Fine Classification of strongly minimal sets
SIU Logic Seminar

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Overview

1. Classifying strongly minimal sets by their acl-geometries
2. Quasi-groups and Steiner systems
3. Coordinatization by varieties of algebras
4. Finer Classification: $\mathrm{dcl}(X)$, $\mathrm{acl}(X)$, sdcl

Thanks to Joel Berman, Gianluca Paolini, Omer Mermelstein, and Viktor Verbovskiy.

Classifying strongly minimal sets and their acl-geometries

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. Zilber Conjecture
    1. unary (lattice of subspaces distributive) (trivial, discrete, disintegrated)
    2. vector space-like (lattice of subspaces modular)
    3. ‘bi-interpretable’ with an algebraically closed field (non-locally modular)
2. Plus
    1. Flat: Combinatorial class
    2. Are there more?

What was the Zilber conjecture?

Conditions on the acl-geometry imply conditions on the algebra of the structure.
Are there finer measures?

Motive 0: The diversity of *ab initio* strongly minimal sets

ab initio: The infinite structure is defined a class $\boldsymbol{K}_0$ of finite structures for a first order theory.

Each of the following may affect properties of an ab intio strongly minimal set that are

- model theoretic
- and/or of wider mathematical interest.

1. the vocabulary τ including number of relations (sorts)
2. axioms T_0 on the class $\boldsymbol{K}_0$ of the finite τ-models
3. the function δ from $\boldsymbol{K}_0$ to $\mathbb{Z}$
4. the way $\boldsymbol{K}_0$ is determined from δ and T_0
5. the function μ from 'good pairs' to $\mathbb{N}$.

A field guide to Hrushovski Constructions [Bal]

Quasi-groups and Steiner systems

Steiner systems

A Steiner k system is collection of points and lines (block) such that

1. two points determine a line
2. each line has exactly k-points.

We work in a vocabulary with one ternary relation R for collinearity. We say the system is $S(2, k, n)$ if there are n-points and lines have length k.

Connections with number theory

For which n's does an $S(2, k, n)$ exist?
for 3 point lines:

Necessity:
$n \equiv 1$ *or* $3 \pmod 6$ is necessary.
Rev. T.P. Kirkman (1847)

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Sufficiency:
$n \equiv 1$ *or* $3 \pmod 6$ is sufficient.
(Bose $6n + 3$, 1939), Skolem ($6n + 1$, 1958)

Linear Spaces

Definition (1-sorted)

The vocabulary τ contains a single ternary predicate R, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems:

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

$\boldsymbol{K}^*$ includes infinite linear spaces.

Groupoids and semigroups

1. A groupoid (magma) is a set A with binary relation $\circ$.
2. A quasigroup is a groupoid satisfying left and right cancellation (Latin Square)
3. A Steiner quasigroup satisfies $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.

existentially closed Steiner Systems

Barbina-Casanovas

[BC1x] Consider the class $\tilde{\boldsymbol{K}}$ of finite structures (A, R) which are the graphs of a Steiner quasigroup.

1. $\tilde{\boldsymbol{K}}$ has ap and jep and thus a limit theory T_{sqg}.
2. T_{sqg} has
   1. quantifier elimination
   2. $2^{\aleph_0}$ 3-types;
   3. the generic model is prime and locally finite;
   4. T_{sqg} has TP_2 and $NSOP_1$.

Strongly minimal linear spaces

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

An easy compactness argument establishes

Corollary

If (M, R) is a strongly minimal linear space for some k, it is a Steiner k-system.

Ab Initio Strongly minimal Steiner Systems

Definition

A *Steiner* $(2, k, v)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP20]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(2, k, \infty)$ Steiner-systems.

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Hrushovski construction for linear spaces

$\mathbf{K}_0^*$ denotes the collection of finite linear spaces in the vocabulary $\tau = \{R\}$.
A line in a linear space is a maximal R-clique
$L(A)$, the lines based in A, is the collections of lines in (M, R) that contain 2 points from A.

Definition: Paolini's δ

[Pao21] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

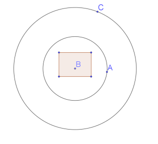
Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. C is a 0-*primitive extension* of A if C is minimal with $\delta(C/A) = 0$.



