Toolboxes and Blackboxes for Solving Polynomial Systems

Jan Verschelde

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
http://www.math.uic.edu/~jan
jan@math.uic.edu

Interactions between Classical and Numerical Algebraic Geometry.
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University of Notre Dame, 22-24 May 2008

Outline

Toolboxes and Blackboxes

1. Solving Polynomial Systems
   - what does *solving* mean?
   - four basic tools
2. Polyhedral Methods
   - recognizing sparse structures
   - tropical algebraic geometry
3. Numerical Irreducible Decomposition
   - witness sets represent components of solutions
   - wrapping software up in interfaces
4. Towards a Polyhedral Method for Curves
   - computing certificates for solution curves
   - some preliminary computational experiments

Solving Polynomial Systems

what does *solving* mean?

Before numerical algebraic geometry:

> *solving systems by numerical homotopy continuation means to compute approximations to all isolated solutions*

What we today understand by solving:

> *a numerical irreducible decomposition gives the irreducible factors for each dimension, along with their multiplicities*

[Leykin, ISSAC 2008]: Numerical Primary Decomposition.

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the cyclic 8-roots system

a well known benchmark problem

a system of 8 equations in 8 unknowns:

$$f(\mathbf{z}) = \left\{ \begin{array}{c} z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 = 0 \\ z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 + z_6 z_7 + z_7 z_0 = 0 \\ i = 3, 4, \ldots, 7 : \displaystyle\sum_{j=0}^{7} \prod_{k=j}^{i} z_{k \bmod 8} = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{array} \right.$$

J. Backelin: "Square multiples n give infinitely many cyclic n-roots". Reports, Matematiska Institutionen, Stockholms Universitet, 1989.
$n = 8$ has 4 as divisor, $4 = 2^2$, so infinitely many roots

how to verify numerically?

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Homotopy Continuation Methods

a numerical way to solve polynomial systems

A geometric way to solve a system:

1. the system is a specific instance of a problem class
2. deform the specific instance into a generic, easier problem
3. solve the generic, easier problem
4. track solutions of generic to the specific problem

Four basic tools:

1. scaling and projective transformations
2. root counting and start systems
3. deforming systems and path tracking
4. root refining and endgames

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Multihomogeneous Structures

scaling and projective transformations

Consider the algebraic eigenvalue problem:

$$A\mathbf{x} = \lambda x, \quad \mathbf{x} \in \mathbb{C}^n,$$

for some n-by-n matrix A.

Ignoring the structure: $(\lambda, \mathbf{x}) \in \mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}$.
Multiprojective space: $(\lambda, \mathbf{x}) \in \mathbb{C} \times \mathbb{C}^n \subset \mathbb{P} \times \mathbb{P}^n$.

A. Morgan and A. Sommese: A homotopy for solving general polynomial systems that respects m-homogeneous structures. *Appl. Math. Comput.*, 24(2):101–113, 1987.

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Multihomogeneous Homotopies

root counting and start systems

Consider $A\mathbf{x} = \lambda\mathbf{x}$, $A \in \mathbb{C}^{n\times n}$. plain Bézout's theorem: $D = 2^n$

Add a hyperplane $c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_0 = 0$ for unique $\mathbf{x}$.

$\{\lambda\}$	$\{x_1, x_2\}$
1	1
1	1
$\vdots$	$\vdots$
0	1

degree table

$\iff$

$\{\lambda\}$	$\{x_1, x_2\}$
$\lambda + \gamma_1$	$\alpha_0 + \alpha_1x_1 + \alpha_2x_2$
$\lambda + \gamma_2$	$\beta_0 + \beta_1x_1 + \beta_2x_2$
$\vdots$	$\vdots$
1	$c_0 + c_1x_1 + c_2x_2$

linear-product start system

The root count $B = 1\cdot 1\cdots 1 + 1\cdot 1\cdots 1 + \cdots + 0\cdot 1\cdots 1 = n$ is exact!

