Algorithm 795: PHCpack: A General-Purpose Solver for Polynomial Systems by Homotopy Continuation

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Polynomial systems occur in a wide variety of application domains. Homotopy continuation methods are reliable and powerful methods to compute numerically approximations to all isolated complex solutions. During the last decade considerable progress has been accomplished on exploiting structure in a polynomial system, in particular its sparsity. In this article the structure and design of the software package PHC is described. The main program operates in several modes, is menu driven, and is file oriented. This package features a great variety of root-counting methods among its tools. The outline of one black-box solver is sketched, and a report is given on its performance on a large database of test problems. The software has been developed on four different machine architectures. Its portability is ensured by the gnu-ada compiler.

Categories and Subject Descriptors: D.3.2 [**Programming Languages**]: Language Classifications—*Ada*; G.1.5 [**Numerical Analysis**]: Roots of Nonlinear Equations—*Systems of equations*; *Polynomials, methods for*; G.2.1 [**Discrete Mathematics**]: Combinatorics—*Counting problems*; G.4 [**Mathematics of Computing**]: Mathematical Software

General Terms: Algorithms, Theory

Additional Key Words and Phrases: Bernshtein's theorem, Bézout number, enumerative geometry, homotopy continuation, mixed volume, polyhedral homotopy, polynomial systems, root count, Schubert calculus, start system

The software originated from the doctoral research of the author and was supported by the Fund for Scientific Research Flanders (Belgium) through Grant G.0261.96 to the NINES Research Group, Department of Computer Science, K.U. Leuven. Several upgrades were made during the author's postdoctoral fellowships at the Department of Computer Science of the Katholieke Universiteit Leuven, at the Department of Mathematics of Michigan State University, and at the Mathematical Sciences Research Institute. Supported in part by NSF under Grant DMS-9804846 at Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA.
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1. INTRODUCTION

The presented software package PHC implements homotopy continuation methods to compute numerically approximations to all isolated solutions of a system of n polynomial equations in n unknowns.

The name *Polynomial Homotopy Continuation* unites the three key concepts of the method. Since we solve *polynomial* systems we exploit the algebraic structure to count the roots and to construct a start system. By *continuation* methods, the known solutions of the start system are extended to the desired solutions of the target system. This deformation is defined by the *homotopy*, i.e., a family of systems connecting start and target system.

The following intuitive reasoning might shed light on the hardness of the problem. Not surprisingly, evaluating a multivariate polynomial can be done in *polynomial time*, i.e., in time proportional to a polynomial function in the dimension, degrees, and number of terms. Computing one solution is *NP-hard*, because we may apply the following nondeterministic algorithm: guess a root by some oracle and verify whether it satisfies the equations. This verification runs in polynomial time. Computing *all* solutions is harder, because there exists no polynomial-time algorithm to verify whether a guessed number of solutions gives the right number of solutions. This counting problem is said to be *#P-hard*. Hence, our problem is intractable [Garey and Johnson 1979] for growing dimension and increasing degrees. Consequently, the computations are restricted to a fixed dimension n, and the complexity is measured in terms of the output size, i.e., the number of solutions. See Blum et al. [1997] for the complexity of Bézout's theorem.

The central concept in polynomial homotopy continuation is the *root count*,[1] because it determines the number of solution paths that need to be traced. Recent research has striven to develop sharp root counts that lead to homotopies with an optimal number of paths. The root count is also a vital instrument in validating numerical results. This term encompasses Bézout numbers, mixed volumes, and combinatorial counts from the Schubert calculus in enumerative geometry.

The history of homotopy continuation for polynomial systems can be roughly divided into two eras, each spanning about one decade. The first decade was focussed on applying Bézout's theorem for counting the solutions. Milestone publications are the introductory paper by Li [1987], the book by Morgan [1987], and the survey by Watson [1986]. Publicly available software packages are CONSOL [Morgan 1987] and HOMPACK [Watson et al. 1987], recently upgraded to Fortran 90 [Watson et al. 1997]. During the last 10 years, root-counting methods have been developed to exploit the structure of a polynomial system. The novel methods are of a symbolic-numeric nature [Emiris 1998]. Progress in homotopy continuation

[1]The term *root count* was coined by Canny and Rojas [1991], who presented the mixed volume as being of important practical significance for solving polynomial systems.

for polynomial systems [Li 1997] benefited from the interaction between combinatorics, algebraic geometry, and applied mathematics [Sturmfels 1998]. The polyhedral methods have brought homotopy continuation into the literature on computational algebraic geometry [Cox et al. 1998]. New, publicly available software packages are Pelican [Huber 1995] and PHC. Recently, optimal homotopies were presented for computing linear subspace intersections in enumerative geometry; see Sottile [1997], Huber et al. [1998], and Verschelde [1998].

The aim of this article is to give an overview on how the algorithms in PHC are used in practice to solve polynomial systems. In the next section related software packages are mentioned. PHC offers a great variety of root-counting methods, as explained in Section 3. The fourth section contains some basics about polynomial continuation. Sections 5–7 describe the general flow, the operation modes of the main program, and the internal structure of the package. The software is portable, and computer-compiler experiences are given in Section 8. In Sections 9 and 10 the outline of the black-box solver is presented, along with its practical performance on a large collection of applications. Before the conclusions, information on how to obtain and install the software is listed.

2. RELATED SOFTWARE

To indicate related software dedicated to solving polynomial systems by homotopy continuation, four different packages that are publicly available are briefly mentioned. We also refer to a program for computing mixed volumes. In closing this section we indicate some large-scale related software projects.

HOMPACK [Morgan et al. 1989; Watson et al. 1987] and CONSOL [Morgan 1987] are written in Fortran 77. HOMPACK is a general package for homotopy continuation with a polynomial driver. It has been parallelized [Allison et al. 1989; Harimoto and Watson 1989] and extended with an end game [Sosonkina et al. 1996]. A Fortran 90 version appeared recently [Watson et al. 1997]. The package POLSYS PLP [Wise et al. 1998] for constructing partitioned linear-product start systems is intended to be used in conjunction with HOMPACK90. The code for CONSOL is contained in Morgan [1987]. Morgan et al. [1991; 1992a; 1992b] developed techniques to handle end-point singularities.

Malajovitch created pss to apply homotopy continuation with verification by α-theory. The program contains facilities for parallel continuation. Originally written in C, the newest version [Malajovich 1996] is programmed in C++. Pelican [Huber 1995; 1996] implements in C the polyhedral methods of Huber and Sturmfels [1995]. Gao has created Fortran software for polyhedral continuation, with facilities to compute the affine roots [Gao et al. 1997].

The computation of mixed volumes is a crucial step in the resolution of sparse polynomial systems. The C program mvlp [Emiris 1994; Emiris and Canny 1995] computes mixed volumes; see Giordano [1996] for a distributed version. For a general resultant-based polynomial-system solver, we refer to Wallack et al. [1998].

In recent years, the attention to software for solving polynomial systems increased largely. FRISCO [The FRISCO Consortium 1996] is a three-year project funded by the European Commission under the Esprit Reactive LTR Scheme (project no. 21.024). A demo of the software produced by the predecessor project PoSSo is available at The Pisa Team of PoSSo [1993].

3. ROOT COUNTS AND START SYSTEMS

The computation of a root count is identified with the resolution of a generic system. In this sense, we call root-counting a symbolic computation mirroring this resolution. The basic root-counting principles for dense, sparse, and determinantal systems are exemplified next.

