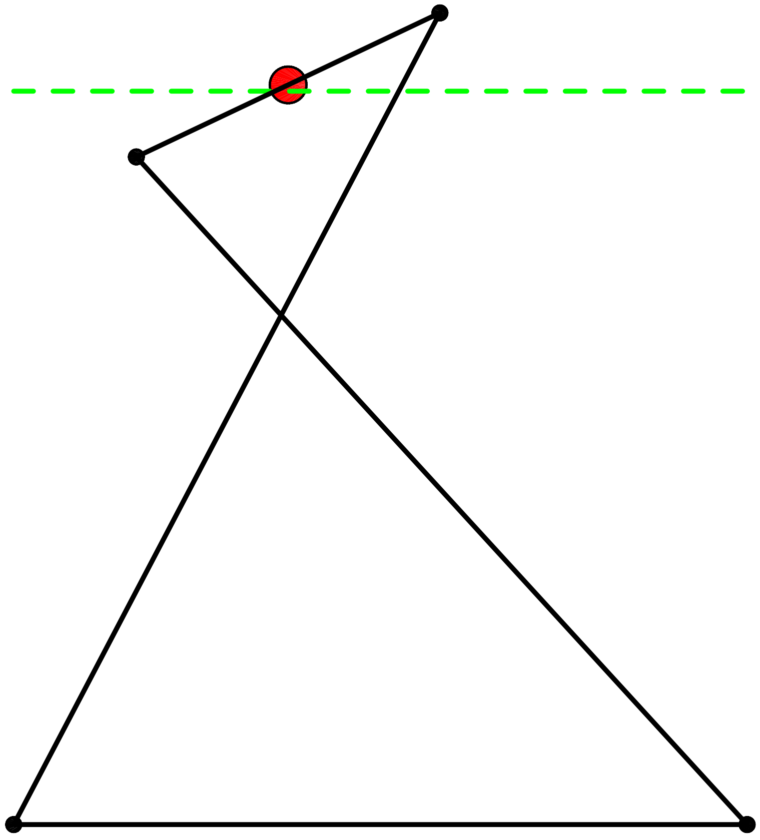

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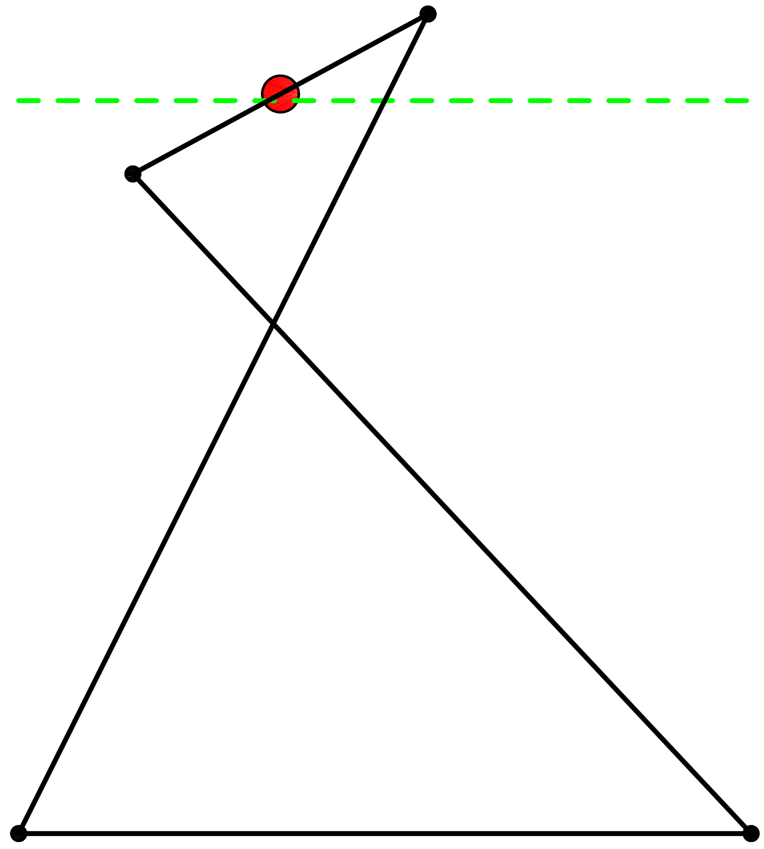

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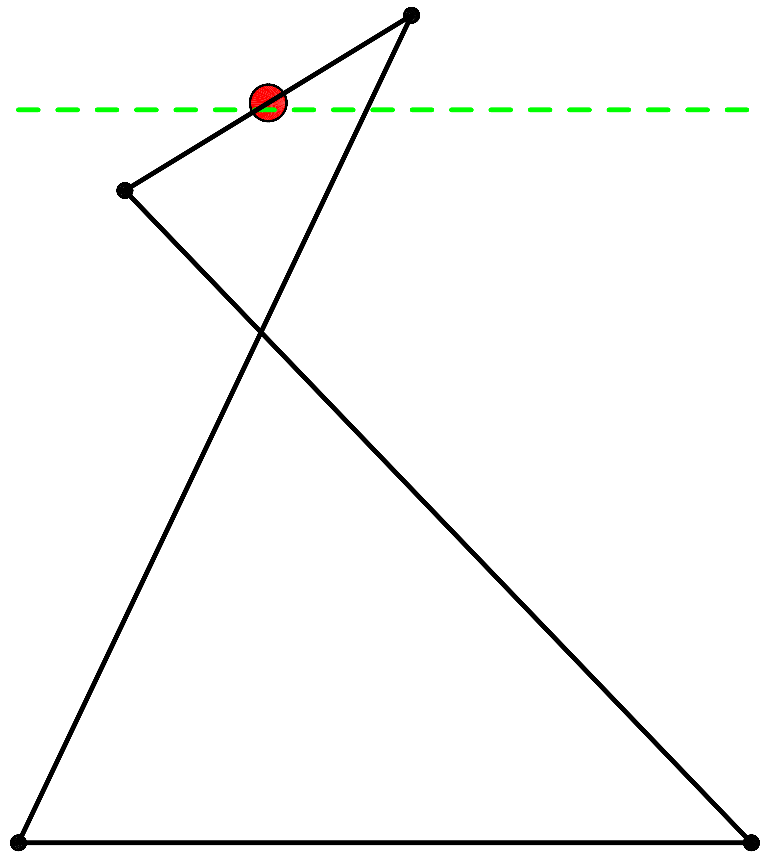

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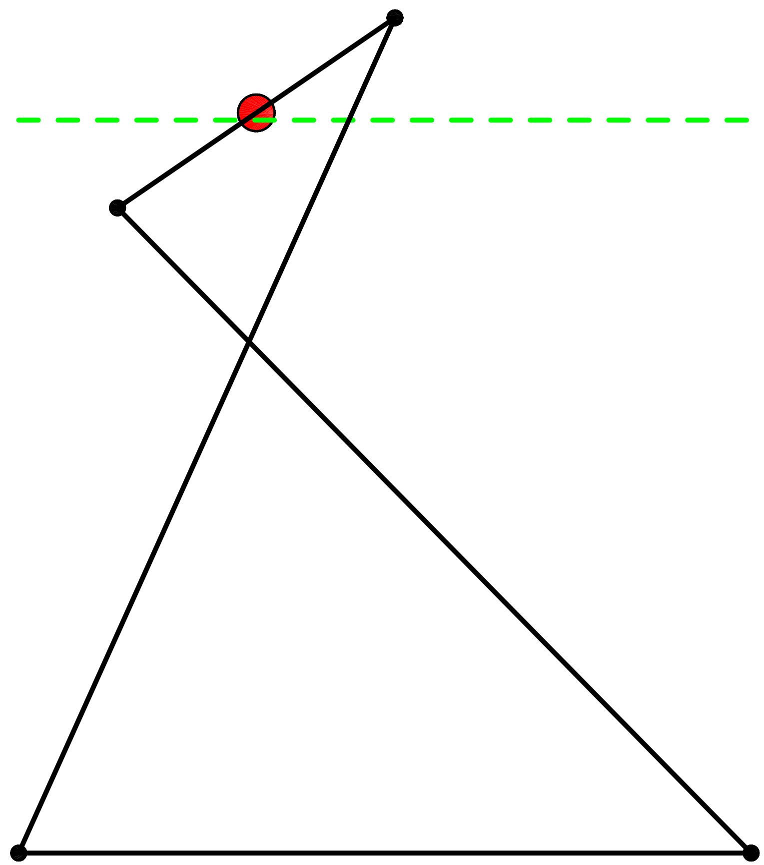

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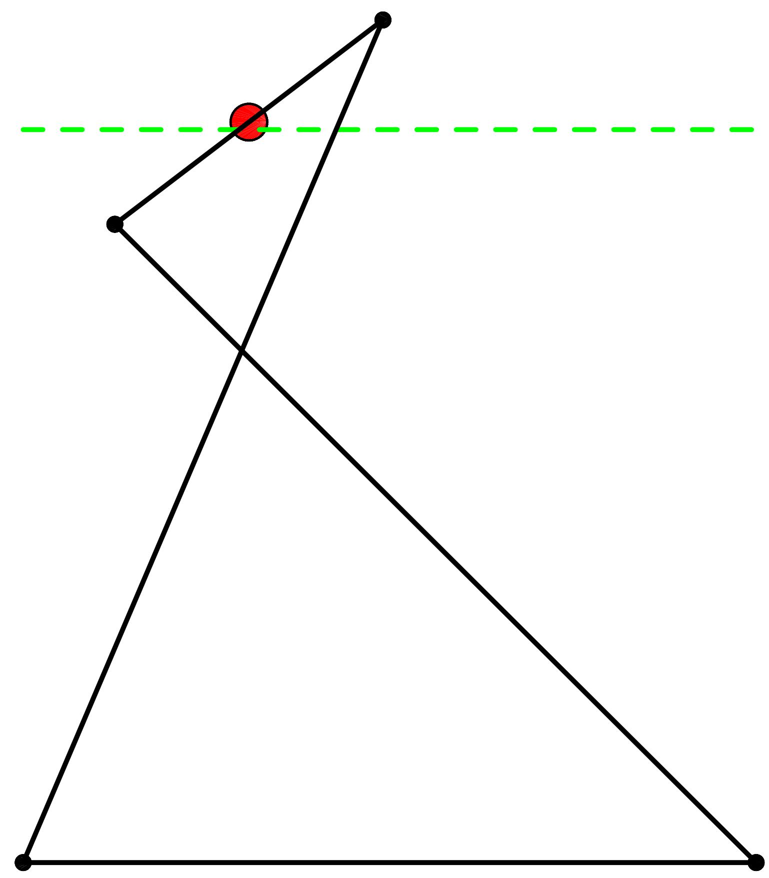

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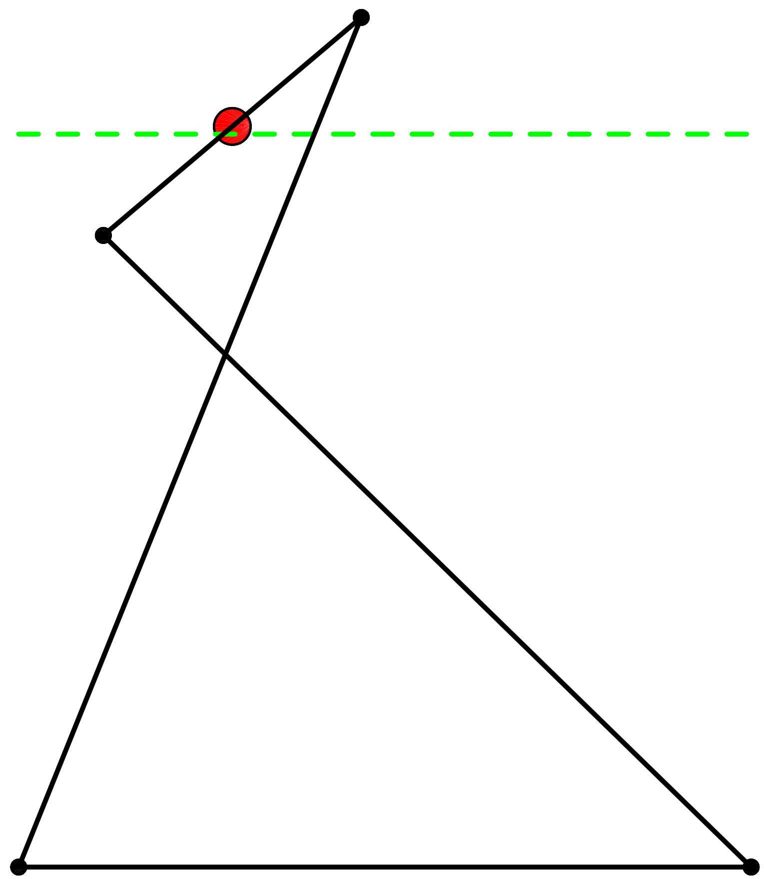

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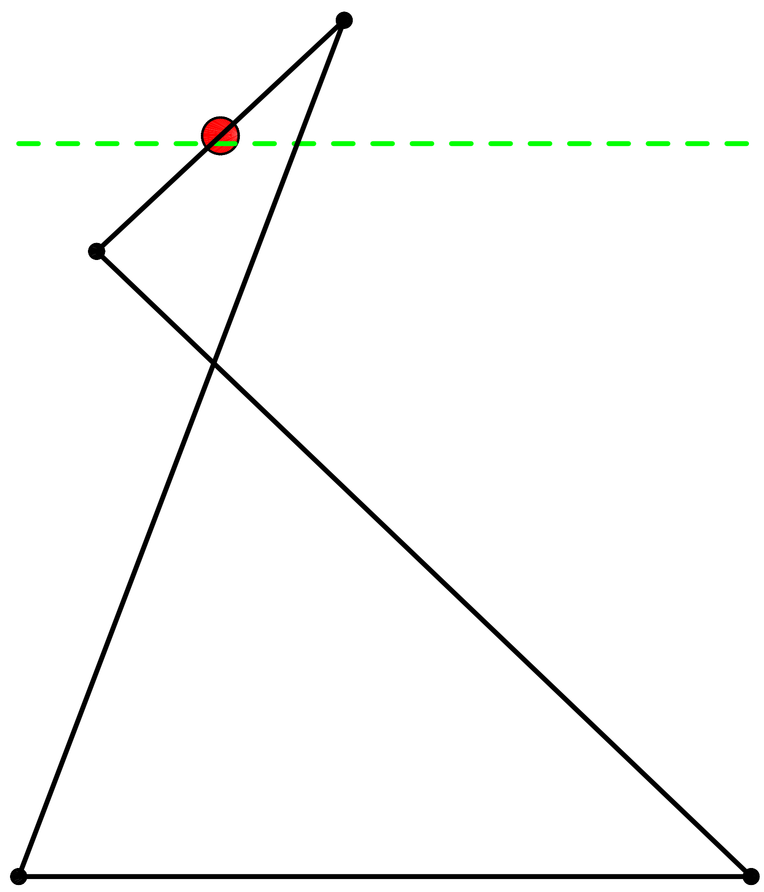

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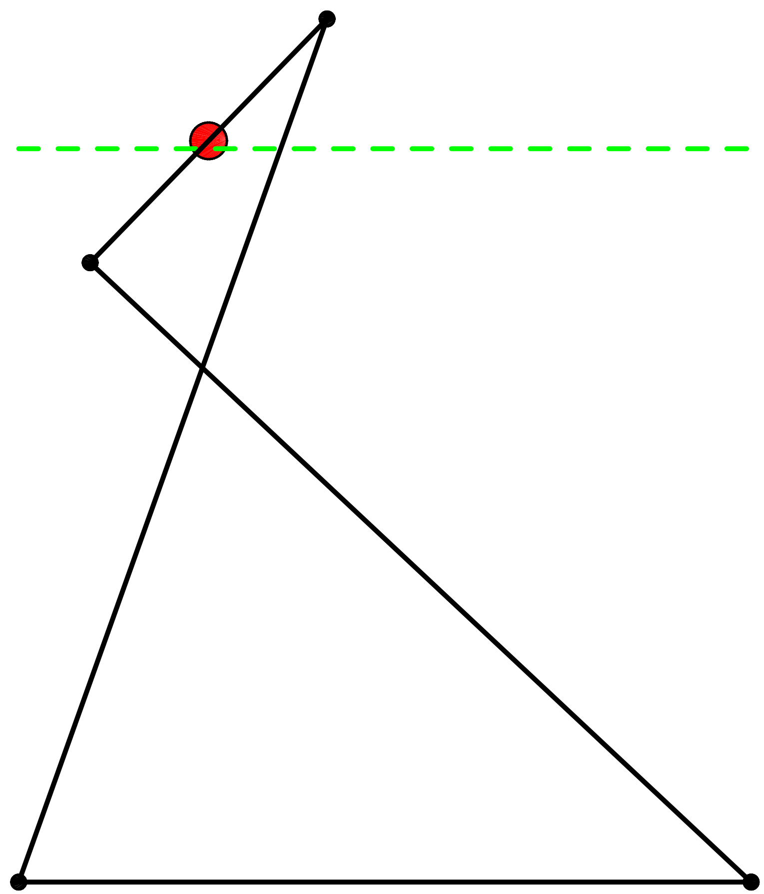

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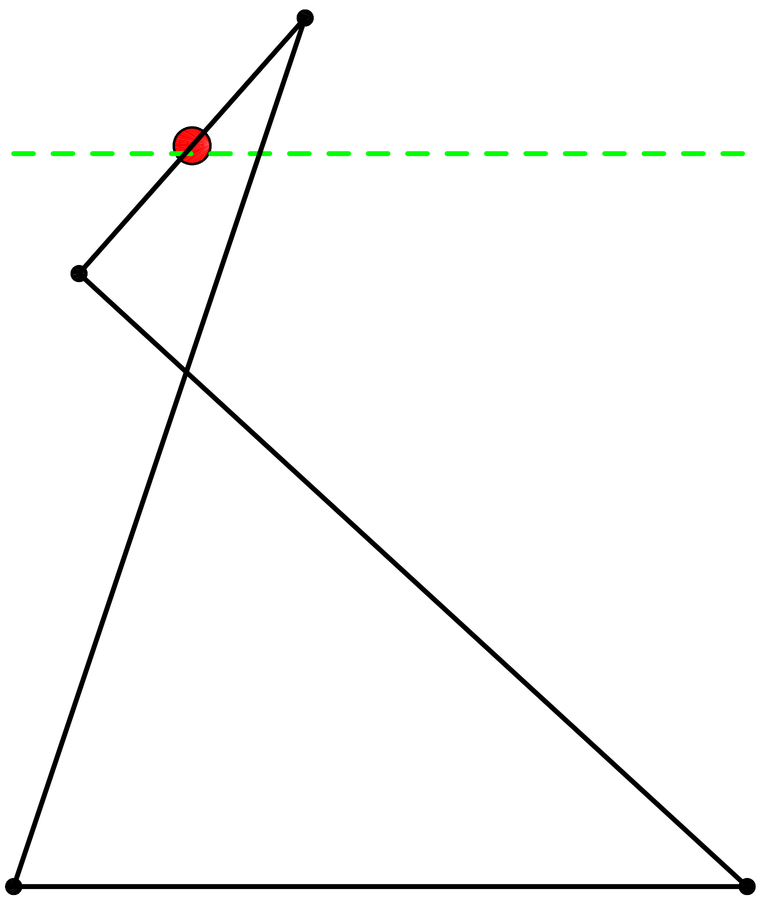

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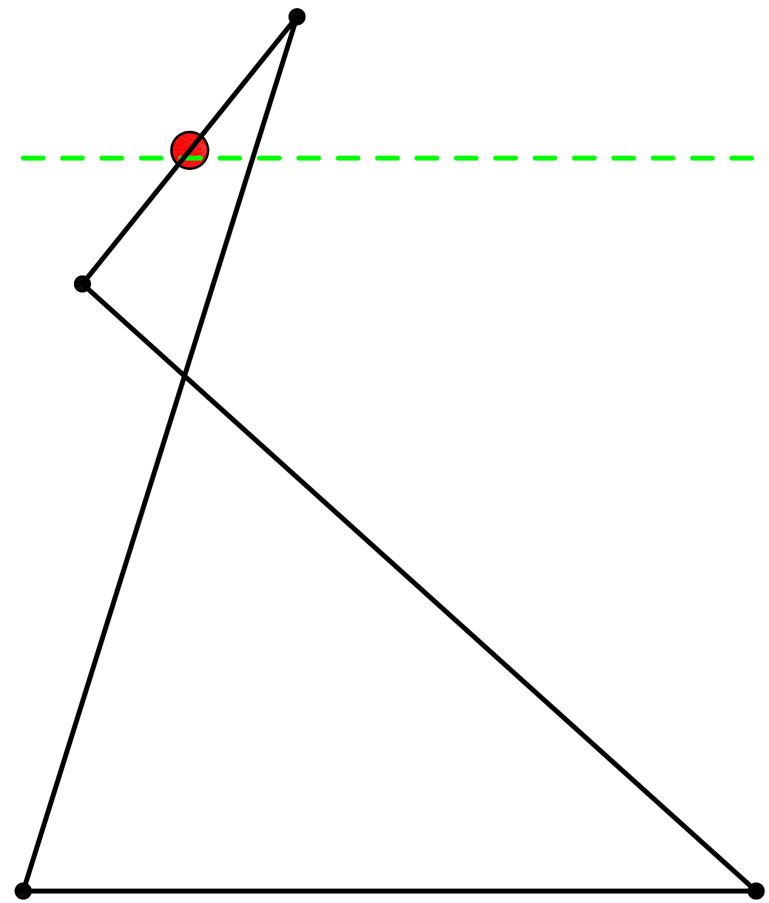

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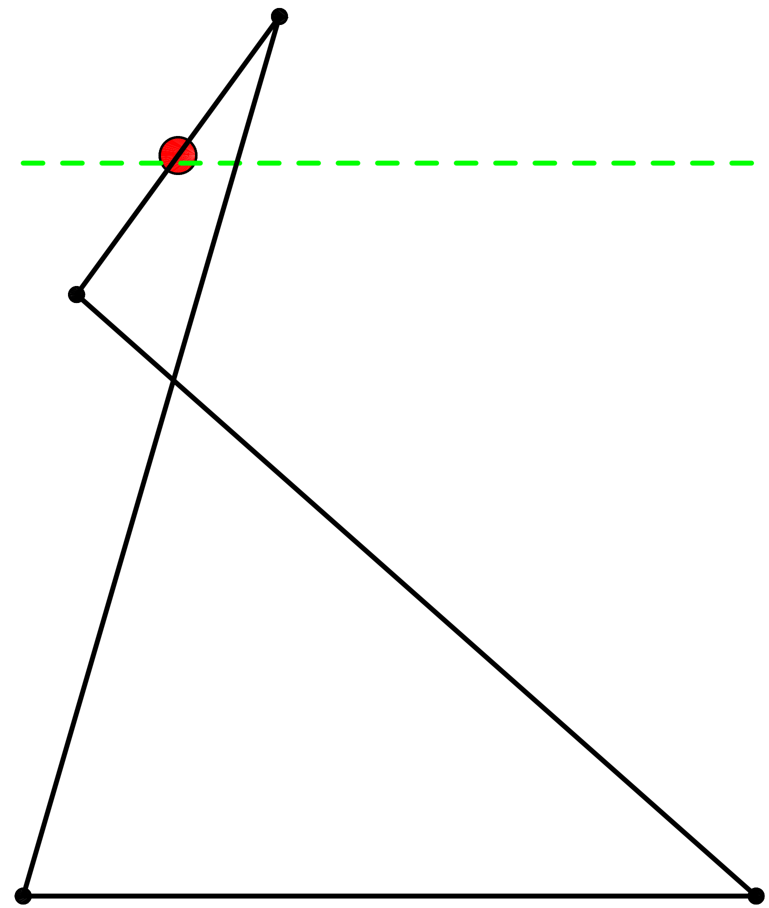

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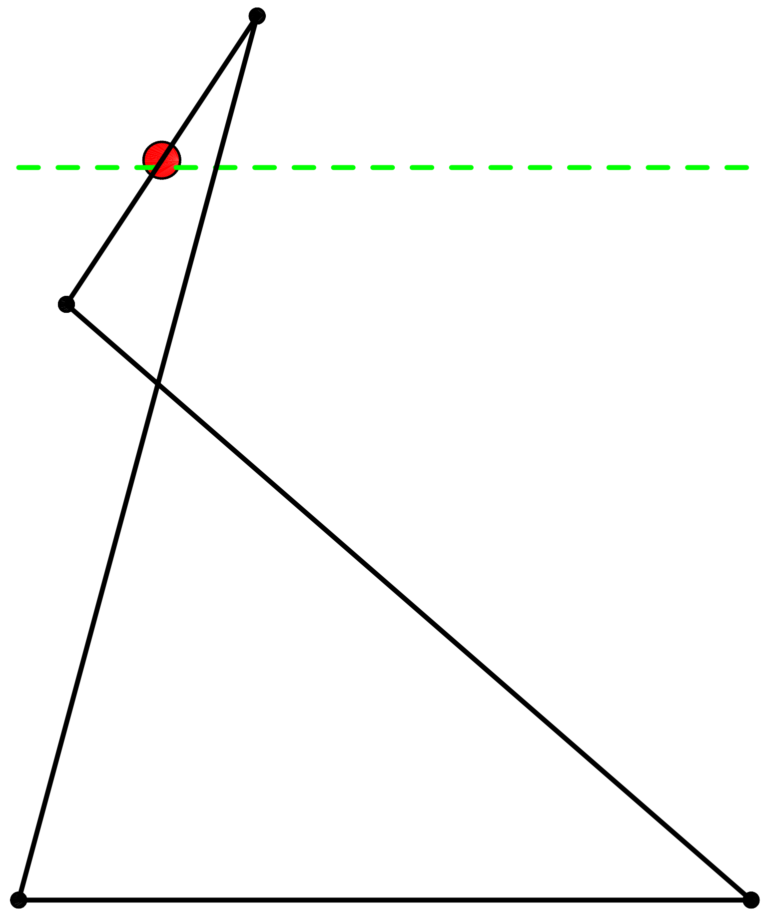

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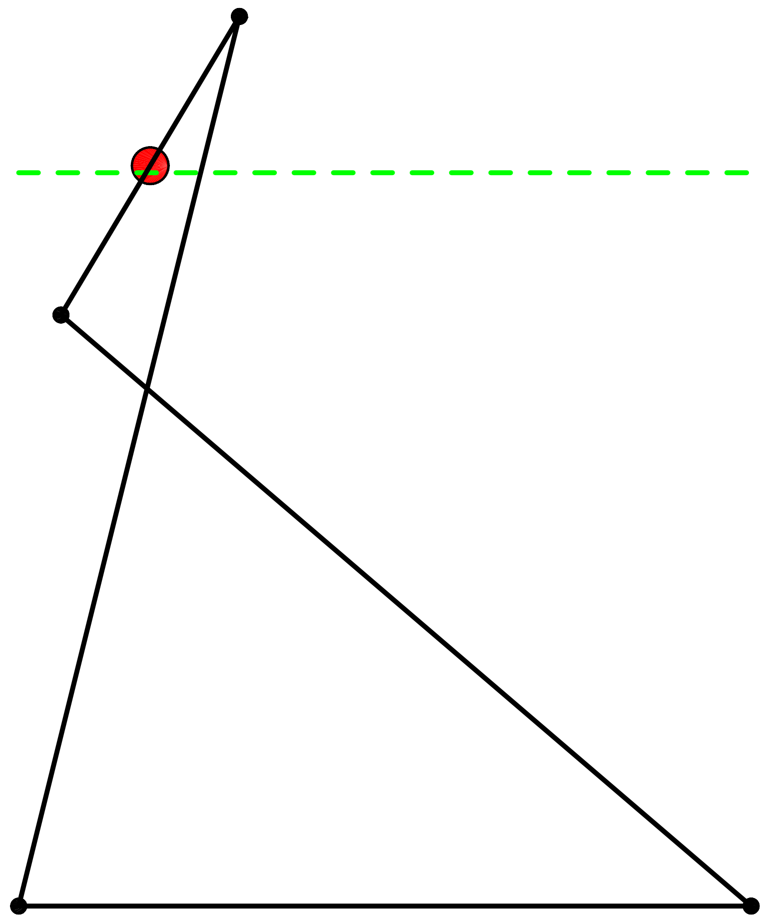

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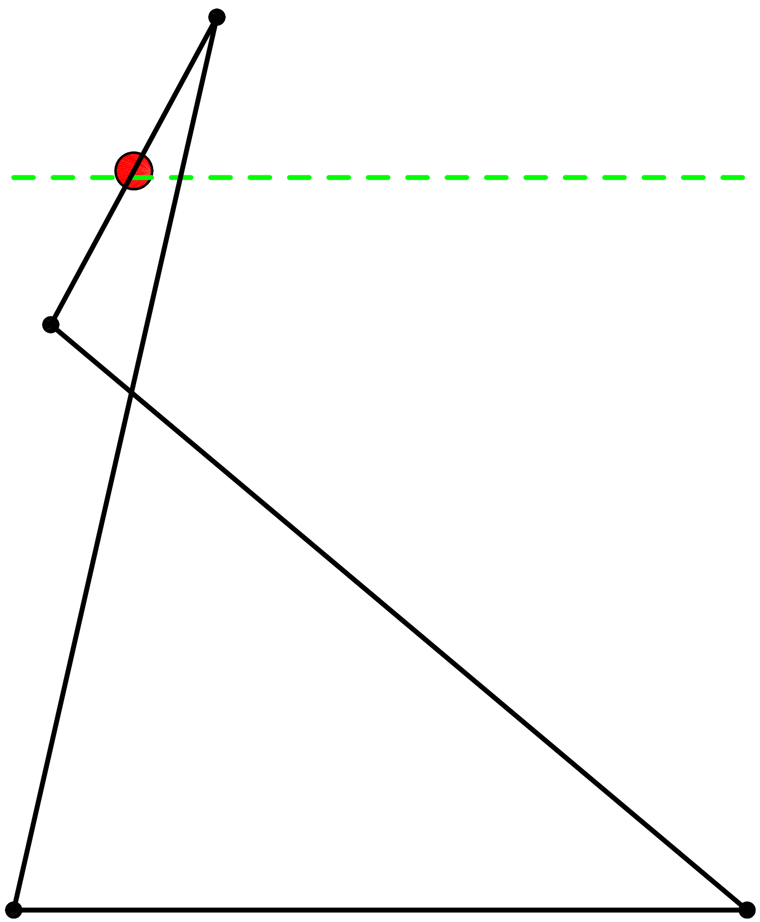

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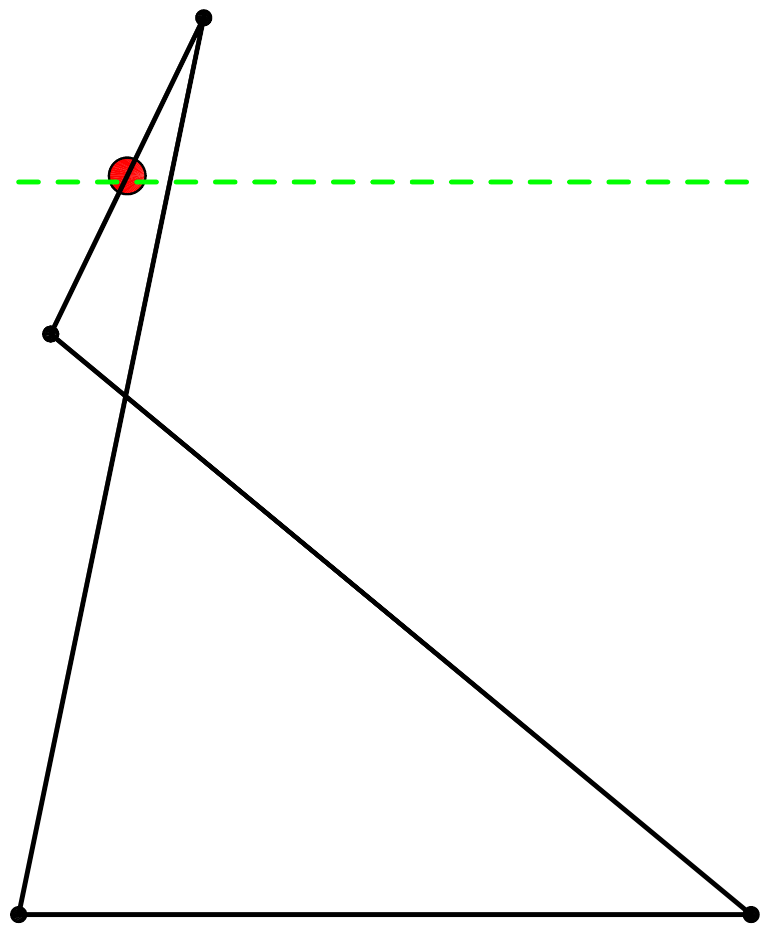

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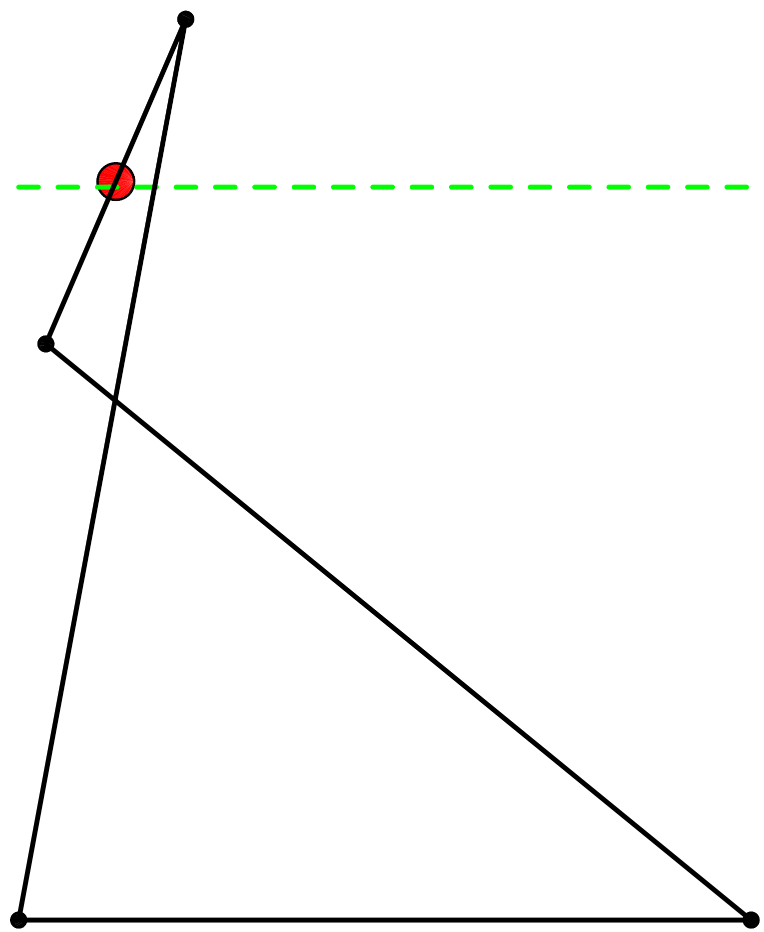

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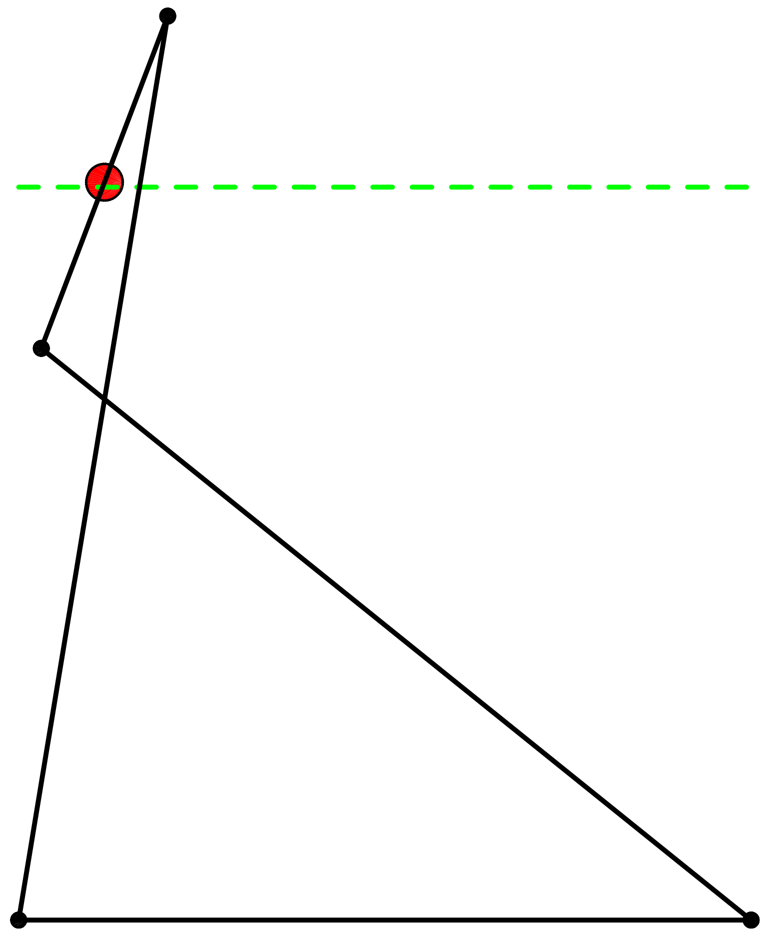

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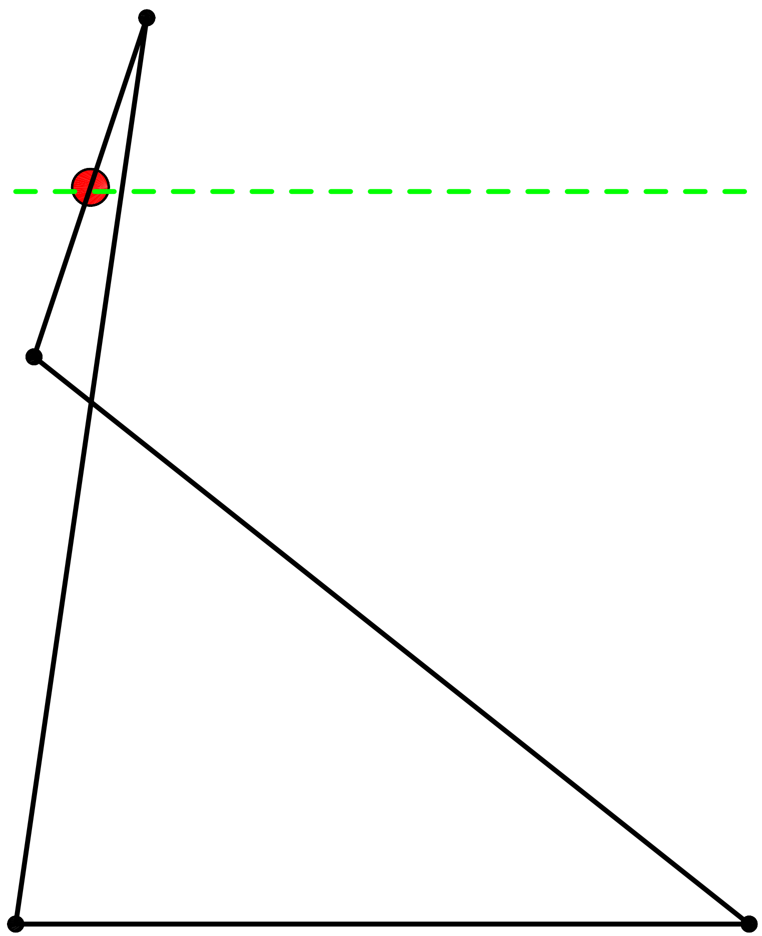

