The Art Gallery Theorem

1. Guarding an Art Gallery
   - the watchman problem
   - triangulations and 3-colorings
2. Simple Polygon Triangulations
   - number of triangles in a triangulation
   - two cases in the proof by induction
3. 3-Colorings
   - coloring the vertices in a triangulation
   - the art gallery theorem

MCS 481 Lecture 7
Computational Geometry
Jan Verschelde, 29 January 2025

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the watchman problem

Given is a floorplan of an art gallery, guarded by rotating cameras.

The floorplan is a simple polygon (no holes), not always convex.
Inner vertices or inner edges are considered holes.

Goal: use as few cameras as possible to the entire gallery is guarded.

Theorem (the art gallery theorem, Vašek Chvátal, 1975)

For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras suffice for any point to be visible.

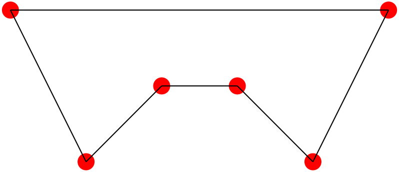
Convex polygons may need fewer cameras, $\lfloor n/3 \rfloor$ is the worst case.

The "*book proof*" follows the 1978 proof by Steve Fisk.

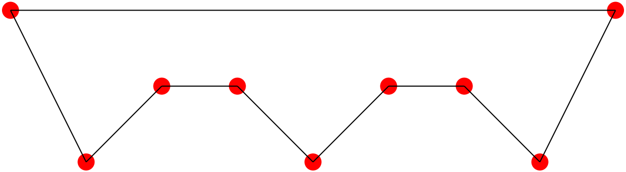
Finding the minimal number of cameras is NP-hard.

comb shaped floorplans

- 6 vertices, 2 cameras needed:



- 9 vertices, 3 cameras needed:



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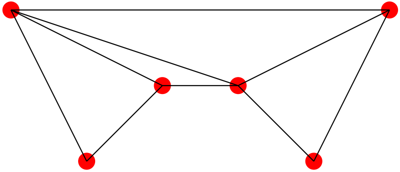
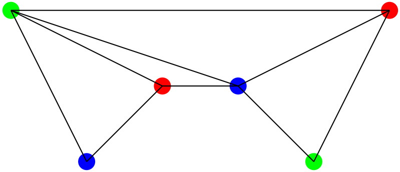
triangulations and 3-colorings

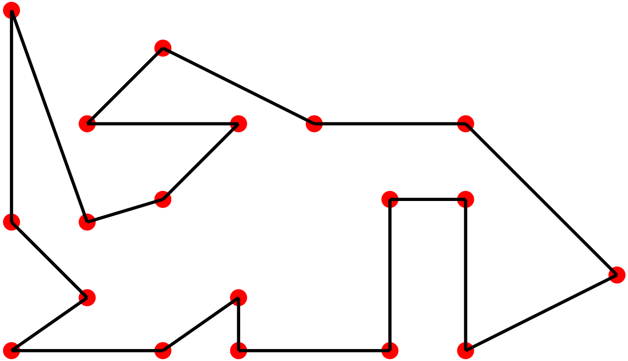
- Consider a triangulation:



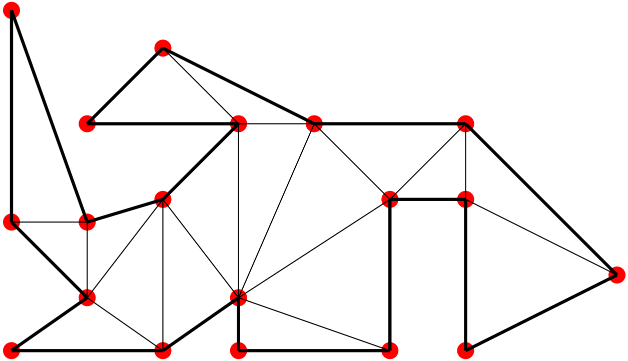
- Assign different colors to each vertex of a triangle:



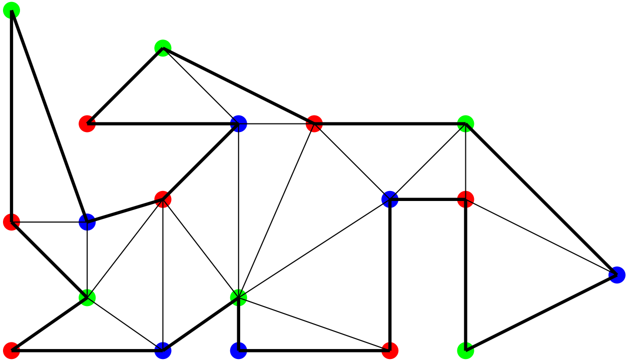
given is a polygonal floorplan



construct a triangulation of the polygon



apply a 3-coloring to the triangulation



all edges are vertical or horizontal

Exercise 1: Consider a simple (no holes) polygon P
with n vertices, where all edges are either vertical or horizontal.

The simplest example is a rectangle and 1 camera suffices.