The use of multihomogenization was proposed in Morgan and Sommese [1987a; 1987b]. Li et al. [1987a; 1987b] introduced random product homotopies, see also Li and Wang [1991].

Example 3.1 Consider a two-dimensional generalized eigenvalue problem, represented by a polynomial in λ with 2-by-2 matrices as coefficients. A linear equation is added to scale the eigenvectors.

$$F(\mathbf{x}) = \begin{cases} \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \lambda^2 + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \lambda + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 & = 0 \end{cases} \tag{1}$$

The total degree D equals $3 \times 3 \times 1$ and overshoots the number of roots. For this problem we see that the components of the eigenvector occur linearly, whereas the degree of the eigenvalue equals two. By separating the unknowns in a partition Z, a 2-homogeneous Bézout number is obtained as follows

$$Z = \{\{\lambda\}, \{x_1, x_2\}\} \quad \begin{bmatrix} \{\lambda\} & \{x_1, x_2\} \\ \overbrace{2} & \overbrace{1} \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \quad B_Z = 2 \times 1 + 2 \times 1 + 0 \times 1 = 4 \tag{2}$$

The matrix contains the degrees of the polynomials with respect to the sets in Z. The Bézout number B_Z is computed as the generalized permanent of this matrix. This computation models the resolution of the following linear-product system:

$$F^{(0)}(\mathbf{x}) = \begin{cases} \overbrace{(\alpha_{11}\lambda + \alpha_{12})(\alpha_{13}\lambda + \alpha_{14})}^{\{\lambda\}}\overbrace{(\alpha_{15}x_1 + \alpha_{16}x_2 + \alpha_{17})}^{\{x_1, x_2\}} & = 0 \\ (\alpha_{21}\lambda + \alpha_{22})(\alpha_{23}\lambda + \alpha_{24})(\alpha_{25}x_1 + \alpha_{26}x_2 + \alpha_{27}) & = 0 \\ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 & = 0 \end{cases} \tag{3}$$

Any random-number generator will yield α-coefficients of the linear-product system $F^{(0)}(\mathbf{x}) = \mathbf{0}$ so that it has exactly four regular solutions. This leads to a homotopy with an optimal number of solution paths.

A heuristic method developed for constructing a good partition Z of the set of unknowns is outlined in Verschelde [1996]. Besides that, an exhaustive enumeration as in Wampler [1992] of all partitions is available in PHC. In case the number of independent roots equals B_Z, interpolation can be used to construct a start system [Verschelde et al. 1991].

The idea of Verschelde and Haegemans [1993] is that not every polynomial should be modeled by the same partition. This leads to partitioned linear-product start systems. General linear-product start systems were constructed in Verschelde and Cools [1993b] and applied to symmetric polynomial systems in Verschelde and Cools [1994]. The key condition is that a linear-product start system must contain all monomials of the target system. Theoretically, these homotopy methods can be considered as a special case of the polyhedral homotopy methods. In practice, we sometimes prefer product start systems, for solving a random linear-product system can be performed much more efficiently than solving a random coefficient system. PHC supports the construction of both partitioned and general linear-product start systems.

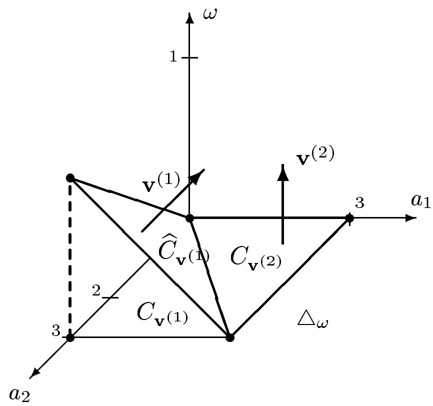
Efficient algorithms to construct general linear-product start systems are elaborated by Li et al. [1996]. Morgan et al. [1995] treated general product decompositions that do not restrict to linear factors. Recent coding efforts on partitioned linear-product start systems are reported by Wise et al. [1998].

The start solutions in linear-product homotopies are obtained by solving linear systems. In polyhedral homotopy methods [Huber and Sturmfels 1995; Verschelde et al. 1994], the start solutions are solutions to binomial systems.

Example 3.2 To solve a system that has two terms in any of its equations, unimodular transformations are applied to transform the system into a triangular structure. For the example below, $\mathbf{x} = \mathbf{y}^U$ abbreviates the substitution $(x_1, x_2) \leftarrow (y_1 y_2^{-1}, y_1^{-1} y_2^2)$.

$$F(\mathbf{x}) = \begin{cases} x_1^2 x_2^1 - 1 & = 0 \\ x_1^4 x_2^3 - 1 & = 0 \end{cases} \quad U = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$F(\mathbf{x} = \mathbf{y}^U) = \begin{cases} y_2 - 1 & = 0 \\ y_1 y_2^2 - 1 & = 0 \end{cases} \tag{4}$$



$$F(x_1, x_2) = \begin{cases} x_1^3 x_2^3 + c_{11} x_1^3 + c_{12} x_2^3 + c_{13} = 0 \\ x_1^3 x_2^3 + c_{21} x_1^3 + c_{22} x_2^3 + c_{23} = 0 \end{cases}$$

$$\widehat{F}(x_1, x_2, t) = \begin{cases} x_1^3 x_2^3 t^0 + c_{11} x_1^3 t^0 + c_{12} x_2^3 t^1 + c_{13} t^0 = 0 \\ x_1^3 x_2^3 t^0 + c_{21} x_1^3 t^0 + c_{22} x_2^3 t^1 + c_{23} t^0 = 0 \end{cases}$$

$$\mathbf{v}^{(1)} = [1, -1, 3]^T \qquad \mathbf{v}^{(2)} = [0, 0, 1]^T$$

Fig. 1. A regular triangulation of the Newton polytope of F with polyhedral homotopy $\hat{F}$.

We see that $F(\mathbf{x}) = \mathbf{0}$ has two regular solutions. Geometrically, we have computed the area of a parallelogram spanned by the origin, the points (2,1) and (4,3), and their sum. This area equals the determinant of the matrix that has in its columns the spanning vectors of the parallelogram. Multiplying by U triangulates this matrix:

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{5}$$

As $\det(U) = 1$, U is called unimodular and preserves volume. Consequently, the transformation $\mathbf{x} = \mathbf{y}^U$ does not change the number of solutions. Note that the total degree and 2-homogeneous Bézout number equal respectively 21 and 10.

The above example is the sparsest case. The fewer monomials the fewer roots we expect [Khovanskii 1991]. In general, we apply Bernshtein's theorem [Bernshtein 1975] and count the number of roots by the mixed volume of the Newton polytopes. By means of a regular subdivision, polyhedral homotopies are constructed that start at systems corresponding to the cells in the subdivision. Figure 1 illustrates the case where all Newton polytopes are the same, which is the case of Kushnirenko's theorem [Kushnirenko 1976]. The root count is obtained as the volume of the Newton polytope shared by all polynomials in the system.