Solve a polynomial system by degeneration:

1. deform each polynomial into a product of linear polynomials
2. compute intersection of hyperplanes: start solutions
3. deform linear-product start system into original problem

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Coefficient-Parameter Polynomial Continuation

using a cheater's homotopy

Consider $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, unknowns $\mathbf{x} \in \mathbb{C}^n$, parameters $\boldsymbol{\lambda} \in \mathbb{C}^m$.

Let $N_{\boldsymbol{\lambda}}$ be the number of regular solutions of $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$. Then:

1. compute $N_{\boldsymbol{\lambda}}$ by solving $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ for generic $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0$,
2. for any $\boldsymbol{\lambda}_1$, $f(\mathbf{x}, (1-t)\boldsymbol{\lambda}_0 + t\boldsymbol{\lambda}_1) = \mathbf{0}$, $t \in [0, 1)$,
   has exactly $N_{\boldsymbol{\lambda}}$ regular roots.

Classical interaction: *principle of conservation of number*.

T.Y. Li, T. Sauer, and J.A. Yorke: The cheater's homotopy:
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Enumerating All Solutions

a pleasingly parallel computation

If we have given:

1. a program to evaluate a family of systems for $(\mathbf{x}, t)$,
2. a function to get the kth start solution, for $t = 0$.

Then we can execute a pleasingly parallel path tracking:

1. track paths independently from each other,
2. no need to keep all solutions in main memory:
   1. write to file as soon as at end of path,
   2. size of main memory is not the bottleneck,
   3. checkpointing: even supercomputers do crash.

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Endgames

dealing with solution paths at the end

At the end of the paths, solutions

1. may diverge to infinity,
2. or converge to a singular solution.

The homotopy $h(\mathbf{z}(s), t(s)) = \mathbf{0}$ defines a path $(\mathbf{z}(s), t(s))$.
At the end, as $t \rightarrow 1$, $s \approx 0$.

For $s \rightarrow 0$: $z_k(s) = c_{k,1} s^{a_{k,1}/\omega} + c_{k,2} s^{(a_{k,1}+1)/\omega} + \cdots, k = 1, 2, \ldots, n,$
is a fractional power series, w is the winding number.
Observe: $a_{k,1} > 0$: $z_k \rightarrow 0$, $a_{k,1} = 0$: $z_k \rightarrow c_{k,1}$, $a_{k,1} < 0$: $z_k \rightarrow \infty$.

A.P. Morgan, A.J. Sommese, and C.W. Wampler:
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Victor Alexandre Puiseux (1820-1883)

In 1850, he gave a first rigorous proof of the convergence of fractional power series, assuming the fundamental theorem of algebra.

V. Puiseux: Mémoirs sur les fonctions algébriques. *J. Math. Pures Appl.* 32, 1851.

Theorem of Puiseux (see Walker's *Algebraic Curves*):
the field of fractional power series over $\mathbb{C}$ is algebraically closed.

back to the cyclic 8-roots problem

applying our basic tools

Recall a system of 8 equations in 8 unknowns:

$$f(\mathbf{z}) = \left\{ \begin{array}{c} z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 = 0 \\ z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_6 + z_6z_7 + z_7z_0 = 0 \\ i = 3, 4, \ldots, 7 : \displaystyle\sum_{j=0}^{7} \prod_{k=j}^{i} z_{k \bmod 8} = 0 \\ z_0z_1z_2z_3z_4z_5z_6z_7 - 1 = 0 \end{array} \right.$$

Product of the degrees: $8! = 40,320 \gg 1,152$ isolated roots.

Enumeration of all 4,140 partitions of $\{z_0, z_1, \ldots, z_7\}$:
$\rightarrow$ no improvement from multihomogeneous root count.

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Newton Polytopes and Mixed Volumes

recognizing the sparse structure of a polynomial system

Most polynomials have few nonzero coefficients:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \neq 0, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The *support* A of f spans the Newton polytope $P = \mathrm{ConvHull}(A)$.
$\mathcal{P} = (P_1, P_2, \ldots, P_n)$ collects the Newton polytopes of a system f.
Remember the principle of conservation of number (classical)
or coefficient-parameter polynomial continuation (numerical):
$N_{\mathbf{c}}$ = the number of solutions for generic coefficients $\mathbf{c}$.

