Numerical Algebraic Geometry

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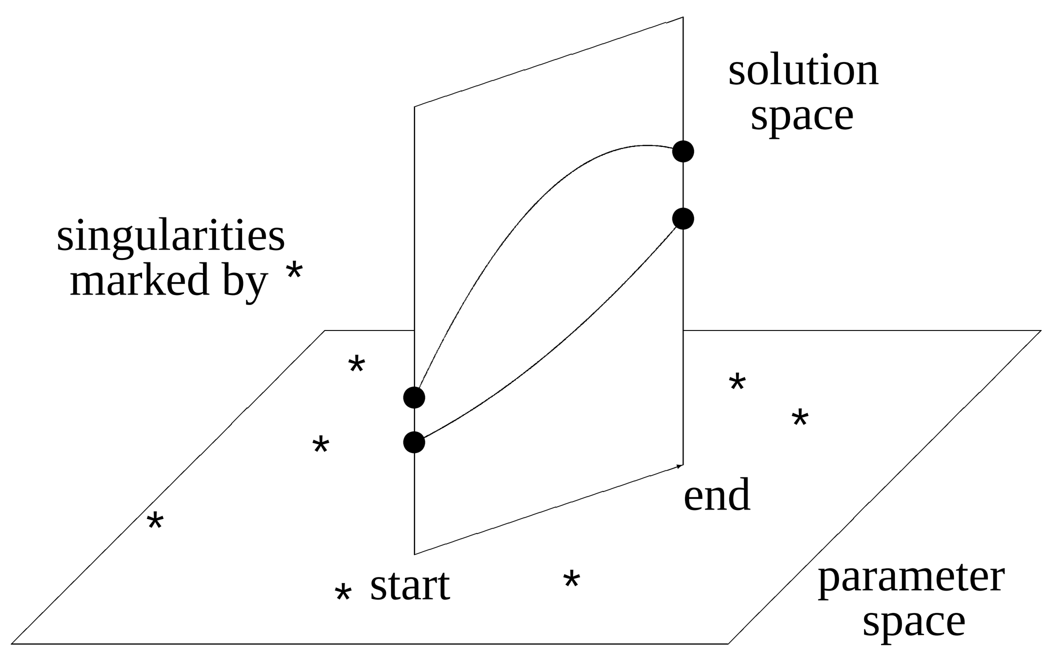
0. The Problem

Solve $f(\boldsymbol{\lambda}, \mathbf{x}) = \mathbf{0}$
for the variables $\mathbf{x}$, where $\boldsymbol{\lambda}$ are parameters;
f is a polynomial system with approximate coefficients.
Solve means: by computer using general methods
$\rightarrow$ so any one can re-solve and verify.
Three categories of solutions:
(1) approximations to all *isolated* solutions;
(2) all irreducible solution components, of all dimensions;
(3) information about the exceptional parameter values.

1. Parameter Continuation



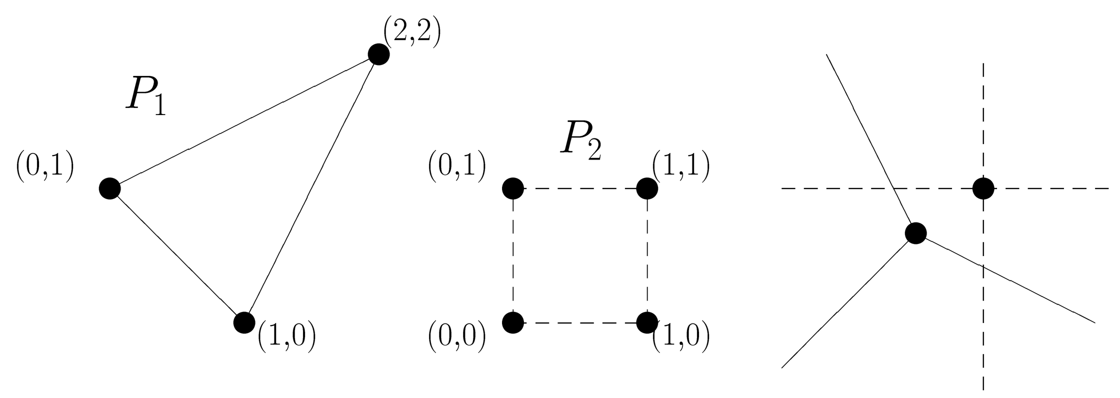
A generic choice for start avoids singularities along the paths.

2. Polyhedral Homotopies

Newton polytopes model sparse structure:

$$f_1(\mathbf{x}) = a_{(2,2)} x_1^2 x_2^2 + a_{(1,0)} x_1 + a_{(0,1)} x_2$$

$$f_2(\mathbf{x}) = b_{(1,1)} x_1 x_2 + b_{(1,0)} x_1 + b_{(0,1)} x_2 + b_{(0,0)}$$



Bernshteǐn: mixed volume bounds #isolated roots in $(\mathbb{C}^*)^n$.
Polyhedral homotopies are optimal for generic coefficients.