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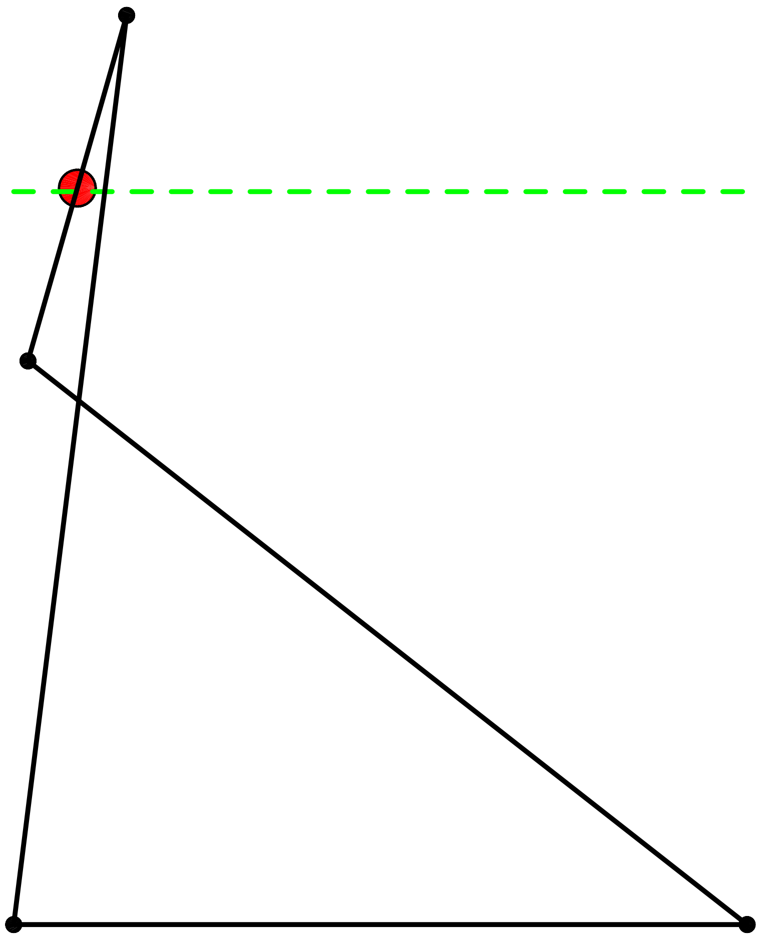

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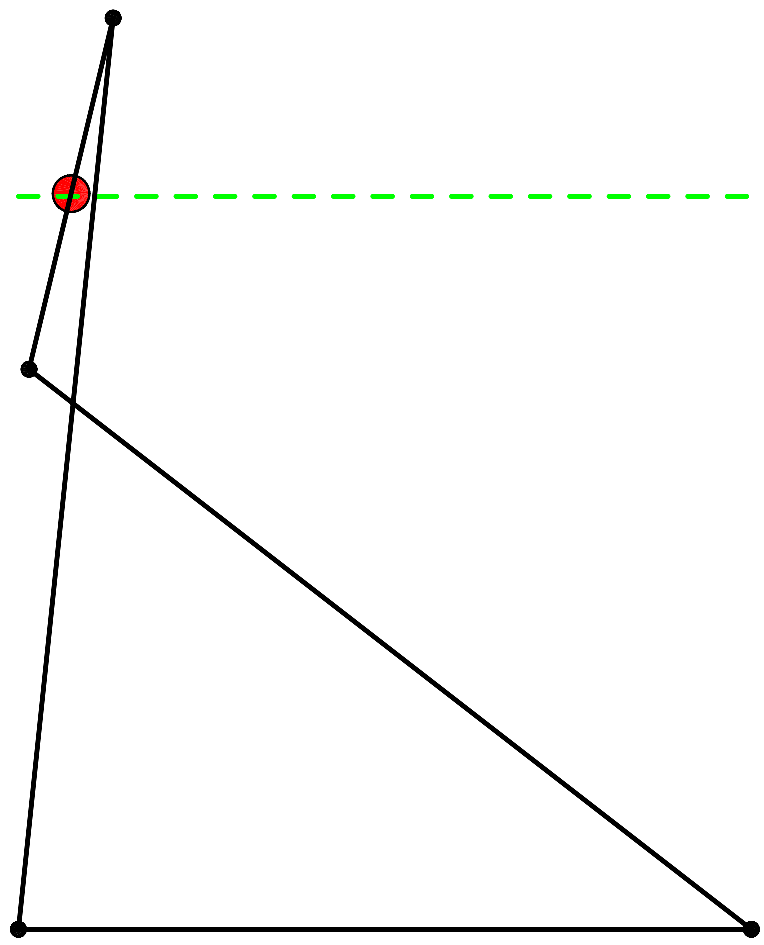

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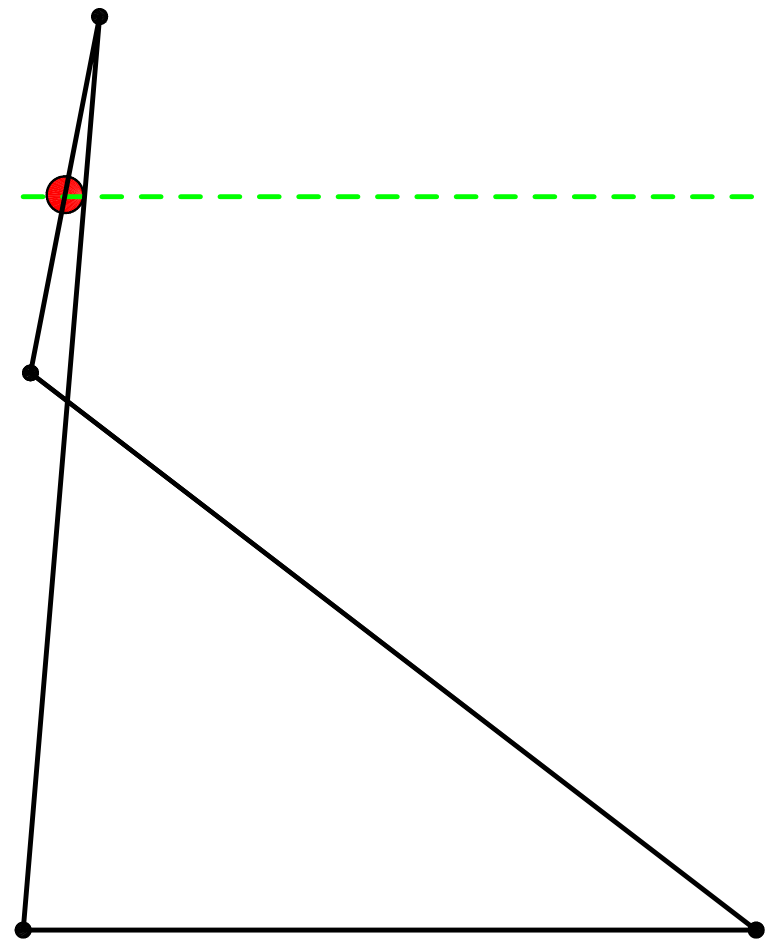

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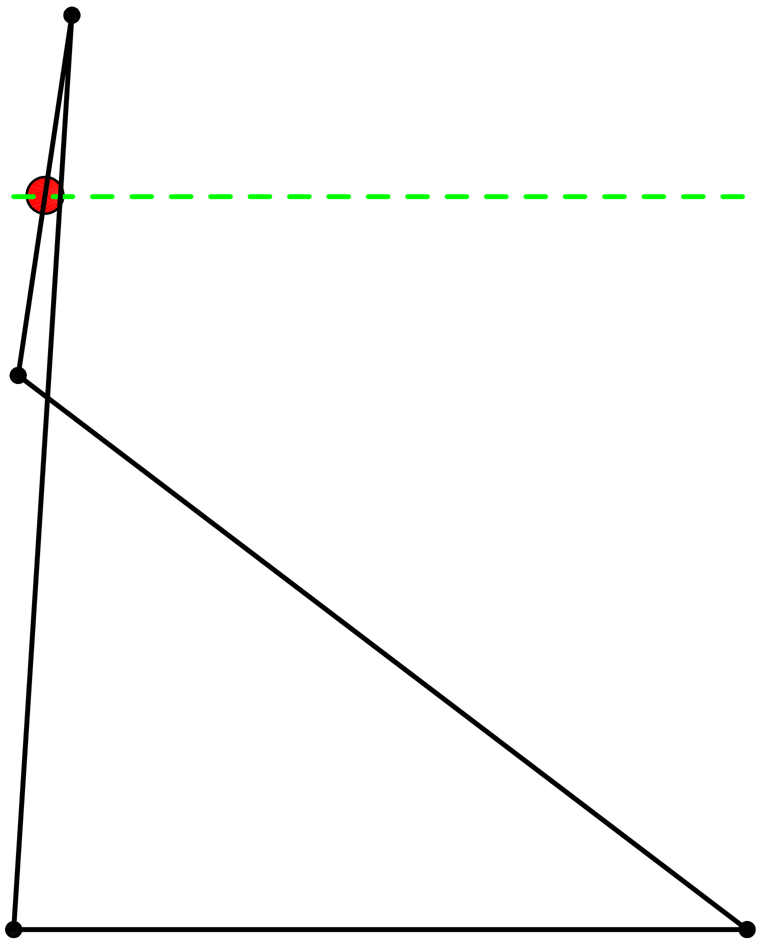

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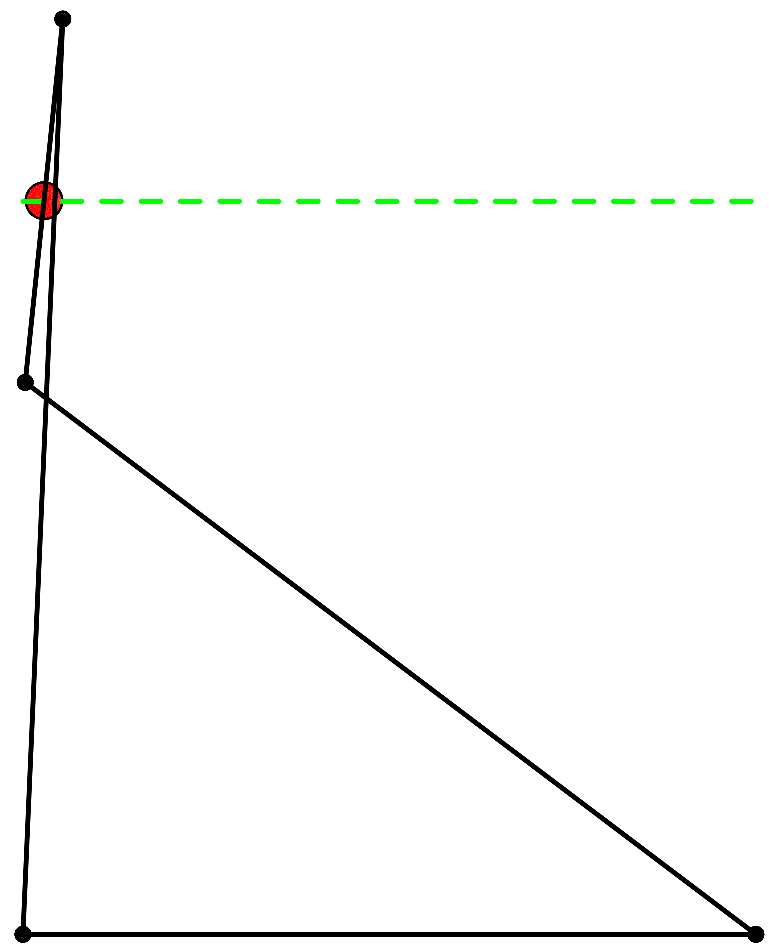

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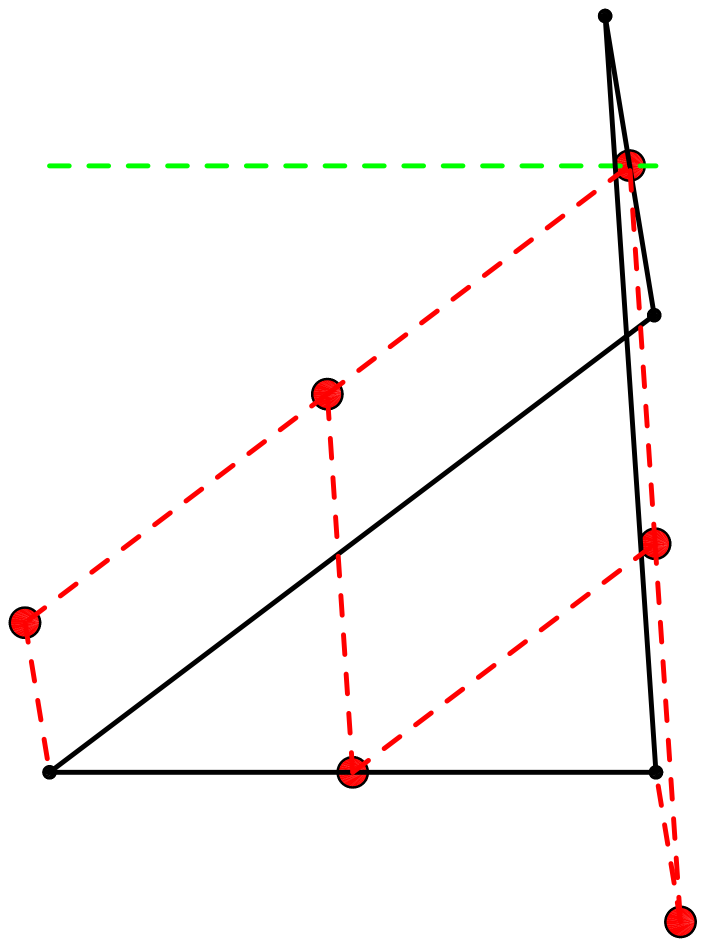

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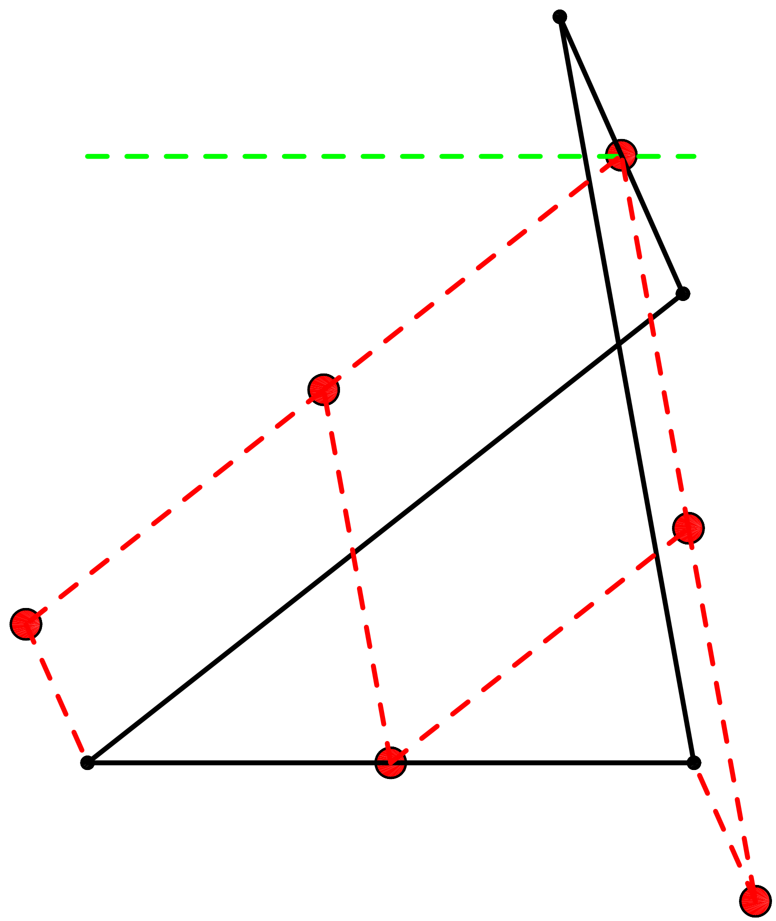

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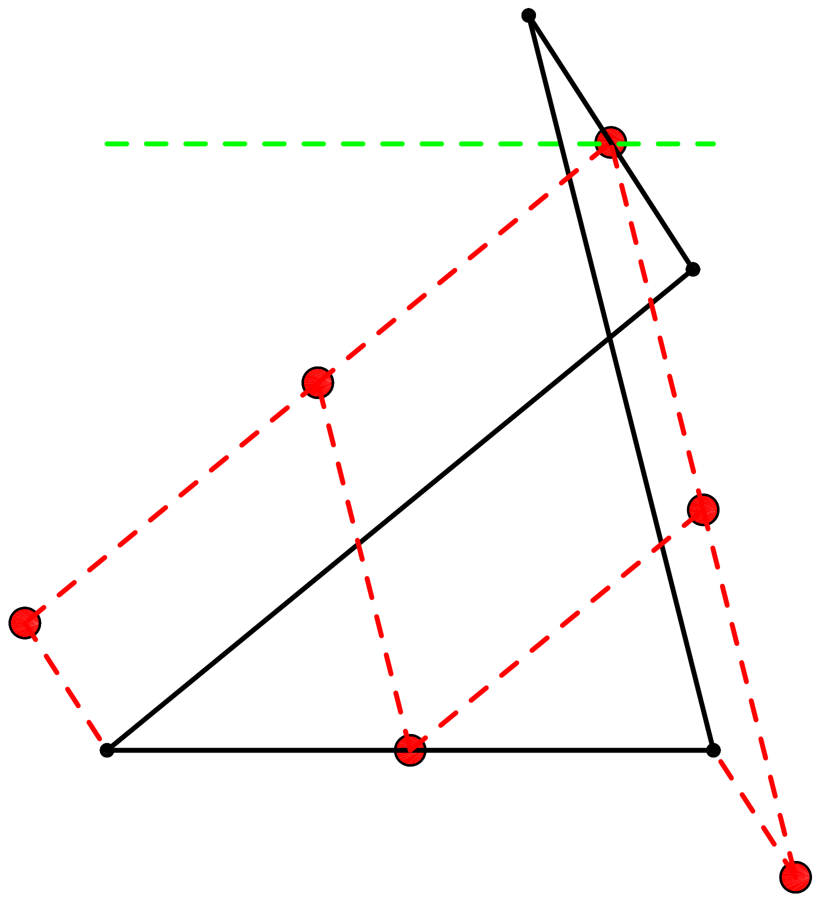

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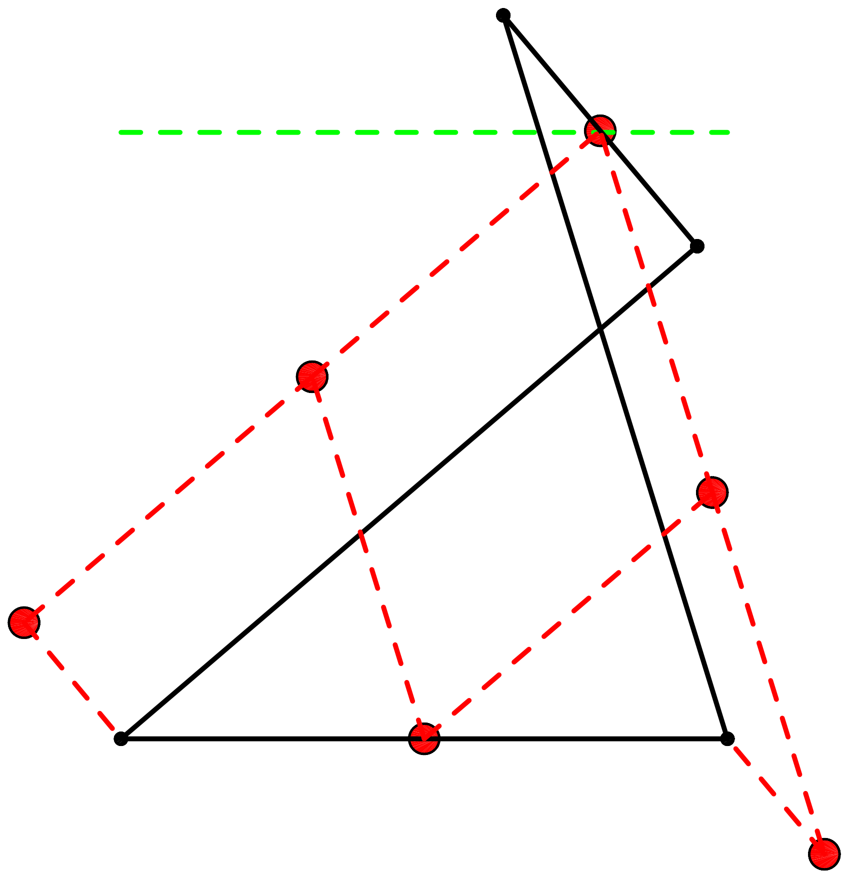

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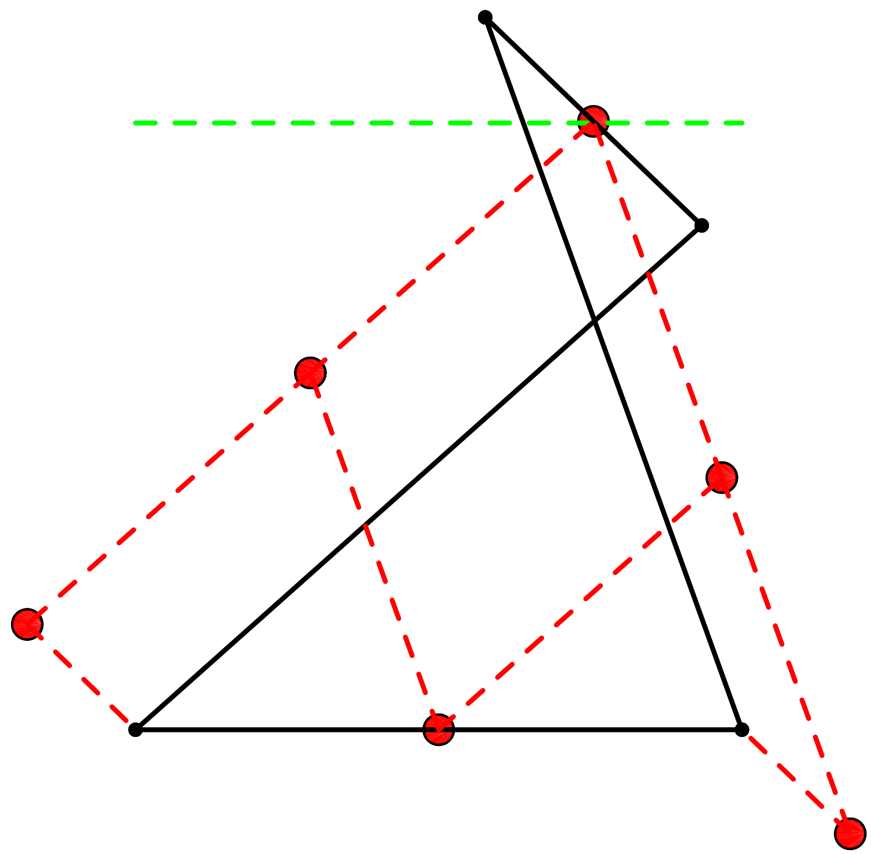

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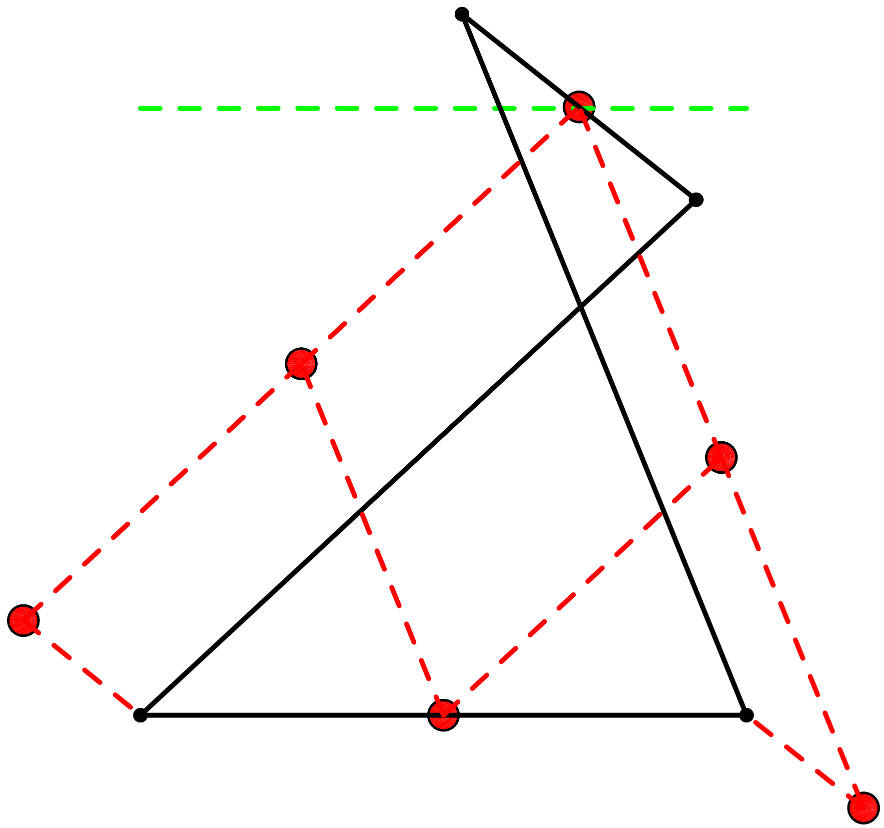

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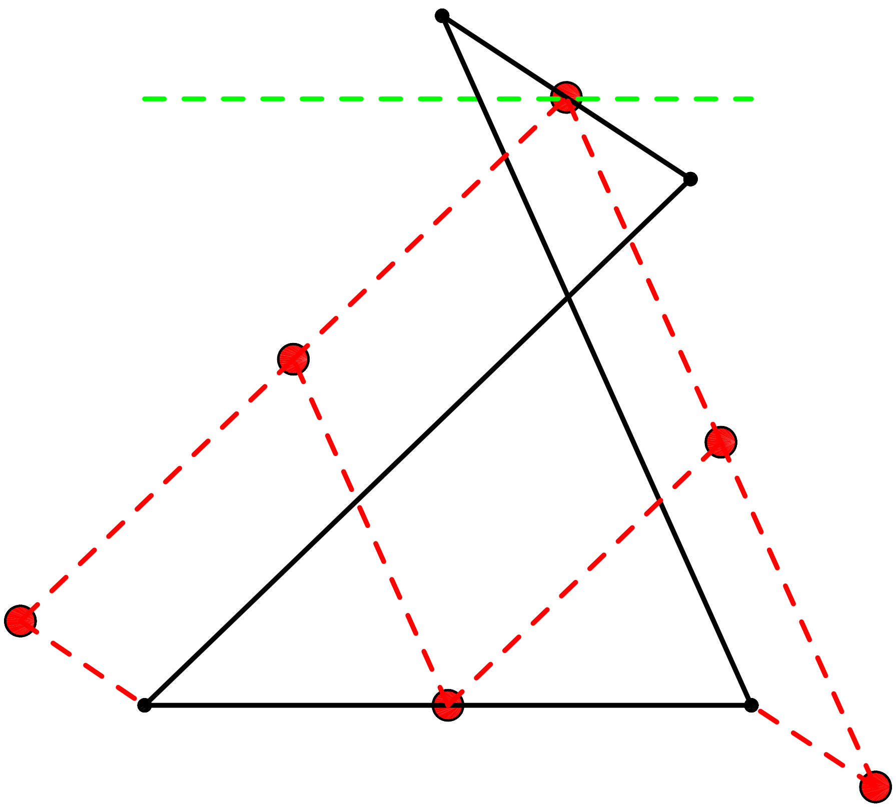

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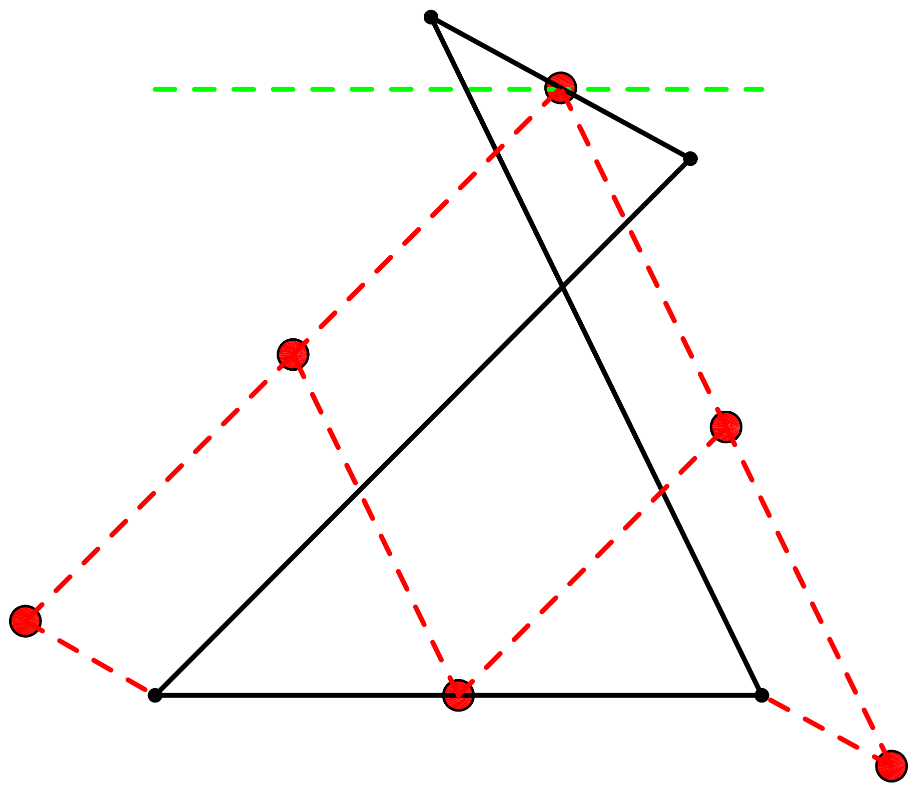

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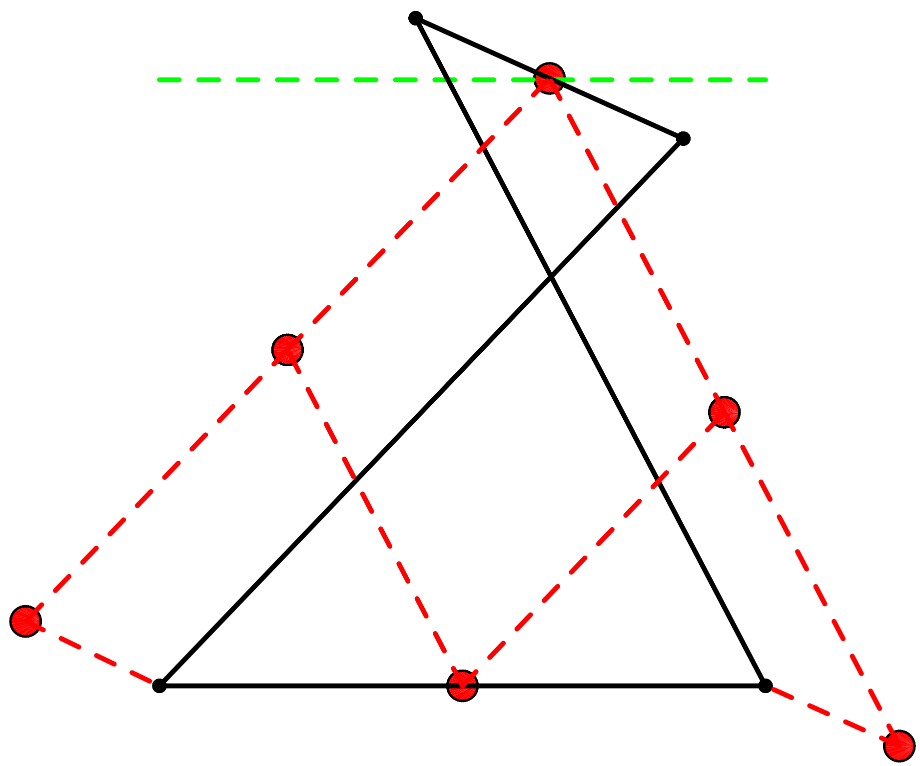

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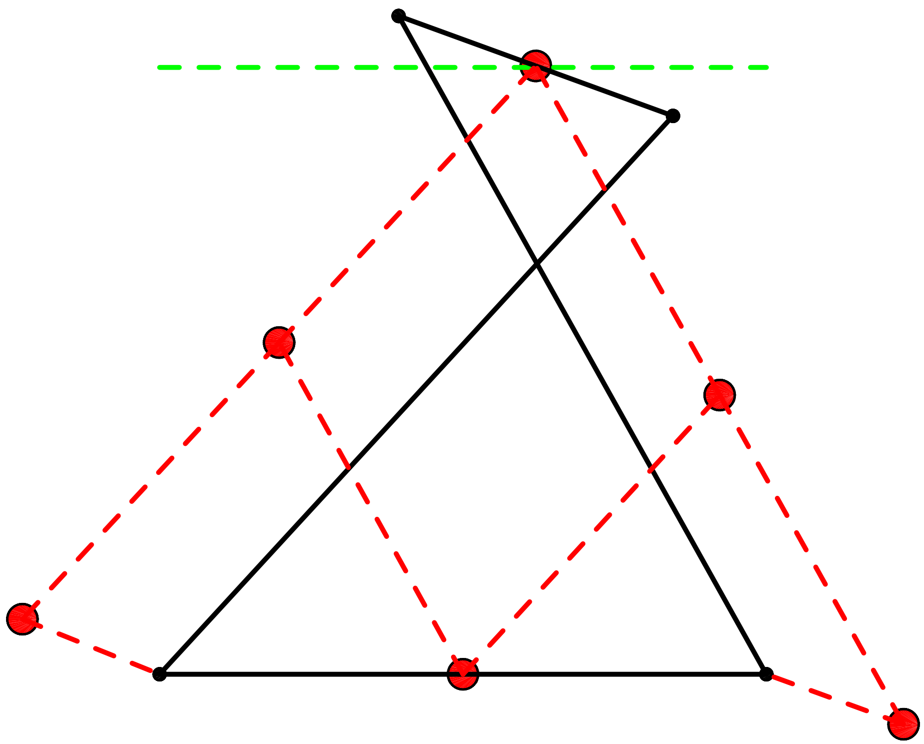

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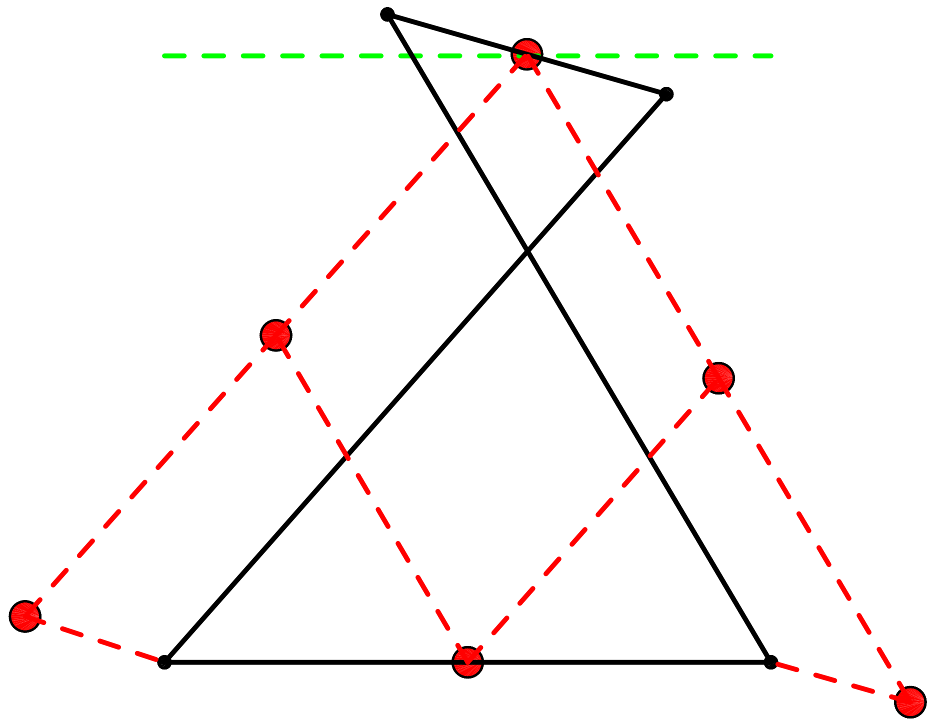