Bernshteĭn's theorem (1975): $N_{\mathbf{c}}$ depends only on $\mathcal{P}$.

In particular: $N_{\mathbf{c}} = V(\mathcal{P})$, the mixed volume of $\mathcal{P}$.
Special case: $P = P_1 = P_2 = \cdots = P_n$: $N_{\mathbf{c}} = n!\mathrm{volume}(P)$.

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The Theorems of Bernshteĭn

Mixed volumes relate volume to surface area:

$$V_n(P_1, P_2, \ldots, P_n) = \sum_{\mathbf{v}} p_1(\mathbf{v})\ V_{n-1}(\partial_{\mathbf{v}} P_2, \ldots, \partial_{\mathbf{v}} P_n),$$

$\mathbf{v} \in \mathbb{Z}^n$, $\gcd(\mathbf{v}) = 1$, $p_1(\mathbf{v}) = \min\limits_{\mathbf{x} \in P_1} \langle \mathbf{x}, \mathbf{v} \rangle$ is a support function
$\partial_{\mathbf{v}} P_k = \{\ \mathbf{x} \in P_k \mid \langle \mathbf{x}, \mathbf{v} \rangle = p_k(\mathbf{v})\ \}$ is a face of P_k.

Theorem A: The number of roots of a generic system equals the mixed volume of its Newton polytopes.

Theorem B: Solutions at infinity are solutions of systems supported on faces of the Newton polytopes.

D.N. Bernshteĭn: The number of roots of a system of equations. *Functional Anal. Appl.* 9(3):183–185, 1975.

F. Minding: Über die Bestimmung des Grades einer durch Elimination hervorgehenden Gleichung. *J. Reine Angew. Math.* 22: 178-183, 1841.

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Polyhedral Homotopies

constructive proofs of Bernshteǐn's theorems

Polyhedral homotopies implement Bernshteǐn's theorems.

An effective complement to the *cheater's* homotopy.

The methods are *optimal* in the sense that every solution path converges to an isolated solution . . .

. . . *provided* the system is sufficiently generic.

B. Huber and B. Sturmfels: A polyhedral method for solving sparse polynomial systems. *Math. Comp.* 64(212): 1541–1555, 1995.

T.Y. Li: Numerical solution of polynomial systems by homotopy continuation methods.
In Volume XI of *Handbook of Numerical Analysis*, pp. 209–304, 2003.

Toolboxes and Blackboxes

Tropical Algebraic Geometry

a new language describing asymptotics of varieties

Polyhedral methods in a tropical world:

1. tropicalizations of polynomials and polytopes
   - introduce t in f: $f(\mathbf{x}, t) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{\omega(\mathbf{a})}$
   - lift supports and polytopes $\widehat{P} = \mathit{ConvHull}(\{ (\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in A \})$

   $\rightarrow$ a tropicalization is an inner normal fan of $\widehat{P}$
2. tropisms
   - are in the intersection of normal cones
     to the edges of the lifted polytopes,
   - give the leading powers to the Puiseux expansions for the start of the solution paths in the polyhedral homotopies.

J. Maurer: Puiseux expansion for space curves.
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a Toolbox for Mixed Volume Computation

and polyhedral homotopies to solve a generic system

T. Gao., T.Y. Li, and M. Wu: Algorithm 846: `MixedVol`:
a software package for mixed-volume computation.
ACM Trans. Math. Softw. 31(4):555–560, 2005.

available in PHCpack:

- version 2.3.13 on 2006-08-25
  Ada translation of `MixedVol` available in `phc -m`
- version 2.3.31 on 2007-07-13
  stable mixed volumes in `phc -m`
  $\rightarrow$ no longer miss solutions with zero components

dynamic enumeration of mixed cells in

- `DEMiCs` by Tomohiko Mizutani and Akiko Takeda
- `HOM4PS-2.0` by Tsung-Lin Lee, T.Y. Li, and Chih-Hsiung Tsai

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A First Blackbox Solver

Source code of PHCpack was first released in August 1997.

toolboxes	via options of the executable `phc` $\rightarrow$ *tools* assume some skill of the user
a blackbox	solver: `phc -b input output` $\rightarrow$ a *solver* has to make assumptions

How `phc -b` works:

1. computes various root counts
2. solves start system with lowest root count
3. track paths to all isolated solutions

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Mixed Volume of cyclic 8-roots

Recall: $8! = 40,320$ as Bézout bound.

