Cost Benefit Analysis

1. ## Present Value of a Future Sum
   - reverse interest calculations
   - borrowing against an inheritance
   - considering payout options

2. ## Life Cycle Savings
   - justifying equipment purchases
   - discounted cash flow analysis

3. ## Proposals of Project Topics
   - is a doctoral degree financially worthwhile?
   - use public transport or your own car?

MCS 472 Lecture 18
Industrial Math & Computation
Jan Verschelde, 19 February 2024

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making investment decisions

We need to make an investment decision.

We assume that any investment

- requires sacrificing current resources, and
- has an expected return.

Is the investment worth it?

The main tool to make investment decisions in
a cost benefit analysis is a reverse interest calculation.

the present value of a future sum

Consider the following two problems.

1. In ten years we will inherit \$100,000.
   Assuming 5% rate with continuous compounding,
   how much can we borrow today?

2. The lottery (or an insurance settlement) offers
   1. either to pay half of the prize p now, that is $p/2$; or
   2. installments of $p/25$ for 25 years.

   What should the interest rate be to break even?

To solve these two problems, we need to answer the following.

1. What is the present value of \$100,000 in ten years?
2. What is the present value of receiving 25 yearly installments?

a reverse interest calculation

What is the present value of $3,000 in five years, assuming an annual interest rate of 8%?

$$\begin{aligned} \$2,042(1.08) &= \$2,205 \\ \$2,205(1.08) &= \$2,381 \\ \$2,381(1.08) &= \$2,572 \\ \$2,572(1.08) &= \$2,778 \\ \$2,778(1.08) &= \$3,000 \end{aligned}$$

Interpretation of the middle number:

$$\$2,572 = \underbrace{\$2,042(1+0.08)^3}_{\text{value of investment after 3 years}} = \underbrace{\$3,000(1+0.08)^{-2}}_{\text{value of investment 2 years before}}$$

Answer: $\$3,000(1+0.08)^{-5} = \$2,042$, rounded to nearest \$.

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borrowing against an inheritance

In ten years we will inherit \$100,000. Assuming 5% rate with continuous compounding, how much can we borrow today?

We start with some notations:

- $t = 10$ is the number of years in the future.
- $P = 100000$ is value of the inheritance.
- $r = 0.05$ is the interest rate.
- L is the amount of the loan.

With continuous interest compounding at rate r,
the loan amount L equals P after t years:

$$Le^{rt} = P \quad \text{or} \quad L = Pe^{-rt}.$$

$L = \$100,000e^{-0.5} = \$60,653.07$ is the highest amount of the loan.

the present value of future money

The formula for the present value P_0 of some future money P, paid N years in the future is

$$P_0 = P e^{-rN},$$

where the interest rate r is called the *discount rate*.

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considering payout options

The lottery (or an insurance settlement) offers

1. either to pay half of the prize p now, that is $p/2$; or
2. installments of $p/25$ for 25 years.

What should the interest rate be to break even?

What is the present value of the 25 installments?

Let r be the interest rate, using continuous compounding:

$$\underbrace{\frac{p}{25}}_{\text{1st installment}} + \underbrace{\frac{p}{25}e^{-r}}_{\text{in 1 year}} + \underbrace{\frac{p}{25}e^{-2r}}_{\text{in 2 years}} + \cdots + \underbrace{\frac{p}{25}e^{-24r}}_{\text{in 24 years}} .$$

a nonlinear equation

The sum of the 25 installments equals:

$$\frac{p}{25}\left(1 + e^{-r} + e^{-2r} + \cdots + e^{-24r}\right) = \frac{p}{25}\left(\frac{1 - e^{-25r}}{1 - e^{-r}}\right).$$

The alternative option is to receive a lump sum of $p/2$.

The break even point is defined by

$$\frac{p}{25}\left(\frac{1 - e^{-25r}}{1 - e^{-r}}\right) = \frac{p}{2}$$

which simplifies into

$$1 - e^{-25r} = 12.5\left(1 - e^{-r}\right)$$

so the break even point does not depend on the prize p.

the break even rate

The lottery (or an insurance settlement) offers

1. either to pay half of the prize p now, that is $p/2$; or
2. installments of $p/25$ for 25 years.

What should the interest rate be to break even?

Answer: 6.74%. (Obtained with NLsolve in Julia.)

The higher the interest rate, the smaller the present value.

using `nlsolve`

The Julia package `NLsolve` provides a nonlinear solver.

```
using NLsolve

"""
    function f!(F, x)

defines F[1] as the input to nlsolve.
"""
function f!(F, x)
    r = x[1]
    F[1] = 1 - exp(-25*r) - 12.5*(1 - exp(-r))
end

sol = nlsolve(f!, [0.05])
```

the output of `nlsolve`

```
sol = nlsolve(f!, [0.05])
```

Printing `sol` shows

```
Results of Nonlinear Solver Algorithm
 * Algorithm: Trust-region with dogleg and autoscaling
 * Starting Point: [0.05]
 * Zero: [0.06737453737435 81]
 * Inf-norm of residuals: 0.000000
 * Iterations: 4
 * Convergence: true
   * |x - x'| < 0.0e+00: false
   * |f(x)| < 1.0e-08: true
 * Function Calls (f): 5
 * Jacobian Calls (df/dx): 5
```

We verify with the evaluation of `sol.zero[1]` using

```
f(r) = 1 - exp(-25*r) - 12.5*(1 - exp(-r)).
```

other considerations and perspectives

Exercise 1:
Assuming the installment increases by 3% each year,
at which rate does the break even point then occur?