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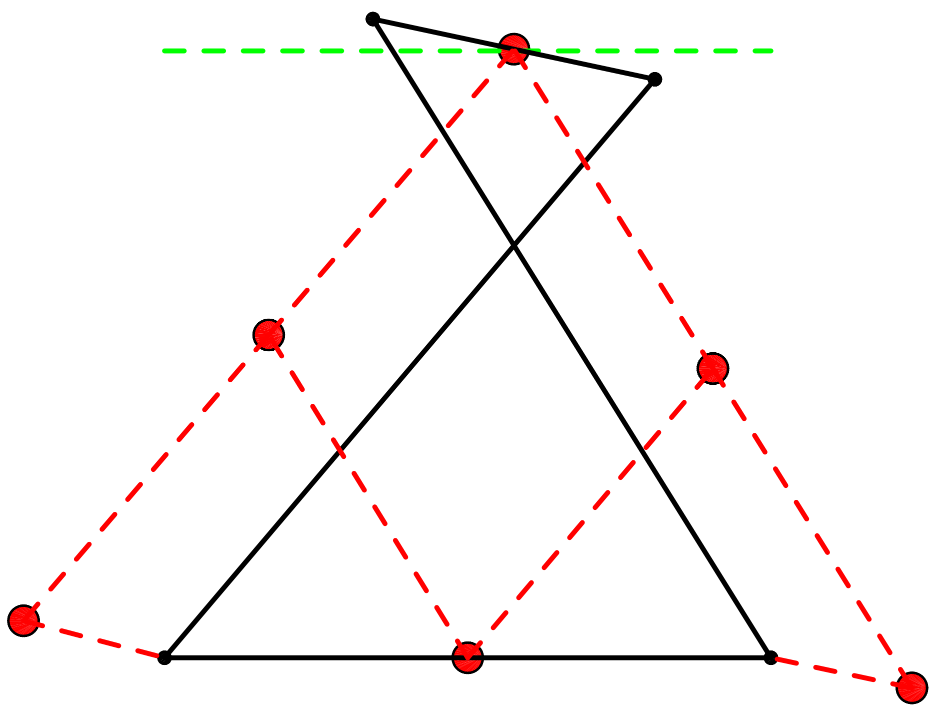

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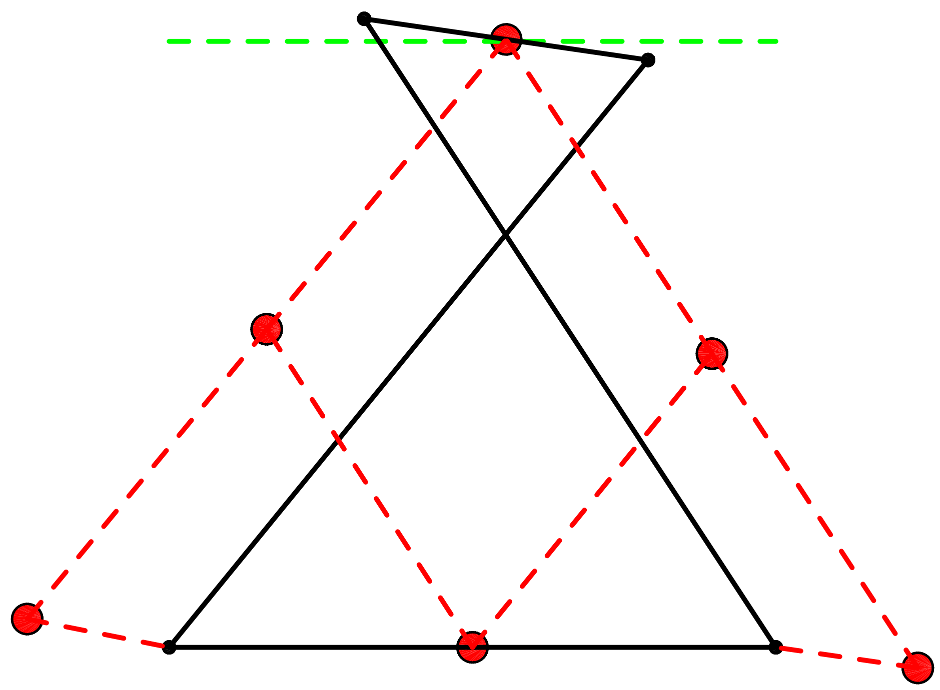

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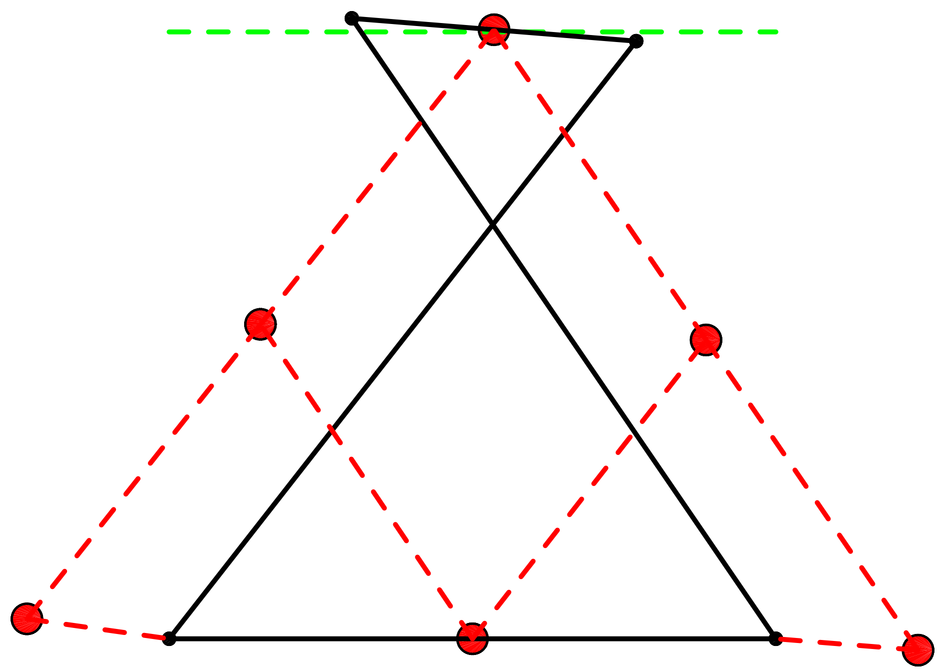

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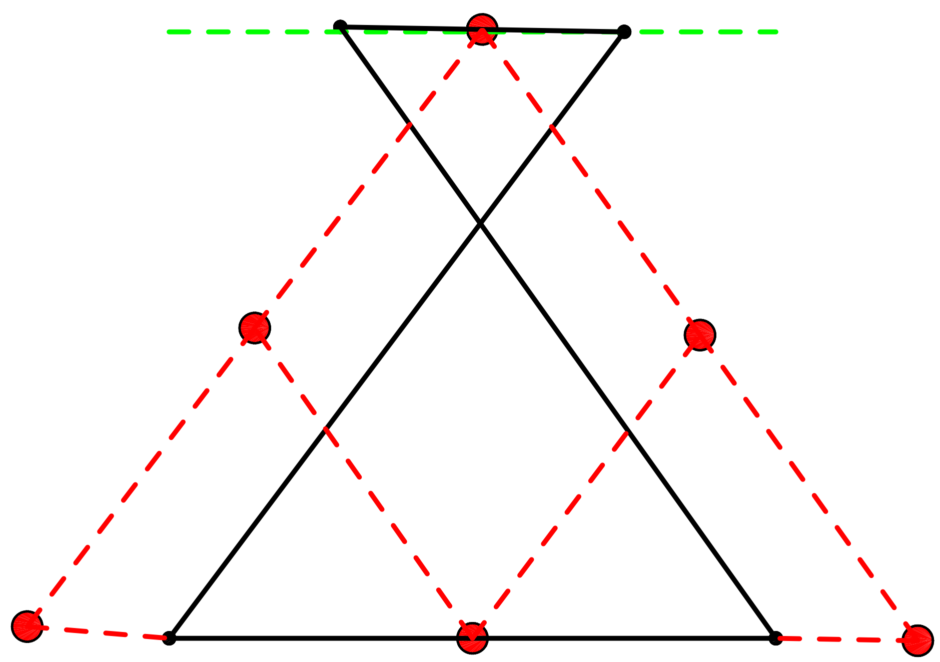

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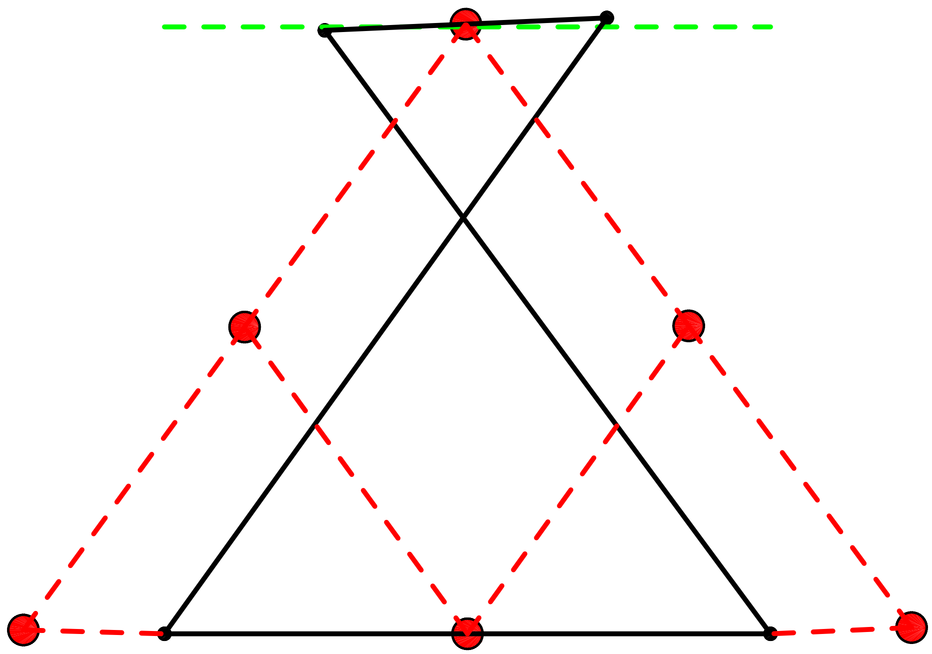

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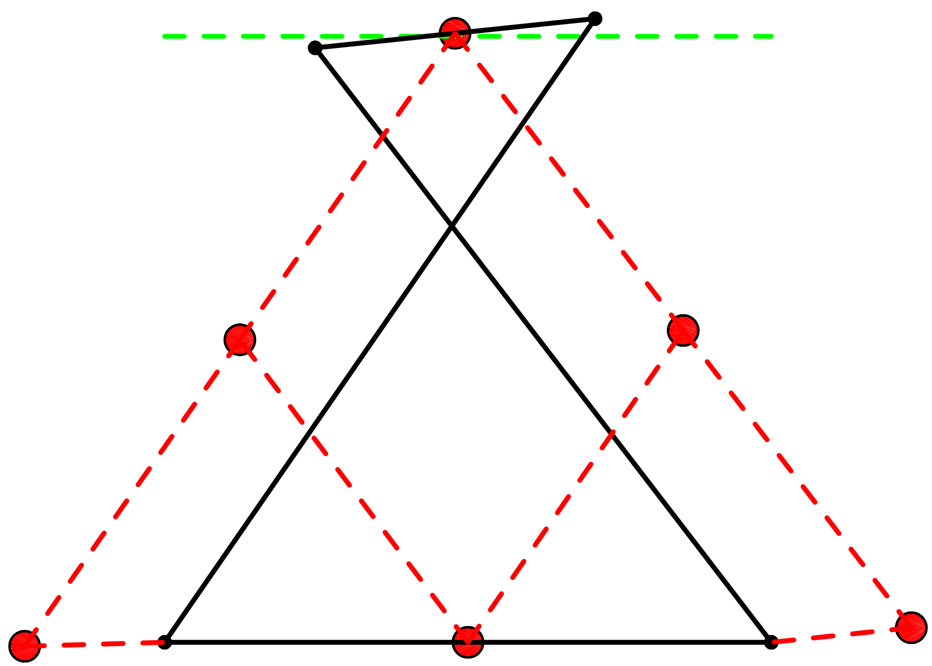

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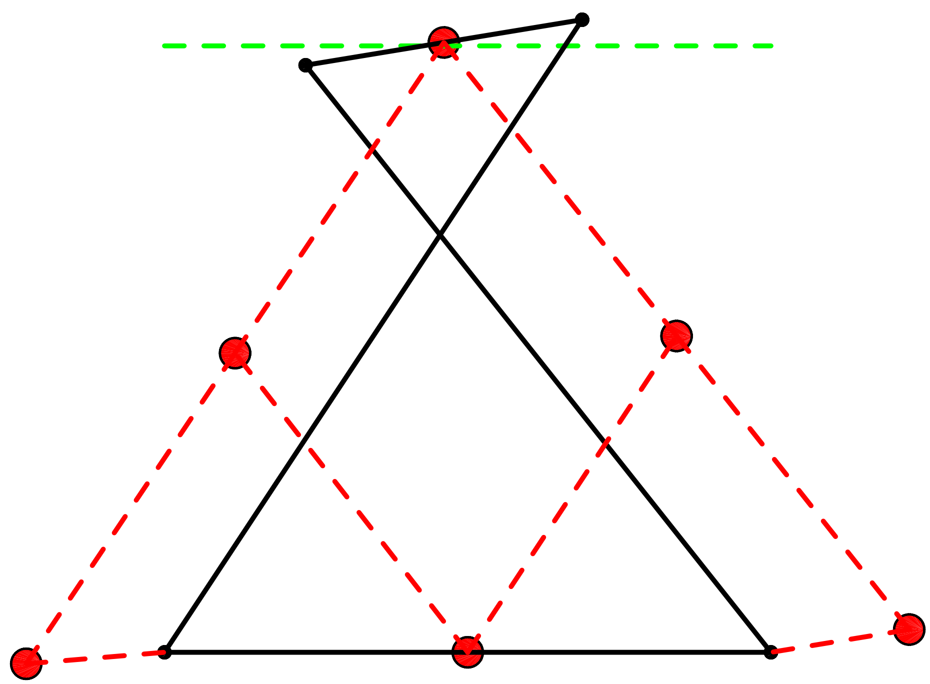

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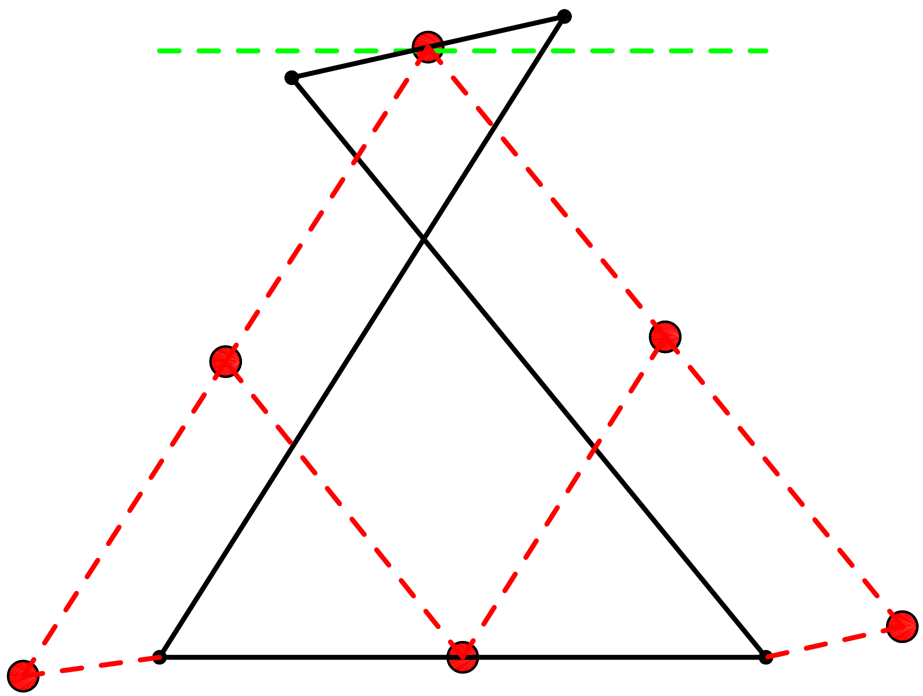

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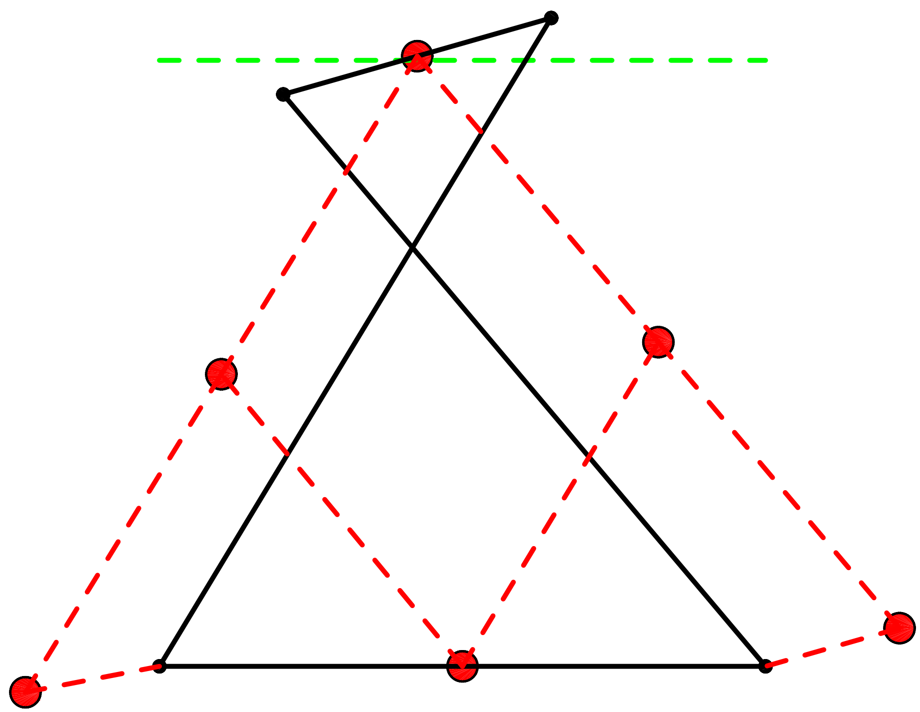

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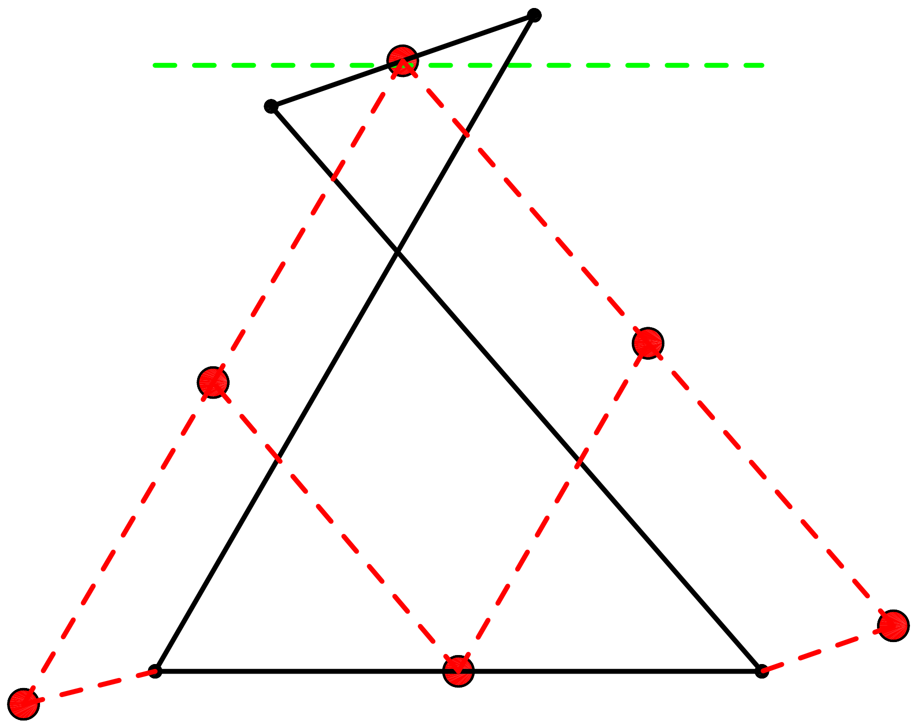

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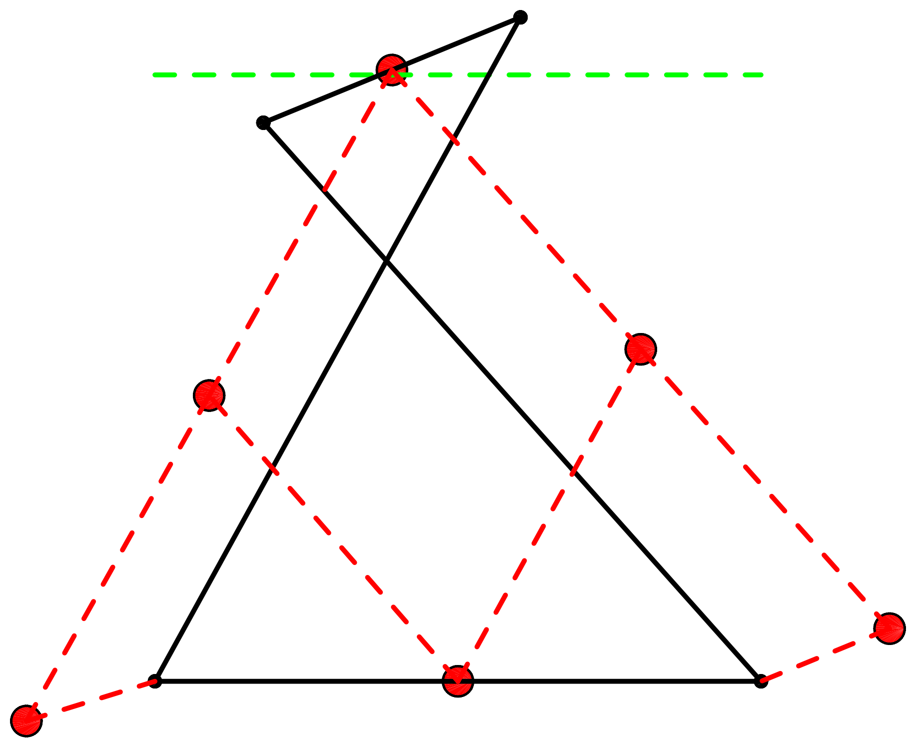

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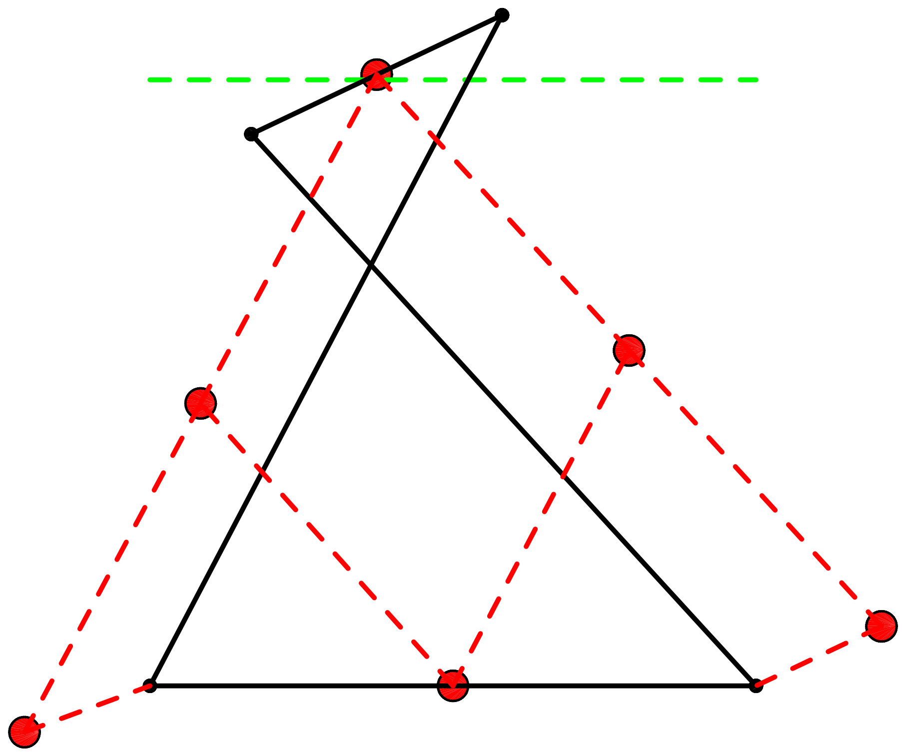

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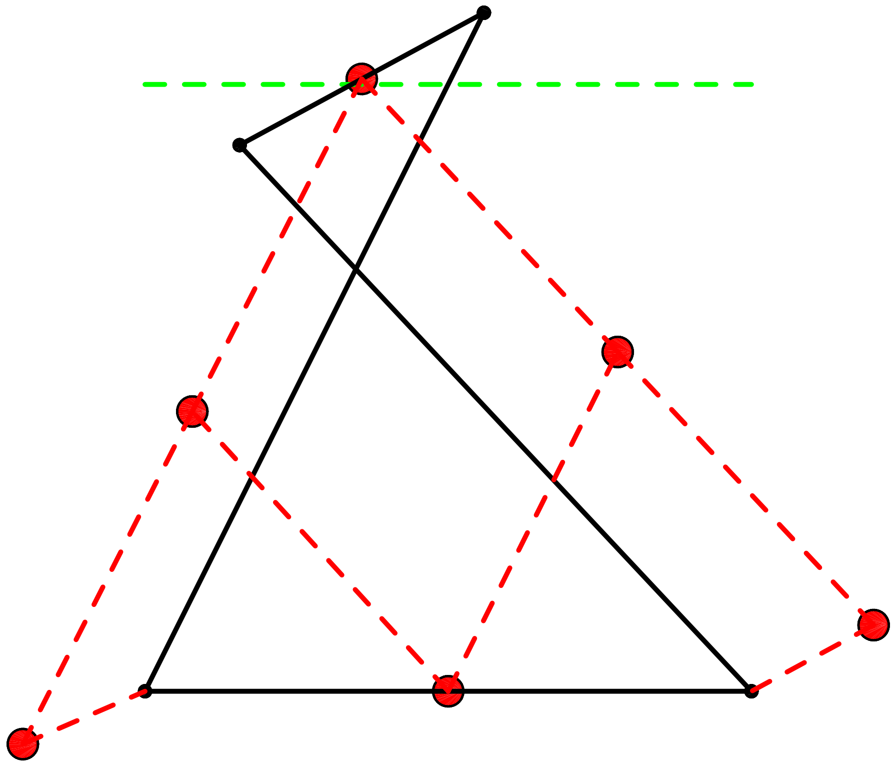

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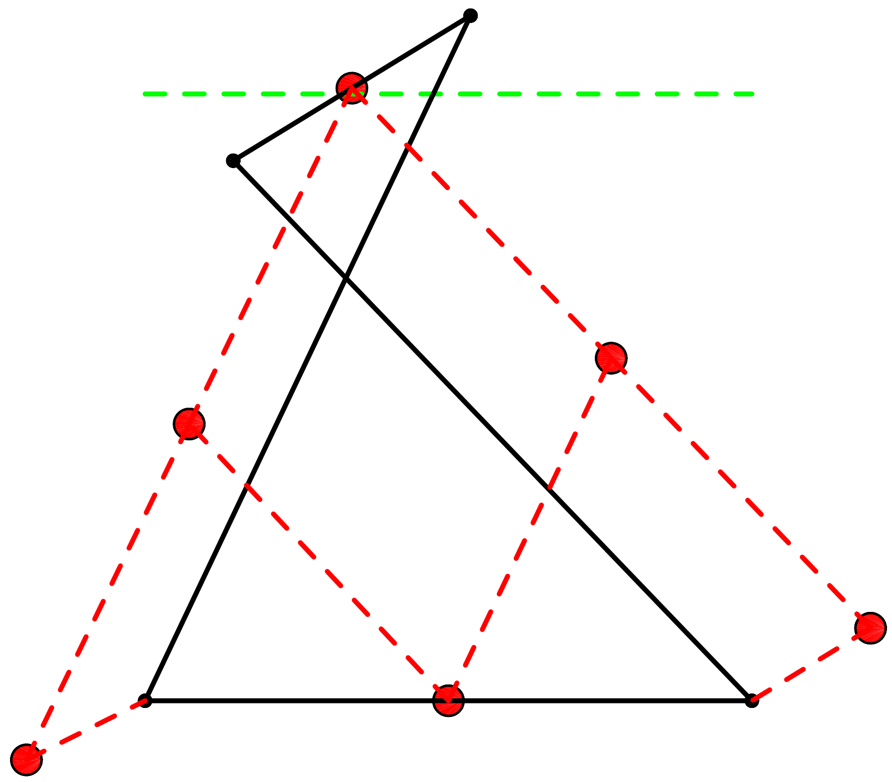

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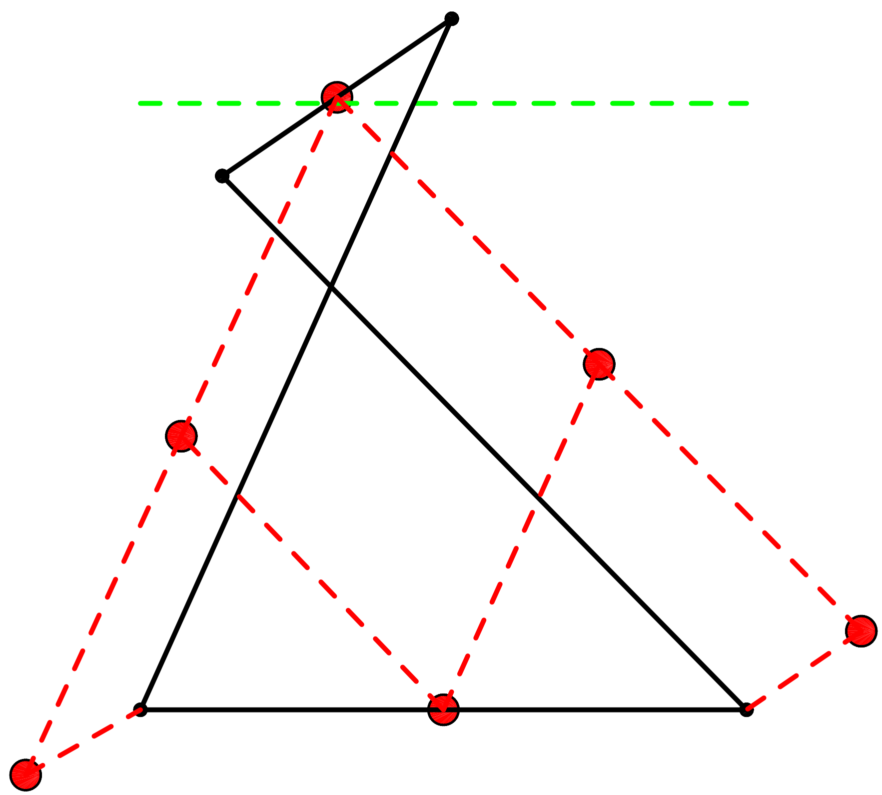

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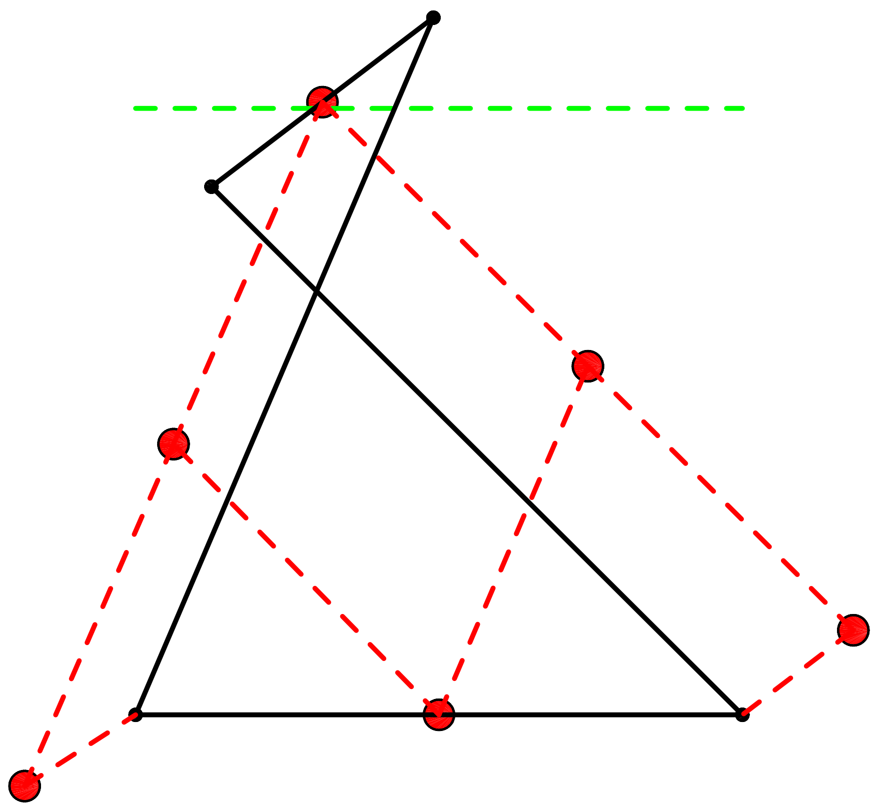

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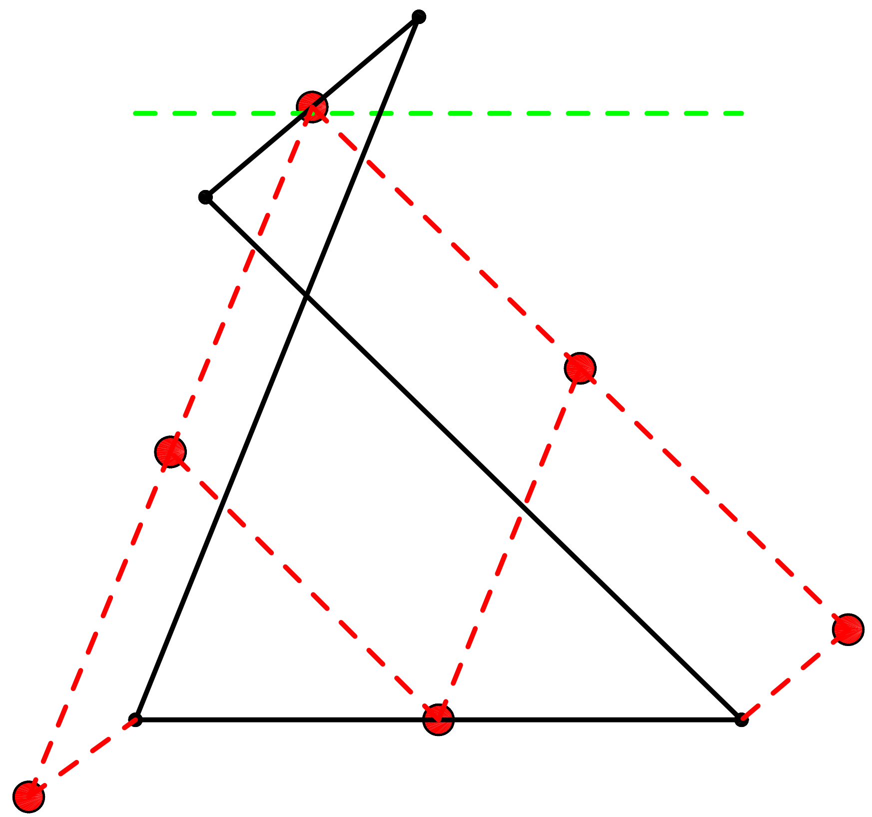

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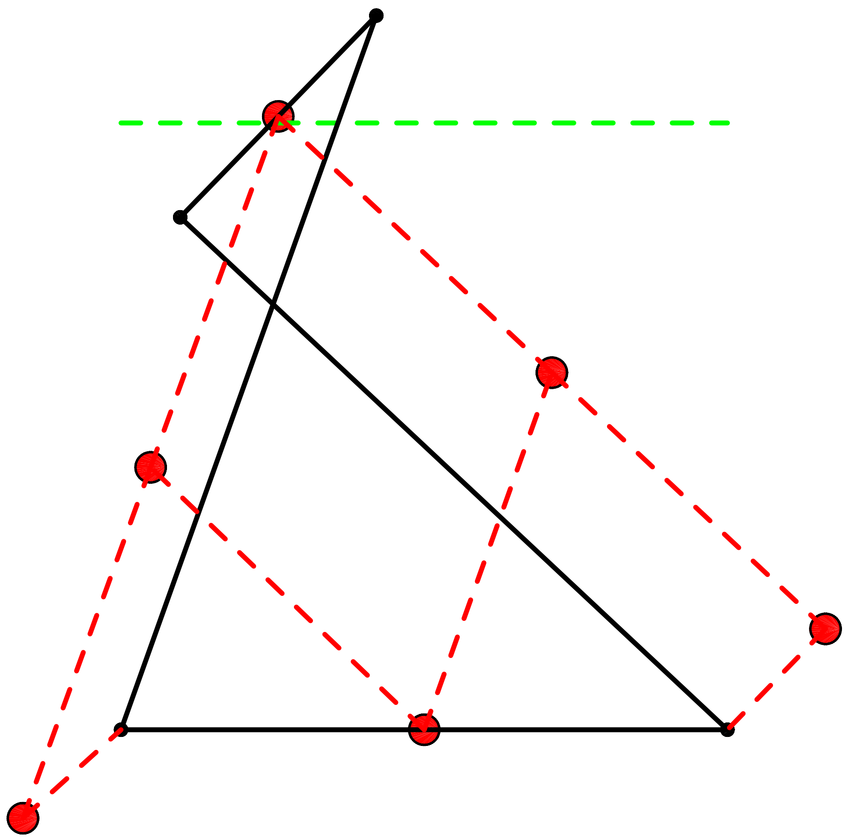

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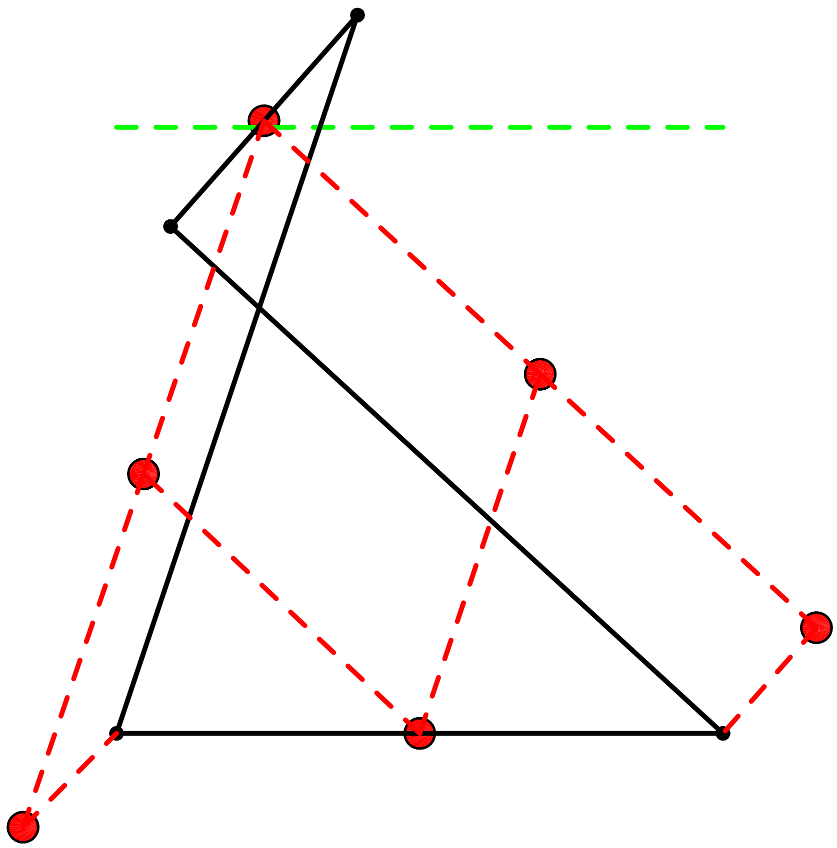