The program features four different lifting methods: implicit, static, dynamic, and symmetric lifting. Implicit lifting refers to the algorithms used in the proof of Bernshtein [1975]. The method in Huber and Sturmfels [1995] is called static, to make the distinction with dynamic lifting, an algorithm that has been developed in Verschelde et al. [1996] to construct regular triangulations of polytopes incrementally with low lifting values. Symmetric lifting was presented in Verschelde and Gatermann [1995]. To construct regular subdivisions, both integer and floating-point lifting functions are available in PHC and elaborated with recursion. The Cayley trick [Gel'fand et al. 1994] defines in its polyhedral version [Sturmfels 1994] a polytope whose volume equals the mixed volume of the considered configuration of polytopes. In PHC, this trick is implemented by means of dynamic lifting. When many polynomials share the same exponents, this method is more efficient than static lifting.

With mixed volumes we restrict the counting to solutions that have all their components different from zero. Extensions to count and compute all isolated affine roots are described in Rojas [1994; 1999], Huber and Sturmfels [1997], Li and Wang [1996], Rojas and Wang [1996], Gao et al. [1997], and in Emiris and Verschelde [1999].

A new class of homotopy methods solves geometric problems whose intersection conditions are modeled by polynomial equations that arise from expanding determinants. Gröbner and SAGBI bases translate questions concerning ideals and subalgebras to monomial equations. The monomial orderings induced by weight vectors provide recipes to set up homotopies that are flat deformations, i.e., preserve the structure of the solution set. These are the key ideas for the Gröbner and SAGBI homotopies introduced in Huber et al. [1998] to enumerate all p-planes that intersect mp given m-planes in general position in $\mathbb{C}^{m+p}$. See Ravi et al. [1996] and Rosenthal and Willems [1998] for the relevance to the pole placement problem in control theory, and for related computational experiments see Rosenthal and Sottile [1998], Sottile [1998], and Verschelde [1998]. A third type of homotopies presented in Sottile [1997] and Huber et al. [1998] has an intrinsic geometric meaning and is briefly described next.

Example 3.3 A classical problem in enumerative geometry [Kleiman and Laksov 1972] deals with finding the two lines in projective 3-space that meet four given lines in general position. For the configurations as in Figure 2 we have to solve the following system:

$$\det(X \mid L_i) = 0, \quad \Leftrightarrow \quad \det\left(\begin{array}{cc|cc} x_{11} & 0 & c_{11}^{(i)} & c_{12}^{(i)} \\ x_{21} & 0 & c_{21}^{(i)} & c_{22}^{(i)} \\ 0 & x_{32} & c_{31}^{(i)} & c_{32}^{(i)} \\ 0 & x_{42} & c_{41}^{(i)} & c_{42}^{(i)} \end{array}\right) \quad i = 1, 2, 3, 4 \tag{6}$$

The special choice of coordinates so that L_2 is spanned by the first two and L_1 by the last two basis vectors admits the choice of local coordinates for the solution X. The best Bézout number for this system equals 6, and the mixed volume equals 4, whereas there are only two solutions. The so-called Pieri homotopy starts at the special configuration, displayed at the left of Figure 2, and moves the third input line in general position. To reach the two solutions of this problem, it suffices to follow the solution paths defined by this homotopy.

The start systems and root counts presented here are optimal for three different classes of polynomial systems. This classification is only a sample

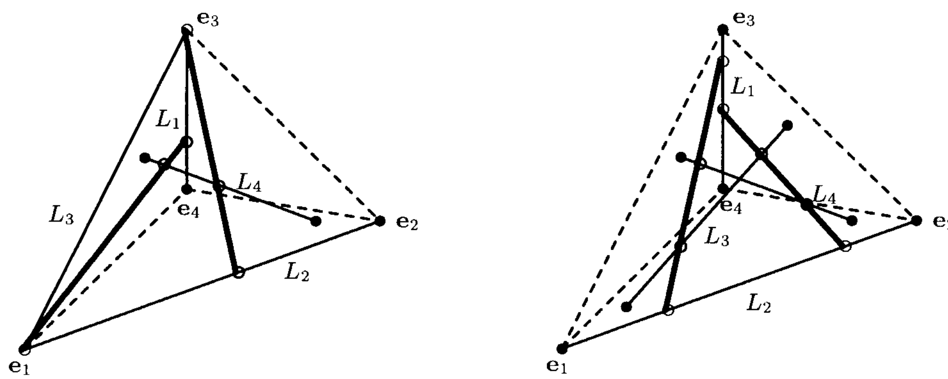
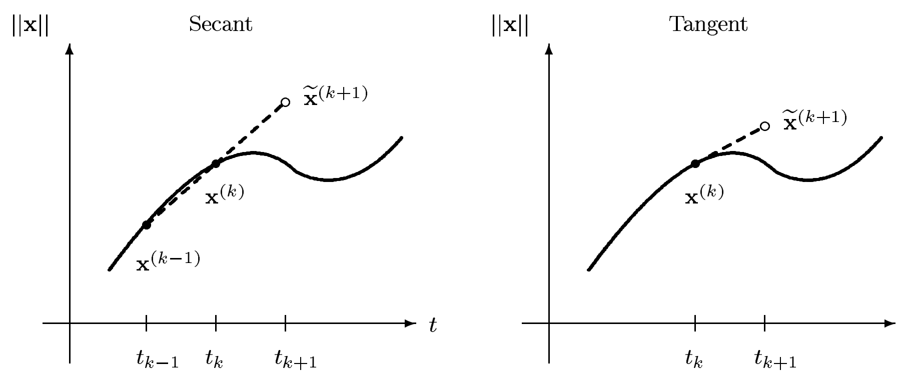


Fig. 2. In $\mathbb{P}^3$ two thick lines meet four given lines L_1, L_2, L_3, and L_4 in a point. At the left we see a special configuration, and the general configuration is at the right.



$$\widetilde{\mathbf{x}}^{(k+1)} := \mathbf{x}^{(k)} + \lambda(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}) \qquad \widetilde{\mathbf{x}}^{(k+1)} := \mathbf{x}^{(k)} + \lambda\frac{d\mathbf{x}(t_k)}{dt}$$

Fig. 3. The secant and tangent predictor with λ as step length.

and by no means exhaustive. We expect that future research developments will extend this list of root counters and homotopies.

4. POLYNOMIAL CONTINUATION AND END GAMES

Solution paths of polynomial homotopies do not turn back as the continuation parameter t increases, due to the regularity of the paths, as discussed in Li and Sauer [1987]. Therefore an increment-and-fix predictor-corrector method is appropriate: after each increase of t, t remains fixed while correcting the solution $\mathbf{x}$ by Newton's method. Figure 3 sketches two possible predictor schemes in the path tracker.

The clustering of solution paths is avoided by tightening the tolerances of the corrector to enforce quadratic convergence of Newton's method in every step.

Only as $t \rightarrow 1$, we may have to deal with paths converging to singular solutions and with paths diverging to infinity. To this end, several *end games* were proposed by Morgan et al. [1991; 1992a; 1992b] and by Sosonkina et al. [1996]. Polyhedral end games [Huber and Verschelde 1998] provide a certificate of divergence that allows us to separate diverging paths from the rest, without first having to compute the actual values of the diverging paths accurately. Next we summarize the idea of Huber and Verschelde [1998].

A solution path is represented by the following power series expansion:

$$\begin{cases} x_i(s) & = a_i s^{\omega_i}(1 + O(s)) \\ t(s) & = 1 - s^m \end{cases} \quad t \approx 1, \quad s \approx 0. \tag{7}$$

The winding number m is lower than or equal to the multiplicity of the solution. We see that for a solution diverging to infinity or to a zero-component solution we have $\omega_i \neq 0$. According to David Bernshtein's second theorem [Bernshtein 1975], this solution corresponds to a solution of the face system defined by the direction ω. This face certifies the divergence.

