Gift Wrapping for Pretropisms

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Graduate Computational Algebraic Geometry Seminar

Outline

1. Pretropisms and Solution Sets
   - an illustrative example
   - pretropisms and tropisms
2. Applying Gift Wrapping
   - computing cones of pretropisms
   - sketching an algorithm in pseudocode
   - running a test program in PHCpack

an illustrative example

$$f(x_1, x_2, x_3) = \left\{ \begin{array}{r} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{array} \right.$$

$$f^{-1}(\mathbf{0}) = Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$$

1. Z_{21} is the sphere $x_1^2 + x_2^2 + x_3^2 - 1 = 0$,
2. Z_{11} is the line $(x_1 = 0.5, x_3 = 0.5^3)$,
3. Z_{12} is the line $(x_1 = \sqrt{0.5}, x_2 = 0.5)$,
4. Z_{13} is the line $(x_1 = -\sqrt{0.5}, x_2 = 0.5)$,
5. Z_{14} is the twisted cubic $(x_2 - x_1^2 = 0, x_3 - x_1^3 = 0)$,
6. Z_{01} is the point $(x_1 = 0.5, x_2 = 0.5, x_3 = 0.5)$.

numerical irreducible decomposition

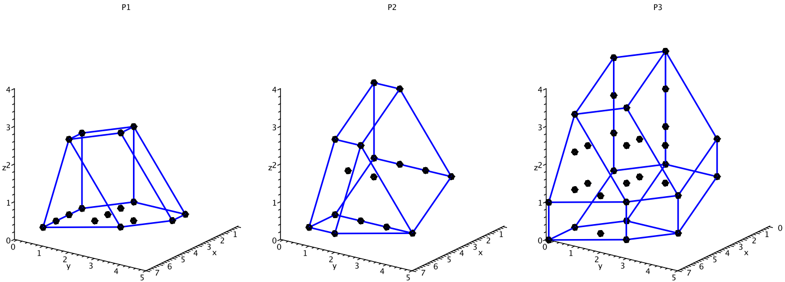
Used in two papers in numerical algebraic geometry:

- first cascade of homotopies: 197 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Numerical decomposition of the solution sets of polynomial systems into irreducible components*. SIAM J. Numer. Anal. 38(6):2022–2046, 2001.

- equation-by-equation solver: 13 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Solving polynomial systems equation by equation*. In Algorithms in Algebraic Geometry, Volume 146 of The IMA Volumes in Mathematics and Its Applications, pages 133–152, Springer-Verlag, 2008.

The mixed volume of the Newton polytopes of this system is 124.
By theorem A of Bernshteĭn, the mixed volume is an upper bound on the number of isolated solutions.

three Newton polytopes



$$f(x_1, x_2, x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

looking for solution curves

The twisted cubic is $(x_1 = t, x_2 = t^2, x_3 = t^3)$.

We look for solutions of the form

$$\begin{cases} x_1 = t^{v_1}, & v_1 > 0, \\ x_2 = c_2 t^{v_2}, & c_2 \in \mathbb{C}^*, \\ x_3 = c_3 t^{v_3}, & c_3 \in \mathbb{C}^*. \end{cases}$$

Substitute $x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3$ into f

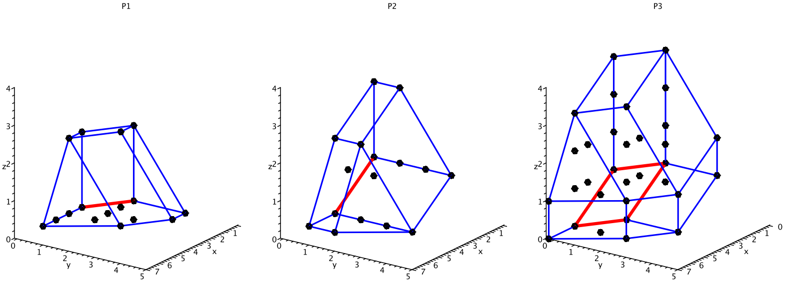
$$f(x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3) = \begin{cases} (0.5c_2 - 0.5)t^2 + O(t^3) = 0 \\ (0.5c_3 - 0.5)t^3 + O(t^5) = 0 \\ 0.5(c_2 - 1.0)(c_3 - 1.0)t^5 + O(t^7) \end{cases}$$

$\rightarrow$ conditions on c_2 and c_3.