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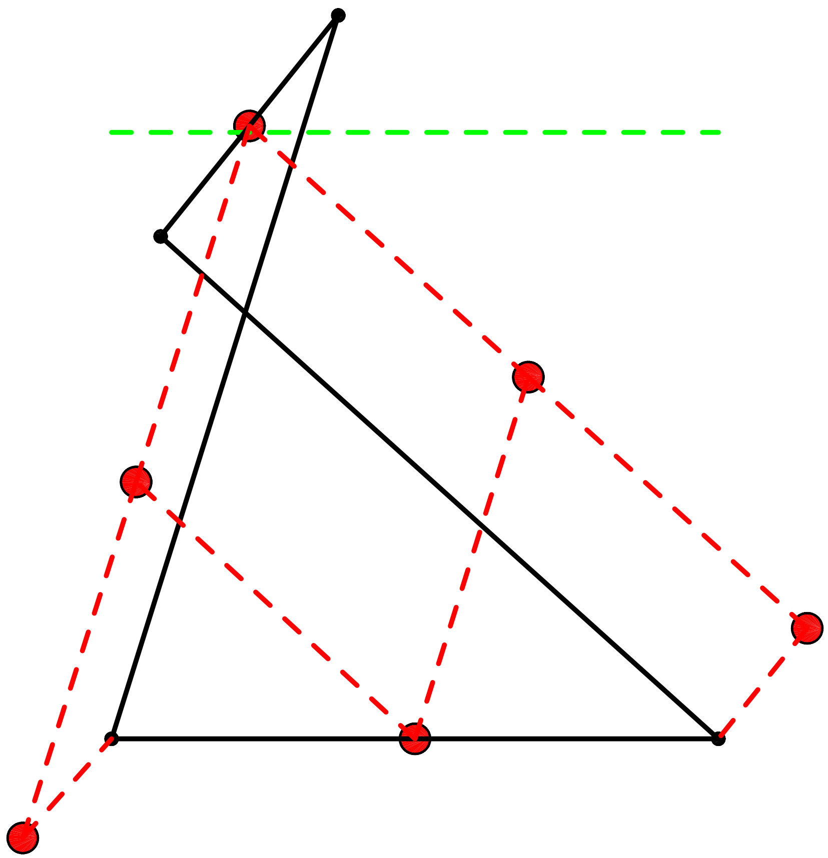

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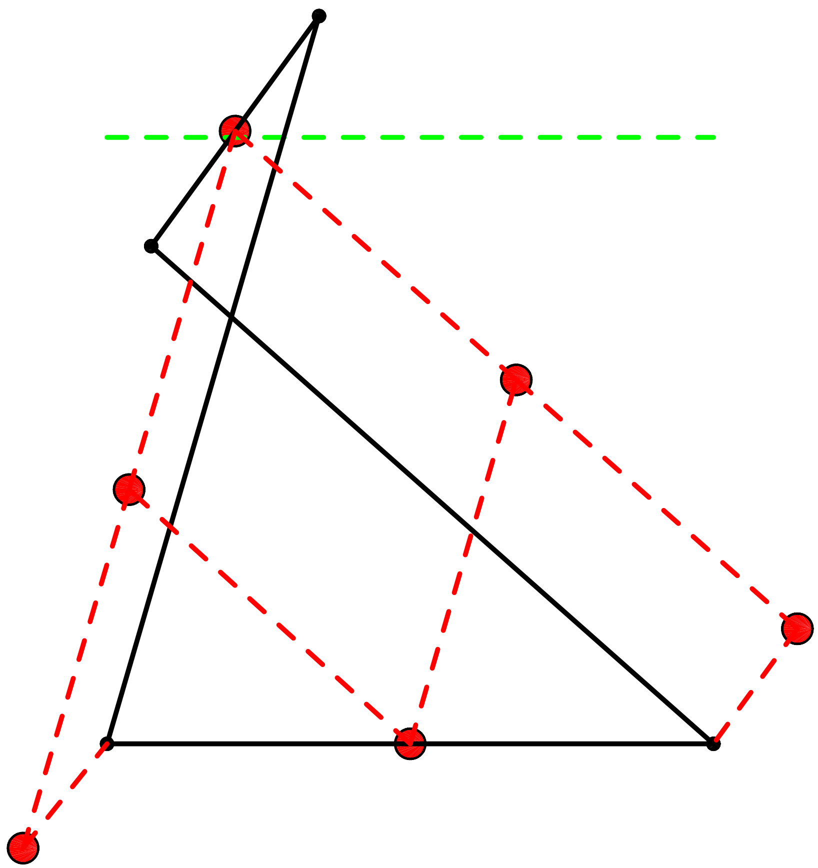

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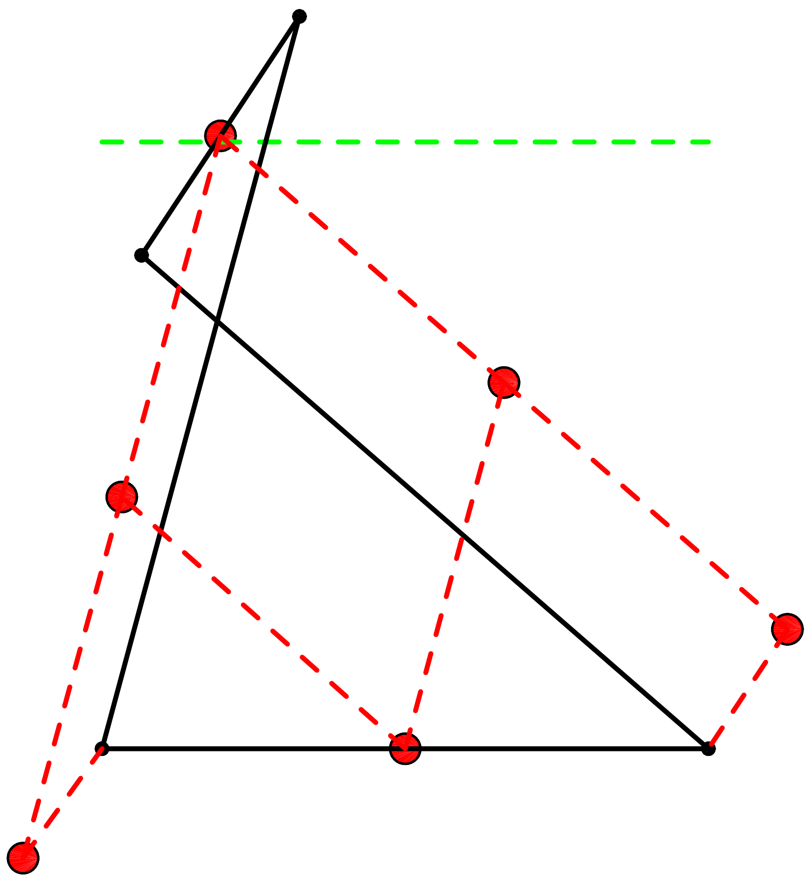

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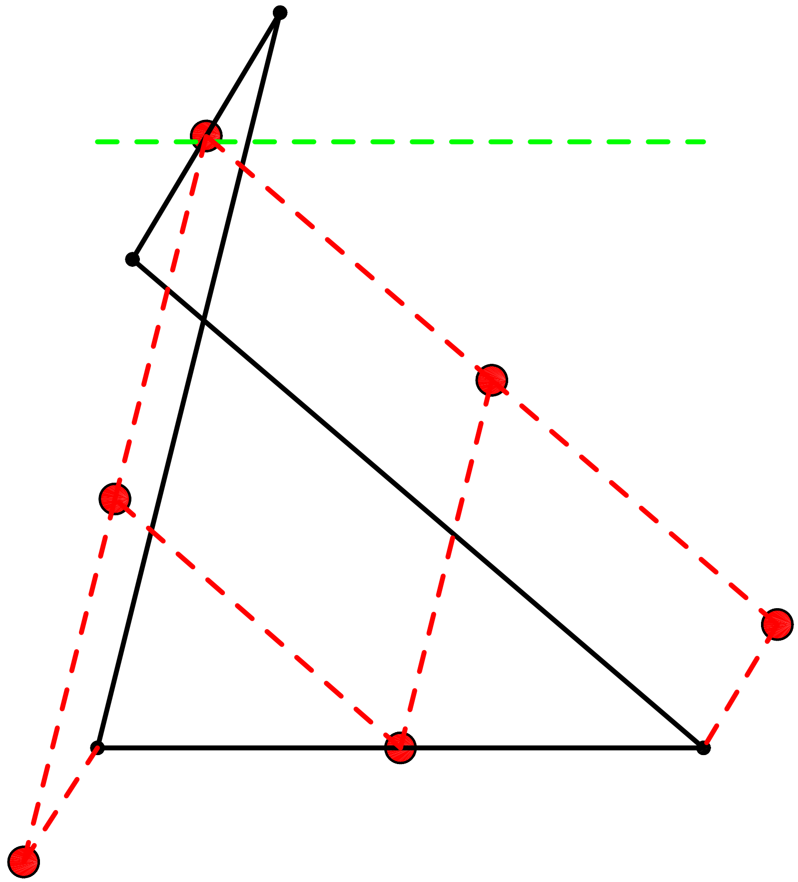

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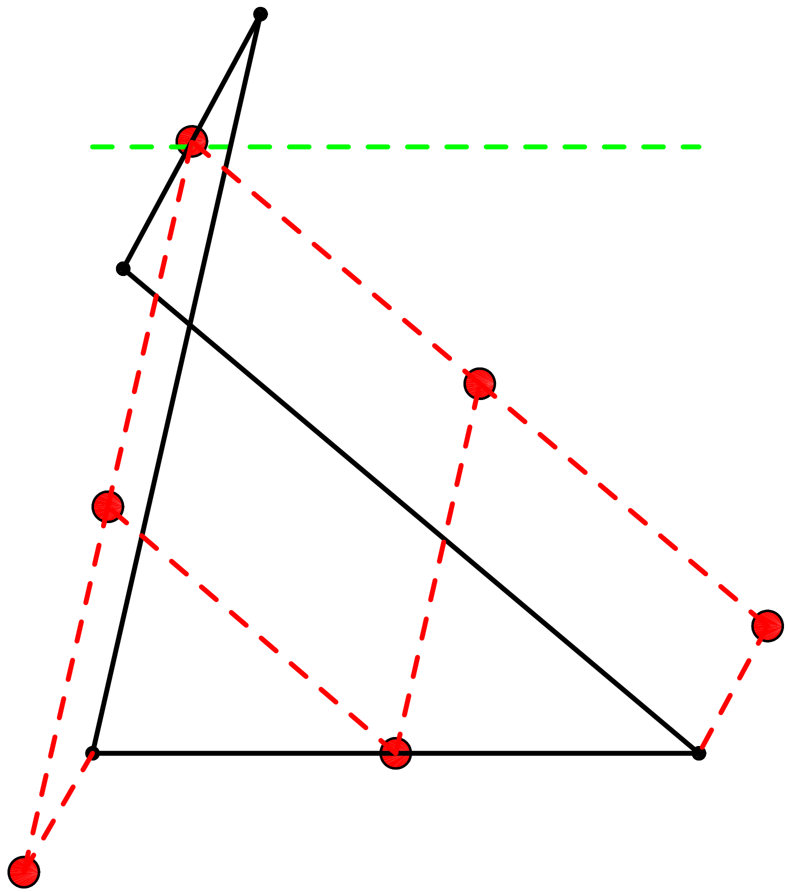

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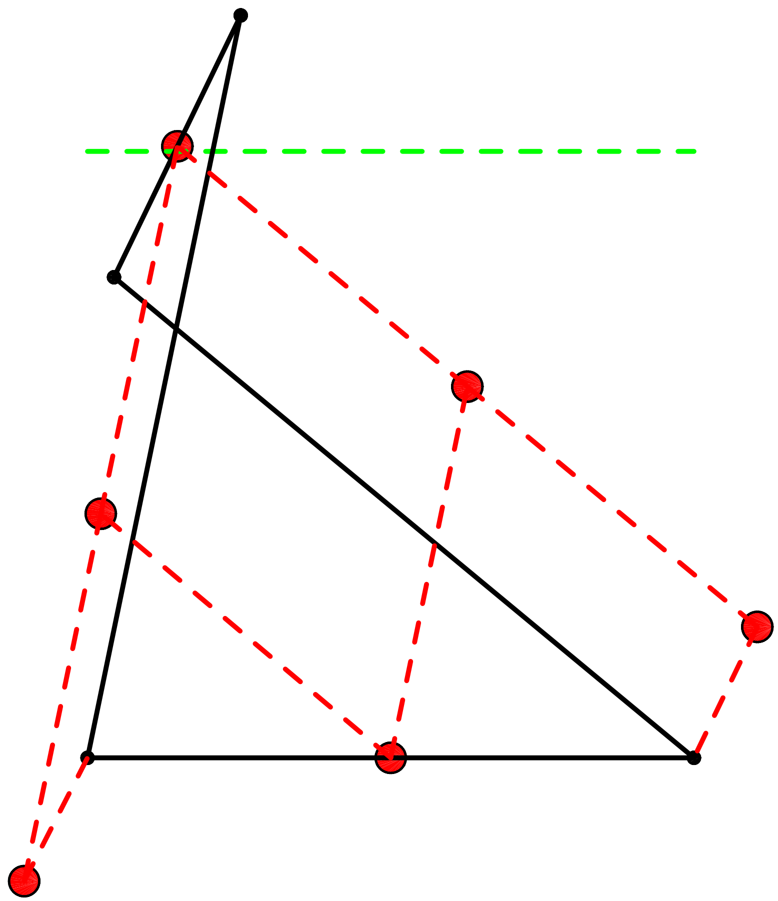

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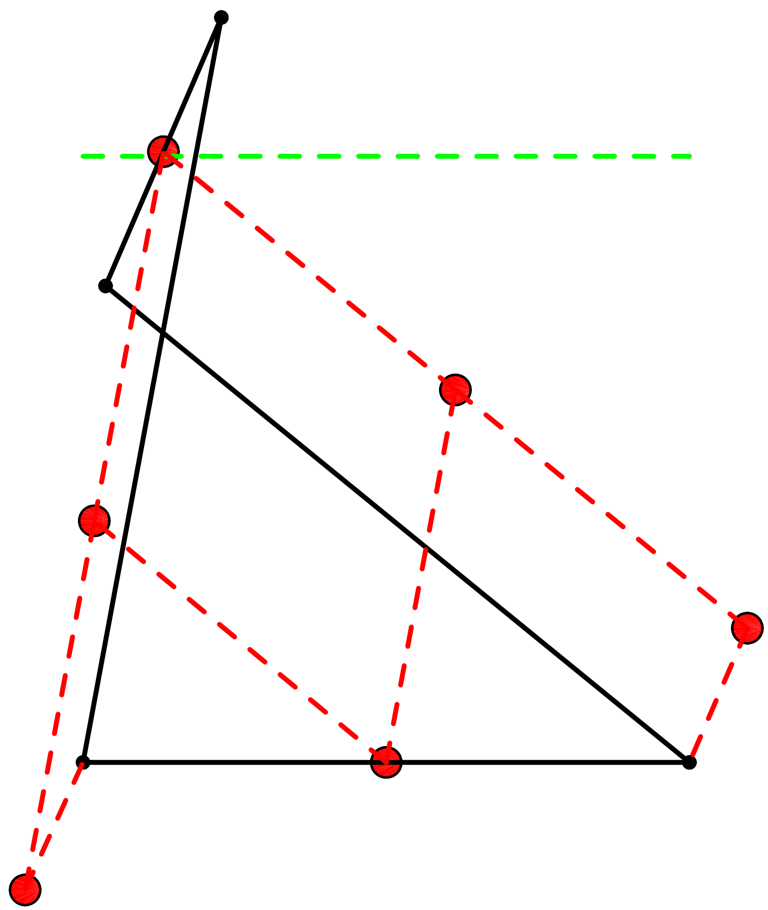

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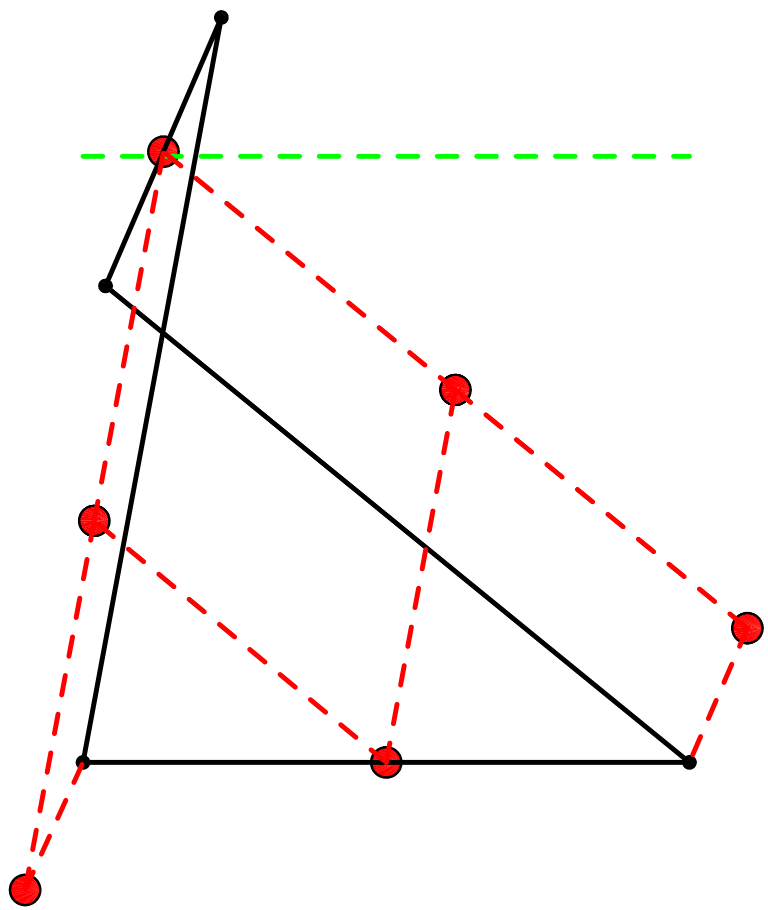

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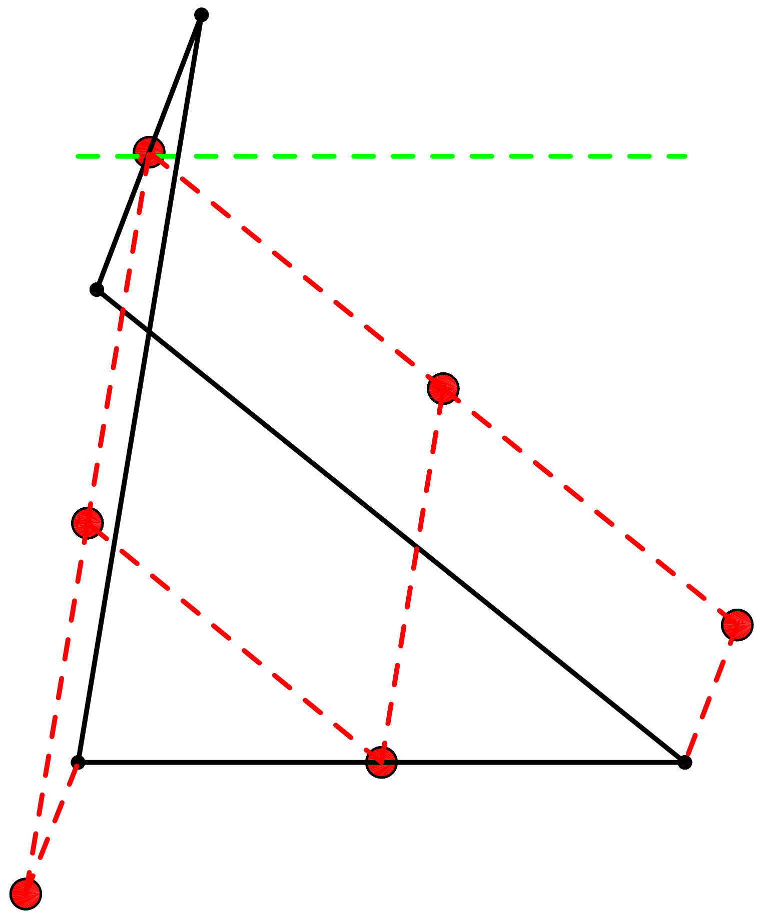

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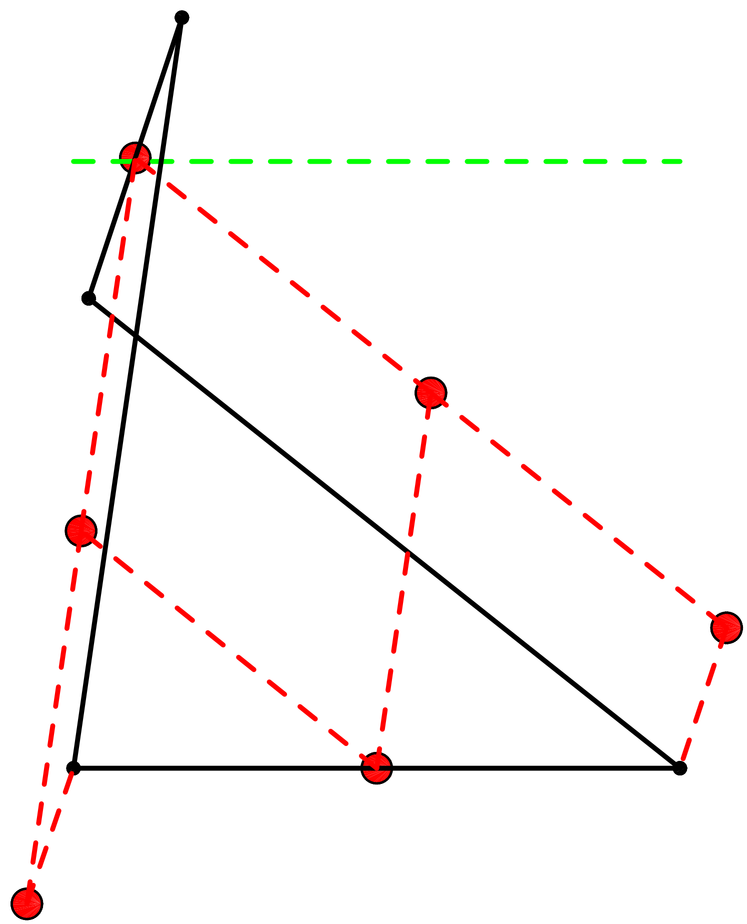

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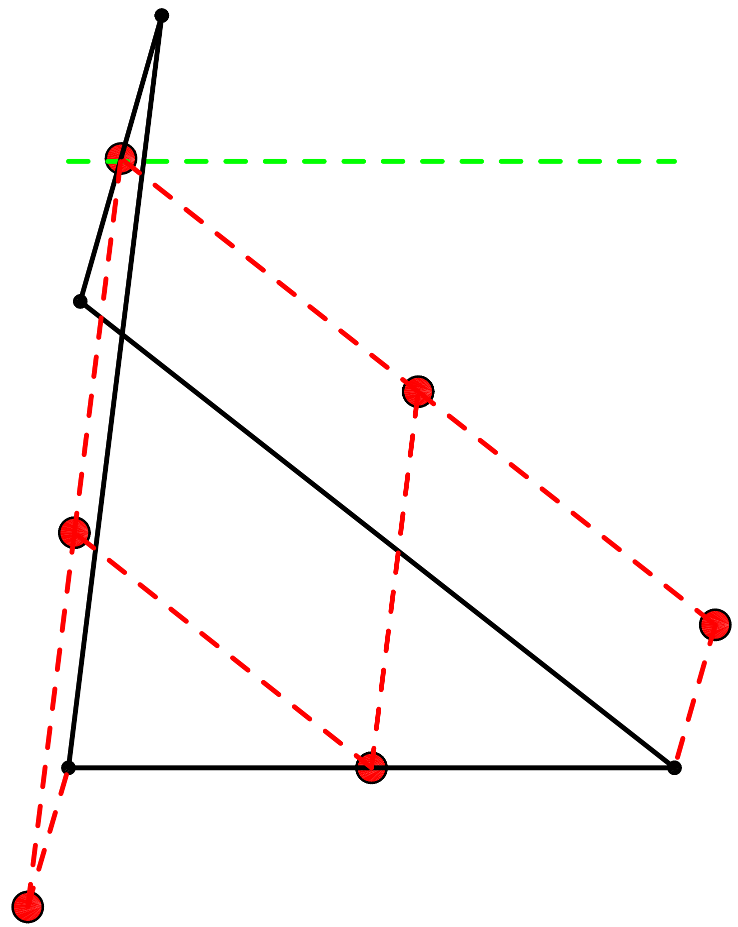

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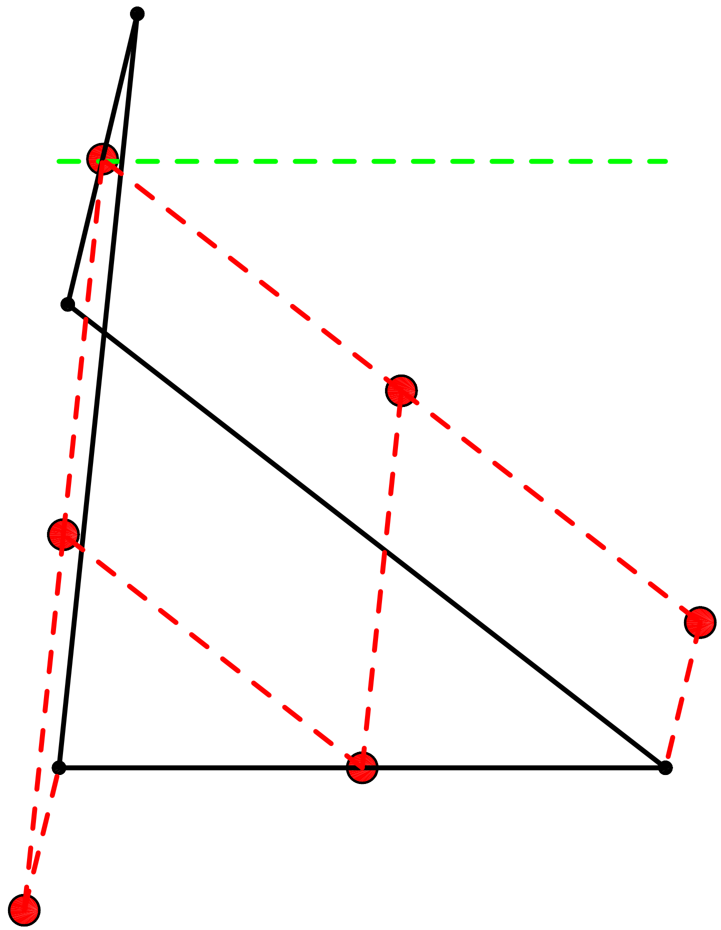

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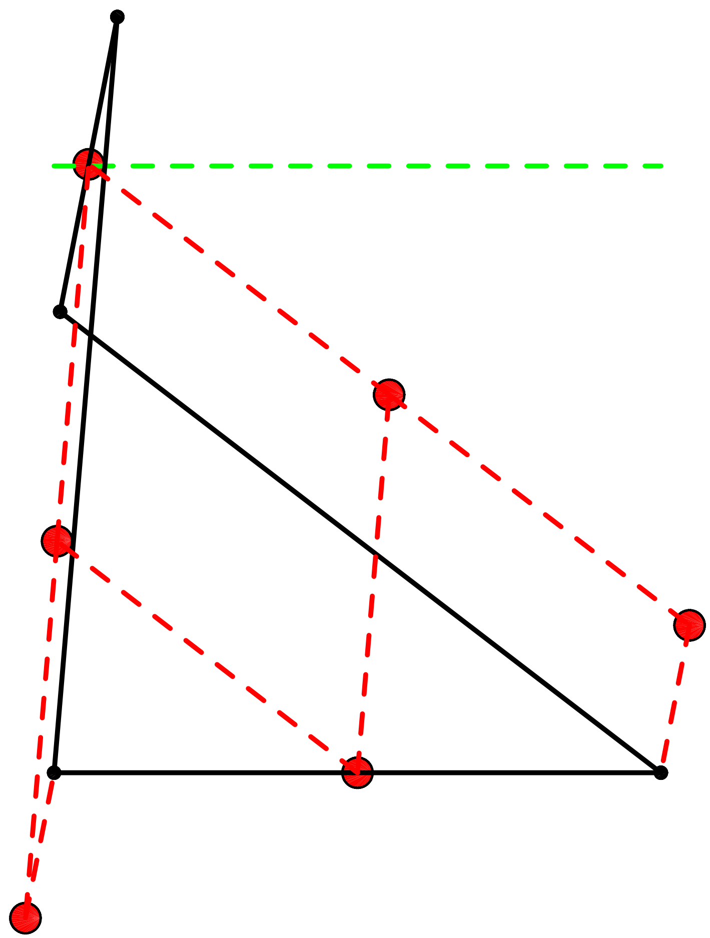

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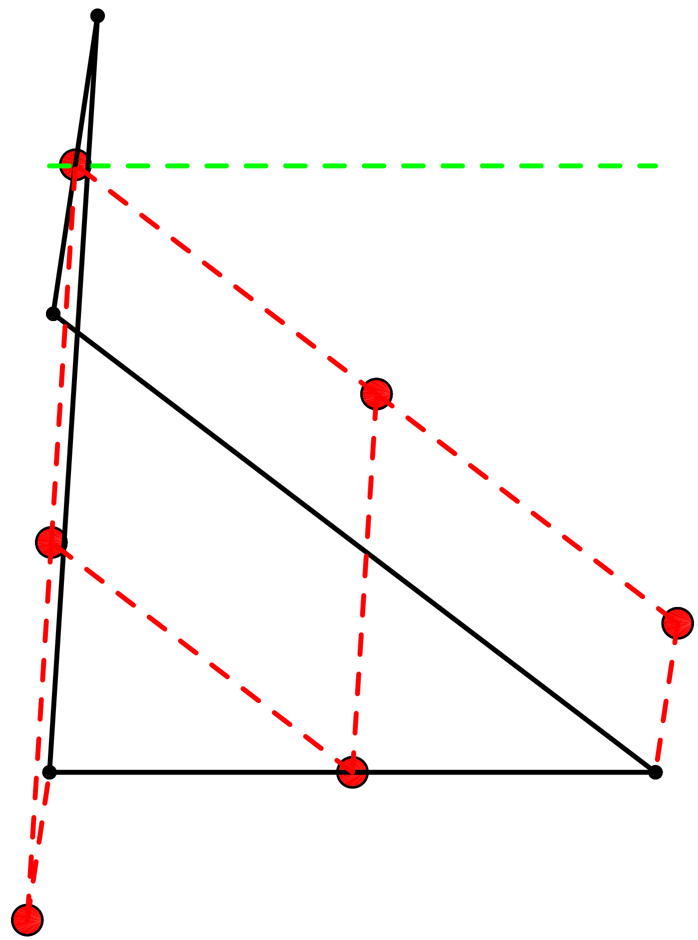

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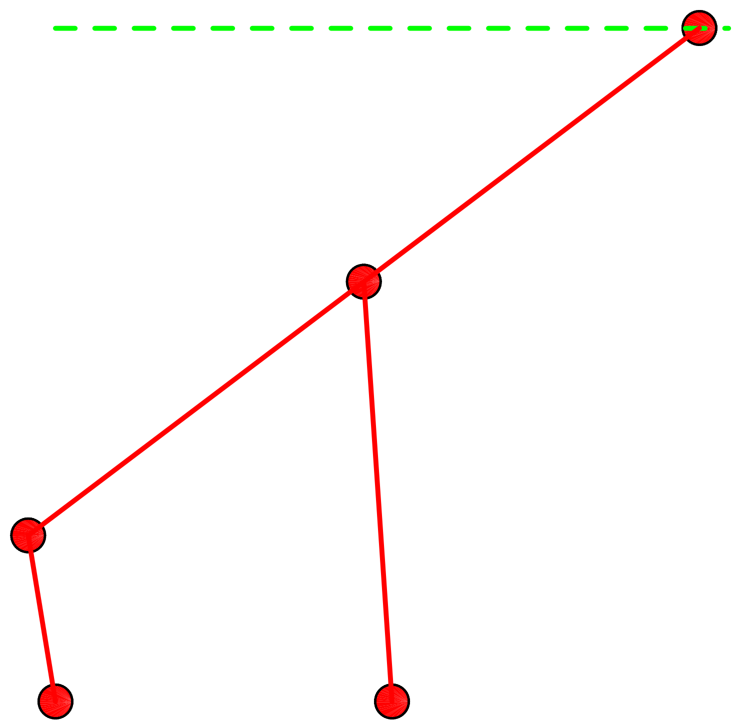

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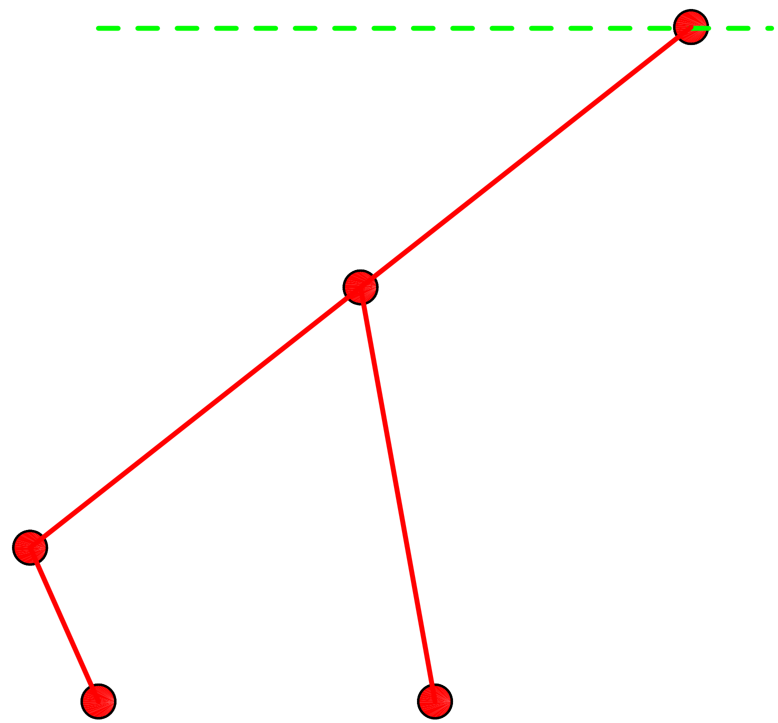

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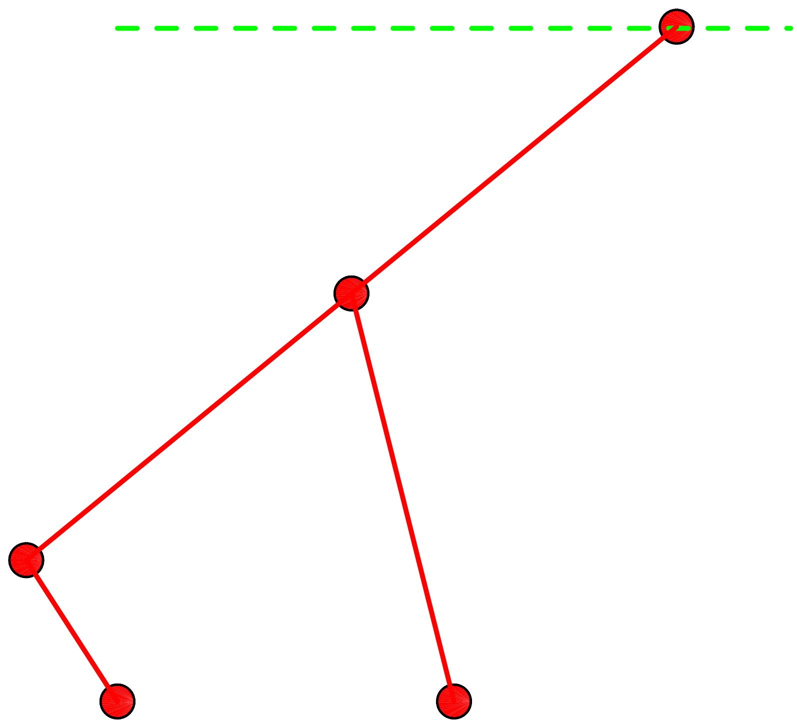

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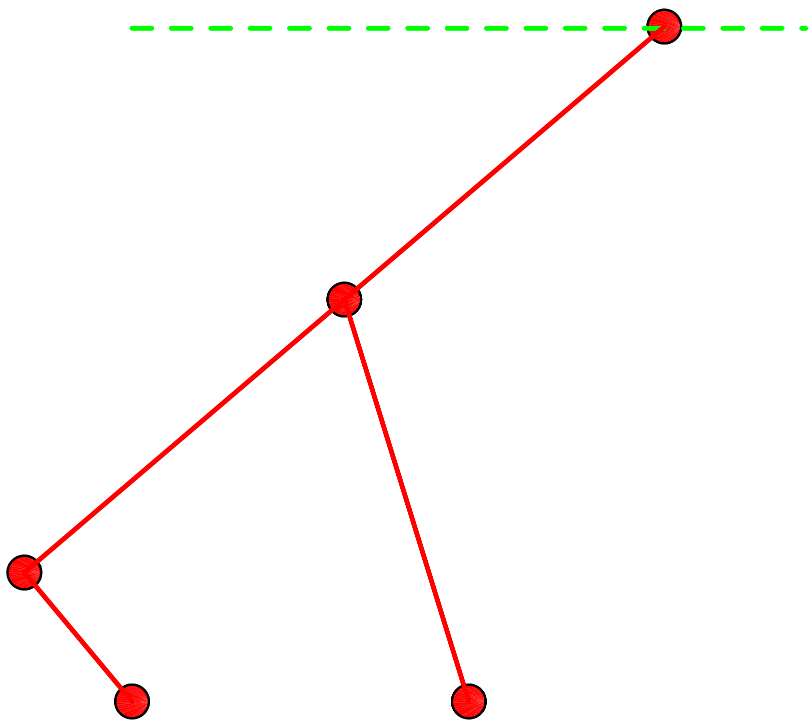