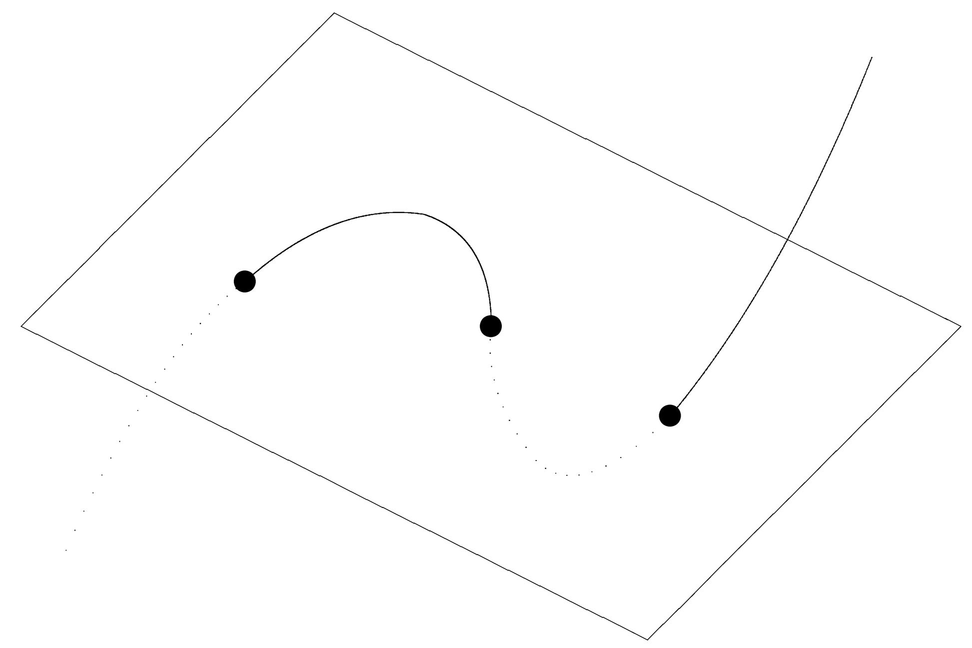
References

[1] T.Y. Li. Numerical solution of multivariate polynomial systems by homotopy continuation methods. *Acta Numerica*, 6:399–436, 1997.

[2] J. Verschelde. Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, 25(2):251–276, 1999. http://www.math.uic.edu/~jan/download.html.

3. Numerical Irreducible Decomposition

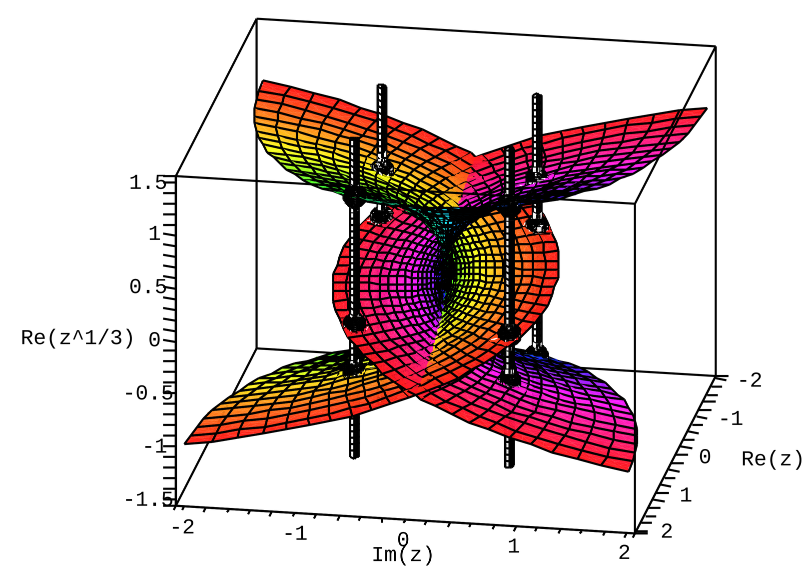
Numerical representations of positive dimensional solution sets.



Cut space curve with a random plane to find its degree.

4. Monodromy Factorization

The Riemann Surface of $z^3 - w = 0$:



Loop around the singular point (0,0) permutes the points.

5. Deflation for Isolated Singularities

Jacobian matrix $J_f(\mathbf{x})$ rank deficient close to $\mathbf{x}^*$;
let R be the numerical rank of $J_f(\mathbf{x}^*)$; and
introduce $R+1$ extra multiplier variables $\boldsymbol{\mu}$.

$$\text{Apply Newton to} \quad \left\{ \begin{array}{rl} f(\mathbf{x}) = \mathbf{0} & \boldsymbol{\mu} \text{ is the multiplier vector} \\ J_f(\mathbf{x}) B \boldsymbol{\mu} = \mathbf{0} & B \text{ is a random matrix} \\ \langle \mathbf{c}, \boldsymbol{\mu} \rangle = 1 & \mathbf{c} \text{ is a random vector} \end{array} \right.$$

Reduced to corank one case. Repeat if necessary.