To check whether a solution path really diverges is equivalent to the test on the value for ω_i. A first-order approximation of ω_i can be computed by

$$\frac{\log|x_i(s_1)| - \log|x_i(s_0)|}{\log(s_1) - \log(s_0)} = \omega_i + O(s_0), \tag{8}$$

with $0 < s_1 < s_0$. The above formula assumes the correct value of the winding number m. To compute m, solution paths are sampled geometrically with ratio h as $s_k = h^{k/m} s_0$. The errors on the estimates for ω_i are

$$e_i^{(k)} = (\log|x_i(s_k)| - \log|x_i(s_{k+1})|) - (\log|x_i(s_{k+1})| - \log|x_i(s_{k+2})|) \tag{9}$$

$$= c_1 h^{k/m} s_0 (1 + O(h^{k/m})). \tag{10}$$

An estimate for m is derived from two consecutive errors $e_i^{(k)}$. Extrapolation improves this estimate.

A parallel development to make resultants deal with situations when the mixed volume overshoots the number of roots is described in Rojas [1997].

5. THE FOUR STAGES OF THE SOLVER

The root count provides important information about the amount of computational work that is required to solve the problem. It suffices to multiply the root count with the estimated time needed to follow one solution path.

In Figure 4, the four stages of the solver are displayed.

The aim of preconditioning is to bring the system in a form more suitable to homotopy continuation. In the second stage, a root-counting method is applied to construct a start system. The tuning of continuation parameters

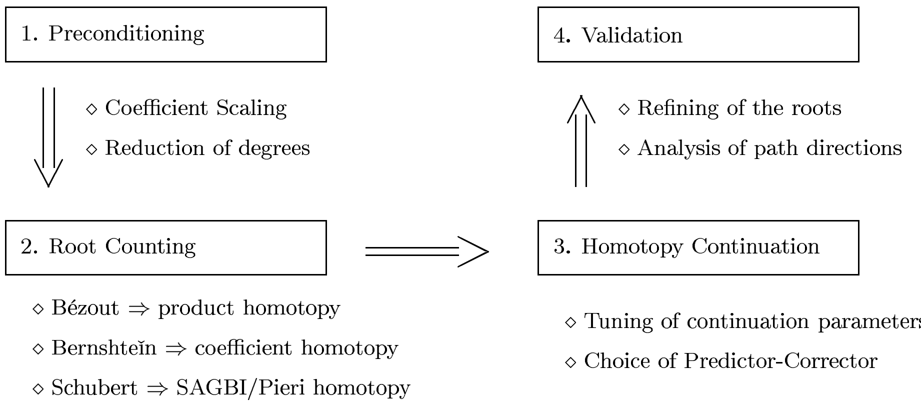


Fig. 4. The four stages in the solver.

and path following by means of predictor-corrector methods is performed in the third stage. The postprocessing stage consists in the validation of the computed results. Basic validation includes for instance the computation of local condition numbers, whereas more elaborate validation procedures eventually require continuation.

6. EXECUTION MODES AND TOOLS

Since we have to respect a strict processing order and may expect computationally lengthy jobs, PHC is organized as a menu-driven and file-oriented program.

The simplest way to solve systems by PHC is to type

```
phc -b input output
```

when `input` is the name of the input file that contains the system. This mode is the so-called *black-box mode* and requires no other input than the polynomial system. Results can be found in the file `output`. One particular choice for a black-box solver is outlined in Section 9.

The second mode is the *full mode*, where PHC runs through all stages of the solver and asks the user to confirm the default settings while giving the opportunity to modify the settings interactively. This mode is invoked by default, just by typing `phc` after the prompt.

Some stages may be skipped, whereas more than one root-counting method can be invoked before the construction of a homotopy. Therefore, the *tool mode* has been created; see Figure 5. Another advantage of working with tools is that intermediate results, such as a mixed subdivision and a random coefficient start system, can be valuable stepping stones in the resolution of a large and difficult system.

Table I gives an overview of the tools and the options of PHC to invoke them.

The need for a separate tool for `mvc` comes from the amount of computational work that is not negligible for computing mixed volumes and

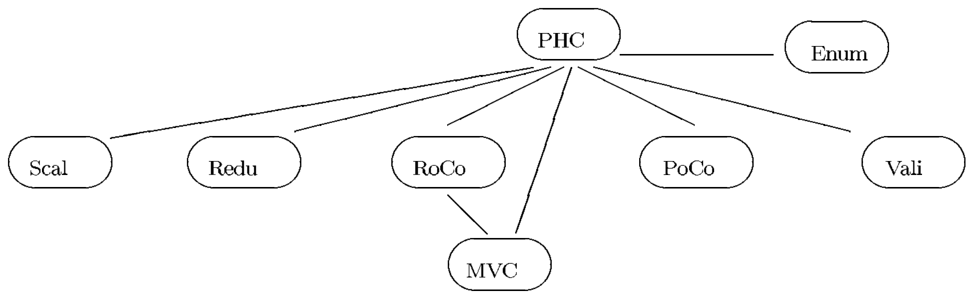


Fig. 5. Schematic overview of the tools offered by the package PHC.

Table I. Overview of Tools, Acronyms, and Options of PHC

Stage	Acronym	Description of the Tool	Option
1	scal	coefficient scaling	phc -s
	redu	reduction of degrees	phc -d
2	roco	root counts and start systems	phc -r
	mvc	mixed-volume computation	phc -m
3	poco	polynomial continuation	phc -p
4	vali	validation of results	phc -v
x	enum	enumerative geometry	phc -e

performing polyhedral continuation. The reduction `redu` tool applies S-polynomials as described in Verschelde and Cools [1992]. This technique generalizes the linear reduction on the coefficient matrix of the system [Morgan 1987].

7. THE INTERNAL DESIGN: THE LIBRARIES OF PHCPACK

There are four large components of the software system: the mathematical library, the homotopy continuation routines, the root-counting methods, and the interface packages.

The sources of PHCpack are organized in the tree shown in Figure 6. The structure reflects the discrete and continuous nature of the program.

The concept of information hiding has been applied more deeply than just separating the four stages of the solver. Next are some examples on how PHC deals with polynomials.

(1) The continuation is not only separated from the choice of the homotopy, but also from the way polynomials are evaluated. This is done by providing the evaluation and differentiation of the homotopy as parameters of the path trackers.

(2) For the evaluation of polynomials, a multivariate Horner scheme is implemented at the level of the polynomial package. The precise definition is hidden to the client procedures that create and evaluate these polynomials.

```
Ada                          : Ada sources of PHC
 |-- System                  : 0. UNIX dependencies, e.g.: timing
 |-- Math_Lib                : 1. general mathematical library
 |      |-- Numbers          : 1.1. number representations
 |      |-- Matrices         : 1.2. matrices and linear-system solvers
 |      |-- Polynomials      : 1.3. multivariate polynomial systems
 |      |-- Supports         : 1.4. support sets and linear programming
 |-- Homotopy                : 2. homotopy and solution lists
 |-- Continuation            : 3. path-tracking routines
 |-- Root_Counts             : 4. root counts and homotopy construction
 |      |-- Product          : 4.1. linear-product start systems
 |      |-- Implift          : 4.2. implicit lifting
 |      |-- Stalift          : 4.3. static lifting
 |      |-- Dynlift          : 4.4. dynamic lifting
 |      |-- Symmetry         : 4.5. exploitation of symmetry relations
 |-- Schubert                : 5. numerical Schubert calculus
 |-- Main                    : 6. main dispatcher
```

Fig. 6. Tree organization showing structure of PHCpack sources.