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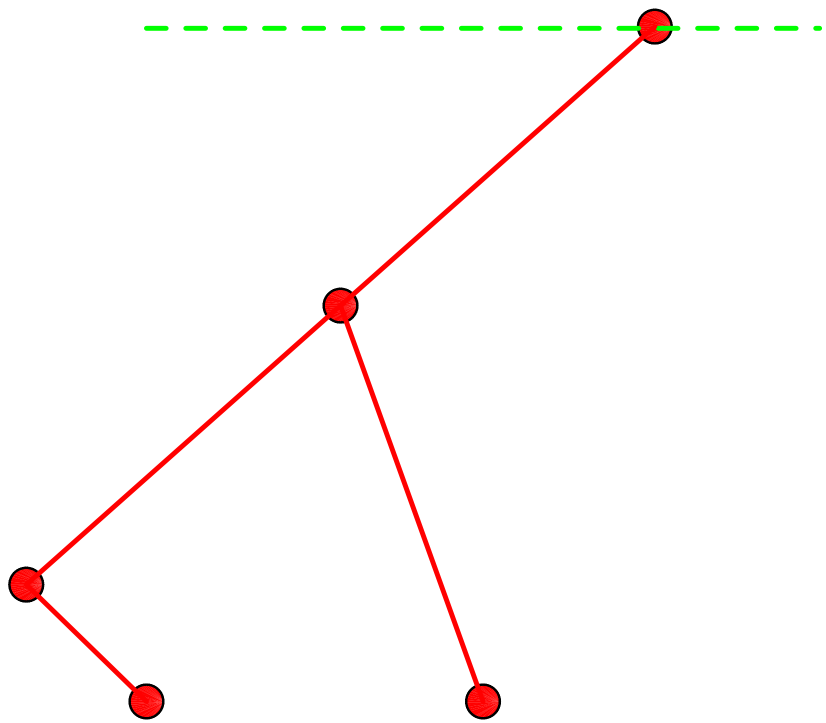

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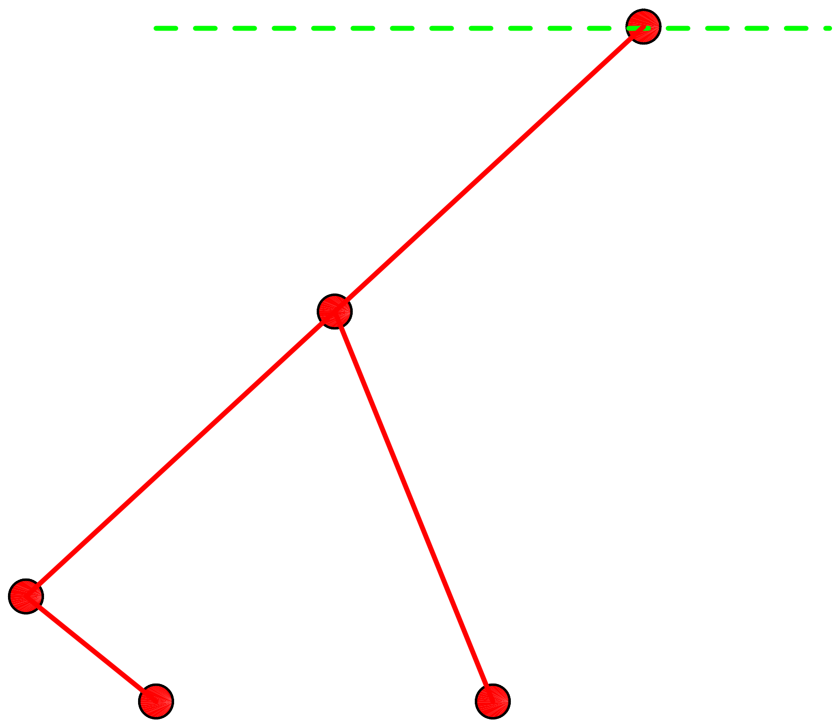

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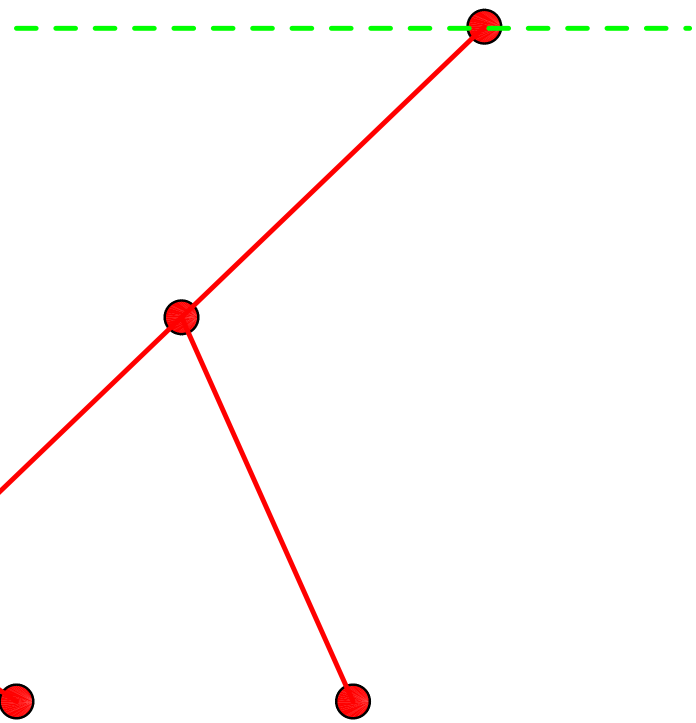

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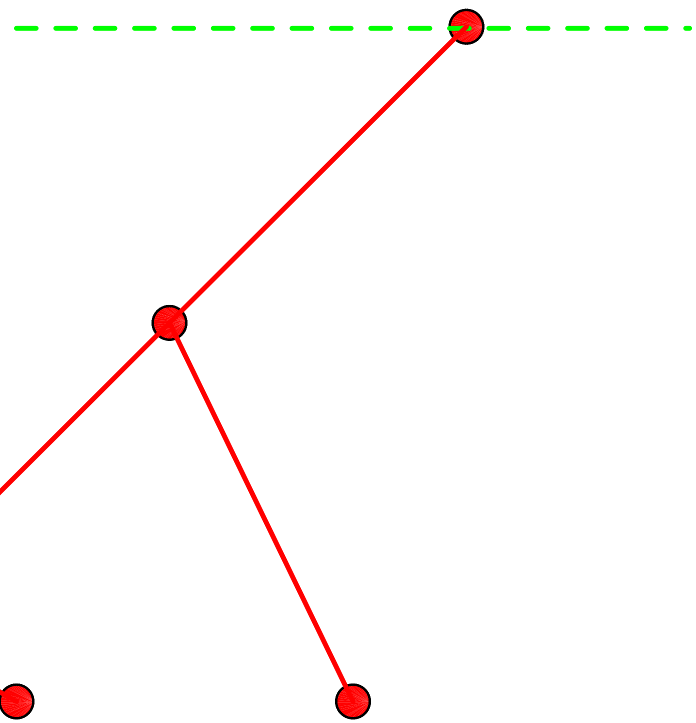

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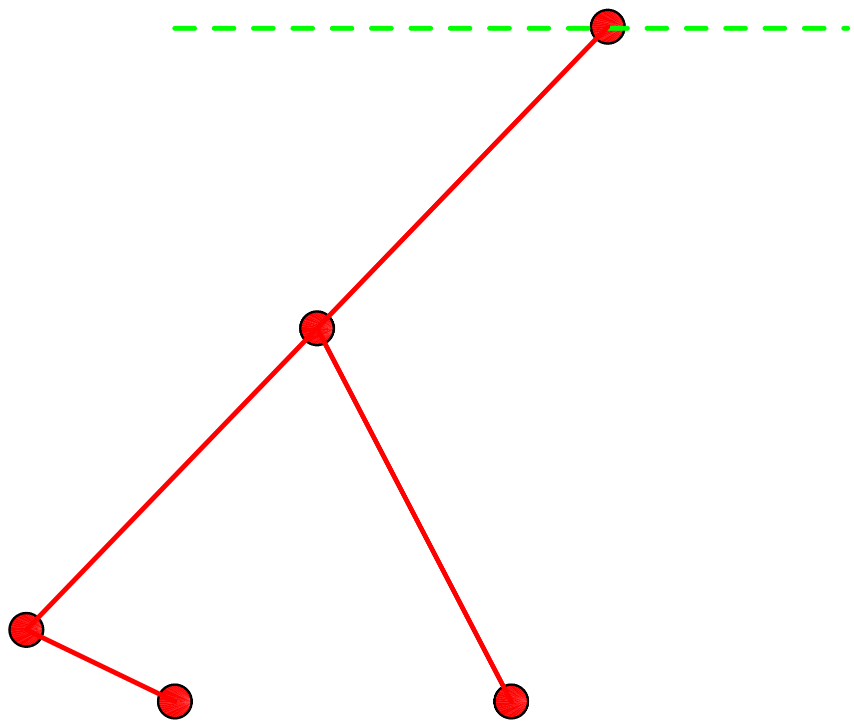

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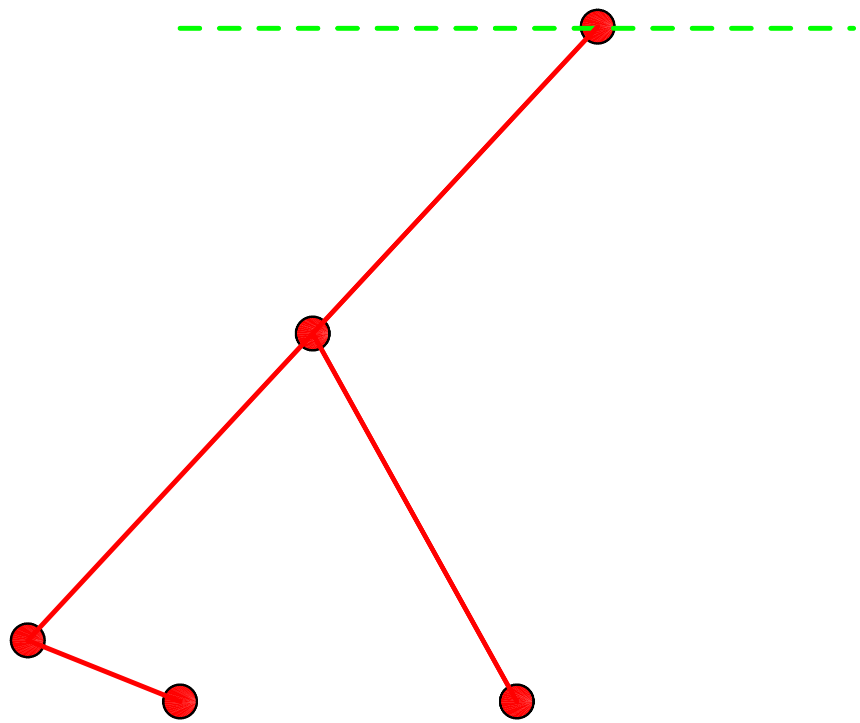

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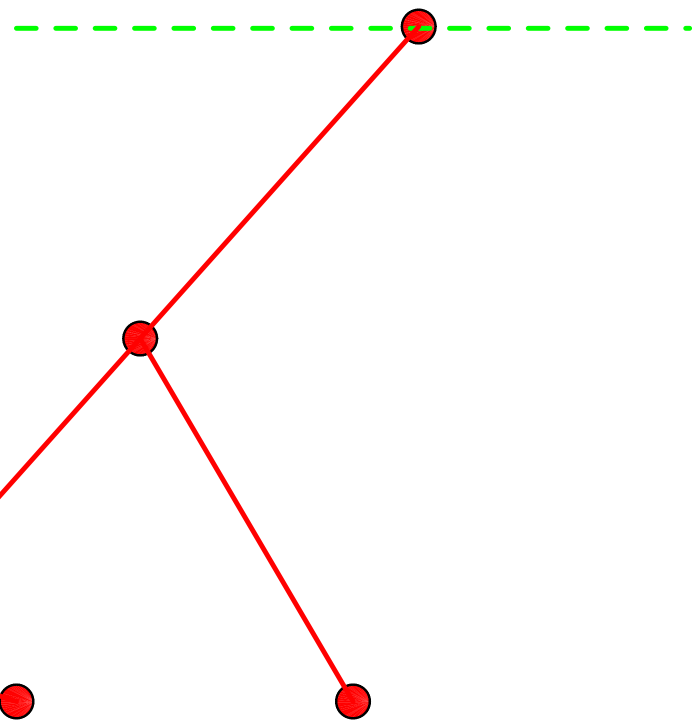

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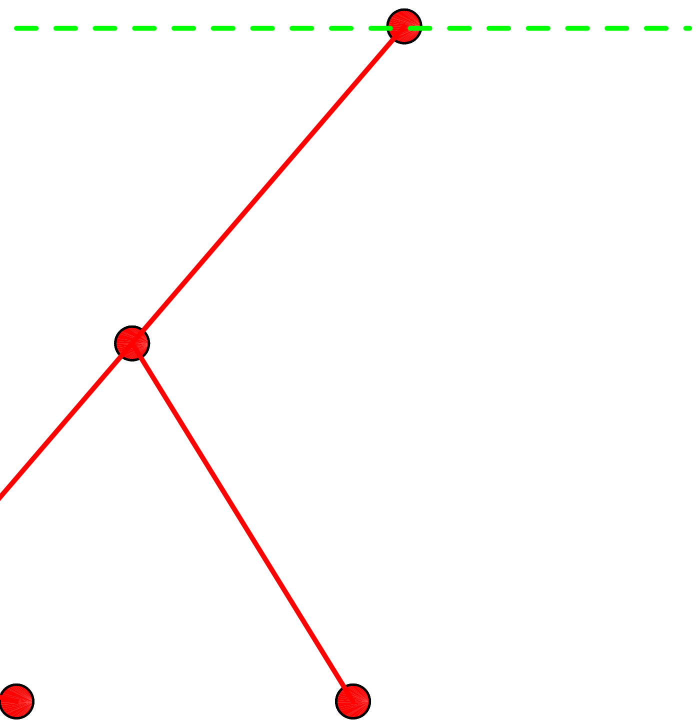

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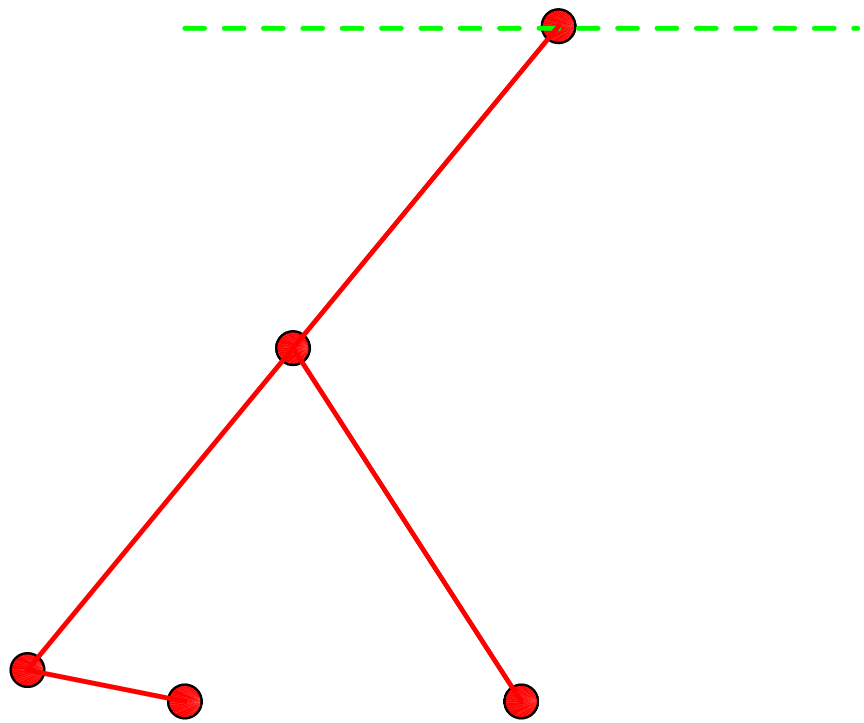

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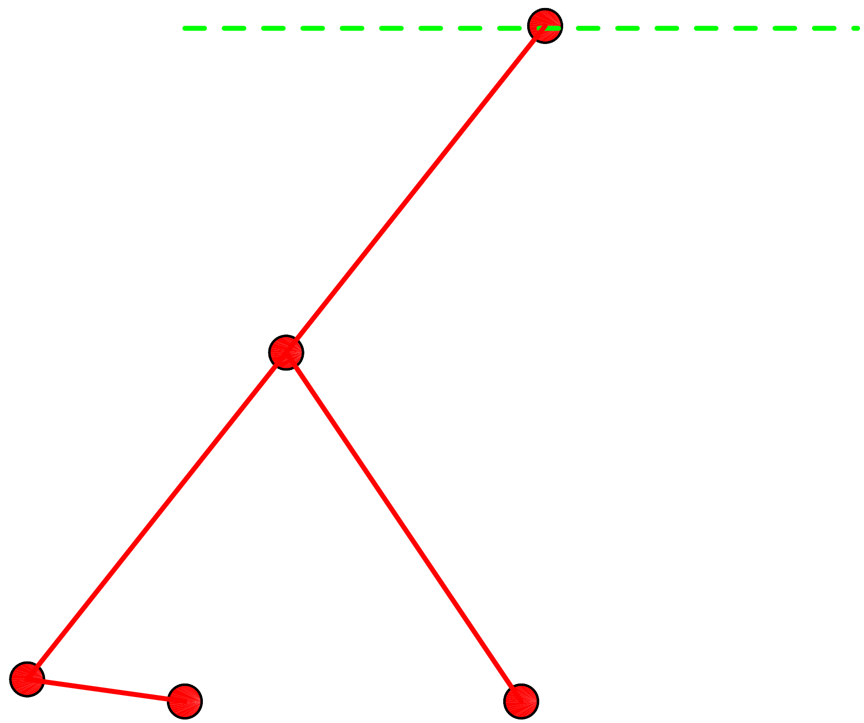

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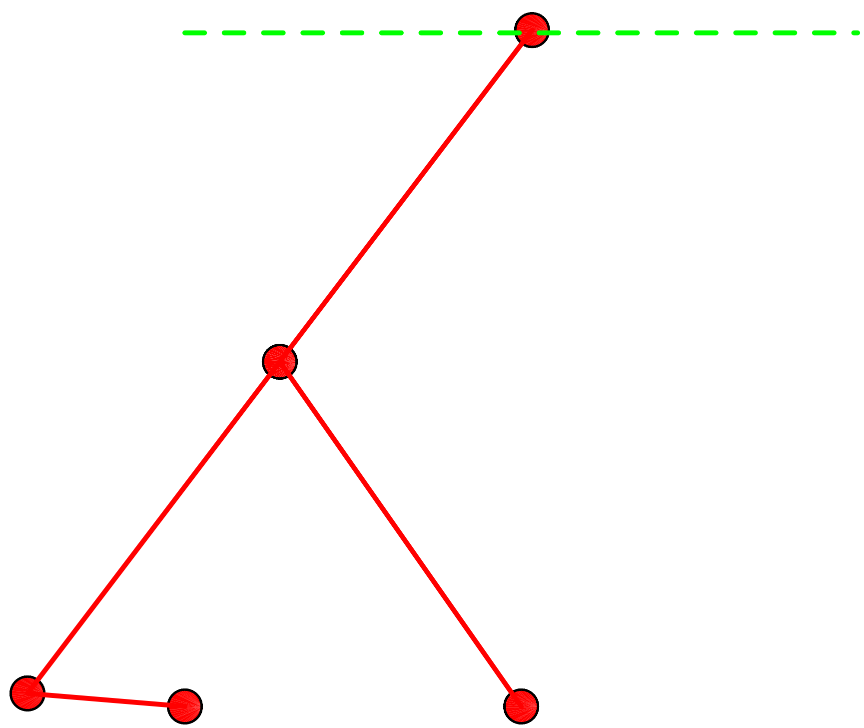

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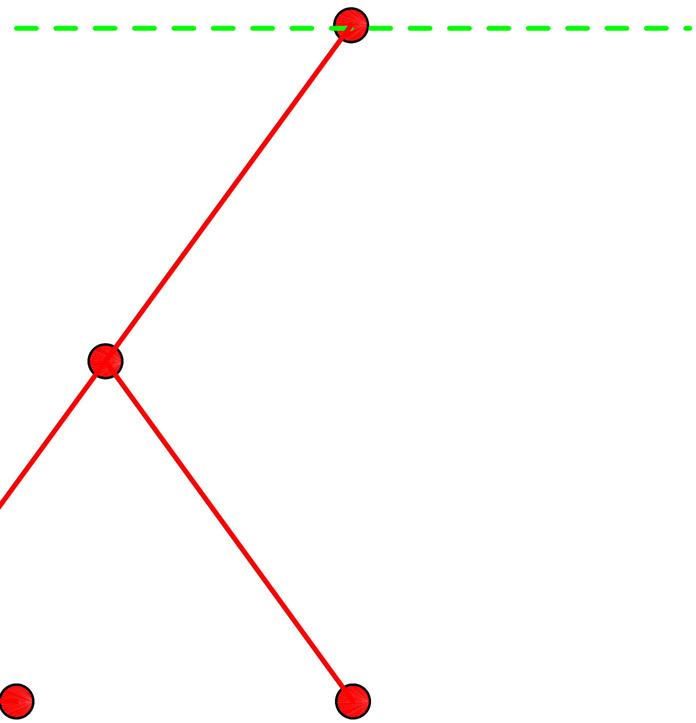

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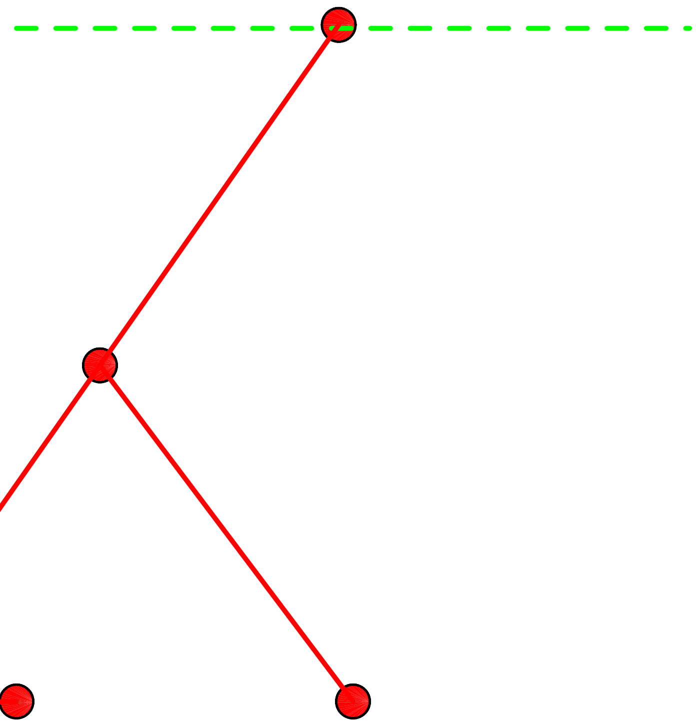

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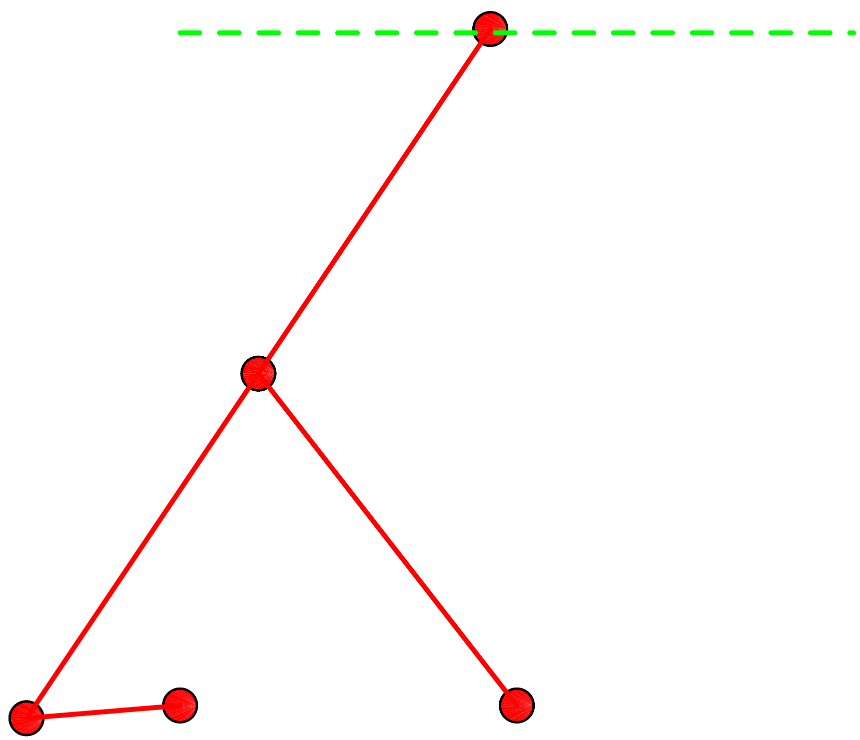

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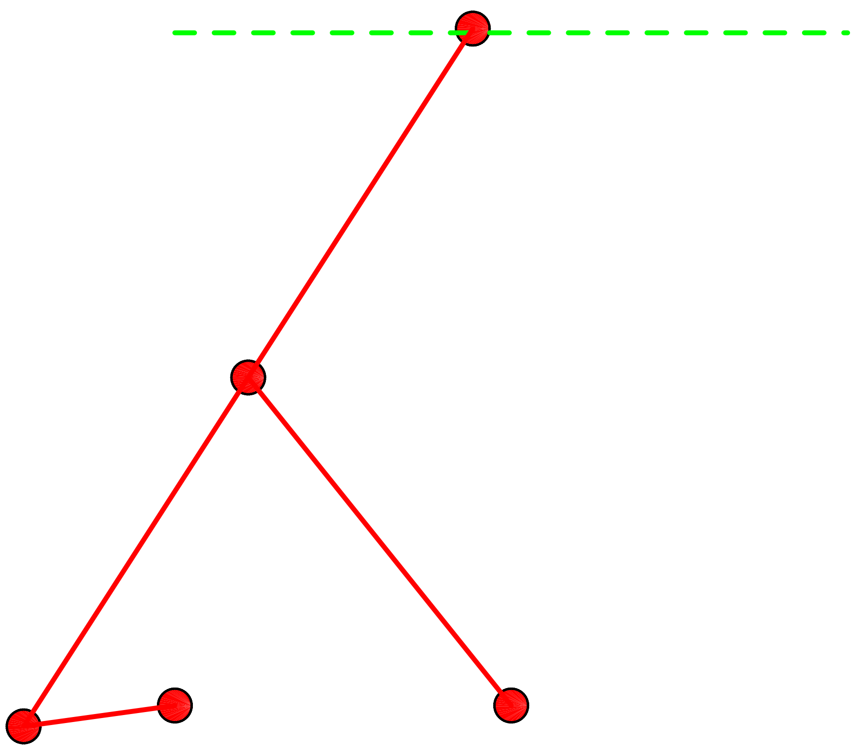

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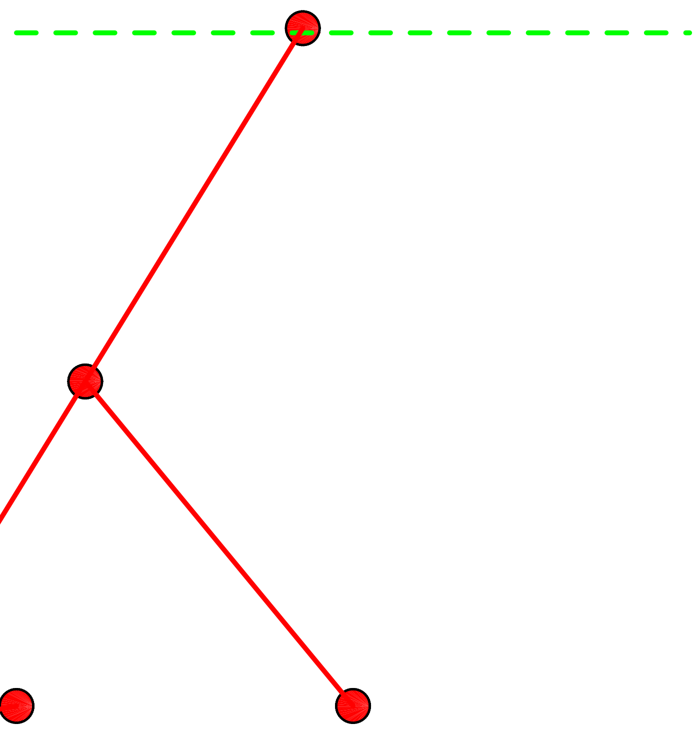

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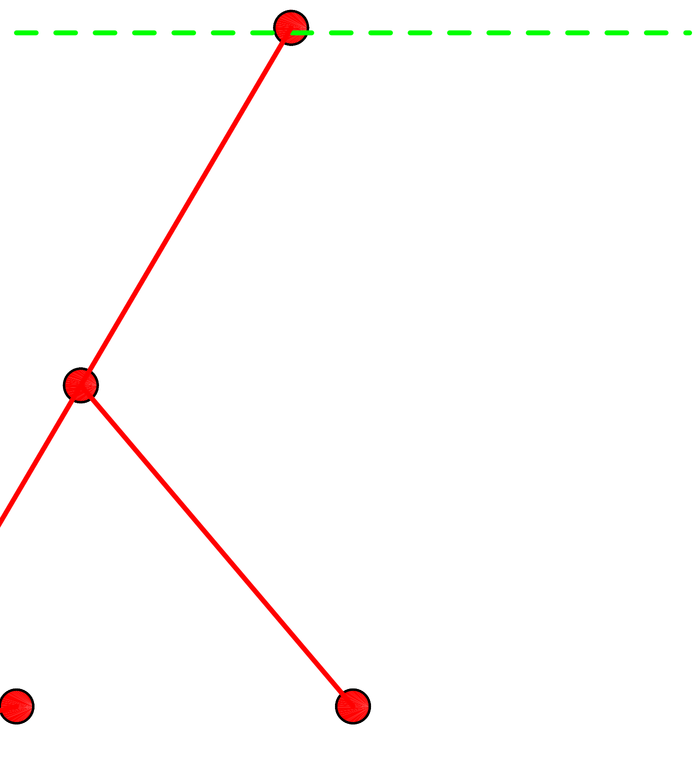

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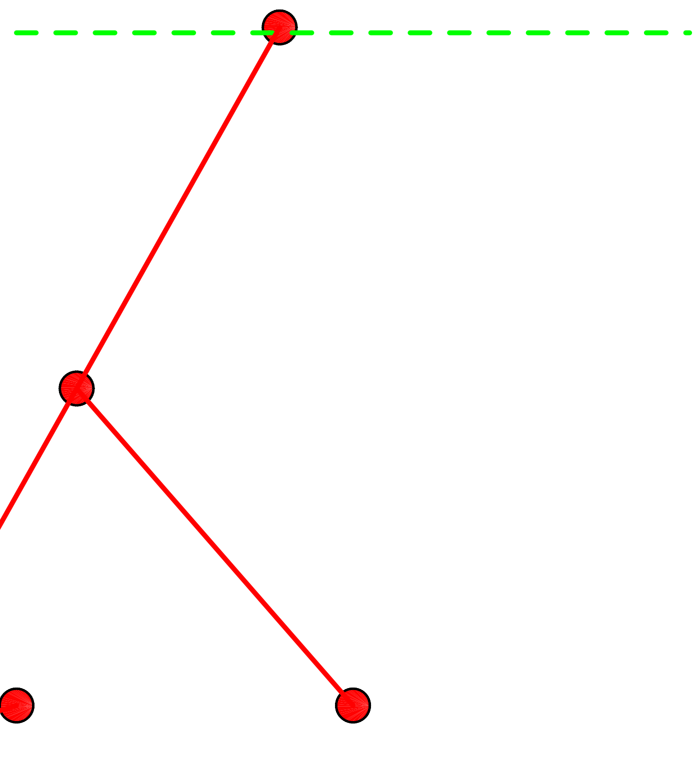

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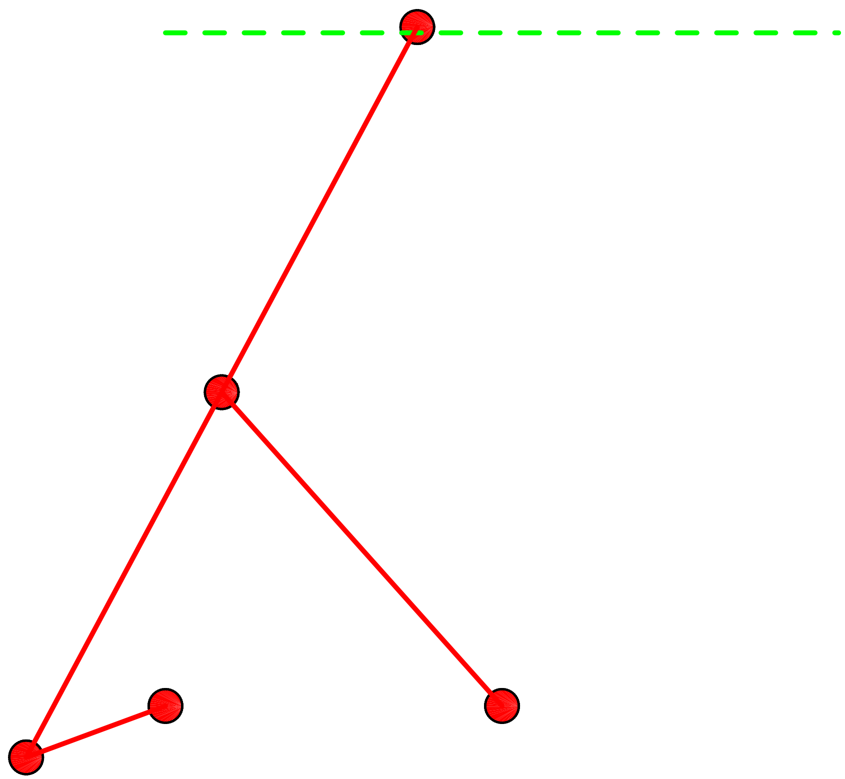

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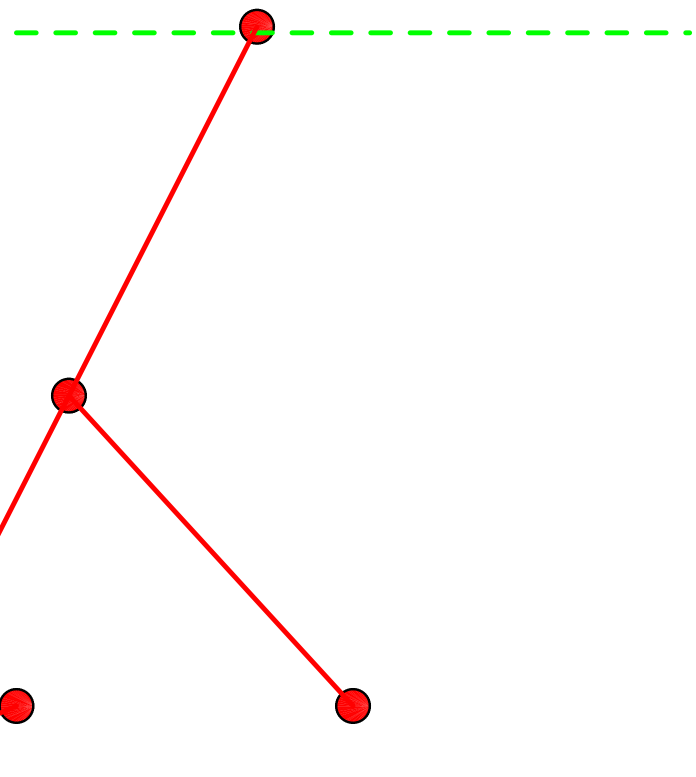

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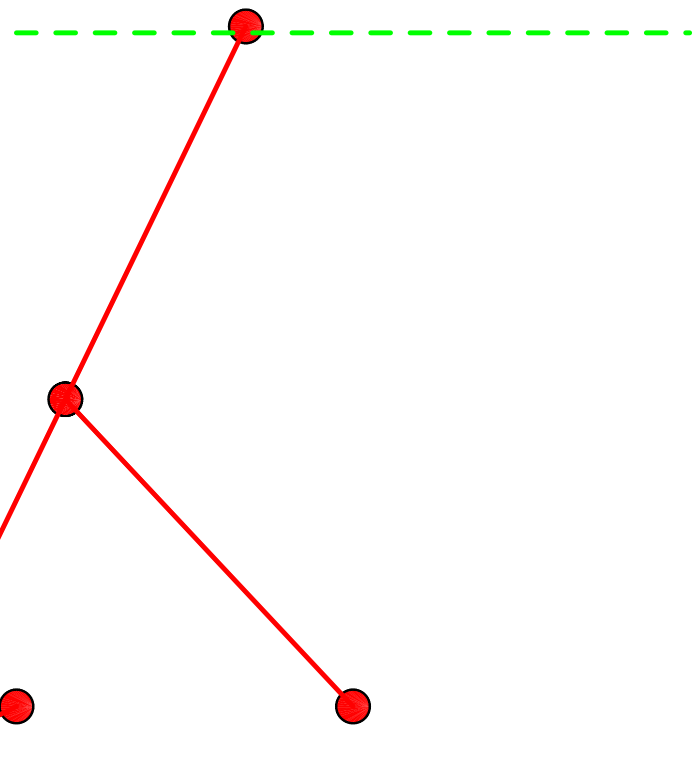