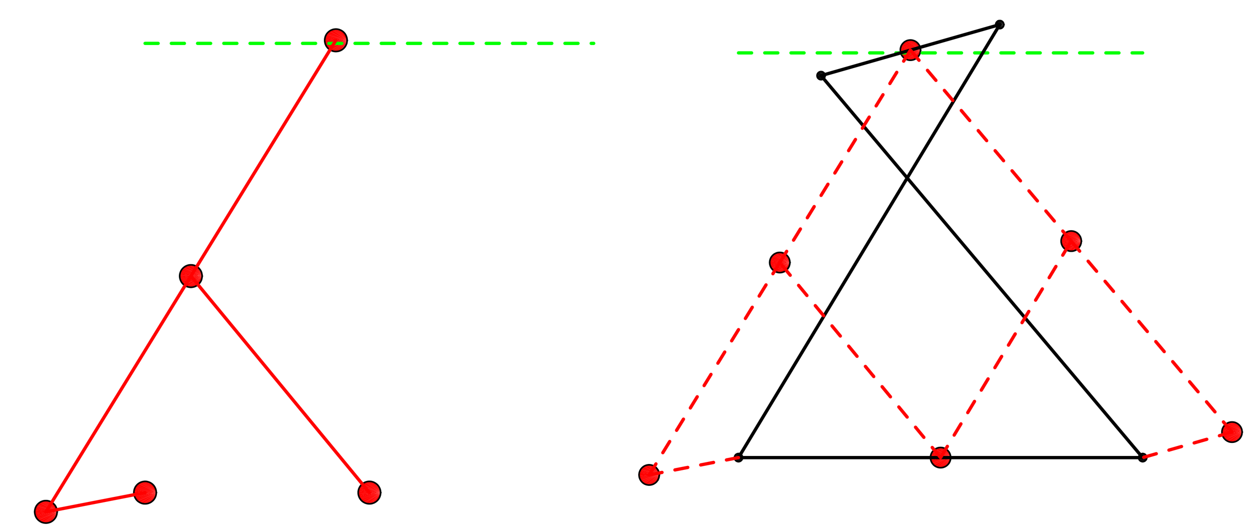
References

[3] A.J. Sommese and C.W. Wampler. *The Numerical solution of systems of polynomials arising in engineering and science.* World Scientific, 2005.

[4] A. Leykin, J. Verschelde, and A. Zhao. Newton's method with deflation for isolated singularities of polynomial systems. *Theoretical Computer Science*, 359(1-3):111–122, 2006.

6. Applications to Mechanisms

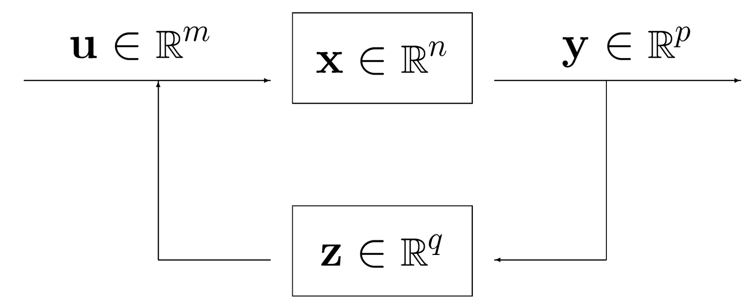
Chebychev's 4-bar mechanism and its cognates:



Given points the mechanism must reach, determine its parameters.

7. Applications to Control

Control of an m-input and p-output plant by a qth order dynamic compensator:



$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases} \qquad \begin{cases} \dot{\mathbf{z}} = F\mathbf{z} + G\mathbf{y} \\ \mathbf{u} = H\mathbf{z} + K\mathbf{y} \end{cases}$$

Every feedback law corresponds to a polynomial map of degree q into the Grassmannian of p-planes in $\mathbb{C}^{m+p}$ that meet $mp + q(m+p)$ given m-planes sampled at $mp + q(m+p)$ interpolation points.

8. Software on Clusters and Supercomputers

PHCpack [2] is available in source form, with binaries for many different computers.
Interfaces: PHCmaple (for Maple), PHClab (for MATLAB and Octave).
PHClib offers C wrappers to treat the code as a library. The parallel path trackers of PHCpack use PHClib and MPI, developed on Rocketcalc personal clusters.
PHCpack is among the experimental packages in SAGE.

References

[5] B. Huber, F. Sottile, and B. Sturmfels. Numerical Schubert calculus. *J. Symbolic Computation* 26(6):767–788, 1998.

[6] J. Verschelde and Y. Wang. Computing Dynamic Output Feedback Laws. *IEEE Transactions on Automatic Control* 49(8):1393–1397, 2004.

[7] A.J. Sommese, J. Verschelde, and C.W. Wampler. Advances in polynomial continuation for solving problems in kinematics. *ASME Journal of Mechanical Design* 126(2):262-268, 2004.

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