(3) A polynomial homotopy can be evaluated more efficiently when the coefficients are parameters to the evaluation routines. The combination with multivariate Horner yields a powerful and flexible coefficient homotopy.

Without using this third data structure, the construction of a random coefficient start system for the cyclic 7-roots problem (924 paths to follow, about 100 cells) by means of polyhedral homotopy continuation took 4h 38m 55s 354ms CPU time. The current version takes only about 15m 49s 391ms CPU time! Polyhedral homotopies are nonlinear in the continuation parameter t, and treating t^ω as just one monomial or as a polynomial of degree ω makes the difference in evaluation. This third data structure was created to deal with floating-point lifting values ω implementing a suggestion of T.Y. Li.

Note that coefficient-parameter polynomial continuation [Morgan and Sommese 1989] or cheater's homotopy [Li et al. 1989; Li and Wang 1992] is a very useful and natural concept.

The computational bottleneck in polynomial continuation is the evaluation of polynomials. The efficiency could improve a lot if the polynomials would be known at compile time, so that optimized in-line evaluators can be used. However, to keep the program user-friendly, compilation of the program must not be required each time a new system has to be solved.

8. ON PORTABILITY: COMPUTERS AND COMPILERS

PHC is written in Ada. Since the gnu-ada compiler (also called GNAT [Ada Core Technologies 1997]) is available for a large number of platforms, the software is portable. The myth that Ada is "big and slow" is disproved in Syiek [1995]: Ada versions even have a slight edge over their C counterparts.

Table II. Compiler Efficiency (on SPARCserver-1000): Compilation Timings and Sizes of Auxiliary Files and Executable

Compiler Options	User Time	System Time	Total Time	Auxiliary Files	Executable
VADS: -O -S	40m 9s	5m 8s	45m 17s	42.1MB	4.0MB
GNAT: -O3 -gnatp	45m 46s	2m 59s	48m 47s	4.1MB	2.9MB

Table III. Performance of Generated Code, on the Cyclic 7-Roots Problem with the Black-Box Solver Compiled with VADS and GNAT. Timings are listed for root-counting, construction of start system, the continuation to the target system, and the total time.

phc	Root Counts	Start System	Continuation	Total Time
VADS	1m 26s 866 ms	24m 20s 589ms	38m 17s 667ms	1h 4m 42s 129ms
GNAT	1m 15s 74 ms	15m 49s 391ms	27m 50s 521ms	45m 21s 434ms

Initially, the VADS (Verdix Ada Development System) compiler was used on three different machine architectures. The implementation started [Verschelde 1990] on a SUN3/280 and moved [Verschelde and Cools 1993a] to a DECStation 5240. A version for an IBM RS/6000 workstation was made available [Verschelde 1995]. Thirdly, SUN-SPARC machines were used [Verschelde and Cools 1996], along with the gnu-ada compiler version 3.03 in Verschelde [1996].

Next a report is given on compiler experiences. Because runtime efficiency is crucial, compilation is done with full optimization and with a suppression of runtime checks. Table II contains experimental data comparing VADS 6.2.3b against gnu-ada 3.09.

In Table III the performance of the generated code is illustrated on one of the benchmark examples.

Although this comparison is by no means thorough, the gnu-ada compiler seems to be the winner, both in compiling and runtime efficiency. Currently, the gnu-ada compiler is maintained by a privately held company Ada Core Technologies (ACT), founded by the creators of the gnu-ada compiler. ACT is committed to provide publicly free releases of their compiler. The most recent public version is numbered 3.11p.

The fourth platform used to develop PHC is a Pentium PC running Linux. The mathematical kernel of PHC has been rewritten using concepts of Ada 95 to incorporate multiprecision facilities. The other major change in the new version is the availability of SAGBI [Verschelde 1998] and Pieri homotopies.

9. PUTTING IT ALL TOGETHER: ONE BLACK-BOX SOLVER

Here the key ingredients are presented to build an overall, general-purpose solver that is reliable and efficient. This allows to solve polynomial systems simply by typing

```
phc -b input output
```

after the prompt.

The outline of the black-box solver is as follows:

1. *Homotopy Construction*:
   1.1. Computation of root counts:
   - D: total degree;
   - B_Z: multihomogeneous Bézout number, based on a heuristically generated partition;
   - B_S: general linear-product Bézout number, based on a heuristically generated set structure;
   - V: mixed volume, by dynamic lifting, when the number of different supports is less than or equal to $n/2$, by static lifting otherwise.

   1.2. Construction of start system, corresponding to the minimal and least expensive root count.

2. *Polynomial Continuation*:
   2.1. Coefficient and variable scaling.
   2.2. Tracking of the solution paths.
   2.3. Root refinement on the descaled solutions.

The computation of root counts by four different methods provides already important information about the problem class. Even though the mixed volume always yields the lowest bound, the construction of the corresponding start system requires continuation and is computationally much more expensive to solve than a linear-product start system. The latter start system is preferred when any of the Bézout numbers equals the mixed volume.

Practical experiments show that scaling the coefficients really helps to smoothen the continuation. For a discussion of scaling see Morgan et al. [1989] or Morgan [1987].

The output file contains the following:

(1) The root counts D, B_Z, B_S, and V.

(2) Start system with start solutions.

(3) Settings of the path tracker.

(4) Results of polynomial continuation on the scaled system.

(5) Solutions of original system as output of the root refiner.

(6) Timings for the stages and timing summary at the end.

End games are not invoked in the black-box solver, because the determination of the end game operation range tends to be problem dependent. The idea behind this black-box solver is to get quickly an idea of the complexity of the system while providing many root counts. To examine singularities and other degeneracies we recommend switching to the tool mode of PHC.

10. THE TEST DATABASE OF POLYNOMIAL SYSTEMS

The progress on solving polynomial systems has always been motivated by the poor performance of the existing technology on practical examples. To make the benchmarking more meaningful and relevant, most systems are taken from practical applications, cited in the literature. We refer to Traverso [1993] and Bini and Mourrain [1998] for similar collections.

In Tables IV and V an overview of the application database is given. To save space, the algebraic description is omitted. Either the reference or the PHC distribution file can be consulted. Some systems appear in different guises, such as the cyclic n-roots and the economics problem. It allows us to see how the particular formulation of a problem can influence the solution process.

Some important characteristics of the systems are listed in Tables VI and VII. Most systems arising from practical applications are *deficient*, i.e., have fewer roots than the root count. The improvement of the mixed volume compared to the Bézout bounds is in many cases really significant. The black-box solver of PHC does not contain facilities to treat affine roots properly; therefore it may miss some isolated solutions with zero components.

The parameters of the black-box solver have been set to handle all examples of the database. In Tables VIII and IX timings are listed for a SPARCserver-1000. We count 33 little examples that require less than one minute to solve. There are 38 larger examples, for which PHC needs between one minute and an hour. Five big examples require more than one hour to solve.

Timings only have a temporary value. They are only good to measure the difficulty of solving one system compared to another one. The exponential gains from selecting a sharp root count are more important. The black-box solver is certainly not the optimal way to solve a particular system, although the timings give a good impression of the general performance of the package.