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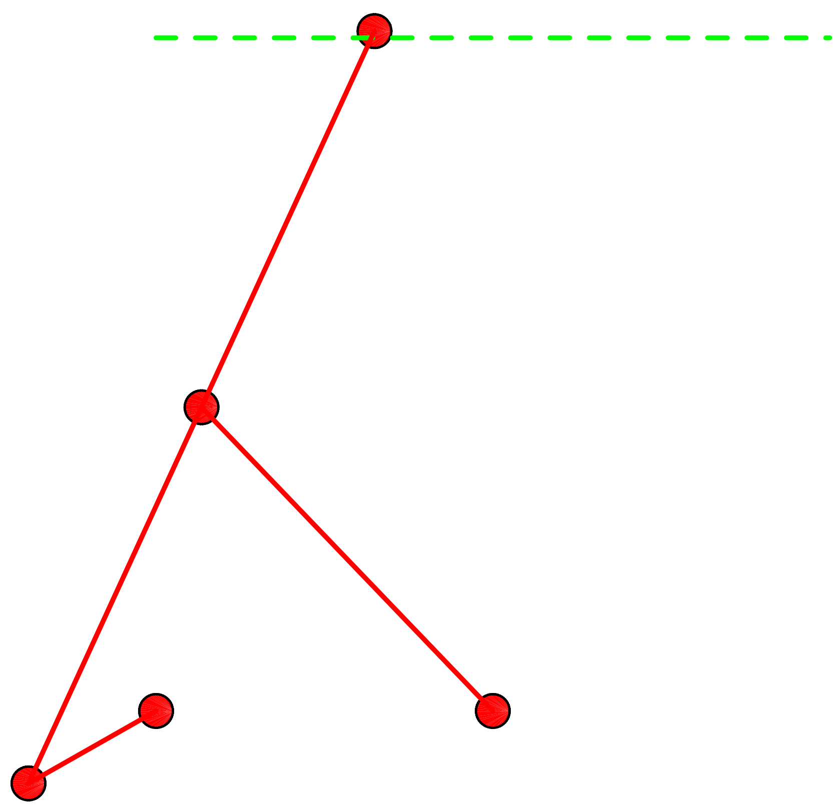

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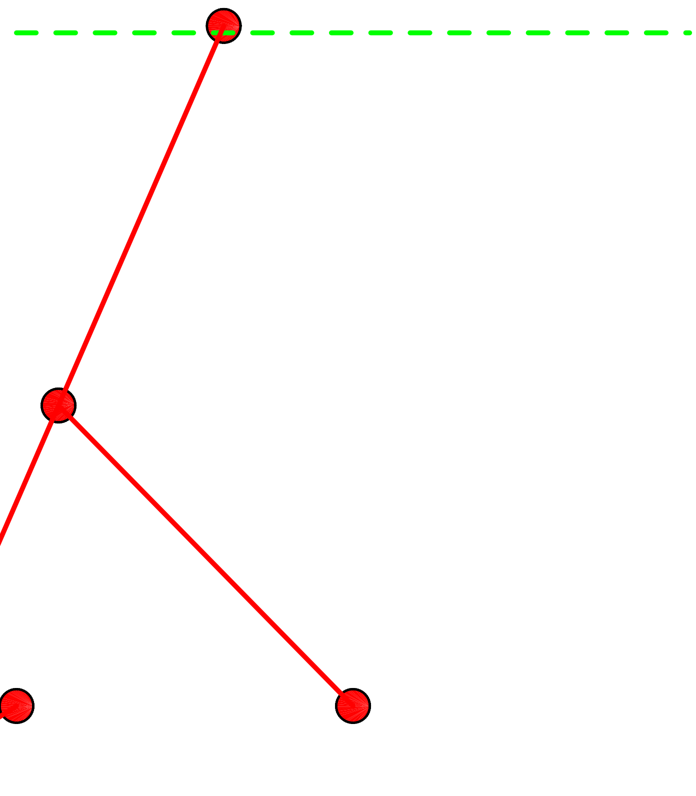

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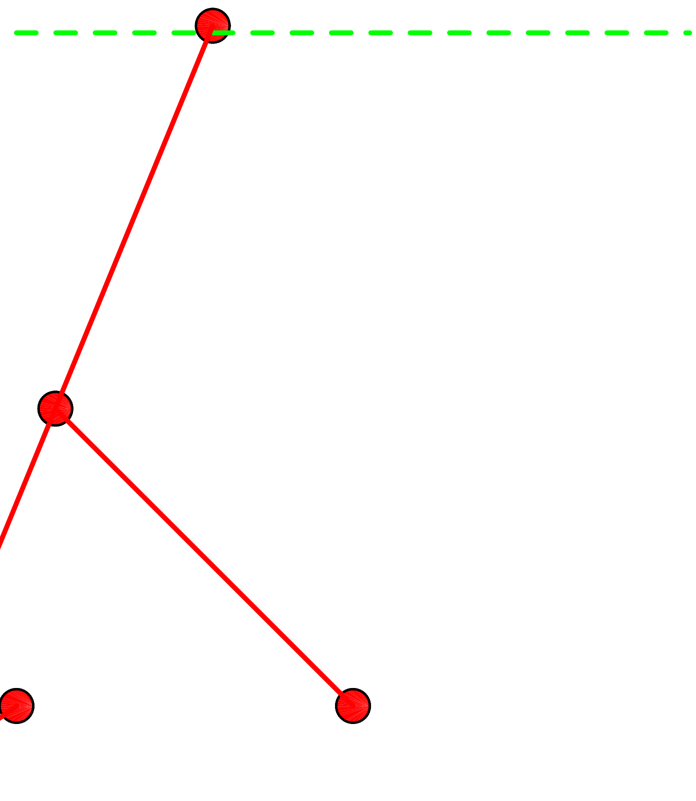

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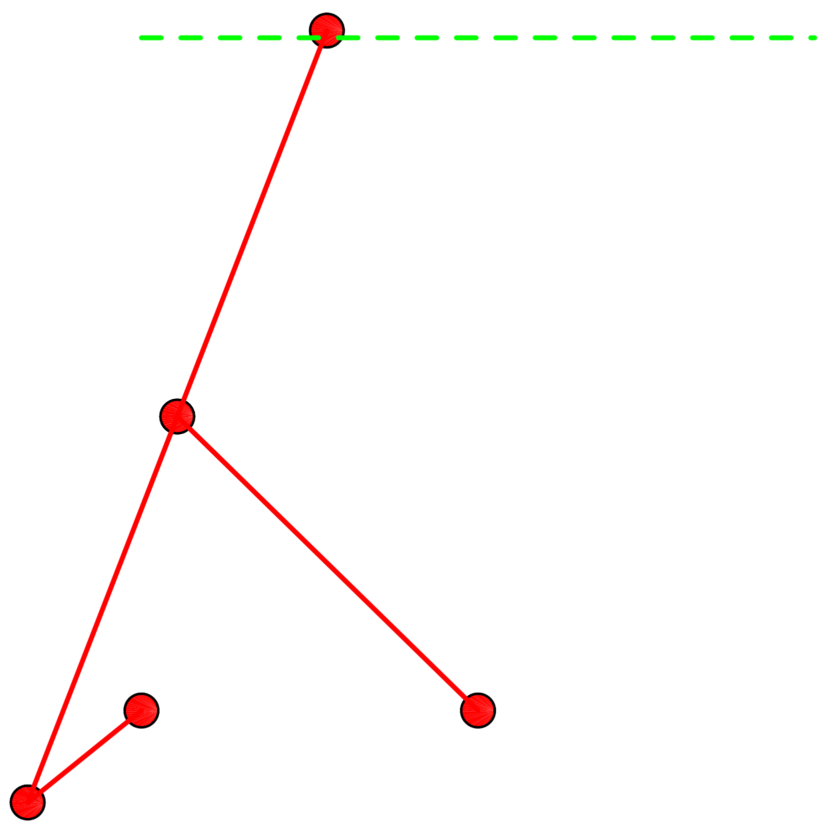

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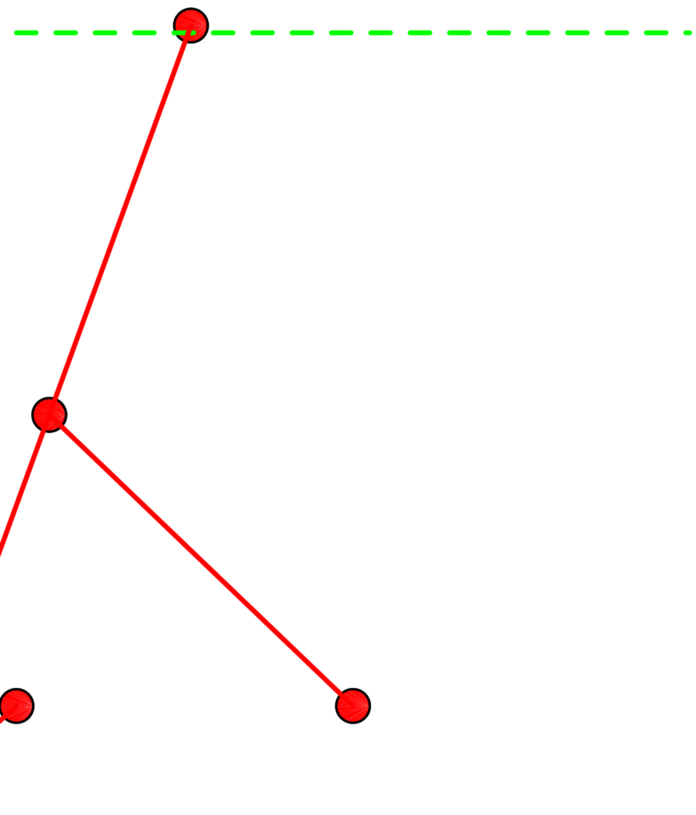

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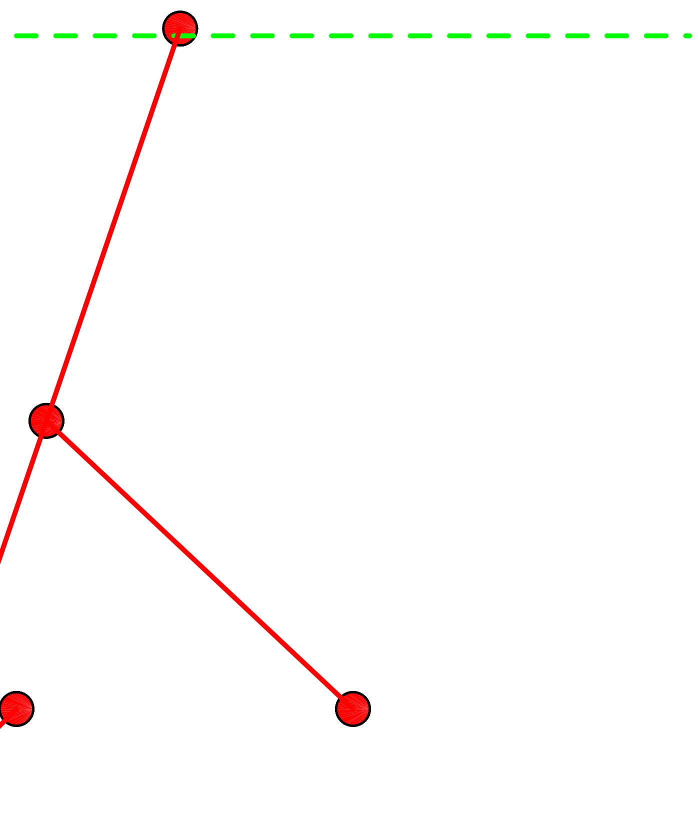

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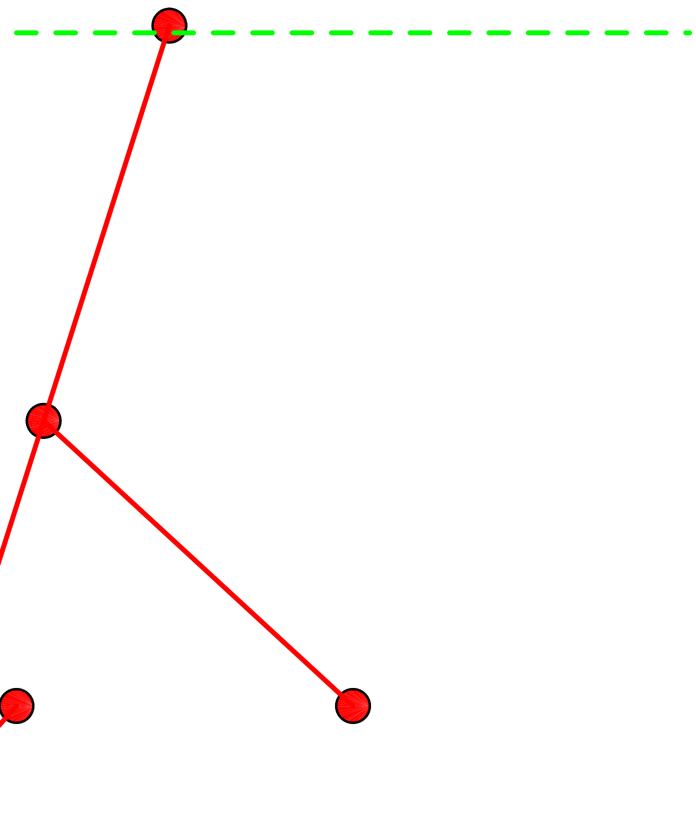

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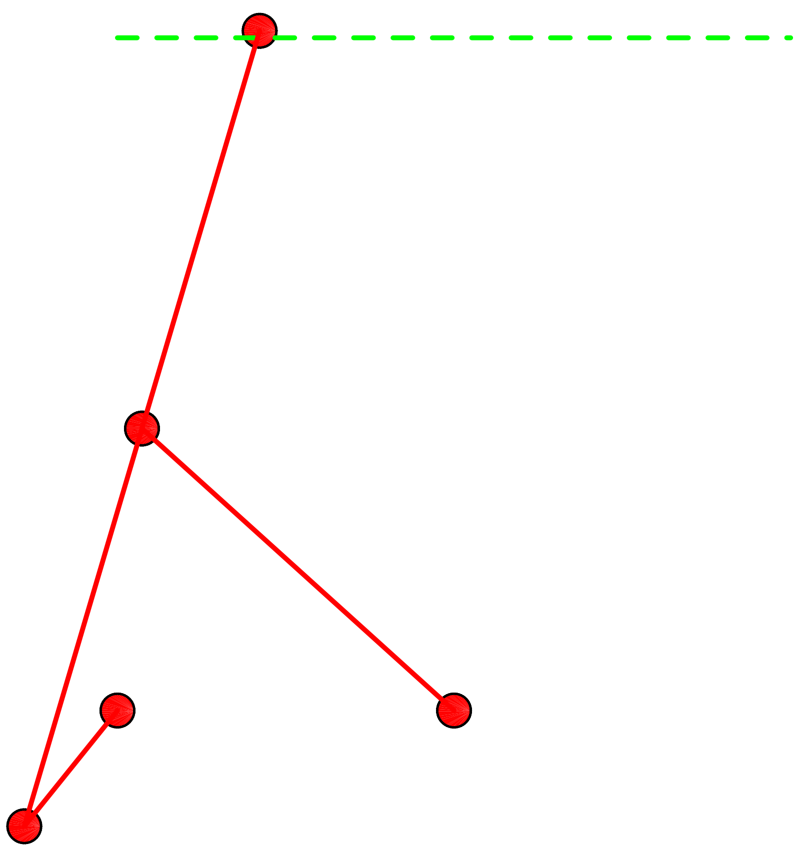

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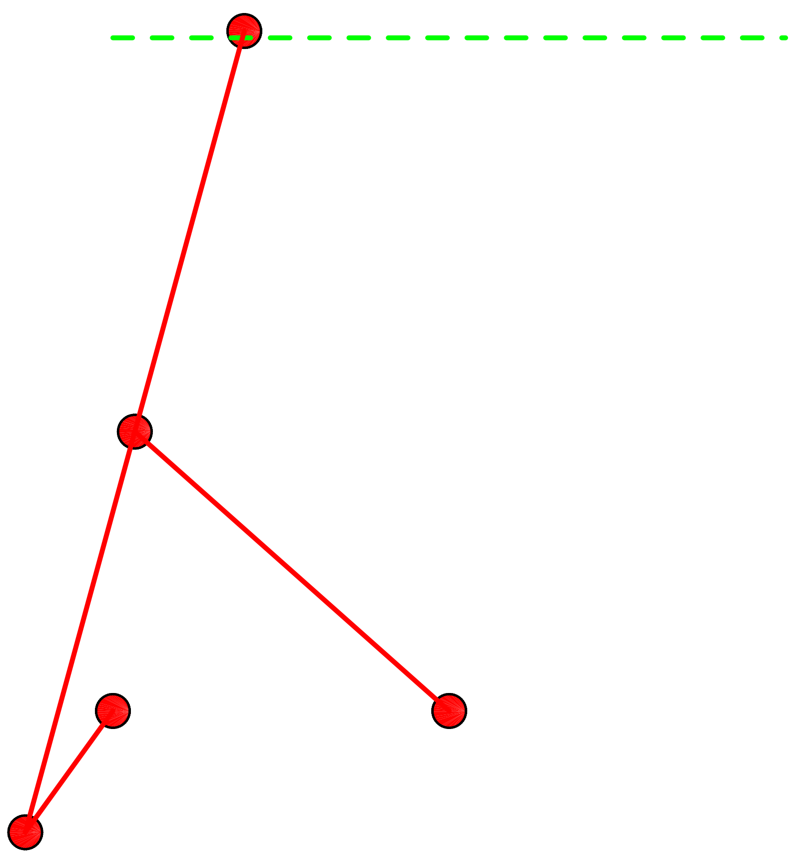

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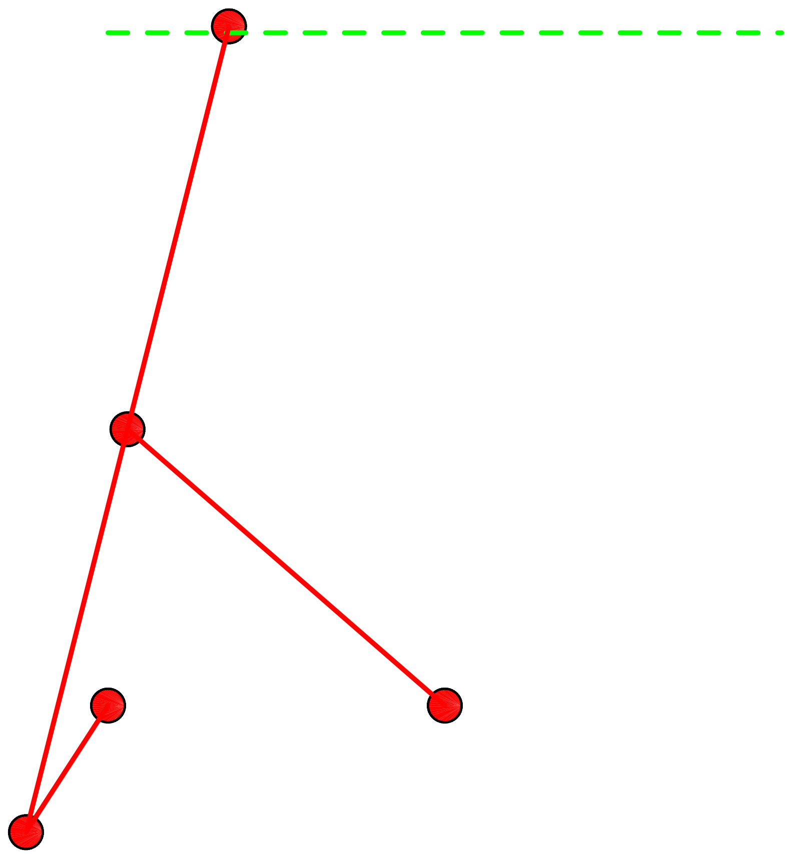

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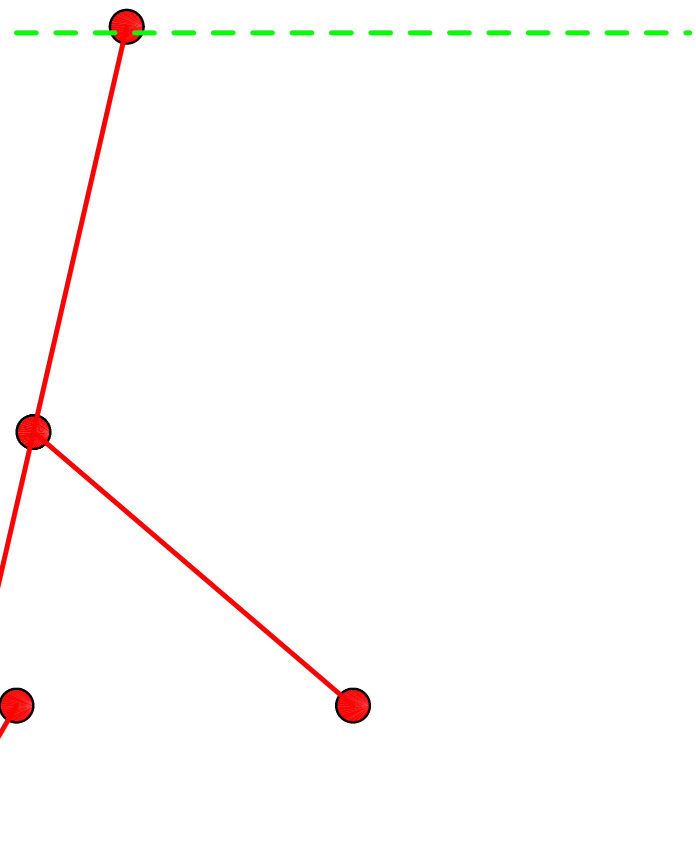

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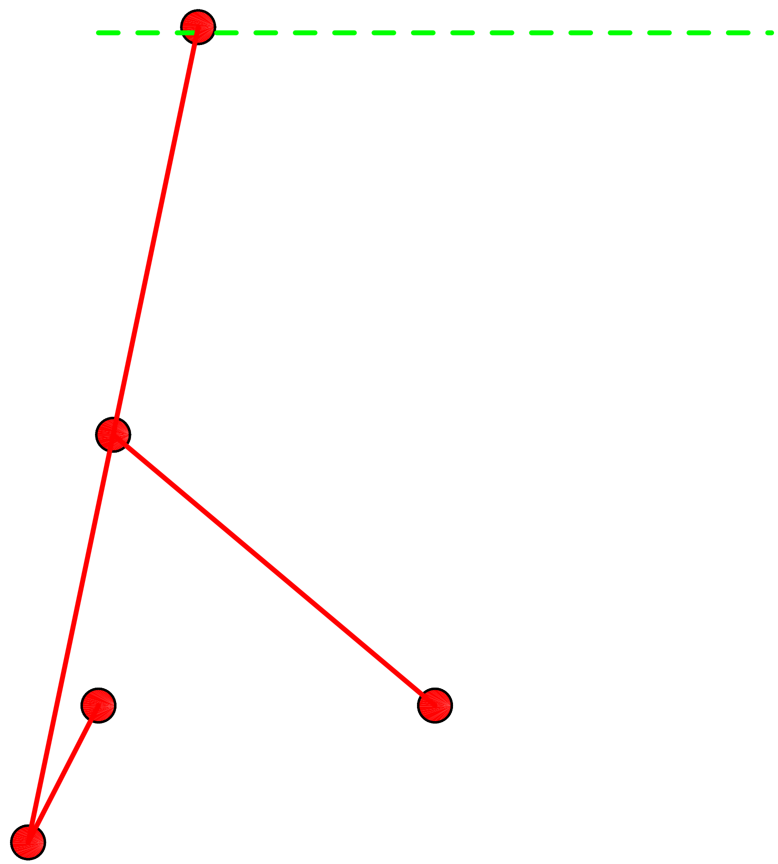

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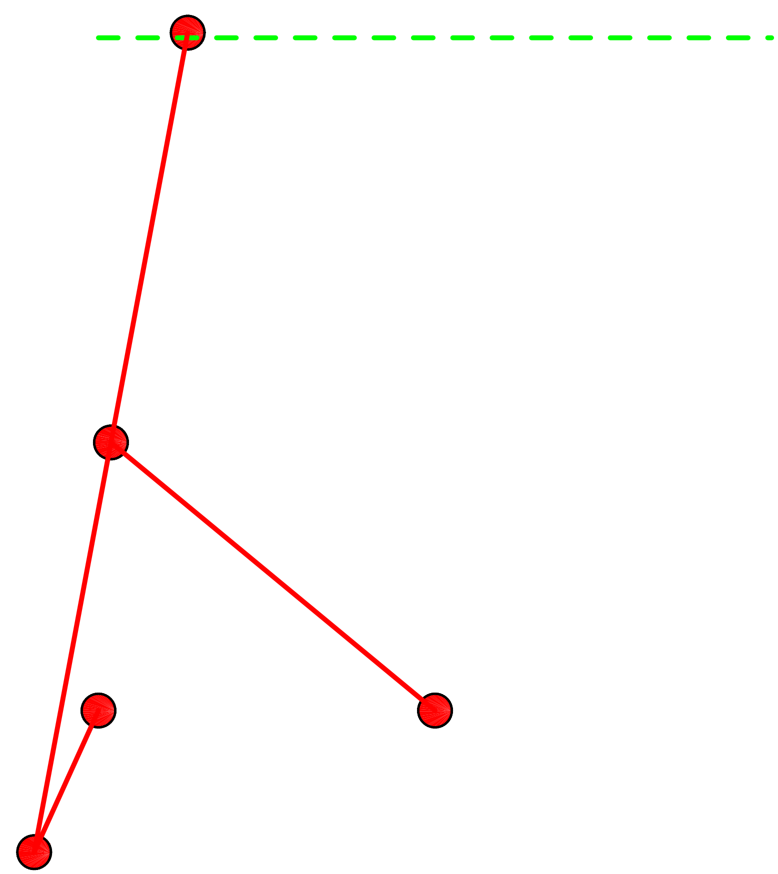

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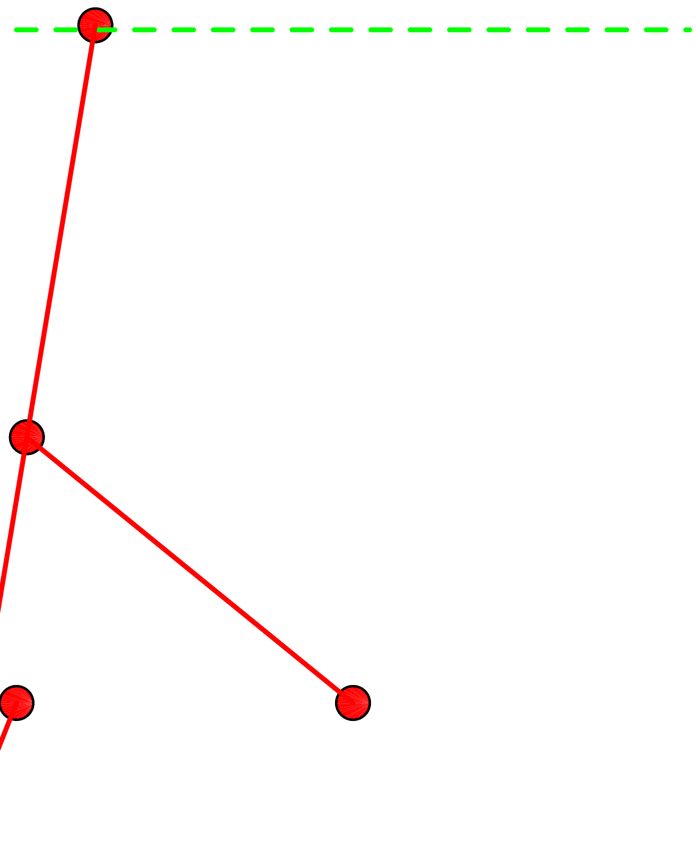

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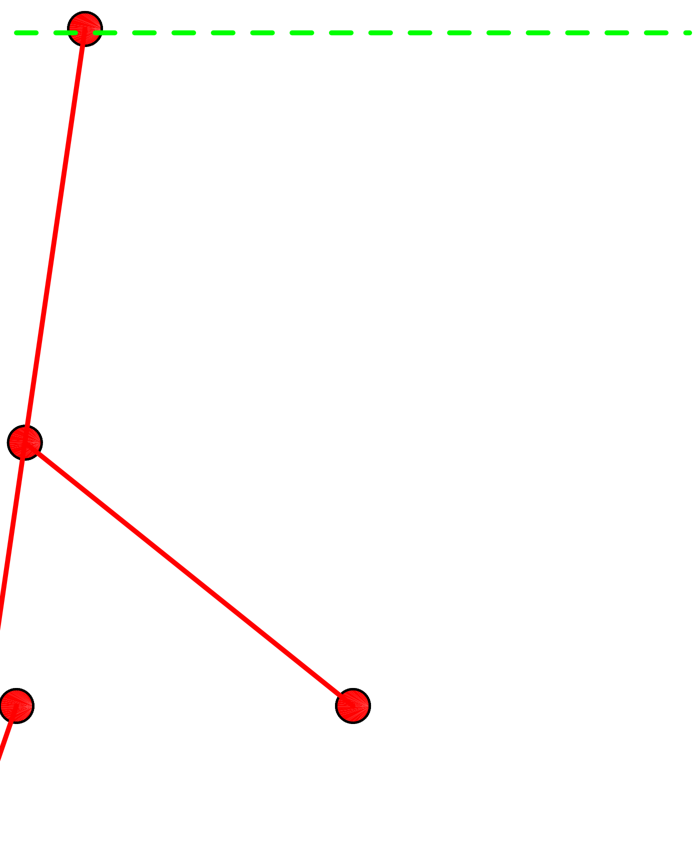

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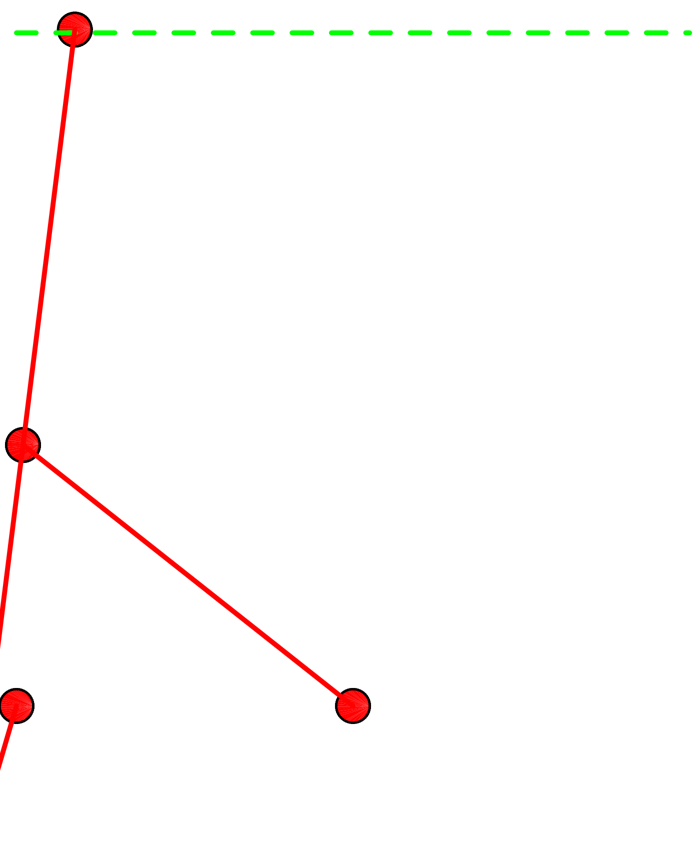

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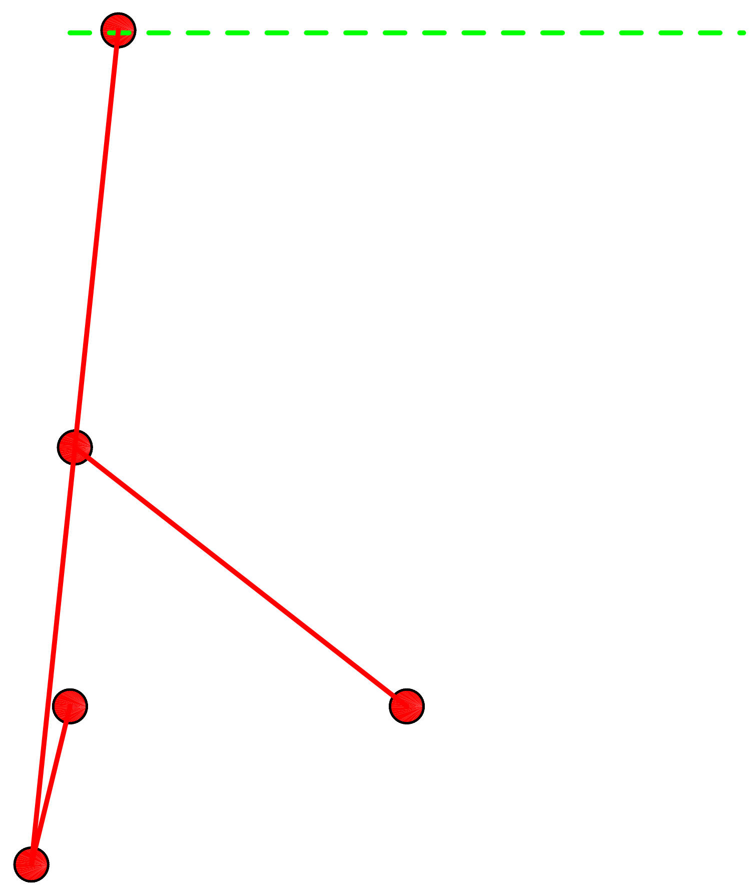

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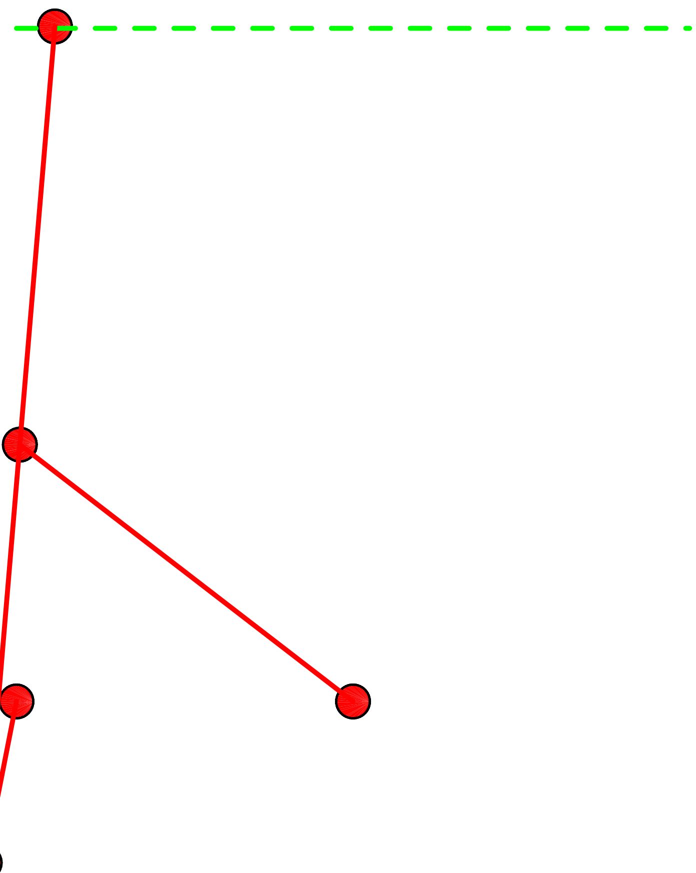

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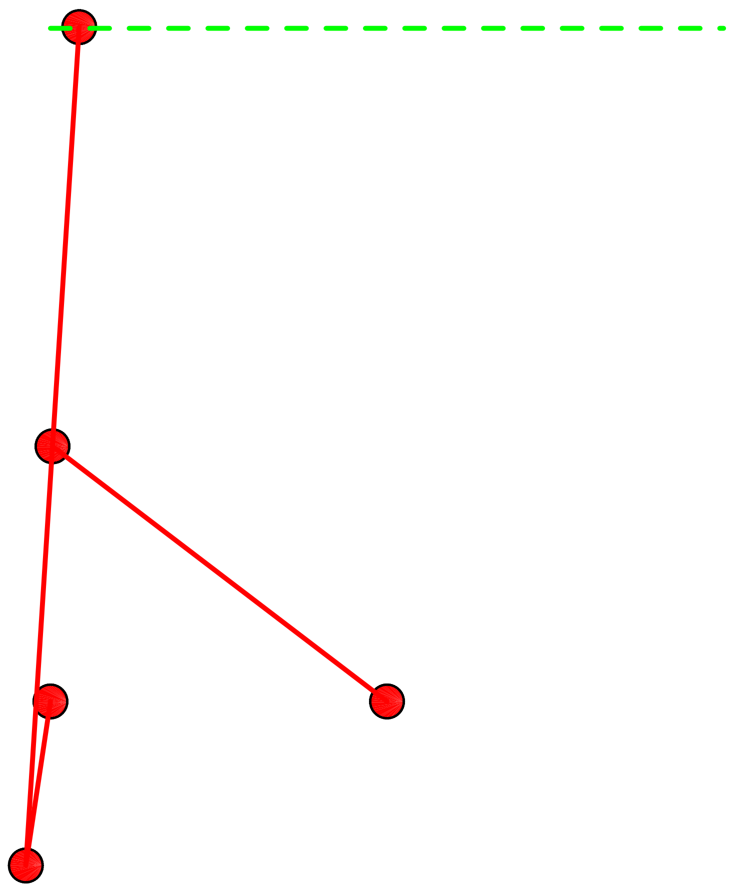

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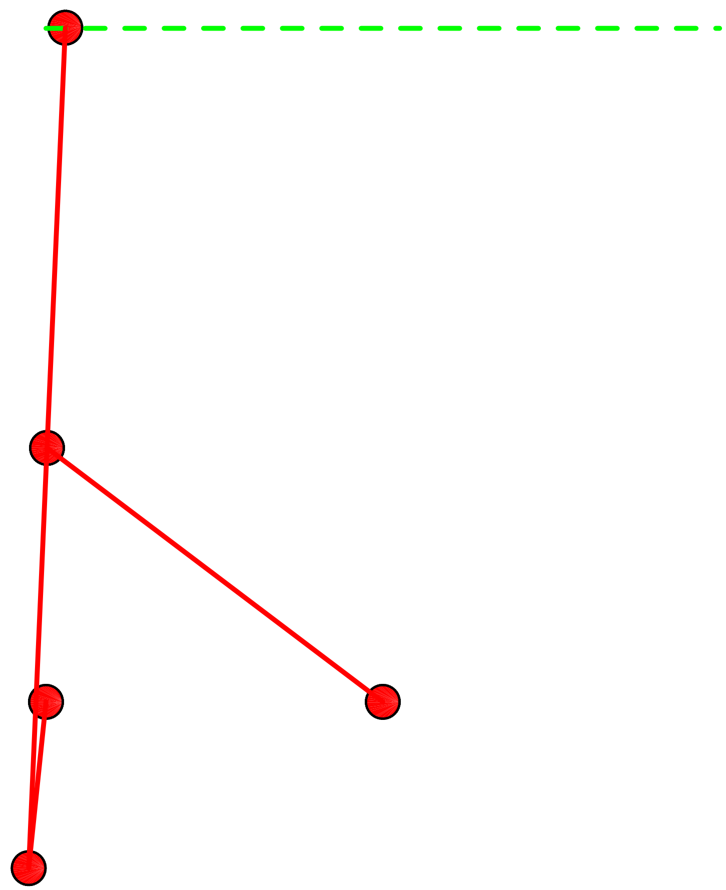