How to find $(v_1, v_2, v_3) = (1, 2, 3)$?

faces of Newton polytopes

Looking at the Newton polytopes in the direction $\mathbf{v} = (1, 2, 3)$:



Selecting those monomials supported on the faces

$$\partial_{\mathbf{v}} f(x_1, x_2, x_3) = \begin{cases} 0.5x_2 - 0.5x_1^2 = 0 \\ 0.5x_3 - 0.5x_1^3 = 0 \\ -0.5x_2x_1^3 - 0.5x_3x_1^2 + 0.5x_3x_2 + 0.5x_1^5 = 0 \end{cases}$$

degenerating the sphere

$$f(x_1,x_2,x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_1 = t \rightarrow 0$:

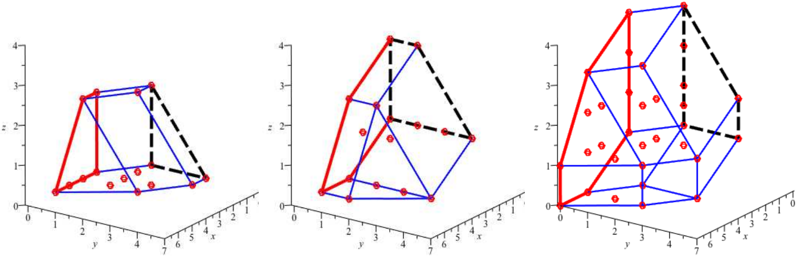
$$\partial_{(1,0,0)} f(x_1,x_2,x_3) \begin{cases} x_2(x_2^2 + x_3^2 - 1)(-0.5) = 0 \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ x_2 x_3 (x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_2 = s \rightarrow 0$:

$$\partial_{(0,1,0)} f(x_1,x_2,x_3) \begin{cases} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) = 0 \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

more faces of Newton polytopes

Looking at the Newton polytopes along $\mathbf{v} = \mathbf{(1,0,0)}$ and $\mathbf{v} = (0, 1, 0)$:



$$\partial_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

$$\partial_{(0,1,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

faces of faces

The sphere degenerates to circles at the coordinate planes.

$$\partial_{(1,0,0)} f(x_1, x_2, x_3) = \begin{cases} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{cases}$$

$$\partial_{(0,1,0)} f(x_1, x_2, x_3) = \begin{cases} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{cases}$$

Degenerating even more:

$$\partial_{(0,1,0)} \partial_{(1,0,0)} f(x_1, x_2, x_3) = \begin{cases} x_2(x_3^2 - 1)(-0.5) \\ x_3(x_3^2 - 1)(-0.5) \\ x_2 x_3(x_3^2 - 1)(x_3 - 0.5) \end{cases}$$

The factor $x_3^2 - 1$ is shared with $\partial_{(1,0,0)} \partial_{(0,1,0)} f(x_1, x_2, x_3)$.

representing a solution surface

The sphere is two dimensional, x_1 and x_2 are free:

$$\left\{ \begin{array}{l} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = 1 + c_1 t_1^2 + c_2 t_2^2. \end{array} \right.$$

For $t_1 = 0$ and $t_2 = 0$, $x_3 = 1$ is a solution of $x^3 - 1 = 0$.

Substituting $(x_1 = t_1, x_2 = t_2, x_3 = 1 + c_1 t_1^2 + c_2 t_2^2)$ into the original system gives linear conditions on the coefficients of the second term: $c_1 = -0.5$ and $c_2 = -0.5$.

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notations

Given $p, q \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, do p and q have a common factor?

For example:

$$\begin{cases} p = (x^2 + y^2 + z^2 - 1)(y - x^2) \\ q = (x^2 + y^2 + z^2 - 1)(z - x^3) \end{cases}$$

In our polyhedral approach, we write p and q as

$$p(x, y, z) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} x^{a_1} y^{a_2} z^{a_3} \quad \text{and} \quad q(x, y, z) = \sum_{\mathbf{b} \in A} c_{\mathbf{b}} x^{b_1} y^{b_2} z^{b_3}$$

where $A = \{ \mathbf{a} \in \mathbb{Z}^3 \mid c_{\mathbf{a}} \neq 0 \}$ and $B = \{ \mathbf{b} \in \mathbb{Z}^3 \mid c_{\mathbf{b}} \neq 0 \}$ are the support sets respectively of p and q.

A spans the Newton polytope $P = \mathrm{conv}(A)$, B spans $Q = \mathrm{conv}(B)$.

inner normals and initial forms

Denote by $\langle \cdot, \cdot \rangle$ the inner product: $\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$.