The demonstration database of polynomial systems is still growing in order to increase the awareness of the importance and relevance of solving polynomial systems to applied mathematics and scientific computing.

In closing, some user applications of PHC are mentioned. PHC was used actively by Charles Wampler [Wampler 1996] to count the roots of various systems in mechanical design. Frank Sottile applied PHC to compute root counts for linear subspace intersections of the Schubert calculus; see Sottile [1998] for various tables. A third example comes from computer graphics. To show that the 12 lines tangent to four given spheres can all be real, Thorsten Theobald used PHC, choosing appropriate parameters in the algebraic formulation set up by Cassiano Durand.

11. OBTAINING AND INSTALLING PHC

The current second release of PHC is available at the Web pages of the author. The distribution contains the Ada sources with makefiles to install

Table IV. An Overview of the Test Database, Part I. Besides the name of the polynomial system, a reference to the literature and a short description is mentioned.

Name	Reference	Title with Description of the Application
boon	Boon [1992]	neurophysiology, posted by Sjirk Boon
butcher	Traverso [1993]	Butcher's problem, from PoSSo test suite
butcher8	Boege et al. [1986]	8-variable version of Butcher's problem
camera1s	Emiris [1997]	camera displacement between 2 positions, frame 1
caprasse	Traverso [1993]	the system Caprasse of the PoSSo test suite
cassou	Li et al. [1996]	the system of Pierrette Cassou-Noguès
chemequ	Meintjes and Morgan [1990]	chemical equilibrium of hydrocarbon combustion
cohn2	Cohn and Deutch [1988]	modular equations for algebraic number fields
cohn3	Galligo and Traverso [1989]	modular equations for algebraic number fields
comb3000	Morgan [1987]	Model A combustion chemistry example
conform1	Emiris [1997]	conformal analysis of cyclic molecules, instance 1
cpdm5	Gatermann [1990]	5-dimensional system of Caprasse and Demaret
cyclic5	Björk and Fröberg [1991]	cyclic 5-roots problem
cyclic6	Björk and Fröberg [1991]	cyclic 6-roots problem
cyclic7	Backelin and Fröberg [1991]	cyclic 7-roots problem
cyclic8	Björk and Fröberg [1994]	cyclic 8-roots problem
d1	Van Hentenryck et al. [1997]	a sparse system, known as benchmark D1
des18_3	Nauheim [1998]	a "dessin d'enfant," called des18_3
des22_24	Nauheim [1998]	a "dessin d'enfant," called des22_24
discret3s	Traverso [1993]	system discret3, scaled by average coefficients
eco5	Morgan [1987]	5-dimensional economics problem
eco6	Morgan [1987]	6-dimensional economics problem
eco7	Morgan [1987]	7-dimensional economics problem
eco8	Morgan [1987]	8-dimensional economics problem
extcyc5	Verschelde and Gatermann [1995]	extended cyclic 5-roots to exploit symmetry
extcyc6	Verschelde and Gatermann [1995]	extended cyclic 6-roots to exploit symmetry
extcyc7	Verschelde and Gatermann [1995]	extended cyclic 7-roots to exploit symmetry
extcyc8	Verschelde and Gatermann [1995]	extended cyclic 8-roots to exploit symmetry
fourbar	Morgan and Wampler [1990]	four-bar design problem, so-called 5-point problem
fbrfive4	Wampler [1996]	four-bar linkage through 5 points, n=4 version
fbrfive12	Wampler [1996]	four-bar linkage, coupler curve through 5 points
gaukwa2	Stroud and Secrest [1966]	Gaussian quadrature formula 2 knots, 2 weights
gaukwa3	Stroud and Secrest [1966]	Gaussian quadrature formula 3 knots, 3 weights
gaukwa4	Stroud and Secrest [1966]	Gaussian quadrature formula 4 knots, 4 weights
geneig	Chu et al. [1988]	generalized eigenvalue problem
heart	Nelsen and Hodgkin [1981]	heart-dipole problem
i1	Van Hentenryck et al. [1997]	benchmark i1 from Interval Arithmetic Benchmarks
ipp	Morgan and Sommese [1987b]	six-revolute-joint problem of mechanics

with the gnu-ada compiler, the database of test examples, and executable versions for Unix workstations SUN, SGI, and PCs running Linux and Solaris. For other platforms the gnu-ada compiler is needed.

The affiliations of the author will change. In case the current Web addresses are obsolete, or to report other (installation) problems, please

Table V. An Overview of the Test Database, Part II. Besides the name of the polynomial system, a reference to the literature and a short description is mentioned.

Name	Reference	Title with Description of the Application
ipp2	Wampler and Morgan [1991]	6R inverse position problem
katsura5	Boege et al. [1986]	a problem of magnetism in physics
kinema	Bellido [1992]	robot kinematics problem
kin1	Van Hentenryck et al. [1997]	kinematics problem
ku10	Steenkamp [1982]	10-dimensional system of Ku
lorentz	Li [1987]	equilibrium of 4-dimensional Lorentz attractor
lumped	Li and Wang [1991]	lumped-parameter chemically reacting system
mickey	Verschelde and Cools [1996]	Mickey-mouse example as illustration
noon3	Noonburg [1989]	neural network, Lotka-Volterra system, n=3
noon4	Noonburg [1989]	neural network, Lotka-Volterra system, n=4
noon5	Noonburg [1989]	neural network, Lotka-Volterra system, n=5
proddeco	Morgan et al. [1995]	system with a product-decomposition structure
puma	Morgan and Shapiro [1987]	hand position and orientation of PUMA robot
quadfor2	Verschelde and Gatermann [1995]	Gaussian quadrature with 2 knots and weights
quadgrid	Sweldens [1994]	interpolating quadrature formula on a grid
rabmo	Moore and Jones [1977]	optimal multidimensional quadrature formulas
rbpl	Mourrain [1993]	parallel robot, the so-called left-hand problem
rbpl24	Mourrain [1996]	parallel robot with 24 real solutions
redcyc5	Emiris [1994]	reduced cyclic 5-roots problem
redcyc6	Emiris [1994]	reduced cyclic 6-roots problem
redcyc7	Emiris [1994]	reduced cyclic 7-roots problem
redcyc8	Emiris [1994]	reduced cyclic 8-roots problem
redeco5	Morgan [1987]	reduced 5-dimensional economics problem
redeco6	Morgan [1987]	reduced 6-dimensional economics problem
redeco7	Morgan [1987]	reduced 7-dimensional economics problem
redeco8	Morgan [1987]	reduced 8-dimensional economics problem
rediff3	Iserles (personal communication 1995)	3-dimensional reaction-diffusion problem
reimer5	Traverso [1993]	The 5-dimensional system of Reimer
rose	Traverso [1993]	a general economic equilibrium model
s9_1	Nauheim [1998]	small system from constructive Galois theory
sendra	Traverso [1993]	the system sendra of the PoSSo test suite
solotarev	Traverso [1993]	the system solotarev of the PoSSo test suite
sparse5	Verschelde and Gatermann [1995]	5-dimensional sparse symmetric polynomial system
speer	Gatermann [1990]	the system of E.R. Speer
trinks	Traverso [1993]	system of Trinks from the PoSSo test suite
virasoro	Schrans and Troost [1990]	the construction of Virasoro algebras
wood	Moré et al. [1981]	system derived from optimizing Wood function
wright	Wright [1985]	system of A.H. Wright

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12. CONCLUSIONS AND FUTURE DEVELOPMENTS

PHC offers a general-purpose solver for polynomial systems that features recent research advances in root-counting methods. The software package

Table VI. Characteristics of the Polynomial Systems, Part I. The dimension is n. D is the total degree of the system. B_Z is an m-homogeneous Bézout number, based on a partition generated by a heuristic method. B_S is a generalized linear-product Bézout number, based on a set structure generated by a heuristic method. V is the mixed volume. #sols is number of isolated solutions found by PHC.