Mixed volume: 2,560 > 1,152 = #isolated roots.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani: PHoM – a polyhedral homotopy continuation method for polynomial systems. *Computing* 73(4): 55–77, 2004.

applied to cyclic 13-roots: mixed volume = 2,704,156 = #paths

but what about components of solutions?

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Numerical Irreducible Decomposition

what solving a polynomial system means

input: $f(\mathbf{x}) = \mathbf{0}$ a polynomial system with $\mathbf{x} \in \mathbb{C}^n$

- **Stage 1**: represent the k-dimensional solutions Z_k, $k = 0, 1, \ldots$

  output: sequence $[W_0, W_1, \ldots, W_{n-1}]$ of ***witness sets***
  $W_k = (E_k, E_k^{-1}(\mathbf{0}) \setminus J_k)$, $\deg Z_k = \#(E_k^{-1}(\mathbf{0}) \setminus J_k)$
  $E_k = f + k$ random hyperplanes, J_k = "junk"

- **Stage 2**: decompose Z_k, $k = 0, 1, \ldots$ into irreducible factors

  output: $W_k = \{W_{k1}, W_{k2}, \ldots, W_{kn_k}\}$, $k = 1, 2, \ldots, n-1$
  n_k irreducible components of dimension k

output: a numerical irreducible decomposition of $f^{-1}(\mathbf{0})$
is a sequence of partitioned witness sets

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Computing Witness Sets for $f^{-1}(\mathbf{0})$

two toolboxes for a witness set computation

Witness set $W_k = (E_k, E_k^{-1}(\mathbf{0}) \setminus J_k)$ for $Z_k \subset f^{-1}(\mathbf{0})$, $k = \dim Z_k$,
consists of $E_k = f + k$ random hyperplanes
and its solutions, $\#(E_k^{-1}(\mathbf{0}) \setminus J_k) = \deg Z_k$.

- **top down**: use a cascade of homotopies
  - $+$ benefits from existing blackbox solver
  - $-$ requires top dimension on input
- **bottom up**: with an equation-by-equation solver
  - $+$ requires no guess for top dimension
  - $-$ performance depends on order of equations

Computing Witness Sets for $f^{-1}(\mathbf{0})$

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Example of a Homotopy in the Cascade

To compute numerical representations of the twisted cubic and the four isolated points, as given by the solution set of one polynomial system, we use the following homotopy:

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{c} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] + t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) + z_1 \end{array} \right] = \mathbf{0}$$

At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$.
At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the system, and z_1 is forced to zero.

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A Cascade of Homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde): *J. Complexity* 16(3):572–602, 2000

1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

solution paths defined by

$$H_i(\mathbf{x}, \mathbf{z}, t) = t\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + (1-t)\begin{pmatrix} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{pmatrix} = \mathbf{0}$$

starting at $t = 1$ with all solutions with $z_i \neq 0$
reach at $t = 0$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

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A refined version of Bézout's theorem

Observe: The linear equations added to $f(\mathbf{x}) = \mathbf{0}$ in the cascade of homotopies do not increase the total degree.

Let $f = (f_1, f_2, \ldots, f_n)$ be a system of n polynomial equations in N variables, $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

Bézout bound: $$\prod_{i=1}^{n} \deg(f_i) \geq \sum_{j=0}^{N} \mu_j \deg(W_j),$$

where W_j is a j-dimensional solution component of $f(\mathbf{x}) = \mathbf{0}$ of multiplicity μ_j.

Note: $j = 0$ gives the "classical" theorem of Bézout.

#paths for cascade on cyclic 8-roots

The flow chart below summarizes the number of solution paths traced in the cascade of homotopies.