Draw examples to justify that $\lfloor n/4 \rfloor$ cameras suffice.

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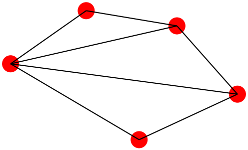
number of triangles

Theorem (Simple Polygon Triangulations)

A simple polygon P with n vertices admits a triangulation with $n - 2$ triangles.

An easy case: P is convex.



Take the leftmost vertex v as the vertex of every triangle.
Starting at the leftmost vertex,

- take two adjacent vertices to form a triangle with v,
- walk to the next vertex, form next triangle, swapping vertices.

proof by induction on n

Theorem (Simple Polygon Triangulations)

A simple polygon P with n vertices admits a triangulation with $n - 2$ triangles.

Proof. By induction on n.
Base case: $n = 3$, one triangle, $1 = 3 - 2$.

Induction hypothesis: Every simple polygon with k vertices, $k < n$, admits a triangulation with $k - 2$ triangles.

Let v be the leftmost highest vertex of P with n vertices.
Removing v from P results in a polygon with $n - 1$ vertices and then we can apply the induction hypothesis.

We must show to apply the induction hypothesis correctly.

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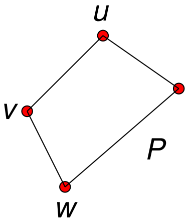
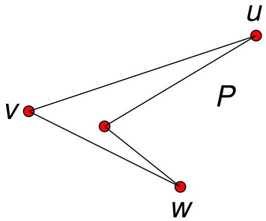
two cases

Let u and w be the neighbors of v. Consider:





There are two cases to consider:

1. The line segment (u, w) lies in P.
2. The line segment (u, w) does not lie entirely in P.

the line segment (u, w) lies in P

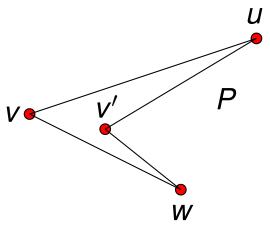
1. Consider the triangle (u, v, w).
2. Apply the induction hypothesis to $P \setminus \{v\}$:
    1. $P \setminus \{v\}$ has $n - 1 - 2 = n - 3$ triangles,
    2. and we add the triangle (u, v, w),

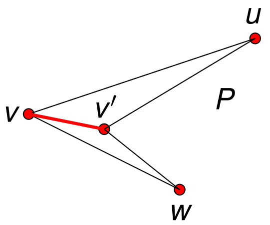
   to obtain a triangulation of P with $n - 2$ triangles.

the line segment (u, w) does not lie entirely in P

There must be at least one vertex at the left of (u, w),
a vertex, different from v, and to the right of v.

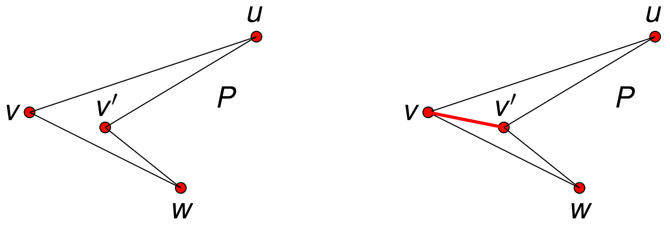
Consider the vertex v' farthest to the left of (u, w):





The line segment (v, v') lies entirely in P
because v and v' are leftmost, they intersect no edge.

application of the induction hypothesis



The edge (v, v') splits P in two pieces P_1 and P_2, respectively with m_1 and m_2 vertices:
$m_1 < n$, $m_2 < n$, $m_1 + m_2 = n + 2$ (the $+2$ counts v and v').

We can now apply the induction hypothesis:

1. P_1 has a triangulation with $m_1 - 2$ triangles.
2. P_2 has a triangulation with $m_2 - 2$ triangles.

So the total number of triangles is
$m_1 - 2 + m_2 - 2 = m_1 + m_2 - 4 = n - 2$.

Q.E.D.

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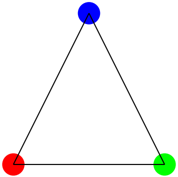
3-colorings

Definition (3-coloring)

Given a triangulation, a *3-coloring* assigns a color (red, green, or blue) to each vertex so that vertices that are adjacent have a different color.

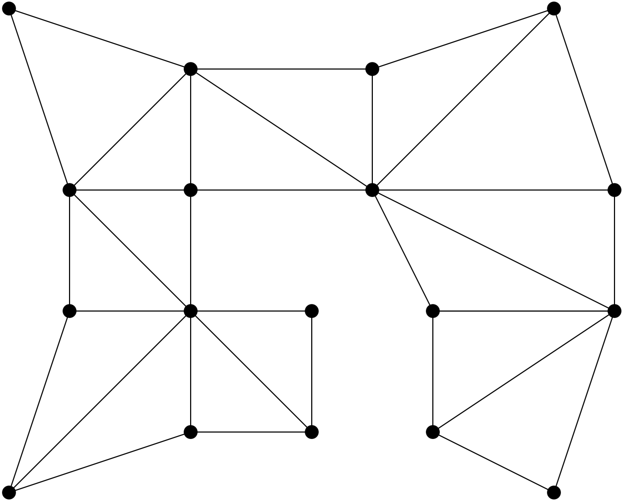
Does a 3-coloring exist?

