Univariate and Multivariate Polynomials

1. Polynomials and Expressions
   - coefficients, monomials, polynomials
   - numeric, exact, and symbolic factorizations

2. Polynomials in Several Variables
   - recursive representations of multivariate polynomials
   - ordering the monomials in several variables

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Introduction to Symbolic Computation
Jan Verschelde, 24 June 2024

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Polynomials and Expressions

A polynomial is a special type of mathematical expression.

Definition (univariate polynomial)

A *univariate polynomial* is a finite sum of terms, where

- every term is a coefficient multiplied with a monomial,
- a monomial is a power of the same variable, and
- all coefficients are of the same type.

The set of all polynomials in x with rational coefficients is $\mathbb{Q}[x]$.

For a coefficient number field K,
denote $K[x]$ as the set of polynomials in x with coefficients in K.

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Numeric, Exact, and Symbolic Factorizations

Polynomials factor in three different ways, as classified below.

1. The *numeric* factorization is defined over the *complex* numbers.

   By the fundamental theorem of algebra,
   every polynomial of degree d has d complex roots and
   can therefore be written as a product of d linear factors.

2. The *exact* factorization is defined over the *rational* numbers.

   The number of linear factors equals the number of rational roots.

3. The *symbolic* factorization is defined over the *algebraic* numbers.

   For every nonlinear factor in the exact factorization,
   add sufficiently as many algebraic numbers as needed,
   so the polynomial over the extended number field
   is a product of linear factors.

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Recursive Representations of Multivariate Polynomials

Definition (multivariate polynomial)

A *multivariate polynomial* is a finite sum of terms, where

- every term is a coefficient multiplied with a monomial,
- a monomial is a product of powers of variables, and
- all coefficients are of the same type.

The set of all polynomials in x and y with rational coefficients is $\mathbb{Q}[x, y]$.

Over any coefficient number field K,
a polynomial in x and y, in $K[x, y]$, can be viewed as

- $\in K[y][x]$, a polynomial in x with coefficients as polynomials in y,

or

- $\in K[x][y]$, a polynomial in y with coefficients as polynomials in x.

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Ordering Monomials in Several Variables

In one variable x, for two natural numbers d and e, we have

1. $d < e \Rightarrow x^d < x^e$,
2. $d = e \Rightarrow x^d = x^e$, or
3. $d > e \Rightarrow x^d > x^e$.

In one variable, monomials of the same degree are the same.

In two variables x and y, the above statement does not hold:

- x^2y and xy^2 both have the same degree three,
- x^2y and xy^2 are *different* monomials.

We use a lexicographic tie breaker:

$$x^2y > xy^2 \text{ because } x \text{ comes before } y \text{ in the alphabet.}$$

Monomial Orders

Definition (pure lexicographic)

In the *pure lexicographic order*
we sort the variables in each monomial in lexicographic order:

- For two monomials, if all variables appear with the same power, then the two monomials are the same.
- Otherwise, the first different power decides the order.

Example: $x^2 >_{\text{lex}} xy^3$.

Definition (degree lexicographic)

In the *degree lexicographic order* we sort as follows:

- Monomials with the larger degree are larger than monomials with a smaller degree.
- Apply the pure lexicographic order for same degree monomials.