4,176 paths	$\longrightarrow$	57 paths to infinity		
		144 solutions with $z_1 = 0$	$\longrightarrow$	W_1 witness set
		3,975 solutions with $z_1 \neq 0$		
3,975 paths	$\longrightarrow$	2,328 paths to infinity		
		1,647 converging paths	$\longrightarrow$	$\widehat{W}_0$ witness **superset**

The set $\widehat{W}_0$ contains, in addition to the 1,152 isolated roots, also points on the solution curve. The points in $\widehat{W}_0$ which lie on the curve are considered **junk** and must be filtered out.

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   - what does *solving* mean?
   - four basic tools
2. Polyhedral Methods
   - recognizing sparse structures
   - tropical algebraic geometry
3. Numerical Irreducible Decomposition
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4. Towards a Polyhedral Method for Curves
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   - some preliminary computational experiments

Interfaces to PHCpack

A first simple Maple interface appeared in

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Numerical irreducible decomposition using PHCpack. In *Algebra, Geometry, and Software Systems*, pp. 109–130, Springer, 2003.

Accessing PHCpack in scripting environments:

- PHCmaple (with Anton Leykin): Maple tools
- PHClab (with Yun Guan) for MATLAB and Octave (MPITB)

Benefits: visualization, symbolic manipulation, high level parallelism.

Programmer's interfaces:

- PHClib: C interface to MPI
- PHCpy (with Kathy Piret): Python module in release 2.3.41

Benefits to open source mathematics software development.
PHCpack is one of the optional packages in Sage,
thanks to Marshall Hampton, Kathy Piret, and William Stein.

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Certifying Solution Components

some problems with current approach

Witness sets are good numerical representations for solution sets, but:

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  But Bézout bounds are often too large for many systems.
- Adding hyperplanes and slack variable increases mixed volume.

  Examples: cyclic 8 roots: 2,560 $\rightarrow$ 4,176,
  cyclic 12 roots: 500,352 $\rightarrow$ 983,952.
- Need certificates, cheaper than witness sets.

Tropical view: look at infinity, look at sparser systems.

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Computing a Series Expansion

a staggered approach to find a certificate for a solution curve

tropicalization

$\downarrow$

compute tropisms $\bigcirc$ $\searrow$ no tropism

$\Rightarrow$ no root at ∞

solve initial forms $\bigcirc$ $\searrow$ no root at ∞

$\Rightarrow$ no series

compute 2nd term $\bigcirc$ $\searrow$ no series

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Tropisms coming from Endgames

joint work with Birk Huber, Numerical Algorithms 18(1):91–108, 1998

Directions of diverging paths for cyclic 8-roots:

tropisms	m	accuracy	#paths
$\pm(-1,1,-1,1,-1,1,-1,1)$	1	10^{-3}	32
$\pm(-1,0,0,1,0,-1,1,0)$	1	10^{-7}	8
$\pm(0,-1,0,0,1,0,-1,1)$	1	10^{-6}	8
$\pm(1,0,-1,0,0,1,0,-1)$	1	10^{-7}	8
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$\pm(0,-1,1,0,-1,0,0,1)$	1	10^{-6}	8
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$\pm(0,1,0,-1,1,0,-1,0)$	1	10^{-6}	8
$\pm(0,0,1,0,-1,1,0,-1)$	1	10^{-6}	8

Every tropism $\mathbf{v}$ defines an initial form $\partial_{\mathbf{v}} f$.
Every equation in $\partial_{\mathbf{v}} f$ has at least two monomials
$\Rightarrow$ admits a solution with all components nonzero.