2. C is good over $B \subseteq A$ if B is minimal contained in A such that C is a 0-*primitive extension* of B. We call such a B a *base*.

In Hrushovski's examples the base is unique. But not in linear spaces.

α is the isomorphism type of $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Instances of α determine a line in linear spaces.

The μ function

Context

Let $\mathcal{U}$ be a collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:

1. (i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;
2. (ii) a number $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$

T_μ is the theory of a strongly minimal Steiner $(\mu(\boldsymbol{\alpha}) + 2)$-system

If $\mu(\boldsymbol{\alpha}) = 1$, T_μ is the theory of a Steiner triple system, bi-interpretable with a Steiner quasigroup.

Key features of constructed model

$A \leq M$ if $A \subseteq B \subseteq M$ implies $\delta(A/B) \geq 0$

When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

Realization of good pairs

1. A good pair A/B *well-placed* by $\mathcal{D}$ in a model M, if $B \subseteq \mathcal{D} \leq M$ and A is 0-primitive over $\mathcal{D}$.
2. For any good pair (A/B), $\chi_M(A/B)$ is the maximal number of disjoint copies of A over B appearing in M.

If C/B is well-placed by $\mathcal{D} \leq M$, $\chi_M(B, C) = \mu(B/C)$.

Primitive singleton

α is the isomorphism class of the good pair $(\{a, b\}, \{c\})$ with $R(a, b, c)$.

Classes of Structures: Steiner

Stages of the construction

1. $\boldsymbol{K}_0^*$: all finite linear τ-spaces.
2. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily ≥ 0.
3. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: $\chi_M(A, B) \leq \mu(A, B)$ μ bounds the number of disjoint realizations of a 'good pair'.
   Now Fraïssé.
4. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).
5. X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0$.).
6. Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \boldsymbol{K}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ implies M is d-closed in N.

T_μ is the common theory of the models in $\boldsymbol{K}_d^\mu$.

Main existence theorems

When $(\boldsymbol{K}_0, \leq)$ has joint embedding and amalgamation there is unique countable generic.

Theorem: Paolini [Pao21]

There is a generic model for $\boldsymbol{K}_0$; it is ω-stable with Morley rank ω.

Theorem (Baldwin-Paolini)[BP20]

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory T_μ.
If $\mu(\alpha) = k$, all lines in any model of T_μ have cardinality $k+2$.
Thus each model of T_μ is a Steiner k-system and $\mu(\alpha)$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. α).

Coordinatization by varieties of algebras

Coordinatizing Steiner Systems

Weakly coordinatized

A collection of algebras V "weakly coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Many results showing arithmetical conditions for the existence of $(S(t, k, n)$ that could be weakly coordinatized. Conclusion: t better be 2.

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Coordinatized

A collection of algebras V "coordinatizes" a class $\mathcal{S}$ of $(2, k)$-Steiner systems if
in addition the algebra operation is definable in the Steiner system.

Euclid, Descartes, Hilbert

Euclid used the Archimedean axiom to define proportionality of segments.

Descartes used Euclid's theorem of the 4th proportional to define multiplication.

Hilbert inverted the procedure to define multiplication (and so interpret a field) in any Desarguesian plane and then define proportionality.

Thus Descartes 'weakly coordinatizes' any plane (over 'Euclidean' subfields of the reals)

Hilbert coordinatizes all Euclidean planes (by Euclidean fields)

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality $q = p^n$ and an element $a \in F$, define a multiplication $*$ on F by

$$x * y = y + (x - y)a.$$

An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$ (a quasigroup).

Motive 1

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality k, each $A \in V$ determines a Steiner k-system. (The 2-generated subalgebras.)
And for prime power k, each strongly minimal Steiner k-system admits a weak coordinatization by V.

Question 1

Can this coordinatization be definable in the strongly minimal (M, R)?

Is there even a definable binary function?

Forcing a prime power

Theorem

If a k-Steiner system is weakly coordinatized, k is a prime power q^n.

Proof: As, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

Strongly minimal block algebras $(M, R, *)$

Theorem: Baldwin

For every prime power q there is a strongly minimal Steiner q-system (M, R) whose theory is interpretable in a strongly minimal block algebra $(M, R, *)$.