Exercise 2:
Consider the payout plan of the lottery from the state's perspective.

What is the purchase price of q of an annuity that

1. pays out $p/25$ for 25 years to the winner,
2. the remaining balance on the annuity grows at rate r annually.

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justifying equipment purchases

Consider the purchase of newer, more efficient equipment.

- The purchase requires a present sum of money, a cost.
- The purchase represents a saving, or a benefit.

The cost occurs now, the benefit later.

Compute the justification for the purchase of the new equipment.

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discounted cash flow analysis

The approach to justify equipment purchases is

discounted cash flow analysis.

1. For each year of the projected life of the equipment, compute

   + the projected savings,
   − less costs for that year.

   This gives the yearly *net cash flow*.

2. Discount the yearly net cash flow back to the present value.

3. Compute the *life cycle saving*:

   + all net cash flow present values over the lifetime,
   − less the first costs.

first costs, return on investment, payback period

Some definitions:

- The *first costs* are the costs for the new equipment.
- The *return on investment* is the discount rate that yields zero life cycle savings.
- The *payback period* is the time before the accumulating present value of the net savings surpasses the initial investment.

the discount rate

We decide to invest in better equipment, which costs q.

The investment

- will save us p
- over n years.

We compute the present value of p at discount rate r:

$$p \left(\frac{1 - e^{-r\,n}}{1 - e^{-r}} \right)$$

against the cost q of the investment.

The discount rate $r = d - i$, consists of

1. d is the interest rate of a safe investment,
2. i is the inflation rate.

.

a numerical example

We consider the purchase of new equipment.

1. The life span of the equipment is 12 years.
2. Each year, the new equipment will save $1052.
3. We use a discount rate $r = d - i = 0.02$.

How much may the equipment cost to justify the purchase?

To answer this question,

- we use a dataframe,
- assuming the equipment costs $10,000.

See the posted Jupyter notebook.

the solution with a dataframe

	year	savings	costs	values	balance
	Int64	Int64	Int64	Float64	Float64
1	0	0	-10000	0.0	-10000.0
2	1	1052	0	1031.17	-8968.83
3	2	1052	0	1010.75	-7958.08
4	3	1052	0	990.736	-6967.34
5	4	1052	0	971.118	-5996.23
6	5	1052	0	951.889	-5044.34
7	6	1052	0	933.04	-4111.3
8	7	1052	0	914.565	-3196.73
9	8	1052	0	896.455	-2300.28
10	9	1052	0	878.704	-1421.57
11	10	1052	0	861.305	-560.267
12	11	1052	0	844.25	283.982
13	12	1052	0	827.533	1111.51
14	0	12624	-10000	11111.5	0.0

what is the return on investment?

Exercise 3:

Consider the previous numerical example,

1. assuming \$7,000 as the purchase cost, and
2. inflation rate at $i = 3\%$.

What is the return on investment for this problem?

Exercise 4:

Consider the previous numerical example,
assuming the cost of the equipment is borrowed

1. with a 15% down payment,
2. at an interest rate of 5%, compounded continuously,
3. with a loan over 12 years.

Under these conditions of the loan,
how much may the equipment cost to justify the purchase?

summary and bibliography

Calculating the present value of a future sum,
we provided a quantified justification for investment decisions.

This lecture follows Chapter 7 of our text book.

- Charles R. MacCluer:
  Industrial Mathematics. Modeling in Industry, Science, and Government. Prentice Hall, 2000.

Available online through our UIC library is a cost benefit analysis description from the management perspective.

- Peter Eichhorn and Ian Towers:
  Principles of Management: Efficiency and Effectiveness in the Private and Public Sector. Springer 2018.

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1. is a doctoral degree financially worthwhile?

Is four years of study beyond the master's to obtain
a doctoral degree financially worthwhile?

Consider the following questions:

- Does the added income throughout the career pay back
  1. the cost of the education, and
  2. the loss of income during the four years of study?
- Is there is a difference between academia and industry?

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2. use public transport or your own car?

For the daily commute, compare the cost of using public transport versus using your own car.

- In your study consider the normal life span of a car.
- The cost of public transport includes a fixed fare, subject to annual fare hikes.
- The cost of a car includes not only the purchase price, but also taxes, insurance, fuel, repairs, and depreciation cost.

Consider the following questions:

- What is the total saving of using public transport?
- Explain how the annual increase in saving could be used to justify an annual fare hike, that is then also fair...