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An Initial Form of the cyclic 8-roots system

For the tropism $\mathbf{v} = (-1, 0, 0, +1, 0, -1, +1, 0)$:

$$\partial_{\mathbf{v}} f(\mathbf{z}) = \left\{ \begin{array}{r} z_0 + z_5 = 0 \\ z_0 z_1 + z_4 z_5 + z_7 z_0 = 0 \\ z_0 z_1 z_2 + z_7 z_0 z_1 = 0 \\ z_5 z_6 z_7 z_0 + z_7 z_0 z_1 z_2 = 0 \\ z_4 z_5 z_6 z_7 z_0 + z_5 z_6 z_7 z_0 z_1 = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 + z_4 z_5 z_6 z_7 z_0 z_1 + z_5 z_6 z_7 z_0 z_1 z_2 = 0 \\ z_4 z_5 z_6 z_7 z_0 z_1 z_2 + z_7 z_0 z_1 z_2 z_3 z_4 z_5 = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{array} \right.$$

Observe: for all $\mathbf{z}^{\mathbf{a}}$: $\langle \mathbf{a}, \mathbf{v} \rangle = -1$,
except for the last equation: $\langle \mathbf{a}, \mathbf{v} \rangle = 0$.

Transforming Coordinates

to eliminate one variable

The tropism $\mathbf{v} = (-1, 0, 0, +1, 0, -1, +1, 0)$ defines a change of coordinates:

$$\left\{ \begin{array}{l} z_0 = x_0^{-1} \\ z_1 = x_0^{0} x_1 \\ z_2 = x_0^{0} x_2 \\ z_3 = x_0^{+1} x_3 \\ z_4 = x_0^{0} x_4 \\ z_5 = x_0^{-1} x_5 \\ z_6 = x_0^{+1} x_6 \\ z_7 = x_0^{0} x_7 \end{array} \right. \qquad \partial_{\mathbf{v}} f(\mathbf{x}) = \left\{ \begin{array}{r} 1 + x_5 = 0 \\ x_1 + x_4 x_5 + x_7 = 0 \\ x_1 x_2 + x_7 x_1 = 0 \\ x_5 x_6 x_7 + x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 + x_5 x_6 x_7 x_1 = 0 \\ x_1 x_2 x_3 x_4 x_5 + x_4 x_5 x_6 x_7 x_1 \\ + x_5 x_6 x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 x_1 x_2 + x_7 x_1 x_2 x_3 x_4 x_5 = 0 \\ x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 = 0 \end{array} \right.$$

After clearing x_0, $\partial_{\mathbf{v}} f$ consists of 8 equations in 7 unknowns.

Solving an overconstrained Initial Form

Choose eight random numbers $\gamma_k \in \mathbb{C}$, $k = 1, 2, \ldots, 8$,
to introduce a slack variable s:

$$\partial_{\mathbf{v}} f(\mathbf{x}, s) = \left\{ \begin{array}{r} 1 + x_5 + \gamma_1 s = 0 \\ x_1 + x_4 x_5 + x_7 + \gamma_2 s = 0 \\ x_1 x_2 + x_7 x_1 + \gamma_3 s = 0 \\ x_5 x_6 x_7 + x_7 x_1 x_2 + \gamma_4 s = 0 \\ x_4 x_5 x_6 x_7 + x_5 x_6 x_7 x_1 + \gamma_5 s = 0 \\ x_1 x_2 x_3 x_4 x_5 + x_4 x_5 x_6 x_7 x_1 + x_5 x_6 x_7 x_1 x_2 + \gamma_6 s = 0 \\ x_4 x_5 x_6 x_7 x_1 x_2 + x_7 x_1 x_2 x_3 x_4 x_5 + \gamma_7 s = 0 \\ x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 + \gamma_8 s = 0 \end{array} \right.$$

The mixed volume of this system is 25 and is exact.
Among the 25 solutions, there are 8 with $s = 0$.

The first Term of a Puiseux Expansion

For $f(\mathbf{x}) = \partial_{\mathbf{e}} f(x) + O(x_0)$, $\mathbf{e} = (1, 0, 0, 0, 0, 0, 0, 0)$,
we use a solution as the leading term of a Puiseux expansion:

$$\begin{cases} x_0 = t^1 \\ x_1 = (\ 0.5 + 0.5i\ )\ t^0 & +y_1\ t \\ x_2 = (\ 1 + i\ )\ t^0 & +y_2\ t \\ x_3 = (\ -i\ )\ t^0 & +y_3\ t \\ x_4 = (\ -0.5 - 0.5i\ )\ t^0 & +y_4\ t \\ x_5 = (\ -1\ )\ t^0 & +y_5\ t \\ x_6 = (\ i\ )\ t^0 & +y_6\ t \\ x_7 = (\ -1 - i\ )\ t^0 & +y_7\ t \end{cases} \qquad i = \sqrt{-1}.$$