We modify the collection of R-structures $\boldsymbol{K}_{\mu}$ to a collection $\boldsymbol{K}_{\mu'}$ of $(R, *)$ structures so that the generic is a strongly minimal quasigroup that induces a Steiner system.

Theorem: Baldwin-Verbovskiy

But for $k > 3$ the coordinatization CAN NOT BE defined in the strongly minimal (M, R).

In searching for a proof we find a related problem.

Finer Classification: $\mathrm{dcl}(X)$, $\mathrm{acl}(X)$, sdcl

Motive 2

Definition

A theory T admits *elimination of imaginaries* if for every model M of T, for every formula $\varphi(\overline{x}, \overline{y})$ and for every $\overline{a} \in M^n$ there exists $\overline{b} \in M^m$ such that

$$\{f \in \mathrm{aut(M)} \mid \mathrm{f}|\overline{\mathrm{b}} = \mathrm{id}_{\overline{\mathrm{b}}}\} = \{\mathrm{f} \in \mathrm{aut(M)} \mid \mathrm{f}(\varphi(\mathrm{M}, \overline{\mathrm{a}},)) = \varphi(\mathrm{M}, \overline{\mathrm{a}})\}.$$

Let T_μ be theory of the basic Hrushovski example.

Question 2

Does T_μ admit elimination of imaginaries?

Baizanov, Verbovskiy

dcl^* and definability of functions

*-closure

$\mathrm{dcl}(X)$ and $\mathrm{acl}(X)$ are the definable and algebraic closures of set X.
$b \in \mathrm{dcl}^*(X)$ means $b \notin \mathrm{dcl}(U)$ for any proper subset of X.

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Fact

Let I be two independent points in M.
If $\mathrm{dcl}^*(I) = \emptyset$ then no binary function is $\emptyset$-definable in M.

That is, dcl is *generically 2-trivial*: if a, b are independent $\mathrm{dcl}(\{a, b\}) = \mathrm{dcl}(a) \cup \mathrm{dcl}(b)$.

Finite Coding

Definition

A finite set $F = \{\overline{a}_1, \ldots, \overline{a}_k\}$ of tuples from M is said to be coded by $S = \{s_1, \ldots, s_n\} \subset M$ over A if

$$\sigma(F) = F \Leftrightarrow \sigma | S = \mathrm{id}_S \quad \text{for any } \sigma \in \mathrm{aut(M/A)}.$$

We say $T = \mathrm{Th}(M)$ has *the finite set property* if every finite set of tuples F is coded by some set S over $\emptyset$.

If $\mathrm{dcl}^*(I) = \emptyset$, T does not have the finite set property.

dcl^* and elimination of imaginaries

Fact: Elimination of imaginaries

A theory T admits *elimination of imaginaries* if its models are closed under definable quotients.

ACF: yes; locally modular: no; combinatorial: no (here)

Fact

If T admits weak elimination of imaginaries then T satisfies the finite set property if and only T admits elimination of imaginaries.

Since every strongly minimal theory has weak elimination of imaginaries, we have

If a strongly minimal T has no definable binary functions it does not admit elimination of imaginaries.

Group Action and Definable Closure

Fix I as two independent points in the generic model M of T_μ.

2 groups

Let $G_{\{I\}}$ be the set of automorphisms of M that fix I setwise and G_I be the set of automorphisms of M that fix I pointwise.

Note that $\mathrm{dcl}^*(X)$ consists of those elements are fixed by G_I but not by G_X for any $X \subsetneq I$.

symmetric definable closure

The *symmetric definable closure* of X, $\mathrm{sdcl(X)}$, is those a that are fixed by every $g \in G_{\{X\}}$.
$b \in \mathrm{sdcl}^*(\mathrm{X})$ exactly when $b \in \mathrm{sdcl(S)}$ but $b \notin \mathrm{sdcl(U)}$ for any proper subset U of X.

No definable binary function/elimination of imaginaries: Sufficient

Lemma

Let $I = \{a_0, a_1\}$ be an independent set with $I \leq M$ and M is a generic model of a strongly minimal theory.