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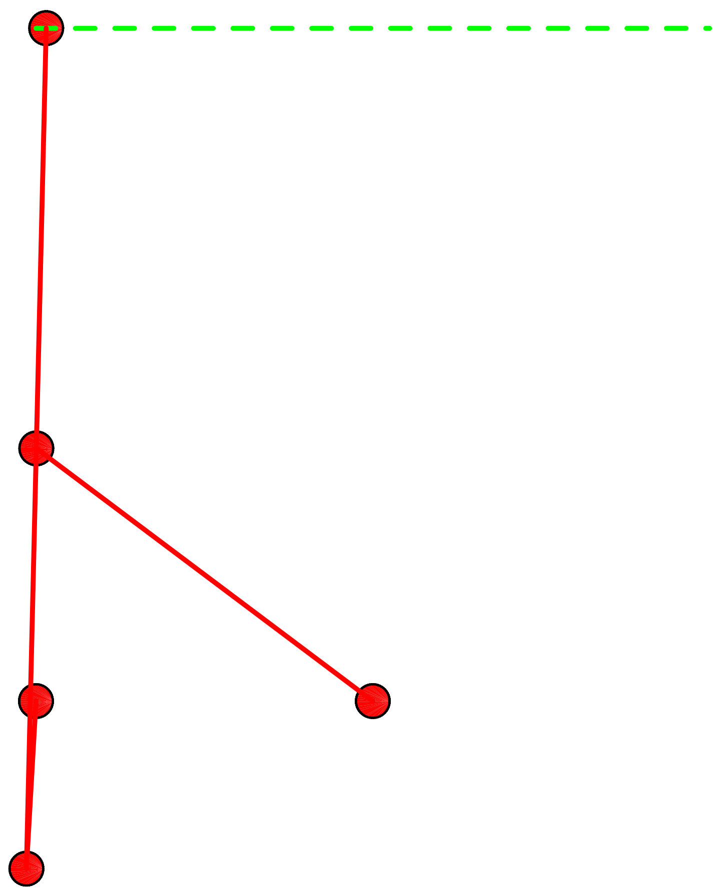

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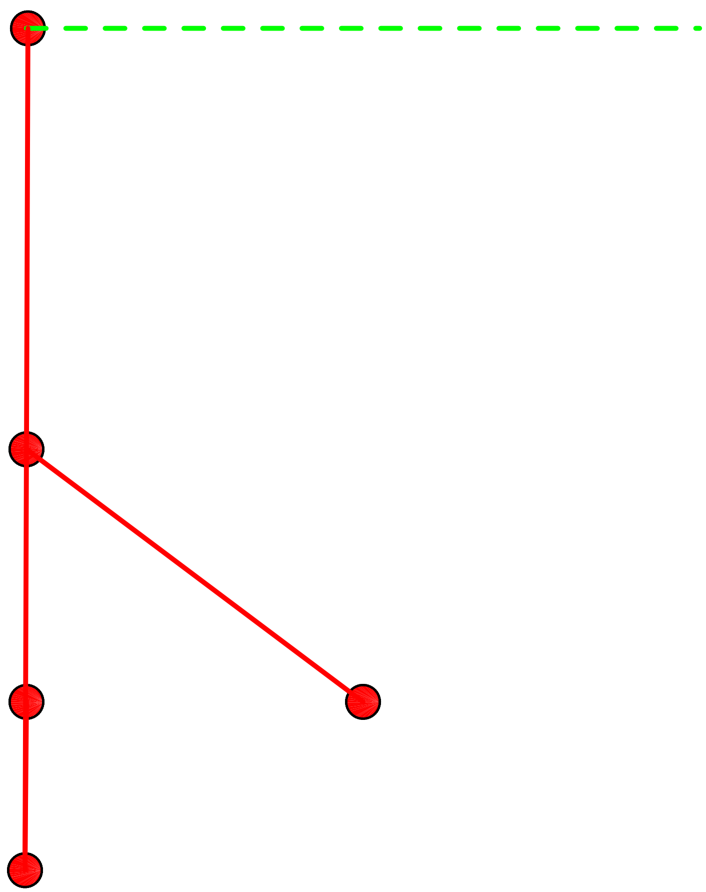

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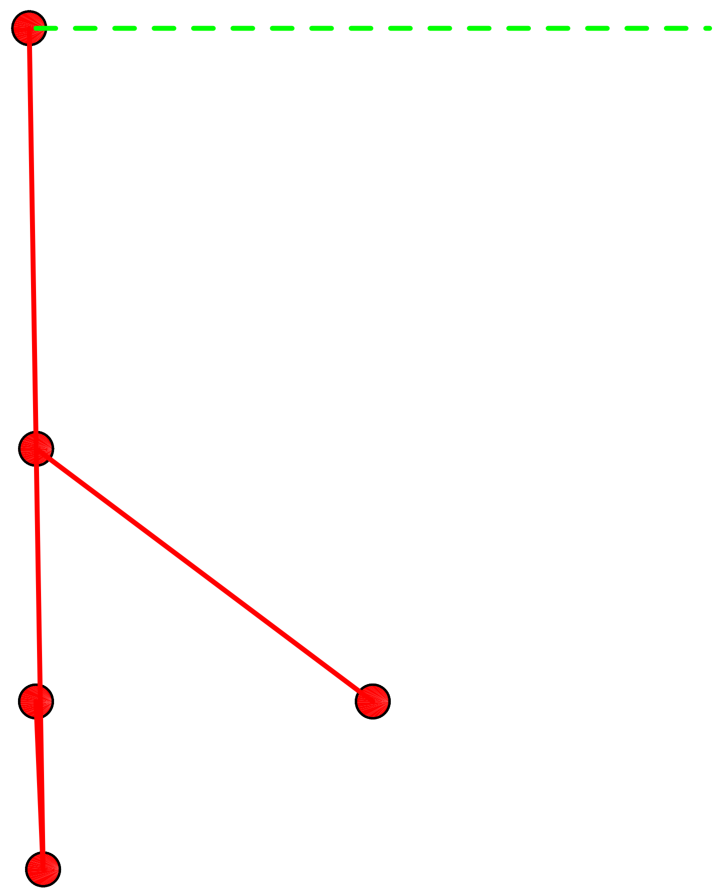

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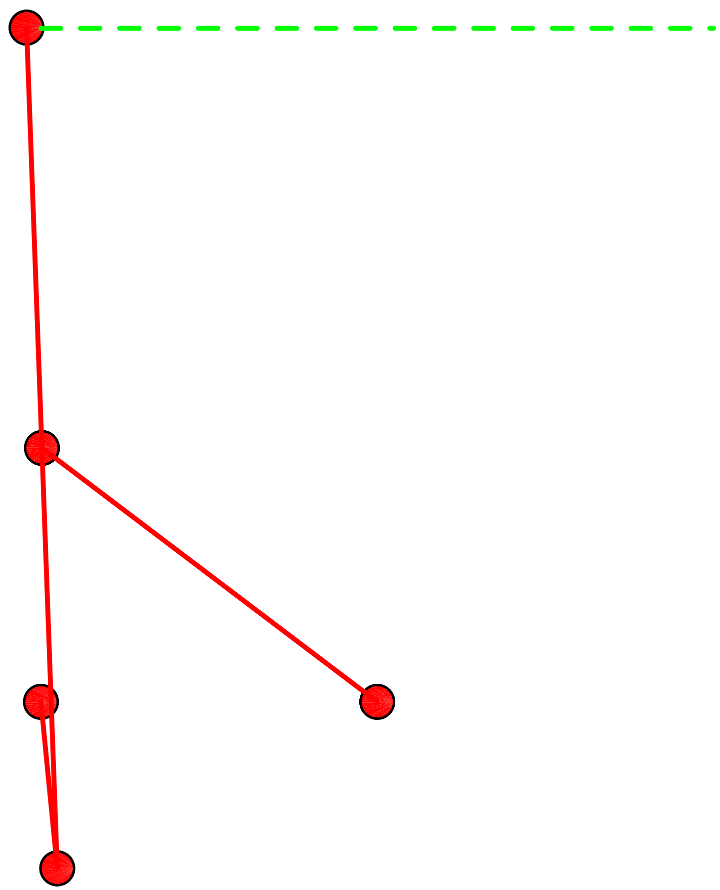

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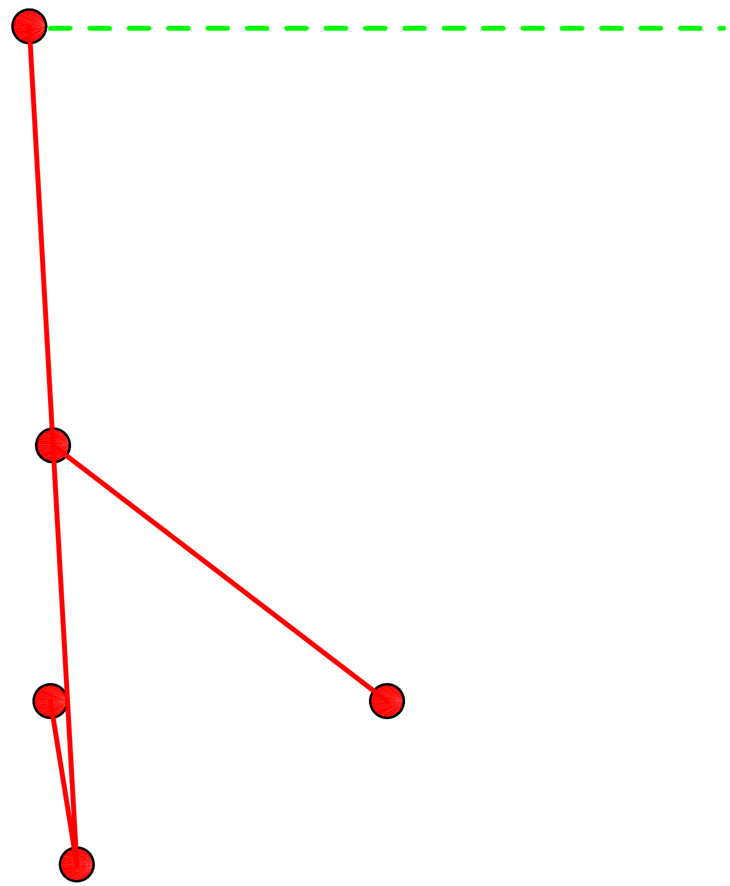

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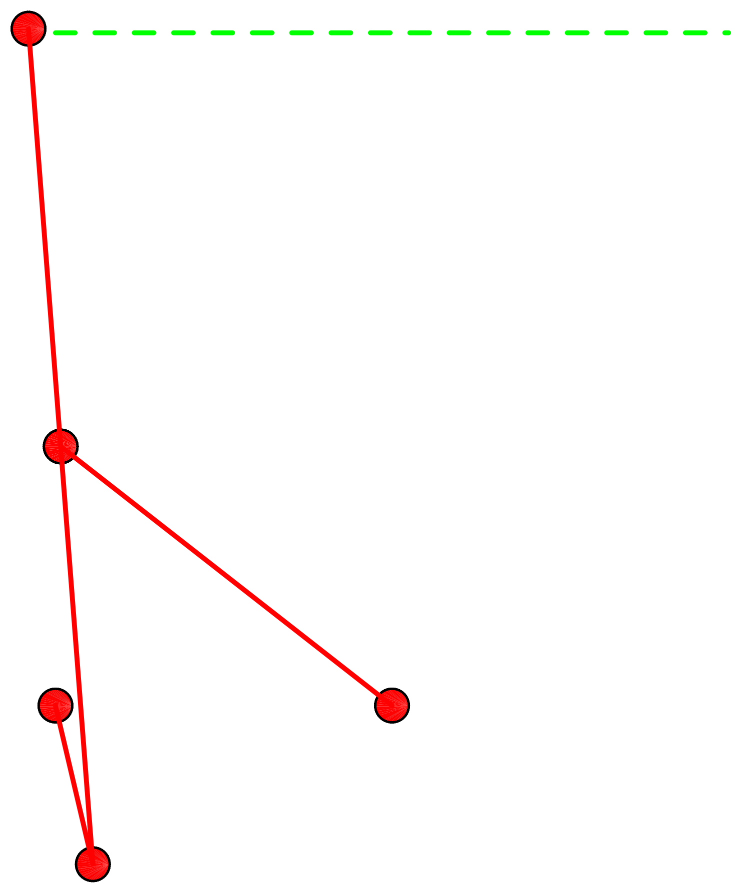

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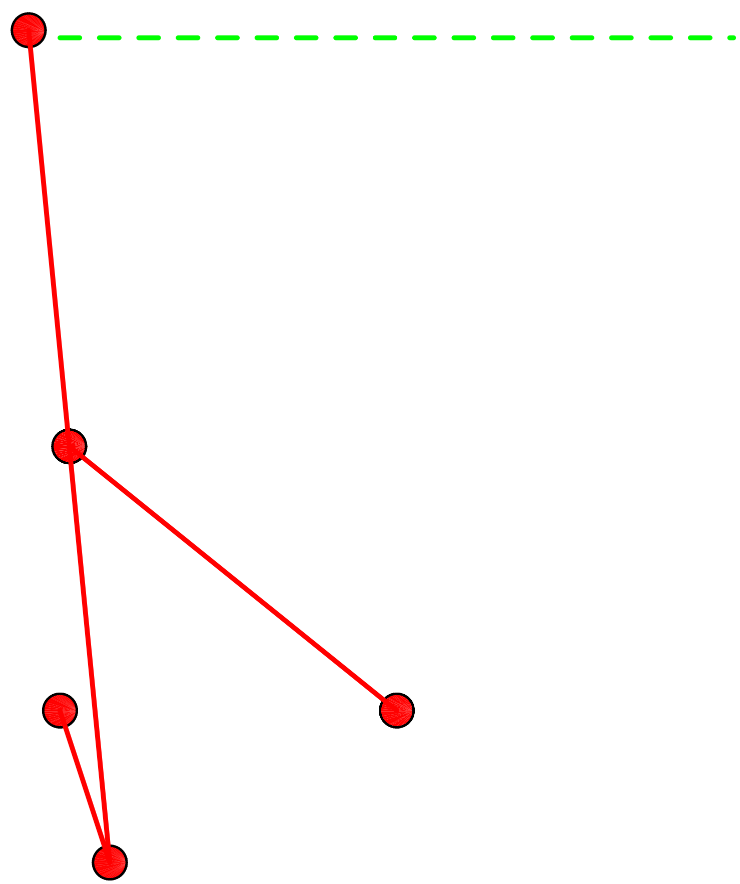

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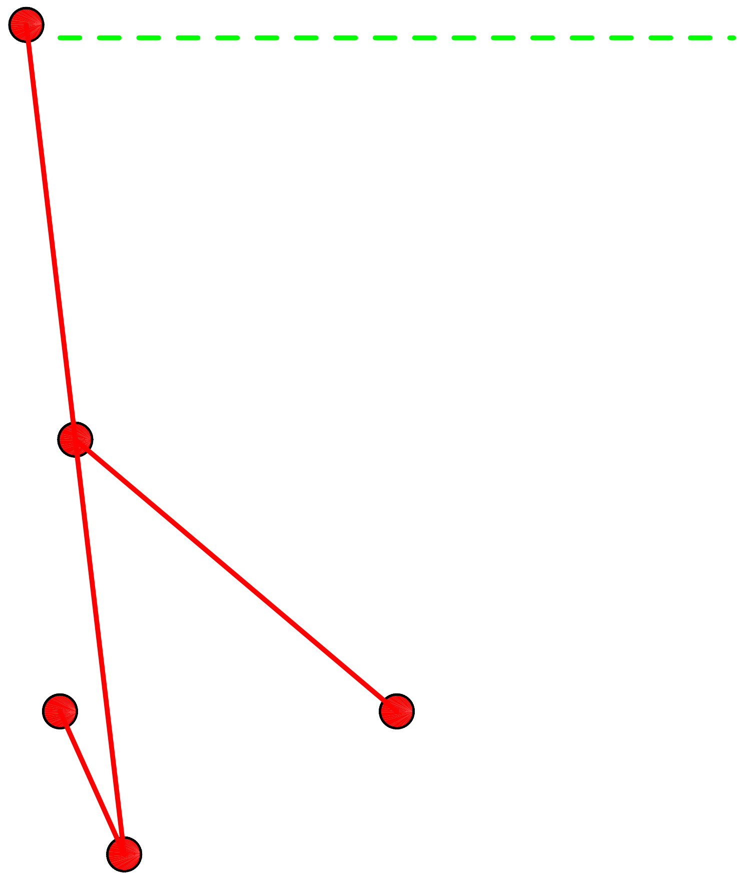

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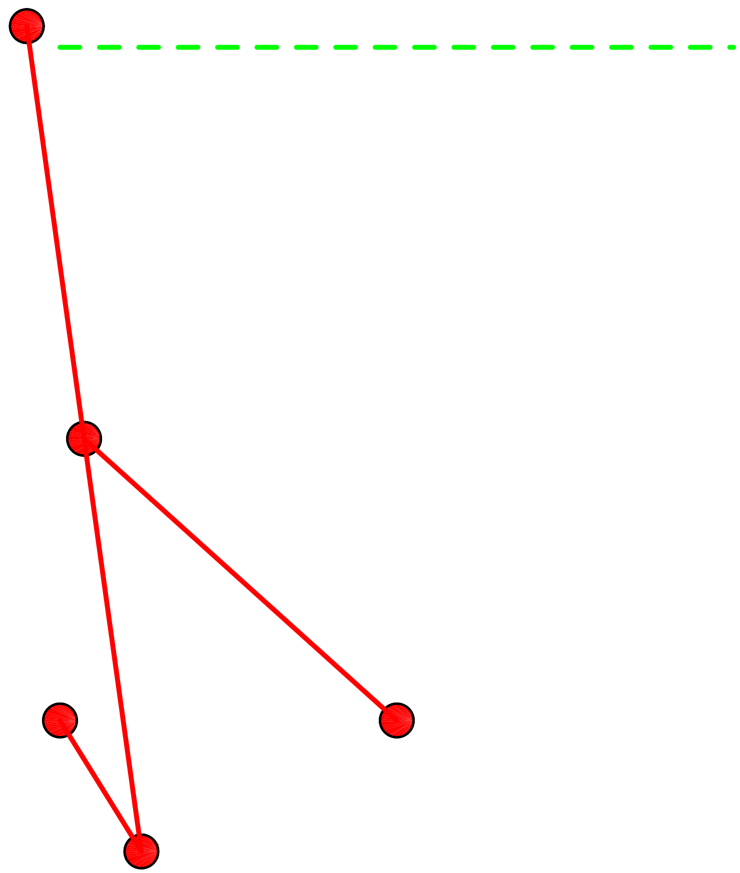

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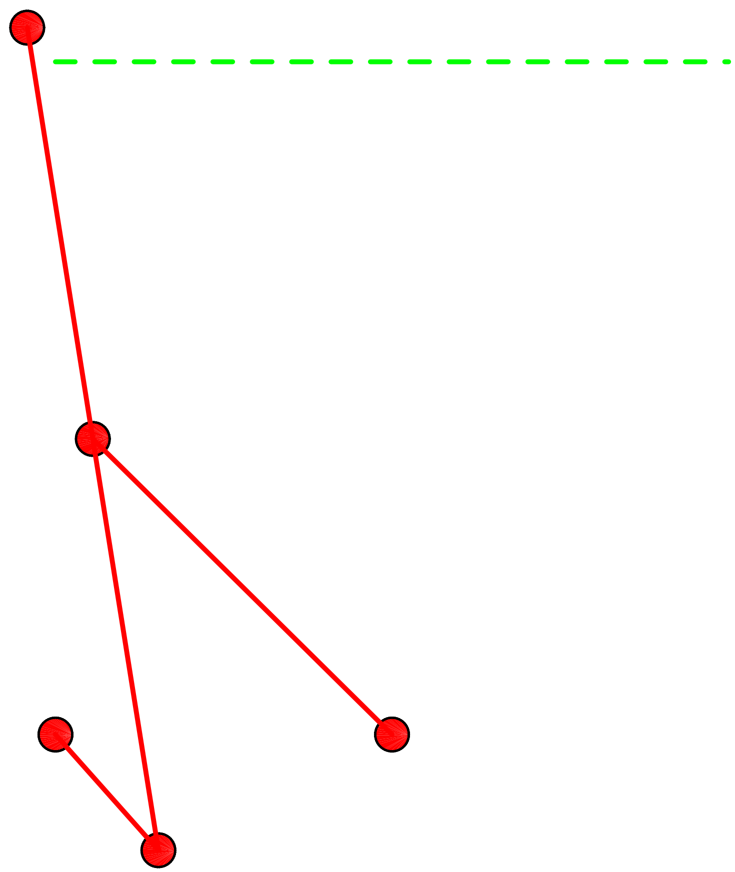

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Five-Point Path Synthesis

Design a 4-bar linkage = design trajectory of coupler point.

Input:	coordinates of points on coupler curve.
Output:	lengths of the bars of the linkage.

C.W. Wampler: **Isotropic coordinates, circularity and Bezout numbers: planar kinematics from a new perspective.** *Proceedings of the 1996 ASME Design Engineering Technical Conference.* Irvine, CA, Aug 18–22, 1996.

A.J. Sommese and C.W. Wampler: **The Numerical Solution of Systems of Polynomials Arising in Engineering and Science.** World Scientific, 2005.

Isotropic Coordinates

- A point $(a, b) \in \mathbb{R}^2$ is mapped to $z = a + ib$, $i = \sqrt{-1}$.
- $(z, \bar{z}) = (a + ib, a - ib) \in \mathbb{C}^2$ are *isotropic coordinates.*
- Observe $z \cdot \bar{z} = a^2 + b^2$.
- Rotation around $(0, 0)$ through angle θ is multiplication by $e^{i\theta}$.
  Multiply by $e^{-i\theta}$ to invert the rotation.
- Abbreviate a rotation by $\Theta = e^{i\theta}$,
  then its inverse $\Theta^{-1} = \bar{\Theta}$, satisfying $\Theta\bar{\Theta} = 1$.

The Loop Equations

Let $A = (a, \bar{a})$ and $B = (b, \bar{b})$ be the fixed base points.

Unknown are $(x, \bar{x})$ and $(y, \bar{y})$, coordinates of the other two points in the 4-bar linkage.

For given precision points $(p_j, \bar{p}_j)$, assuming $\theta_0 = 1$,

$$\left\{ \begin{array}{c} (p_j + x\theta_j + a)(\bar{p}_j + \bar{x}\bar{\theta}_j + \bar{a}) = (p_0 + x + a)(\bar{p}_0 + \bar{x} + \bar{a}) \\ (p_j + y\theta_j + b)(\bar{p}_j + \bar{y}\bar{\theta}_j + \bar{b}) = (p_0 + y + b)(\bar{p}_0 + \bar{y} + \bar{b}) \end{array} \right.$$

Since the angle θ_j corresponding to each $(p_j, \bar{p}_j)$ is unknown, five precision points are needed to determine the linkage uniquely.

Adding $\theta_j\bar{\theta}_j = 1$ to the system leads to 12 equations in 12 unkowns: $(x, \bar{x})$, $(y, \bar{y})$, and $(\theta_j, \bar{\theta}_j)$, for $j = 1, 2, 3, 4$.

```
12
theta[1]*Theta[1]-1;
theta[2]*Theta[2]-1;
theta[3]*Theta[3]-1;
theta[4]*Theta[4]-1;
-.4091256991*x*theta[1]-1.061607555*I*x*theta[1]+1.157260179-.3374636810*X+.1524877812*I*X
   -.3374636810*x-.1524877812*I*x-.4091256991*X*Theta[1]+1.061607555*I*X*Theta[1];
.4011300738*x*theta[2]-1.146477955*I*x*theta[2]+1.338182778-.3374636810*X+.1524877812*I*X
   -.3374636810*x-.1524877812*I*x+.4011300738*X*Theta[2]+1.146477955*I*X*Theta[2];
.3705985316*x*theta[3]-1.454067014*I*x*theta[3]+2.114519894-.3374636810*X+.1524877812*I*X
   -.3374636810*x-.1524877812*I*x+.3705985316*X*Theta[3]+1.454067014*I*X*Theta[3];
.3188425748*x*theta[4]-.850446965*I*x*theta[4]+.6877863684-.3374636810*X+.1524877812*I*X
   -.3374636810*x-.1524877812*I*x+.3188425748*X*Theta[4]+.850446965*I*X*Theta[4];
-1.742137552*y*theta[1]-.3932004150*I*y*theta[1]+1.524665181+.9955481716*Y+.8208949212*I*Y
   +.9955481716*y-.8208949212*I*y-1.742137552*Y*Theta[1]+.3932004150*I*Y*Theta[1];
-.9318817788*y*theta[2]-.4780708150*I*y*theta[2]-.5680292799+.9955481716*Y+.8208949212*I*Y
   +.9955481716*y-.8208949212*I*y-.9318817788*Y*Theta[2]+.4780708150*I*Y*Theta[2];
-.9624133210*y*theta[3]-.7856598740*I*y*theta[3]-.1214837957+.9955481716*Y+.8208949212*I*Y
   +.9955481716*y-.8208949212*I*y-.9624133210*Y*Theta[3]+.7856598740*I*Y*Theta[3];
-1.014169278*y*theta[4]-.1820398250*I*y*theta[4]-.6033068118+.9955481716*Y+.8208949212*I*Y
   +.9955481716*y-.8208949212*I*y-1.014169278*Y*Theta[4]+.1820398250*I*Y*Theta[4];
```

Numerical Homotopy Continuation Methods

If we wish to solve $f(\mathbf{x}) = \mathbf{0}$, then we construct a system $g(\mathbf{x}) = \mathbf{0}$ whose solutions are known. Consider the ***homotopy***

$$H(\mathbf{x}, t) := (1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}.$$

By ***continuation***, we trace the paths starting at the known solutions of $g(\mathbf{x}) = \mathbf{0}$ to the desired solutions of $f(\mathbf{x}) = \mathbf{0}$, for t from 0 to 1.

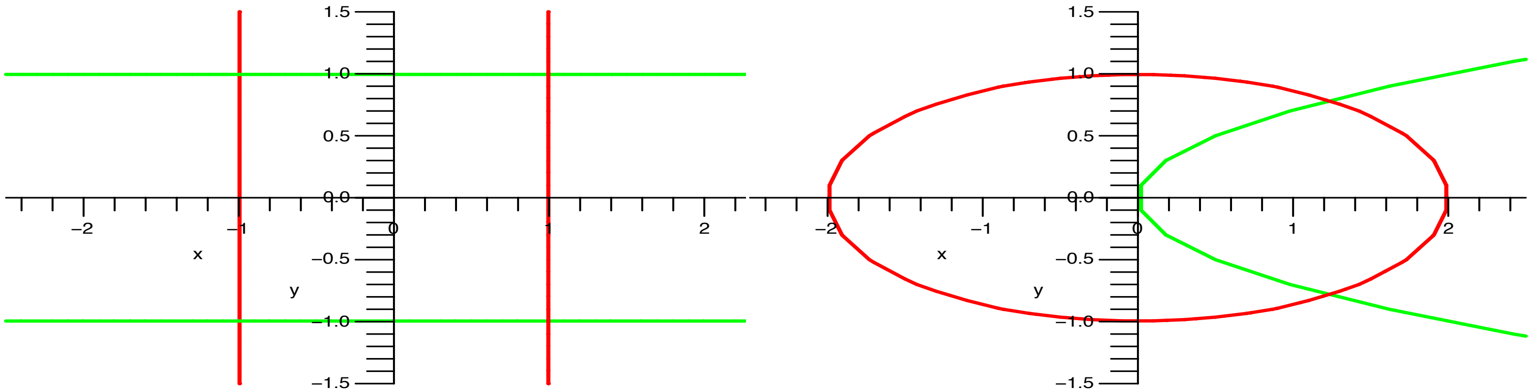
homotopy continuation methods are ***symbolic-numeric***:

homotopy methods treat polynomials as algebraic objects,
continuation methods use polynomials as functions.

geometric interpretation: move from general to special,

solve special, and move solutions from special to general.

Product Deformations



$$\gamma \left(\underbrace{\left\{ \begin{array}{r} x^2 - 1 = 0 \\ y^2 - 1 = 0 \end{array} \right.}_{\text{start system}} \right) (1-t) \quad + \quad \left(\underbrace{\left\{ \begin{array}{r} x^2 + 4y^2 - 4 = 0 \\ 2y^2 - x = 0 \end{array} \right.}_{\text{target system}} \right) t, \quad \gamma \in \mathbb{C}$$

The theorem of Bézout

$f = (f_1, f_2, \ldots, f_n)$

$d_i = \deg(f_i)$

total degree D :

$$D = \prod_{i=1}^{n} d_i$$

$$g(\mathbf{x}) = \left\{ \begin{array}{ll} \alpha_1 x_1^{d_1} - \beta_1 = 0 & \text{start} \\ \alpha_2 x_2^{d_2} - \beta_2 = 0 & \text{system} \\ \vdots & \alpha_i, \beta_i \in \mathbb{C} \\ \alpha_n x_n^{d_n} - \beta_n = 0 & \text{random} \end{array} \right.$$

Theorem: $\boldsymbol{f(\mathbf{x}) = 0}$ ***has at most D isolated solutions in $\mathbb{C}^n$, counted with multiplicities.***

Sketch of Proof: $V = \{\ (f, \mathbf{x}) \in \mathbb{P}(\mathcal{H}_D) \times \mathbb{P}(\mathbb{C}^n) \mid f(\mathbf{x}) = \mathbf{0}\ \}$

$\Sigma' = \{(f, \mathbf{x}) \in V \mid \det(D_{\mathbf{x}} f(\mathbf{x})) = 0\}$, $\Sigma = \pi_1(\Sigma'), \pi_1 : V \to \mathbb{P}(\mathcal{H}_D)$

Elimination theory: Σ is variety $\Rightarrow \mathbb{P}(\mathcal{H}_D) - \Sigma$ is connected.

Thus $h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}$ avoids $\Sigma, \forall t \in [0,1)$.

Implicitly defined curves

Consider a homotopy $h_k(x(t), y(t), t) = 0$, $k = 1, 2$.

By $\frac{\partial}{\partial t}$ on homotopy: $\frac{\partial h_k}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h_k}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial h_k}{\partial t}\frac{\partial t}{\partial t} = 0$, $k = 1, 2$.

Set $\Delta x := \frac{\partial x}{\partial t}$, $\Delta y := \frac{\partial y}{\partial t}$, and $\frac{\partial t}{\partial t} = 1$.

$$
\begin{array}{ll}
\text{Increment} & t := t + \Delta t \\
\text{Solve} & \left[\begin{array}{cc} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{array} \right] \left[\begin{array}{c} \Delta x \\ \Delta y \end{array} \right] = - \left[\begin{array}{c} \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial t} \end{array} \right] (Newton) \\
\text{Update} & \left\{ \begin{array}{l} x := x + \Delta x \\ y := y + \Delta y \end{array} \right.
\end{array}
$$

Predictor-Corrector Methods

loop

1. predict $\begin{cases} t_{k+1} := t_k + \Delta t \\ \mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + \Delta \mathbf{x} \end{cases}$

2. correct with Newton

3. if convergence

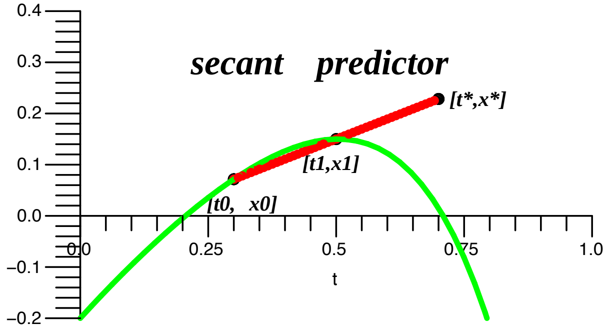
   then enlarge Δt

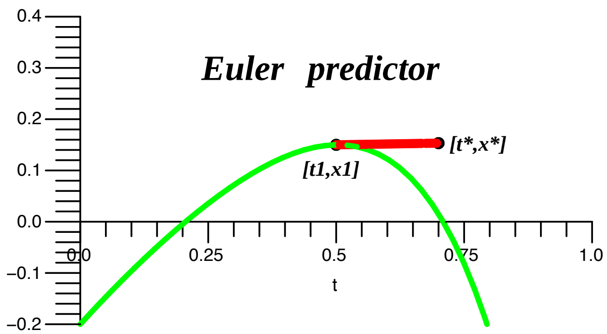
   continue with $k+1$

   else reduce Δt

   back up and restart at k

until $t = 1$.





Robustness of Continuation Methods

sure to find **all** *roots at the end of the paths?*

- dealing with curve jumping:
  1. fix #Newton steps to force quadratic convergence;
  2. rerun clustered paths with same discretization of t.
- Robust step control by interval methods, see

R.B. Kearfott and Z. Xing: **An interval step control for continuation methods.** *SIAM J. Numer. Anal.* 31(3): 892–914, 1994.

- Root of multiplicity μ will appear at the end of the paths as a cluster of μ roots.

  Use "endgames", eventually in multi-precision arithmetic.

Complexity Issues

The Problem: a hierarchy of complexity classes

P	:	evaluation of a system at a point
NP	:	find one root of a system
$\#P$	:	find **all** roots of a system (*intractable!*)

Complexity of Homotopies: for bounds on #Newton steps in a linear homotopy, see

L. Blum, F. Cucker, M. Shub, and S. Smale: **Complexity and Real Computation**. Springer 1998.

M. Shub and S. Smale: **Complexity of Bezout's theorem V: Polynomial Time**. *Theoretical Computer Science* 133(1):141–164, 1994.

On average, we can find an approximate zero in polynomial time.

Solving the Loop Equations

Recall: 12 equations in 12 unknowns.

All equations are quadratic, so the total degree is $2^{12} = 4,096$.

+ Tracking 4,096 takes 25 minutes on a modern computer.

− Of the 4,096 paths, only 36 will converge.

$\longrightarrow$ *4,060 wasted paths!*

Singularities are keeping us in business

numerical analysis: bifurcation points and endgames

Rall (1966); Reddien (1978); Decker-Keller-Kelley (1983);
Griewank-Osborne (1981); Hoy (1989);
Deuflard-Friedler-Kunkel (1987); Kunkel (1989, 1996);
Morgan-Sommese-Wampler (1991); Li-Wang (1993, 1994);
Allgower-Schwetlick (1995); Pönisch-Schnabel-Schwetlick (1999);
Allgower-Böhmer-Hoy-Janovský (1999); Govaerts (2000)

computer algebra: standard bases (SINGULAR)

Mora (1982); Greuel-Pfister (1996); Marinari-Möller-Mora (1993)

numerical polynomial algebra: multiplicity structure

Möller-Stetter (1995); Mourrain (1997);
Stetter-Thallinger (1998); Dayton-Zeng (2005)

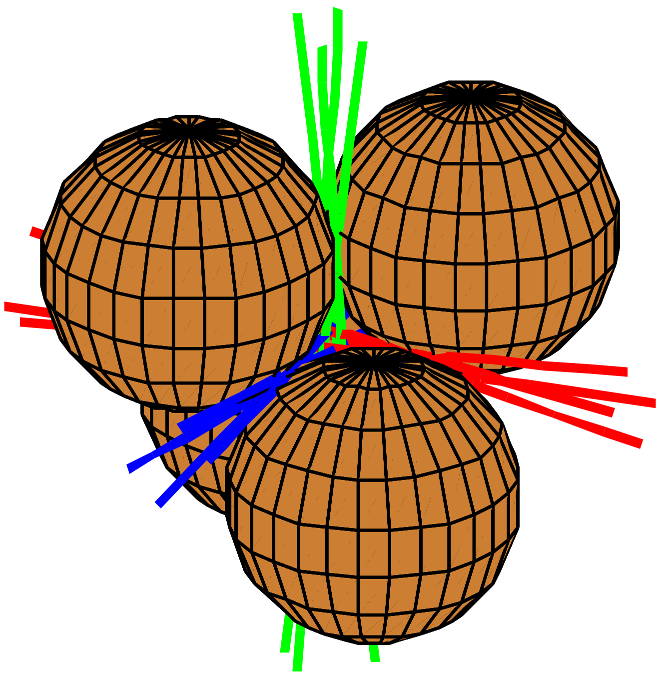
deflation: Ojika-Watanabe-Mitsui (1983); Lecerf (2003)

Twelve lines tangent to four spheres

Frank Sottile and Thorsten Theobald: Lines tangents to $2n-2$ spheres in $\mathbb{R}^n$

Trans. Amer. Math. Soc. 354

pages 4815-4829, 2002.



Problem:

Given 4 spheres,

find all lines tangent

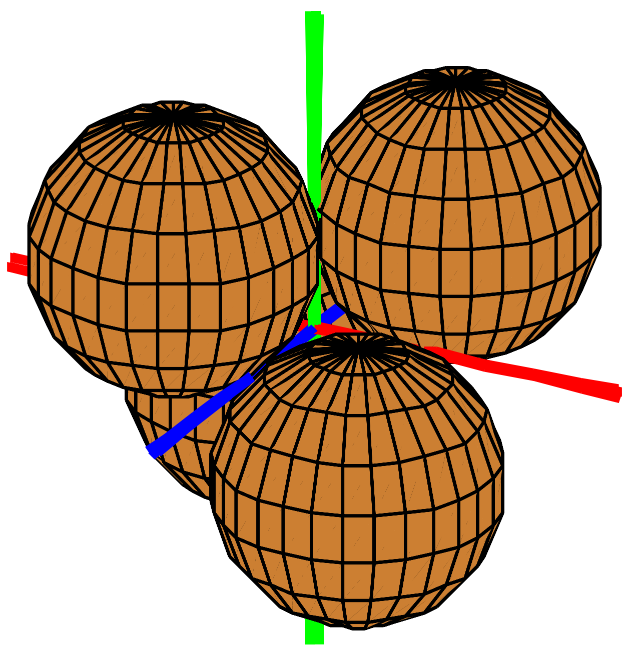
to all 4 given spheres.

Observe:

12 solutions in groups of 4.