Decide whether solution is isolated: substitute series in $f(\mathbf{x}) = \mathbf{0}$ and solve for y_k, $k = 1, 2, \ldots, 7$ in lowest order terms of t.
$\rightarrow$ solve an overdetermined linear system in the coefficients of the 2nd term of the Puiseux expansion.

The second Term of a Puiseux Expansion

Because we find a nonzero solution for the y_k coefficients, we use it as the second term of a Puiseux expansion:

$$\left\{ \begin{array}{ll} x_0 = t^{1} & \\ x_1 = (\ 0.5 + 0.5i\ )\ t^{0} & + (\ -0.5i\ )\ t \\ x_2 = (\ 1 + i\ )\ t^{0} & + (\ -i\ )\ t \\ x_3 = (\ -i\ )\ t^{0} & + (\ 1 - i\ )\ t \\ x_4 = (\ -0.5 - 0.5i\ )\ t^{0} & + (\ 0.5i\ )\ t \\ x_5 = (\ -1\ )\ t^{0} & + (\ 0\ )\ t \\ x_6 = (\ i\ )\ t^{0} & + (\ -1 + i\ )\ t \\ x_7 = (\ -1 - i\ )\ t^{0} & + (\ i\ )\ t \end{array} \right. \qquad i = \sqrt{-1}.$$

Substitute series in $f(\mathbf{x})$: result is $O(t^2)$.

the cyclic 12-roots problem

According to J. Backelin, also here infinitely many solutions.

Mixed volume 500,352 and increases to 983,952
by adding one random hyperplane and slack variable.

Like for cyclic 8, $\mathbf{v} = (-1,+1,-1,+1,-1,+1,-1,+1,-1,+1,-1,+1)$
is a tropism. Mixed volume of $\partial_{\mathbf{v}} f(\mathbf{x}, s) = \mathbf{0}$ is 49,816.
One of the solutions is

$$
\begin{array}{ll}
x_0 = t & x_1 = 0.5 - 0.866025403784439i \\
x_2 = -1.0 & x_3 = -0.5 - 0.866025403784439i \\
x_4 = -0.5 + 0.866025403784439i & x_5 = 0.5 + 0.866025403784439i \\
x_6 = -1.0 & x_7 = -0.5 + 0.866025403784438i \\
x_8 = 1.0 & x_9 = 0.5 + 0.866025403784438i \\
x_{10} = 0.5 - 0.866025403784439i & x_{11} = -0.5 - 0.866025403784439i
\end{array}
$$

It satisfies not only $\partial_{\mathbf{v}} f$, but also f itself.

An Exact Solution for cyclic 12-roots

For the tropism $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$:

$$
\begin{array}{ll}
z_0 = t^{-1} & z_1 = t\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_2 = -t^{-1} & z_3 = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_4 = t^{-1}\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) & z_5 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_6 = -t^{-1} & z_7 = t\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_8 = t^{-1} & z_9 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_{10} = t^{-1}\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) & z_{11} = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)
\end{array}
$$

makes the system entirely and exactly equal to zero.

Numerical Algebraic Geometry

and its ramifications

Numerical Algebraic Geometry applies numerical analysis to solve problems in algebraic geometry.

An inspiration for several research developments:

- Numerical Schubert Calculus
  Birk Huber, Frank Sottile, and Bernd Sturmfels
  $\rightarrow$ homotopies for problems in enumerative geometry
- Numerical Jet Geometry
  Greg Reid and Wenyuan Wu
  $\rightarrow$ a new way for solving differential algebraic equations
- Numerical Polynomial Algebra
  Hans Stetter; Barry Dayton and Zhonggang Zeng
  $\rightarrow$ symbolic-numeric algorithms for polynomials