Name	n	D	B_Z	B_S	V	# sols
boon	6	1024	344	216	20	8
butcher	7	4608	2090	605	24	5
butcher8	8	4608	1461	587	26	16
camera1s	6	64	20	20	20	20
caprasse	4	144	62	94	48	48
cassou	4	1344	368	361	24	16
chemequ	5	108	56	44	16	16
cohn2	4	900	468	358	124	18
cohn3	4	1080	484	358	213	102
comb3000	10	96	66	28	16	16
conform1	3	64	16	16	16	16
cpdm5	5	243	243	243	242	157
cyclic5	5	120	120	106	70	70
cyclic6	6	720	720	588	156	156
cyclic7	7	5040	5040	4200	924	924
cyclic8	8	40320	40320	30365	2560	1152
d1	12	4068	320	896	192	48
des18_3	8	324	544	241	46	46
des22_24	10	256	128	82	42	42
discret3s	8	256	128	128	128	128
eco5	5	54	20	16	8	8
eco6	6	162	48	36	16	16
eco7	7	486	112	80	32	32
eco8	8	1458	256	176	64	64
extcyc5	5	120	120	106	70	70
extcyc6	6	720	720	588	156	156
extcyc7	7	5040	5040	4200	924	924
extcyc8	8	40320	40320	30365	2560	1152
fbrfive12	12	4096	96	96	36	36
fbrfive4	4	256	96	194	36	36
fourbar	4	256	96	96	80	36
gaukwa2	4	24	11	11	5	2
gaukwa3	6	720	225	225	49	6
gaukwa4	8	40320	6769	6769	729	24
geneig	6	243	10	10	10	10
heart	8	576	193	193	121	4
i1	10	59049	452	437	66	66
ipp	8	256	96	96	64	48

is portable via the gnu-ada compiler. Practical evidence for the performance of its black-box solver has been given on a large set of applications.

A recent exciting development concerns homotopies that solve problems in enumerative geometry; see Sottile [1997] and Huber et al. [1998]. This class of homotopies is already incorporated in the current second release of the package. As future (maybe futuristic) long-term project we dream of a comprehensive framework for constructing homotopies, integrating the

Table VII. Characteristics of the Polynomial Systems, Part II. The dimension is n. D is the total degree of the system. B_Z is an m-homogeneous Bézout number, based on a partition generated by a heuristic method. B_S is a generalized linear-product Bézout number, based on a set structure generated by a heuristic method. V is the mixed volume. #sols is number of isolated solutions found by PHC.

Name	n	D	B_Z	B_S	V	# sols
ipp2	11	1024	576	848	288	16
katsura5	6	32	32	32	32	32
kin1	12	4608	320	896	192	48
kinema	9	64	240	64	64	40
ku10	10	1024	2	2	2	2
lorentz	4	16	14	12	12	11
lumped	4	16	8	11	7	4
mickey	2	4	4	4	4	4
noon3	3	27	29	21	21	21
noon4	4	81	81	73	73	73
noon5	5	243	243	233	233	233
proddeco	4	256	96	96	26	6
puma	8	128	16	32	16	16
quadfor2	4	24	11	11	4	2
quadgrid	5	120	10	10	10	5
rabmo	9	36000	22740	7090	136	16
rbpl	6	486	160	160	160	150
rbpl24	9	576	80	80	80	40
redcyc5	4	24	24	19	14	14
redcyc6	5	120	96	83	26	26
redcyc7	6	720	720	511	132	132
redcyc8	7	5040	3960	3107	320	144
redeco5	5	8	12	8	8	8
redeco6	6	16	28	16	16	16
redeco7	7	32	64	32	32	32
redeco8	8	64	144	64	64	64
rediff3	3	8	8	8	7	7
reimer5	5	720	720	720	720	144
rose	3	216	144	136	136	136
s9_1	8	16	41	10	10	10
sendra	2	49	49	46	46	46
solotarev	4	36	10	8	6	6
sparse5	5	100000	3840	3840	160	160
speer	4	625	384	246	96	43
trinks	6	24	24	18	10	10
virasoro	8	256	3072	256	200	200
wood	4	36	25	16	9	9
wright	5	32	32	32	32	32

PACKage approach of numerical analysis with the general-purpose solvers of computer algebra.

ACKNOWLEDGMENTS

The author acknowledges the technical assistance, usage of PHC, concern, comments, and advice from Yvan Barbaix, Marc Beckers, Anita Ceulemans, Ronald Cools, Dirk Craeynest, Cassiano Durand, Karel de Vlaminck,

Table VIII. Part I of Timing Summary on SPARCserver-1000, with the Black-Box Version Made with the gnu-ada Compiler. The CPU time of the process is expressed in hours, minutes, seconds, and milliseconds.