Twelve lines tangent to four spheres

Frank Sottile and Thorsten Theobald: Lines tangents to $2n-2$ spheres in $\mathbb{R}^n$
Trans. Amer. Math. Soc. 354
pages 4815-4829, 2002.



Problem:

Given 4 spheres,
find all lines tangent
to all 4 given spheres.

Observe:

3 lines of multiplicity 4.

An Input Polynomial System

```
x0**2 + x1**2 + x2**2 - 1;
x0*x3 + x1*x4 + x2*x5;
x3**2 + x4**2 + x5**2 - 0.25;
x3**2 + x4**2 - 2*x2*x4 + x2**2 + x5**2 + 2*x1*x5 + x1**2 - 0.25;
x3**2 + 1.73205080756888*x2*x3 + 0.75*x2**2 + x4**2 - x2*x4 + 0.25*x2**2
+ x5**2 - 1.73205080756888*x0*x5 + x1*x5
+ 0.75*x0**2 - 0.86602540378444*x0*x1 + 0.25*x1**2 - 0.25;
x3**2 - 1.63299316185545*x1*x3 + 0.57735026918963*x2*x3
+ 0.66666666666667*x1**2 - 0.47140452079103*x1*x2 + 0.08333333333333*x2**2
+ x4**2 + 1.63299316185545*x0*x4 - x2*x4 + 0.66666666666667*x0**2
- 0.81649658092773*x0*x2 + 0.25*x2**2
+ x5**2 - 0.57735026918963*x0*x5 + x1*x5 + 0.08333333333333*x0**2
- 0.28867513459481*x0*x1 + 0.25*x1**2 - 0.25;
```

Original formulation as polynomial system: **Cassiano Durand**.
Centers of the spheres at the vertices of a tetrahedron: **Thorsten Theobald**.
Algebraic numbers sqrt(3), sqrt(6), etc. approximated by double floats.

The system has 6 isolated solutions, each of multiplicity 4.

Solutions at the End of Continuation

Two solutions in a **cluster**: (real and imaginary parts)

```
solution 1 :
  x0 :  -7.07106803165780E-01    3.77452918725401E-08
  x1 :  -4.08248430737360E-01   -1.83624917064964E-07
  x2 :   5.77350143082334E-01   -8.36140714113780E-08
  x3 :  -2.50000000000000E-01   -1.57896818458518E-16
  x4 :   4.33012701892221E-01   -9.11600174682333E-17
  x5 :   9.56878363411174E-08    1.54062878745083E-07
solution 2 :
  x0 :  -7.07106794356709E-01   -1.29682370414209E-07
  x1 :  -4.08248217029256E-01    1.11010906008961E-07
  x2 :   5.77350304985648E-01   -8.03312536501087E-08
  x3 :  -2.50000000000001E-01   -1.74789416181029E-16
  x4 :   4.33012701892220E-01   -1.00914936462574E-16
  x5 :  -6.07788020445124E-08   -1.39412292964849E-07
```

this is the **input** to our deflation algorithm

Newton's Method for Overdetermined Systems

Singular Value Decomposition of N-by-n Jacobian matrix J_f:

$J_f = U \Sigma V^T$, $\quad$ U and V are orthogonal: $U^T U = I_N, V^T V = I_n$,

and singular values $\boldsymbol{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n}$ as the only nonzero elements on the diagonal of the N-by-n matrix Σ $(N > n)$.

The **condition number** $\mathrm{cond}(J_f(\mathbf{z})) = \frac{\sigma_1}{\sigma_n}$.

$\mathrm{Rank}(J_f(\mathbf{z})) = R \Longleftrightarrow \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_R, 0, \ldots, 0)$.

At a **multiple root** $\mathbf{z}_0$: $\mathrm{Rank}(J_f(\mathbf{z}_0)) = R < n$.

Close to $\mathbf{z}_0$, $\mathbf{z} \approx \mathbf{z}_0 : \sigma_{R+1} \approx 0$, or $\boldsymbol{|\sigma_{R+1}| < \epsilon}$, ϵ is tolerance.

Moore-Penrose inverse: $J_f^+ = V \Sigma^+ U^T$, with $R = \mathrm{Rank}(J_f)$,

and $\Sigma^+ = \mathrm{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_R}, 0, \ldots, 0)$.

Then $\Delta \mathbf{z} = -J_f(\mathbf{z})^+ f(\mathbf{z})$ is the least squares solution.

Dedieu-Shub (1999); Li-Zeng (2005)

Deflation Operator Dfl reduces to Corank One

Consider $f(\mathbf{x}) = \mathbf{0}$, N equations in n unknowns, $N \geq n$.

Suppose $\mathrm{Rank}(A(\mathbf{z}_0)) = R < n$ for $\mathbf{z}_0$ an isolated zero of $f(\mathbf{x}) = 0$.

Choose $\mathbf{h} \in \mathbb{C}^{R+1}$ and $B \in \mathbb{C}^{n \times (R+1)}$ at random.

Introduce $R+1$ new multiplier variables $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{R+1})$.

$$\mathbf{Dfl}(f)(\mathbf{x}, \boldsymbol{\lambda}) := \left\{ \begin{array}{rcl} f(\mathbf{x}) & = & \mathbf{0} \\ A(\mathbf{x})B\boldsymbol{\lambda} & = & \mathbf{0} \\ \mathbf{h}\boldsymbol{\lambda} & = & 1 \end{array} \right. \qquad \begin{array}{c} \mathrm{Rank}(A(\mathbf{x})) = R \\ \Downarrow \\ \mathrm{corank}(A(\mathbf{x})B) = 1 \end{array}$$

Compared to the deflation of Ojika, Watanabe, and Mitsui:
(1) we do not compute a maximal minor of the Jacobian matrix;
(2) we only add new equations, we never replace equations.

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

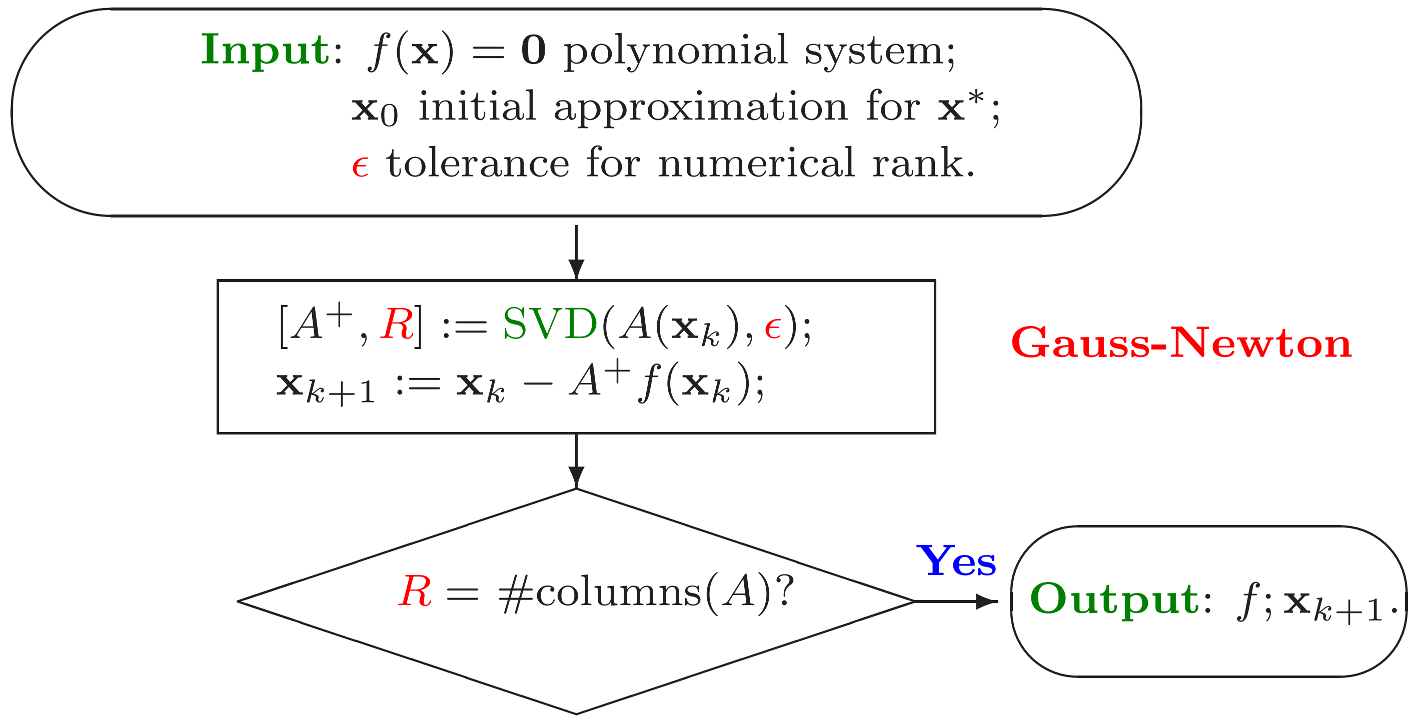
Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

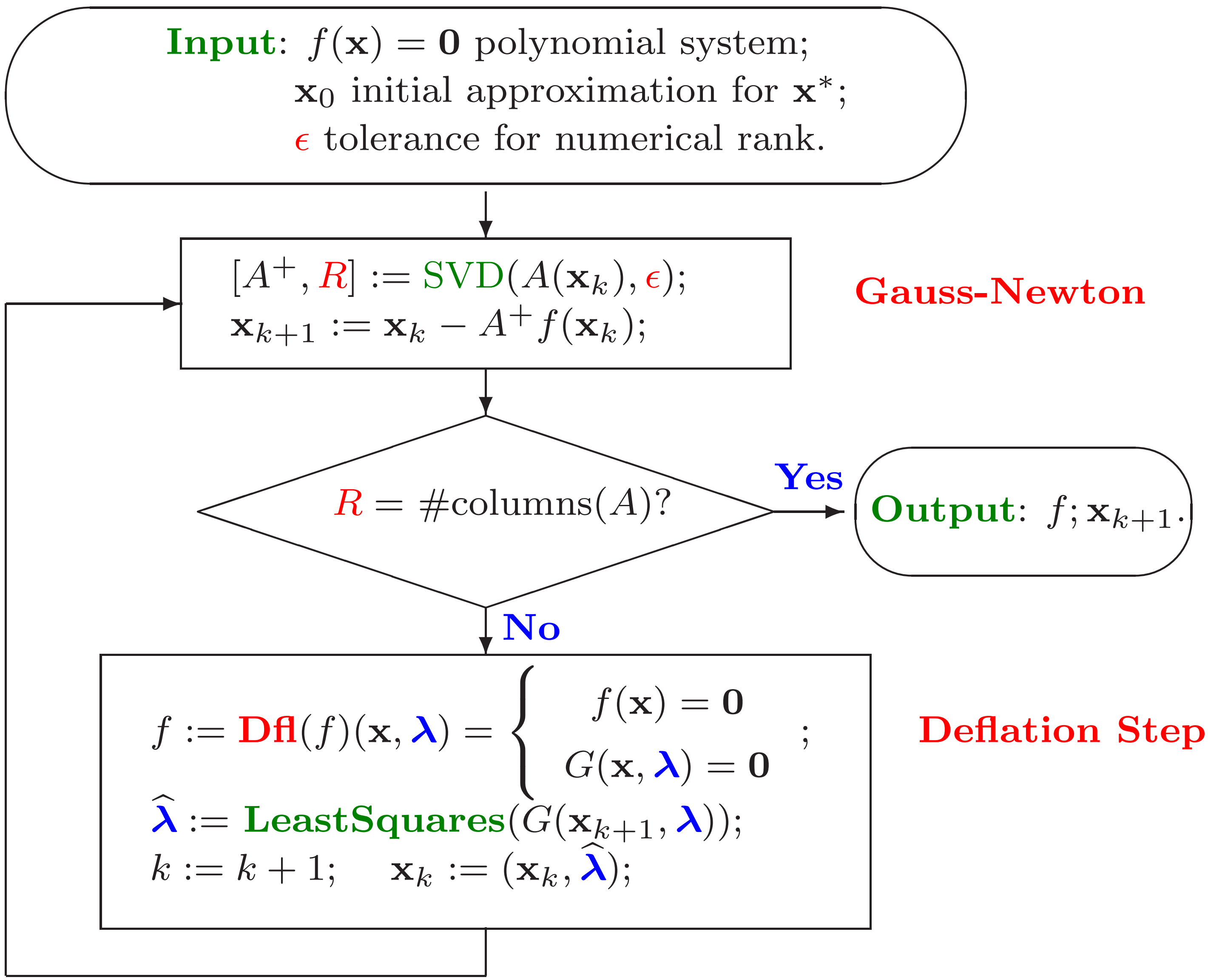
$$[A^+, R] := \mathrm{SVD}(A(\mathbf{x}_k), \epsilon);$$
$$\mathbf{x}_{k+1} := \mathbf{x}_k - A^+ f(\mathbf{x}_k);$$

Gauss-Newton

Newton's Method with Deflation



Newton's Method with Deflation



12 Lines Tangent to 4 Spheres revisited

Continuation methods find 24 solutions, clustered in groups of 4.

The rank at all solutions is 4, corank is 2.

One deflation suffices to restore quadratic convergence.

An average **condition number** drops from 3.4E+8 to 1.1E+2.

We can compute the solutions
with **accuracy close to machine precision**,
on a system with approximate coefficients,
given with double float precision.

A Bound on the Number of Deflations

Theorem (Anton Leykin, JV, Ailing Zhao):

The number of deflations needed to restore the quadratic convergence of Newton's method converging to an isolated solution is strictly less than the multiplicity.

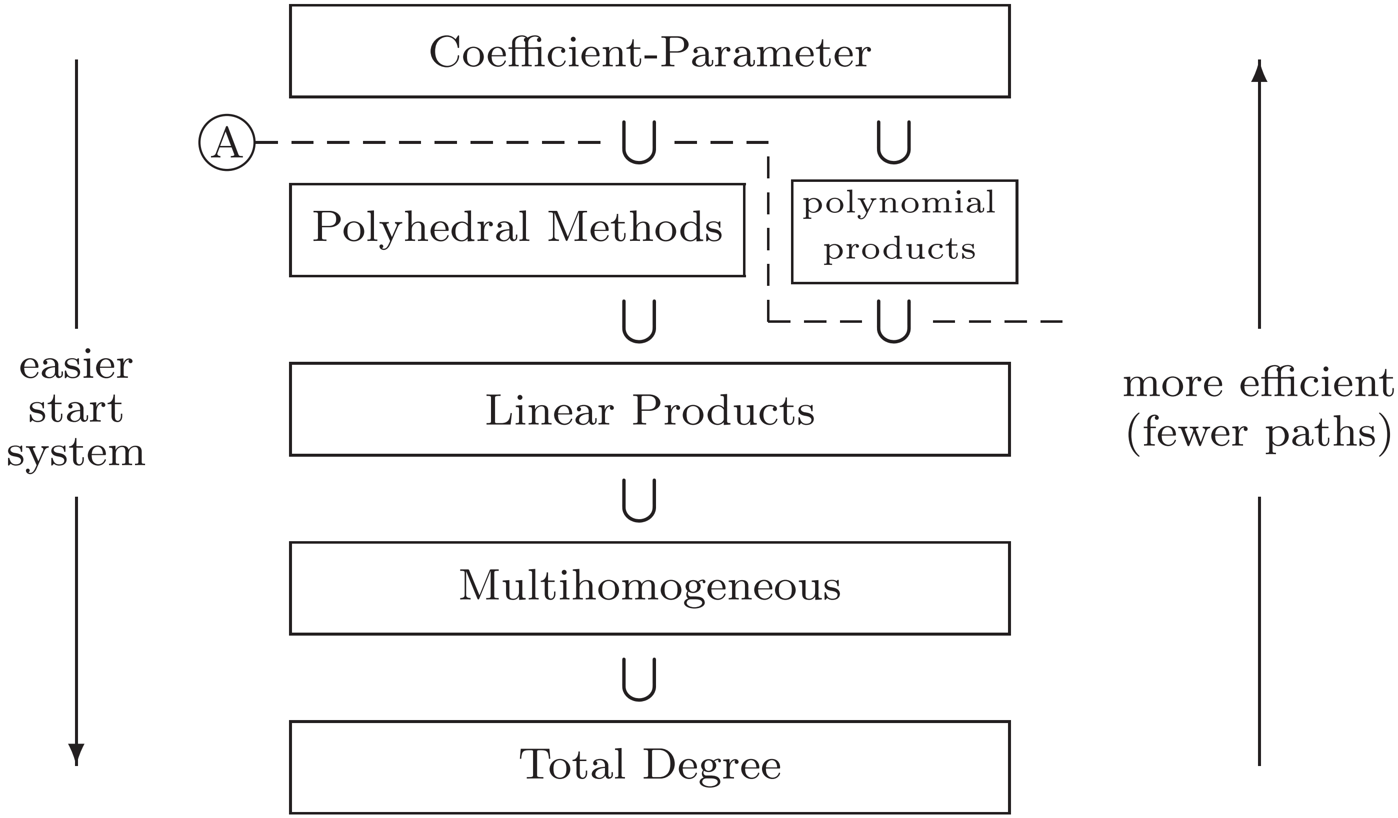
Duality Analysis of Barry H. Dayton and Zhonggang Zeng:

(1) tighter bound on number of deflations; and

(2) special case algorithms, for corank $= 1$.

(Proceedings of ISSAC 2005)

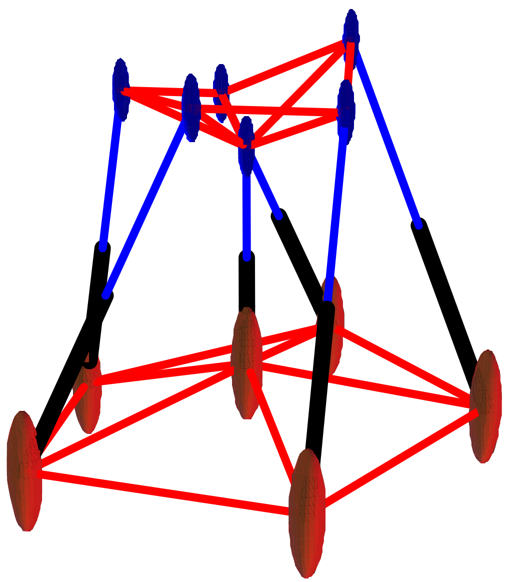
A Hierarchy of Structures



Below line A: solving start systems is done automatically.

Above line A: special ad-hoc methods must be designed.

Assembly of Stewart-Gough Platforms



end plate, the platform

is connected by legs to

a stationary base

Forward Displacement Problem:

Given: position of base and leg lengths.

Wanted: position of end plate.

The Equations for the Platform Problem

workspace $\mathbb{R}^3 \times SO(3)$: position and orientation

$$SO(3) = \{ \ A \in \mathbb{C}^{3\times 3} \mid A^H A = I, \ \det(A) = 1 \ \}$$

more efficient to use Study (or soma) coordinates:

$[e : g] = [e_0 : e_1 : e_2 : e_3 : g_0 : g_1 : g_2 : g_3] \in \mathbb{P}^7$ quaternions on the Study quadric: $f_0(e,g) = e_0 g_0 + e_1 g_2 + e_2 g_2 + e_3 g_3 = 0$, excluding those e for which $ee' = 0$, $e' = (e_0, -e_1, -e_2, -e_3)$

given leg lengths L_i, find $[e : g]$ leads to

$$f_i(e,g) = gg' + (bb_i' + a_i a_i' - L_i^2)ee' + (gb_i'e' + eb_i g') - (ge'a_i' + a_i eg') - (eb_i e'a_i' + a_i eb_i'e') = 0, \quad i = 1,2,\ldots 6$$

$\Rightarrow$ solve $f = (f_0, f_1, \ldots, f_6)$, 7 quadrics in $[e : g] \in \mathbb{P}^7$

expecting $2^7 = 128$ solutions...

Literature on Stewart-Gough platforms

M. Raghavan: **The Stewart platform of general geometry has 40 configurations.** *ASME J. Mech. Design* 115:277–282, 1993.

B. Mourrain: **The 40 generic positions of a parallel robot.** In *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, ed. by M. Bronstein, pages 173–182, ACM 1993.

F. Ronga and T. Vust: **Stewart platforms without computer?** In *Real Analytic and Algebraic Geometry, Proceedings of the International Conference, (Trento, 1992)*, pages 196–212, Walter de Gruyter 1995.

M.L. Husty: **An algorithm for solving the direct kinematics of general Stewart-Gough Platforms.** *Mech. Mach. Theory,* 31(4):365–380, 1996.

C.W. Wampler: **Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates.** *Mech. Mach. Theory* 31(3): 331–337, 1996.

P. Dietmaier: **The Stewart-Gough platform of general geometry can have 40 real postures.** In *Advances in Robot Kinematics: Analysis and Control*, ed. by J. Lenarcic and M.L. Husty, pages 7–16. Kluwer 1998.

Coefficient-Parameter Homotopies

Solve system $f(\mathbf{x}, \mathbf{q}) = \mathbf{0}$ with natural parameters $\mathbf{q} \in \mathbb{C}^m$.

1. solve system once for a generic choice $\mathbf{q}^0$ of the parameters;
2. to move from generic to specific instance $\mathbf{q}^1$, use homotopy

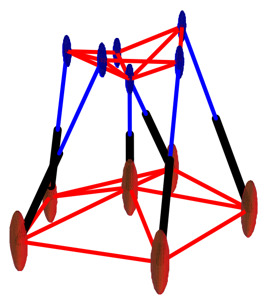
$$f(\mathbf{x}, \mathbf{q}(t)) = f(\mathbf{x}, (1-t)\mathbf{q}^0 + t\mathbf{q}^1) = \mathbf{0}, \quad \text{for } t \text{ from } 0 \text{ to } 1.$$

At $t = 0$, for $\mathbf{q} = \mathbf{q}^0$, we have the maximal number of isolated regular solutions. Singularities can occur only at $t = 1$.

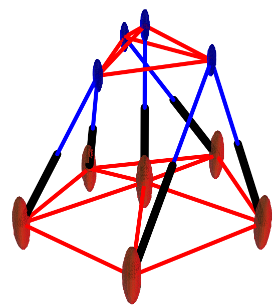
A.P. Morgan and A.J. Sommese: **Coefficient-parameter polynomial continuation.** *Appl. Math. Comput.*, 29(2):123–160, 1989.

A.J. Sommese and C.W. Wampler: **The Numerical Solution of Systems of Polynomials Arising in Engineering and Science.** World Scientific, 2005.

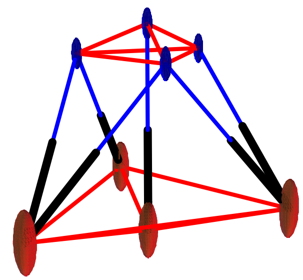
A family of Stewart-Gough platforms



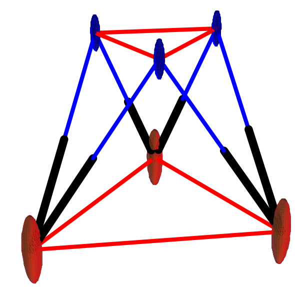
6-6, 40 solutions



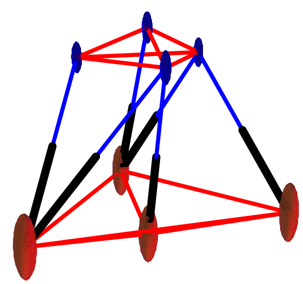
4-6, 32 solutions



4-4a, 16 solutions



3-3, 16 solutions



4-4b, 24 solutions

Multihomogeneous version of Bézout's theorem

Consider the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$, $A \in \mathbb{C}^{n \times n}$.

Add one general hyperplane $\displaystyle\sum_{i=1}^{n} c_i x_i + c_0 = 0$ for unique $\mathbf{x}$.

Bézout's theorem: $D = 2^n \leftrightarrow$ at most n solutions

Embed in multi-projective space: $\mathbb{P} \times \mathbb{P}^n$, separating λ from $\mathbf{x}$.

$\{\lambda\}$	$\{x_1, x_2\}$
1	1
1	1
0	1

degree table

$\Longleftrightarrow$

$\{\lambda\}$	$\{x_1, x_2\}$
$\lambda + \gamma_1$	$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$
$\lambda + \gamma_2$	$\beta_0 + \beta_1 x_1 + \beta_2 x_2$
1	$c_0 + c_1 x_1 + c_2 x_2$

linear-product start system

The root count $B = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 1$ is a permanent.

How to find the best partition?

A multi-homogeneous Bézout number depends on the choice of a partition of the set of unknowns. So, how to choose?

- Knowledge of the application, e.g.: eigenvalue problem.
- Enumerate all partitions and retain the partition with the smallest Bézout number.

#unknowns	1	2	3	4	5	6	7	8	9	···
#partitions	1	2	5	15	52	203	877	4140	21147	···

C.W. Wampler: **Bezout number calculations for multi-homogeneous polynomial systems.** *Appl. Math. Comput.* 51(2–3):143–157, 1992.

- Heuristics based on structures of the monomials are effective in most of the practical cases.

linear-product start systems

$$f(\mathbf{x}) = \begin{cases} x_1 x_2^2 + x_1 x_3^3 - c x_1 + 1 = 0 & c \in \mathbb{C} \\ x_2 x_1^2 + x_2 x_3^2 - c x_2 + 1 = 0 & \\ x_3 x_1^2 + x_3 x_2^2 - c x_3 + 1 = 0 & D = 27 \end{cases}$$

$\{x_1\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	symmetric	
$\{x_2\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$	supporting	$B = 21$
$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	set structure	

Choose 7 random complex numbers $c_1, c_2, \ldots, c_7$ and create

$$g(\mathbf{x}) = \begin{cases} (x_1 + c_1)(c_2 x_2 + c_3 x_3 + c_4)(c_5 x_2 + c_6 x_3 + c_7) = 0 \\ (x_2 + c_1)(c_2 x_1 + c_3 x_3 + c_4)(c_5 x_1 + c_6 x_3 + c_7) = 0 \\ (x_3 + c_1)(c_2 x_1 + c_3 x_2 + c_4)(c_5 x_1 + c_6 x_2 + c_7) = 0 \end{cases}$$

8 generating solutions

Solving the Loop Equations again

- Partition the unknowns in 6 groups:
  $\{\{\theta_1, \bar{\theta}_1\}, \{\theta_2, \bar{\theta}_2\}, \{\theta_3, \bar{\theta}_3\}, \{\theta_4, \bar{\theta}_4\}, \{x, \bar{x}\}, \{y, \bar{y}\}\}$.
- The 6-homogeneous Bézout number is 96
  $\ll$ 4,096 (= total degree).
- Solving takes only 16 seconds $\ll$ 25 minutes,
  but still 60 wasted paths.

Polyhedral Homotopies

D.N. Bernshteĭn. *Functional Anal. Appl.* 1975.

B. Huber and B. Sturmfels. *Math. Comp.* 1995.

T.Y. Li. *Handbook of Numerical Analysis. Volume XI.* 2003.

T. Gao, T.Y. Li, and M. Wu. Algorithm 846: **MixedVol**: A software package for mixed volume computation. *ACM Trans. Math. Softw.* 31(4):555–560, 2005.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani. **PHoM** – a polyhedral homotopy continuation method for polynomial systems. *Computing* 73(4):55–77, 2004.

G. Jeronimo, G. Matera, P. Solernó, and A. Waissbein. Deformation techniques for sparse systems. `arXiv:math.CA/0608714 v1 29 Aug 2006`.

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	desired properties
$L(f) = L(f_2, f_1, \ldots, f_n)$	invariant under permutations
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	shift invariant
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	monotone increasing
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	unimodular invariant
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	root count of product is sum of root counts

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

properties of $L(f)$	$V(\mathcal{P})$ mixed volume
invariant under permutations	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
shift invariant	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
monotone increasing	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
unimodular invariant	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
root count of product is sum of root counts	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	$V(\mathcal{P})$ mixed volume
$L(f) = L(f_2, f_1, \ldots, f_n)$	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

exploit sparsity $\boxed{L(f) = V(\mathcal{P})}$ 1st theorem of Bernshteĭn

3 stages to solve a polynomial system $f(\mathbf{x}) = \mathbf{0}$

1. Compute the mixed volume (aka the BKK bound) of the Newton polytopes spanned by the supports A of f via a **regular mixed-cell configuration Δ_ω**.

2. Given Δ_ω, solve a generic system $g(\mathbf{x}) = \mathbf{0}$, using polyhedral homotopies. Every cell $C \in \Delta_\omega$ defines one homotopy

$$h_C(\mathbf{x}, s) = \sum_{\mathbf{a} \in C} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} + \sum_{\mathbf{a} \in A \setminus C} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} s^{\nu_{\mathbf{a}}}, \quad \nu_{\mathbf{a}} > 0,$$

   tracking as many paths as the mixed volume of the cell C, as s goes from 0 to 1.

3. Use $(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$ to solve $f(\mathbf{x}) = \mathbf{0}$.

Stages 2 and 3 are **computationally most intensive** $(1 \ll 2 < 3)$.

Solving the Loop Equations once more

- The mixed volume equals 36 and is an exact root count.

  584 milliseconds to compute the mixed volume

  1 second 448 milliseconds to solve a random start system

  1 second 800 milliseconds to track 36 paths to target system

- total time: 3 seconds and 868 milliseconds

  better than the 16 seconds with multihomogenization

A Static Distribution of the Workload

joint with Yan Zhuang

manager	worker 1	worker 2	worker 3
Vol(cell 1) = 5	#paths(cell 1) : 5		
Vol(cell 2) = 4	#paths(cell 2) : 4		
Vol(cell 3) = 4	#paths(cell 3) : 4		
Vol(cell 4) = 6	#paths(cell 4) : 1	#paths(cell 4) : 5	
Vol(cell 5) = 7		#paths(cell 5) : 7	
Vol(cell 6) = 3		#paths(cell 6) : 2	#paths(cell 6) : 1
Vol(cell 7) = 4			#paths(cell 7) : 4
Vol(cell 8) = 8			#paths(cell 8) : 8
total #paths : 41	#paths : 14	#paths : 14	#paths : 13

Since polyhedral homotopies solve a **generic** system $g(\mathbf{x}) = \mathbf{0}$,

we **expect** every path to take the same amount of work...

An academic Benchmark: cyclic n-roots

The system

$$f(\mathbf{x}) = \left\{ \begin{array}{l} f_i = \displaystyle\sum_{j=0}^{n=1} \prod_{k=1}^{i} x_{(k+j) \bmod n} = 0, \quad i = 1, 2, \ldots, n-1 \\ f_n = x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0 \end{array} \right.$$

appeared in

G. Björck: **Functions of modulus one on Z_p whose Fourier transforms have constant modulus** In *Proceedings of the Alfred Haar Memorial Conference, Budapest*, pages 193–197, 1985.

very sparse, well suited for polyhedral methods

Results on the cyclic n-roots problem

Problem	#Paths	CPU Time
cyclic 5-roots	70	0.13m
cyclic 6-roots	156	0.19m
cyclic 7-roots	924	0.30m
cyclic 8-roots	2,560	0.78m
cyclic 9-roots	11,016	3.64m
cyclic 10-roots	35,940	21.33m
cyclic 11-roots	184,756	2h 39m
cyclic 12-roots	500,352	24h 36m

Wall time for start systems to solve the cyclic n-roots problems, using 13 workers, with static load distribution.

Dynamic versus Static Workload Distribution

#workers	Static versus Dynamic on our cluster				Dynamic on argo	
	Static	Speedup	Dynamic	Speedup	Dynamic	Speedup
1	50.7021	–	53.0707	–	29.2389	–
2	24.5172	2.1	25.3852	2.1	15.5455	1.9
3	18.3850	2.8	17.6367	3.0	10.8063	2.7
4	14.6994	3.4	12.4157	4.2	7.9660	3.7
5	11.6913	4.3	10.3054	5.1	6.2054	4.7
6	10.3779	4.9	9.3411	5.7	5.0996	5.7
7	9.6877	5.2	8.4180	6.3	4.2603	6.9
8	7.8157	6.5	7.4337	7.1	3.8528	7.6
9	7.5133	6.8	6.8029	7.8	3.6010	8.1
10	6.9154	7.3	5.7883	9.2	3.2075	9.1
11	6.5668	7.7	5.3014	10.0	2.8427	10.3
12	6.4407	7.9	4.8232	11.0	2.5873	11.3
13	5.1462	9.8	4.6894	11.3	2.3224	12.6

Wall time in seconds to solve a start system for the cyclic 7-roots problem.

Design of Serial Chains I

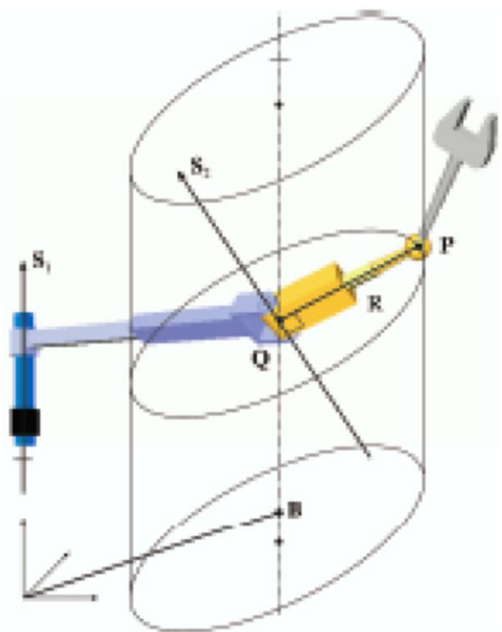


Figure 4.4: The elliptic cylinder reachable by a PRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory*. PhD thesis, University of California, Irvine, 2004.

Design of Serial Chains II

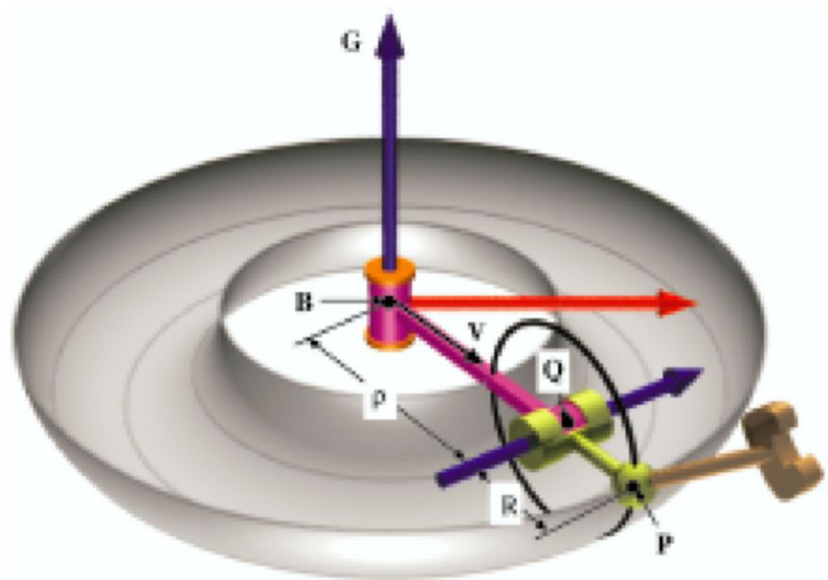


Figure 4.7: The circular torus traced by the wrist center of a "right" RRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory.* PhD thesis, University of California, Irvine, 2004.

Design of Serial Chains III

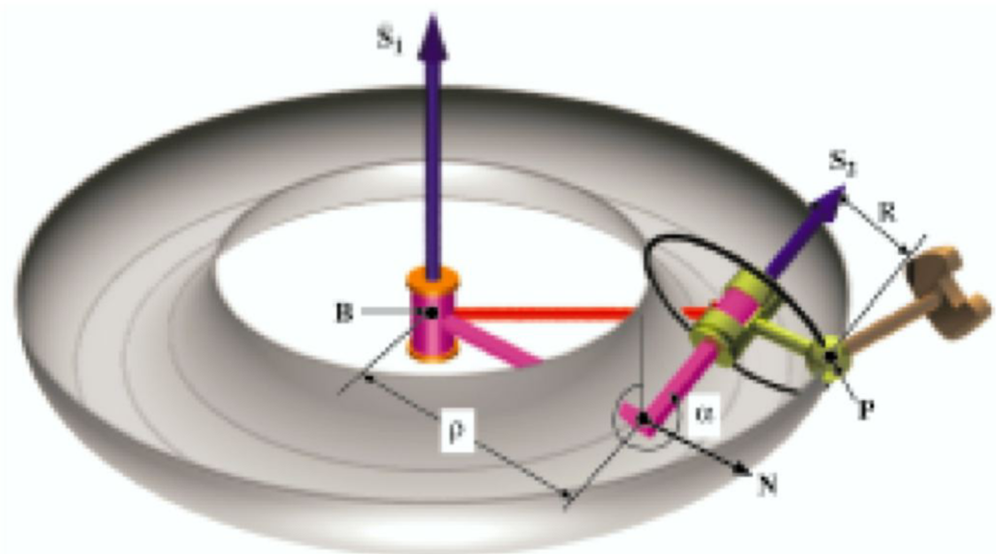


Figure 4.8: The general torus reachable by the wrist center of an RRS serial chain.

H.J. Su. *Computer-Aided Constrained Robot Design Using Mechanism Synthesis Theory*. PhD thesis, University of California, Irvine, 2004.

For more about these problems:

H.-J. Su and J.M. McCarthy: **Kinematic synthesis of RPS serial chains**. In the *Proceedings of the ASME Design Engineering Technical Conferences* (CDROM), Chicago, IL, Sep 2-6, 2003.

H.-J. Su, C.W. Wampler, and J.M. McCarthy: **Geometric design of cylindric PRS serial chains**. *ASME Journal of Mechanical Design* 126(2):269–277, 2004.

H.-J. Su, J.M. McCarthy, and L.T. Watson: **Generalized linear product homotopy algorithms and the computation of reachable surfaces**. *ASME Journal of Information and Computer Sciences in Engineering* 4(3):226–234, 2004.

Results on Mechanical Design Problems

joint with Yan Zhuang

Surface	Bounds on #Solutions			Wall Time	
	Bézout	linear-product	Mixvol	our cluster	on argo
elliptic cylinder	2,097,152	247,968	125,888	11h 33m	6h 12m
circular torus	2,097,152	868,352	474,112	7h 17m	4h 3m
general torus	4,194,304	448,702	226,512	14h 15m	6h 36m

Wall time for mechanism design problems on our cluster and argo.

- Compared to the linear-product bound, polyhedral homotopies cut the #paths about in half.
- The second example is easier (despite the larger #paths) because of increased sparsity, and thus lower evaluation cost.