For $\mathbf{v} \neq \mathbf{0}$, the face $\mathrm{in}_{\mathbf{v}} P$ of $P = \mathrm{conv}(A)$ is spanned by $\mathrm{in}_{\mathbf{v}} A$ with

$$\mathrm{in}_{\mathbf{v}} A = \{ \, \mathbf{a} \in A \mid \langle \mathbf{a}, \mathbf{v} \rangle = \min_{\mathbf{b} \in A} \langle \mathbf{b}, \mathbf{v} \rangle \, \}.$$

We use $\mathrm{in}_{\mathbf{v}} A$ because of *initial forms* of polynomials:

$$\mathrm{in}_{\mathbf{v}} p(x,y,z) = \sum_{\mathbf{a} \in \mathrm{in}_{\mathbf{v}} A} c_{\mathbf{a}} x^{a_1} y^{a_2} z^{a_3} \quad \text{where} \quad p(x,y,z) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} x^{a_1} y^{a_2} z^{a_3}.$$

We may take $\mathbf{v}$, with integer coordinates v_1, v_2, and v_3, normalized so that $\gcd(v_1, v_2, v_3) = 1$.
This normalization gives unique normals to all proper facets.

pretropisms

Let (A,B) be two supports. A *pretropism for* (A,B) is a vector $\mathbf{v} \neq \mathbf{0}$: $\#\mathrm{in}_{\mathbf{v}}A \geq 2$ and $\#\mathrm{in}_{\mathbf{v}}B \geq 2$. Denote by $T(A,B)$ the set $\{ \mathbf{v} \neq 0 \mid \#\mathrm{in}_{\mathbf{v}}A \geq 2 \text{ and } \#\mathrm{in}_{\mathbf{v}}B \geq 2 \}$. A pretropism is a candidate for a tropism. A tropism $\mathbf{v}$ is a pretropism for which a root of the initial form system $\mathrm{in}_{\mathbf{v}}f(\mathbf{x}) = \mathbf{0}$ determines the leading coefficients of a Puiseux series expansion of a solution component of $f(\mathbf{x}) = \mathbf{0}$.

Proposition

If $T(A,B) = \emptyset$, then for any two Laurent polynomials p and q with respective support sets A and B, p and q have no common factor.

Proof. Suppose p and q have a nontrivial common factor f, i.e.: $p = p_1 f$ and $q = q_1 f$. Denote the Newton polytope of f by F, then $P = P_1 + F$ and $Q = Q_1 + F$, where P, P_1, Q, and Q_1 are the respective Newton polytopes of p, p_1, q, and q_1.
All normals to faces of F are tropisms. □

predicting a common factor

Recall the example:

$$\left\{ \begin{array}{l} p = (x^2 + y^2 + z^2 - 1)(y - x^2) \\ q = (x^2 + y^2 + z^2 - 1)(z - x^3) \end{array} \right.$$

The set $\{(1,0,0), (0,1,0), (0,0,1), (-1,-1,-1)\}$ contains all normalized vectors to the facets of the simplex, the Newton polytope of the common factor of p and q.

Among the curves common to p and q we get the equations of the twisted cubic via the initial forms $\mathrm{in}_{(-1,-2,-3)}p = z^2(y - x^2)$ and $\mathrm{in}_{(+1,+2,+3)}p = -1(y - x^2)$.

classifying pretropisms

Redundant tropisms: for curves restrict to first positive component.
For surfaces: every edge of the Newton polytope of a common factor will also be a pretropism.

For two Newton polytopes P and Q, we classify pretropisms as follows:

- A *facet pretropism* is an inner normal to a facet common to both P and Q.
  We call this common facet a *tropical prefacet*.
- An *edge pretropism* is a pretropism not contained in any tropical prefacet.

Edges of P and Q that are parallel to each other are always contained in a larger face tuple $(\mathrm{in}_{\mathbf{v}}P, \mathrm{in}_{\mathbf{v}}Q)$ for some pretropism $\mathbf{v}$.

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applying gift wrapping

Recall the geometric intuition of gift wrapping:

- view a supporting hyperplane as wrapping paper,
- the paper first touches a vertex, then an edge, etc.
- planes supporting facets are rotated along ridges.

Consider as given a tuple of Newton polytopes,
the pretropisms correspond to those

- facets of the Minkowski sum of the Newton polytopes,
- that are spanned by sums of edges of the polytopes.

On the complexity:

- Storing the entire face lattice of a Newton polytope of a sparse polynomial has an acceptable complexity.
- Storing the entire face lattice of the sum of the Newton polytopes is not efficient and most likely not even desirable.

two Newton polytopes in 3-space: facet pretropisms

Given are two support sets A and B, $A \in \mathbb{Z}^{3 \times n_A}$ and $B \in \mathbb{Z}^{3 \times n_B}$.

Could the polynomials p (supported on A) and q (supported on B) have a common factor?

This question is equivalent for two facets:

$$F_A \text{ of } \mathrm{conv}(A) \text{ and } F_B \text{ of } \mathrm{conv}(B)$$

to share the same inner normal.