Name	Root Counts	Start System	Continuation	Total Time
boon	0h 0m 0s 190	0h 0m 5s 868	0h 0m 14s 394	0h 0m 20s 937
butcher	0h 0m 13s 267	0h 0m 29s 23	0h 1m 44s 962	0h 2m 28s 449
butcher8	0h 0m 50s 513	0h 0m 25s 188	0h 4m 20s 602	0h 5m 38s 507
camera1s	0h 0m 8s 258	0h 0m 0s 110	0h 0m 34s 406	0h 0m 44s 682
caprasse	0h 0m 0s 769	0h 0m 10s 4	0h 0m 17s 80	0h 0m 28s 888
cassou	0h 0m 1s 145	0h 0m 10s 688	0h 1m 3s 439	0h 1m 15s 972
chemequ	0h 0m 1s 116	0h 0m 4s 827	0h 0m 6s 886	0h 0m 13s 378
cohn2	0h 0m 3s 989	0h 0m 49s 953	0h 2m 49s 74	0h 3m 46s 619
cohn3	0h 0m 4s 991	0h 1m 12s 618	0h 16m 15s 864	0h 17m 37s 282
comb3000	0h 0m 7s 814	0h 0m 5s 630	0h 0m 18s 162	0h 0m 33s 118
conform1	0h 0m 0s 42	0h 0m 0s 45	0h 0m 3s 880	0h 0m 4s 310
cpdm5	0h 0m 18s 683	0h 2m 27s 225	0h 9m 51s 598	0h 12m 43s 370
cyclic5	0h 0m 0s 562	0h 0m 11s 768	0h 0m 32s 469	0h 0m 45s 993
cyclic6	0h 0m 6s 40	0h 1m 15s 816	0h 2m 44s 292	0h 4m 9s 434
cyclic7	0h 1m 15s 74	0h 15m 49s 391	0h 27m 50s 521	0h 45m 21s 434
cyclic8	0h 14m 41s 38	1h 25m 14s 851	2h 54m 28s 884	4h 35m 54s 367
d1	0h 0m 15s 182	0h 5m 34s 397	0h 13m 30s 348	0h 19m 25s 426
des18_3	0h 3m 51s 209	0h 1m 19s 587	0h 1m 44s 25	0h 6m 57s 913
des22_24	0h 0m 23s 538	0h 0m 50s 660	0h 1m 22s 251	0h 2m 40s 53
discret3s	0h 2m 20s 4	0h 0m 0s 719	0h 56m 20s 922	0h 58m 52s 121
eco5	0h 0m 0s 281	0h 0m 1s 222	0h 0m 2s 829	0h 0m 4s 686
eco6	0h 0m 2s 217	0h 0m 4s 132	0h 0m 6s 771	0h 0m 13s 785
eco7	0h 0m 22s 215	0h 0m 20s 511	0h 0m 34s 19	0h 1m 18s 249
eco8	0h 5m 28s 66	0h 1m 1s 471	0h 1m 55s 472	0h 8m 27s 528
extcyc5	0h 0m 2s 355	0h 0m 36s 607	0h 0m 37s 521	0h 1m 17s 726
extcyc6	0h 0m 30s 15	0h 1m 45s 137	0h 2m 56s 572	0h 5m 15s 706
extcyc7	0h 9m 15s 674	0h 17m 22s 278	0h 30m 29s 581	0h 57m 33s 835
extcyc8	1h 58m 58s 816	1h 36m 57s 179	3h 58m 54s 943	7h 36m 30s 580
fbrfive12	0h 1m 30s 161	0h 1m 6s 686	0h 2m 18s 859	0h 4m 58s 570
fbrfive4	0h 0m 0s 463	0h 0m 17s 283	0h 1m 2s 690	0h 1m 21s 972
fourbar	0h 0m 0s 625	0h 0m 8s 706	0h 2m 1s 20	0h 2m 12s 565
gaukwa2	0h 0m 0s 55	0h 0m 0s 705	0h 0m 1s 460	0h 0m 2s 391
gaukwa3	0h 0m 1s 430	0h 0m 22s 803	0h 0m 52s 615	0h 1m 17s 674
gaukwa4	0h 1m 18s 940	0h 27m 42s 595	0h 52m 30s 671	1h 21m 42s 787
geneig	0h 0m 0s 476	0h 0m 0s 303	0h 0m 11s 314	0h 0m 14s 648
heart	0h 1m 10s 899	0h 3m 14s 236	0h 4m 39s 75	0h 9m 6s 712
i1	0h 0m 37s 869	0h 0m 49s 215	0h 1m 59s 959	0h 3m 29s 913
ipp	0h 1m 4s 541	0h 1m 21s 14	0h 1m 47s 940	0h 4m 16s 287

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The author thanks the referees for many helpful comments and useful suggestions on the first version.

Table IX. Part II of Timing Summary on SPARCserver-1000, with the Black-Box Version Made with the gnu-ada Compiler. The CPU time of the process is expressed in hours, minutes, seconds, and milliseconds.

Name	Root Counts				Start System				Continuation				Total Time			
ipp2	0h	8m	21s	346	0h	11m	59s	944	0h	27m	18s	539	0h	47m	48s	632
katsura5	0h	0m	12s	382	0h	0m	0s	8	0h	0m	27s	511	0h	0m	41s	303
kin1	0h	2m	2s	926	0h	5m	39s	399	0h	11m	37s	403	0h	19m	25s	259
kinema	0h	2m	28s	847	0h	0m	0s	22	0h	3m	11s	205	0h	5m	42s	168
ku10	0h	0m	1s	547	0h	0m	2s	253	0h	0m	3s	611	0h	0m	8s	357
lorentz	0h	0m	0s	173	0h	0m	0s	23	0h	0m	6s	633	0h	0m	7s	256
lumped	0h	0m	0s	289	0h	0m	0s	714	0h	0m	1s	975	0h	0m	3s	255
mickey	0h	0m	0s	8	0h	0m	0s	2	0h	0m	0s	175	0h	0m	0s	248
noon3	0h	0m	0s	58	0h	0m	0s	33	0h	0m	4s	639	0h	0m	5s	154
noon4	0h	0m	0s	399	0h	0m	0s	103	0h	0m	51s	160	0h	0m	53s	163
noon5	0h	0m	2s	922	0h	0m	0s	316	0h	7m	9s	741	0h	7m	17s	834
proddeco	0h	0m	0s	299	0h	0m	10s	135	0h	0m	42s	488	0h	0m	54s	26
puma	0h	0m	2s	457	0h	0m	0s	420	0h	0m	34s	43	0h	0m	40s	105
quadfor2	0h	0m	0s	24	0h	0m	0s	183	0h	0m	0s	910	0h	0m	1s	271
quadgrid	0h	0m	4s	451	0h	0m	0s	159	0h	0m	14s	627	0h	0m	20s	443
rabmo	0h	1m	10s	899	0h	3m	14s	236	0h	4m	39s	75	0h	9m	6s	712
rbpl	0h	1m	32s	693	0h	0m	0s	670	0h	10m	24s	859	0h	12m	4s	957
rbpl24	3h	19m	40s	323	0h	0m	1s	331	0h	9m	55s	836	3h	29m	47s	764
redcyc5	0h	0m	0s	348	0h	0m	2s	767	0h	0m	3s	505	0h	0m	6s	995
redcyc6	0h	0m	3s	559	0h	0m	9s	911	0h	0m	17s	549	0h	0m	31s	863
redcyc7	0h	0m	54s	118	0h	1m	56s	885	0h	3m	12s	794	0h	6m	7s	216
redcyc8	0h	9m	58s	283	0h	10m	54s	911	0h	18m	11s	414	0h	39m	14s	893
redeco5	0h	0m	0s	290	0h	0m	0s	3	0h	0m	3s	109	0h	0m	3s	671
redeco6	0h	0m	2s	389	0h	0m	0s	6	0h	0m	7s	958	0h	0m	10s	860
redeco7	0h	0m	22s	751	0h	0m	0s	10	0h	0m	24s	990	0h	0m	48s	725
redeco8	0h	4m	30s	648	0h	0m	0s	18	0h	1m	20s	991	0h	5m	53s	618
rediff3	0h	0m	0s	28	0h	0m	0s	225	0h	0m	0s	568	0h	0m	1s	2
reimer5	0h	0m	0s	913	0h	0m	0s	65	0h	9m	20s	820	0h	9m	30s	537
rose	0h	0m	0s	86	0h	0m	0s	347	0h	2m	26s	320	0h	2m	30s	262
s9_1	0h	0m	0s	663	0h	0m	0s	65	0h	0m	14s	843	0h	0m	17s	344
sendra	0h	0m	0s	50	0h	0m	0s	131	0h	0m	18s	328	0h	0m	19s	474
solotarev	0h	0m	0s	79	0h	0m	0s	365	0h	0m	1s	37	0h	0m	1s	695
sparse5	0h	0m	0s	396	0h	0m	21s	672	0h	3m	45s	456	0h	4m	11s	36
speer	0h	0m	1s	554	0h	0m	52s	9	0h	13m	35s	109	0h	14m	31s	786
trinks	0h	0m	0s	666	0h	0m	1s	855	0h	0m	5s	962	0h	0m	8s	967
virasoro	3h	40m	50s	731	0h	7m	0s	273	0h	7m	42s	993	3h	55m	41s	856
wood	0h	0m	0s	55	0h	0m	0s	761	0h	0m	2s	860	0h	0m	3s	912
wright	0h	0m	1s	332	0h	0m	0s	6	0h	0m	13s	251	0h	0m	15s	210

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Received: August 1997; accepted: February 1999