The base case in the induction is one triangle:

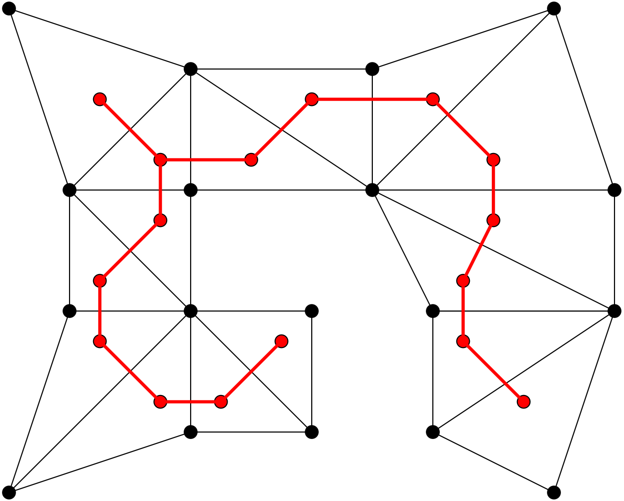


Is there an algorithm to color the vertices in a triangulation?

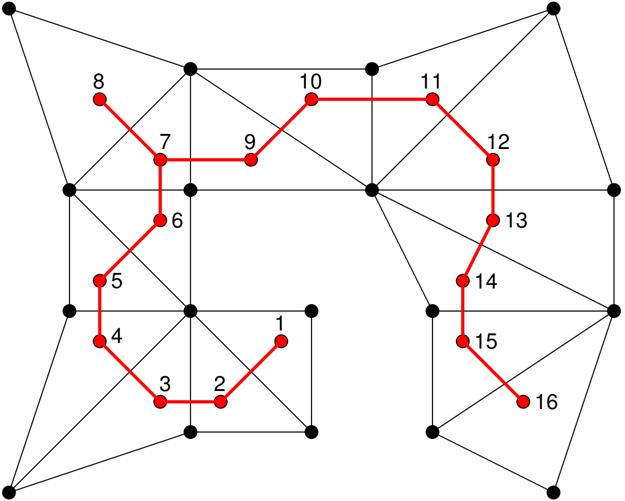
given a triangulation, visit all triangles



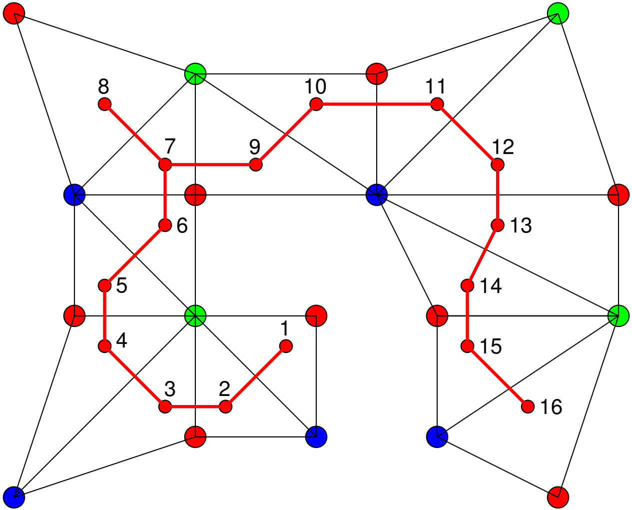
the dual graph



traverse the dual graph in depth first order



apply a 3-coloring



pseudo code for traversing and 3-coloring

Exercise 2: Given a doubly connected edge list which stores a triangulation of a simple polygon,
write pseudo code to visit all triangles in depth first search order.

Exercise 3: Given a doubly connected edge list which stores a triangulation of a simple polygon and a 3-coloring of the first triangle,
write pseudo code to apply a 3-coloring to all vertices.

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the art gallery theorem

Theorem (the art gallery theorem, Vašek Chvátal, 1975)

For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras suffice for any point to be visible.

Proof. A simple polygon has a triangulation with $n - 2$ triangles.

The dual graph of the triangulation has a vertex for each triangle.
Vertices are connected by an edge if their triangles are adjacent.

As the polygon has no holes, the dual graph is a tree,
which we traverse in depth-first search order.

As we traverse all triangles, we apply a 3-coloring to the vertices.
The colors on the first triangle determine the colors of all others.

Determine which color occurs the least number of times.
This least number equals or is less than $\lfloor n/3 \rfloor$.
Position the cameras at the vertices with that color.

Q.E.D.

exercises

We started the third chapter in the textbook, covered section 3.1.

Consider the following activities, listed below.

1. Write the solutions to exercises 1, 2, and 3.
2. Consider the exercises 1,2,3 in the textbook.