If the components of each inner normal vector to a facet are normalized so their greatest common divisor equals one,
then the problem of finding a facet pretropism is reduced to sorting:

1. Sort the inner normals of the facets to A and B lexicographically.
2. Merge the sorted lists of inner normals.

two Newton polytopes in 3-space: edge pretropisms

Given are two support sets A and B, $A \in \mathbb{Z}^{3 \times n_A}$ and $B \in \mathbb{Z}^{3 \times n_B}$.

The search for a pretropism starts at a facet F_A of $\mathrm{conv}(A)$:

- The vertex points that span the facet are ordered: two consecutive vertex points of the facet span an edge e_A.
- Two neighboring facets are connected through exactly one edge: the facet normal $\mathbf{v}$ of F_A and the normal $\mathbf{u}$ to the neighboring facet span the inner normal cone of the connecting edge e_A.

With e_A and its cone spanned by $\{\mathbf{u}, \mathbf{v}\}$, we explore B:

- For random real $\lambda_\mathbf{u}, \lambda_\mathbf{v} > 0$, let $\mathbf{w} = \lambda_\mathbf{u}\mathbf{u} + \lambda_\mathbf{v}\mathbf{v}$, then $\mathrm{in}_\mathbf{w} B = \{\mathbf{b}\}$.
- Run over all edges e_B incident to $\mathbf{b}$ and check the pair (e_A, e_B): if the intersection of the inner normal cones to e_A and e_B is not empty, then their intersection contains a pretropism.

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sketching an algorithm in pseudocode

Input: (A, B), $A \in \mathbb{Z}^{3 \times n_A}$, $B \in \mathbb{Z}^{3 \times n_B}$.
Output: $T_f(A, B)$; $T_e(A, B)$.

1. $F_A := \mathrm{conv}(A)$; $F_B := \mathrm{conv}(B)$;
2. $T_f(A, B) := \{ \mathbf{v} \mid \mathbf{v}$ is normal to $f \in F_A \cap F_B \}$;
3. for all $\mathbf{v}$ normal to facet $f \in F_A$, $\mathbf{v} \not\in T_f(A, B)$ do
3.1 let e_A be edge of A, not visited before;
3.2 let $\mathbf{u}$: $e_A = \mathrm{in}_{\mathbf{u}} A \cap \mathrm{in}_{\mathbf{v}} A$;
3.3 let $\mathbf{b}$ be a vertex of $\mathrm{in}_{\mathbf{u}+\mathbf{v}} B$;
3.4 for all edges e_B incident to $\mathbf{b}$ do
3.4.1 if $\mathbf{w} \perp (e_A, e_B)$ is edge pretropism then
3.4.1.1 $T_e(A, B) := T_e(A, B) \cup \{\mathbf{w}\}$;
3.4.1.2 set $\mathbf{b}$ to unvisited vertex of B; goto 3.4;

a crude cost analysis

Assuming a uniform cost of arithmetic ($\leftrightarrow$ multiprecision):

- The cost of computing all facet pretropisms:

  Let N_A = #facets of $\mathrm{conv}(A)$ and N_B = #facets of $\mathrm{conv}(B)$,
  set $M = \max(N_A, N_B)$,
  then the cost reduces to sorting, which is $O(M \log_2(M))$.

- The cost of computing all edge pretropisms:

  Let N_A = #edges of $\mathrm{conv}(A)$ and N_B = #edges of $\mathrm{conv}(B)$,
  set $M = \max(N_A, N_B)$,
  then the cost reduces to making combinations: $O(M^2)$.

  For a sharper bound, let m be the maximum number of edges per facet, then the cost for all edge pretropisms becomes $O(Mm)$.

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running a test program in PHCpack

The code for pretropisms is not (yet) wrapped to `phcpy`.

Running the test program `ts_pretrop` on the illustrative example:

```
The list of facet pretropisms :
 0 1 0
 0 0 1
 -1 -1 -1
 1 0 0
```

which corresponds to the inner normals of a simplex, the Newton polytope of the common factor $x_1^2 + x_2^2 + x_3^2 - 1$.

pretropisms for the space curves

Take the first two polynomials of the illustrative example:

```
p : (y-x**2)*(x**2 + y**2 + z**2 - 1)*(x - 0.5);
q : (z-x**3)*(x**2 + y**2 + z**2 - 1)*(y - 0.5);
```

The output of the test program `ts_pretrop` contains

```
The edge tropisms via giftwrapping :
 0 0 -1
 -1 -2 -3
 -1 -3 -3
 1 2 3
 0 -1 0
```

We recognize the twisted cubic (t^1, t^2, t^3).