1. *If $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$ then I is not finitely coded.*
2. *If $\mathrm{dcl}^*(I) = \emptyset$ then I is not finitely coded and there is no parameter free definable binary function.*

No definable binary function/elimination of imaginaries

Theorem (B-Verbovskiy)

Suppose T_μ has only a ternary predicate (3-hypergraph) R. If T_μ is either in

1. Hrushovski's original family of examples
2. or one of the B-Paolini Steiner systems

and also satisfies:

1. $\mu \in \mathcal{U}$
2. If $\delta(B) = 2$, then $\mu(B/C) \geq 3$ except
3. $\mu(\boldsymbol{\alpha}) \geq 2$ (for linear spaces)

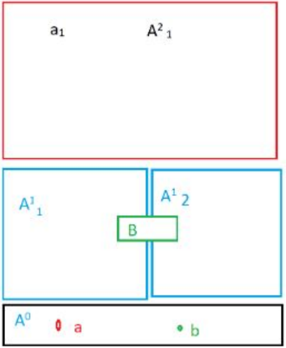
If I is an independent pair $A \leq M \models T_\mu$, then

(I) $\mathrm{dcl}^*(I) = \emptyset$

(II) T_μ does not admit elimination of imaginaries.

Counterexample: $\mathrm{dcl}^*(I) \neq \emptyset$

There are examples (Verbovskiy) of the Hrushovski construction and linear spaces with $\mu(C/B) = 2$ and $|B| = 2$ and $\mathrm{dcl}^*(I) \neq \emptyset$.



A^1_1 and A^1_2 are isomorphic and primitive over A^0. A^2_1 is 0-primitive over B and can be mapped into $A^0 A^1_1 A^1_2$ over B taking a_1 to a. Obviously this is not an isomorphism over A^0.

G-normal sets

Definition

$\mathcal{A} \subseteq M$ is G-normal if

1. $\mathcal{A} \leq M$
2. $\mathcal{A}$ is G-invariant
3. $\mathcal{A} \subset_{<\omega} \mathrm{acl}(I)$.

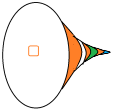
Fact

There are G-decomposable sets.
Namely for any finite U with $d(U/I) = 0$,

$$\mathcal{A} = \mathrm{icl(I \cup G(U))}$$

Constructing a G-decomposition

Linear Decomposition



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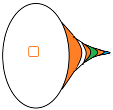
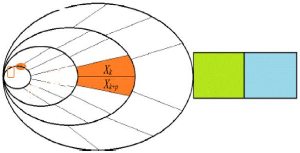


Tree Decomposition



Prove by induction on levels that $\mathrm{dcl}^*(I) = \emptyset$. $(\mathrm{sdcl}^*(\mathrm{I}) = \emptyset)$

First thought: All petals move

To show $\mathrm{dcl}^*(X) = 0$:

Trap: Not all petals are moved by G

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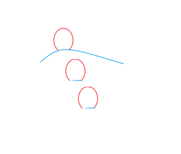


Dual Induction

Lemma

Assume that $\hat{T}_\mu$ satisfies $\mu(A/B) \geq 3$ for any good pair (A/B), where $\delta(B) = 2$. For $m \geq 1$,

1. $\dim_m$: $d(E) = 2$ for any G_I-invariant set $E \subseteq \mathcal{A}^m$, which is not a subset of $\mathcal{A}^0$.
2. moves_m: No $A^m_{f,k}$ is G_I-invariant.

Key step

$\mathrm{dim_m}$ and $moves_{m+1}$ implies $\mathrm{dim_{m+1}}$.

How μ matters

$G_{\{I\}}$-decomposition permits removal of the hypothesis:
If $\delta(B) = 2$, then $\mu(B/C) \geq 3$
and elimination of imaginaries still fails.

Theorem

[BV20] Let T_μ be the theory of the basic Hrushovski construction or a strongly minimal Steiner system
Let $I = \{a, b\}$ be a pair of independent points.

1. $\mathrm{sdcl}^*(\mathrm{I}) = \emptyset$.
2. Further, if for any good pair C/B,
   $\delta(B) = 2$ implies $\mu(C/B) \geq 3$,
   then T_μ is generically-discrete ($\mathrm{dcl}^*(I) = \emptyset$).

Consequently, no such T admits elimination of imaginaries.

In particular the Steiner systems (M, R) do not interpret quasigroups.

The n-ample hierarchy

n-Ample for $1 \leq n < \omega$ ([Eva11, BMPZ14] requires the existence of a sequence of tuple with a certain specified interaction of acl^{eq} and forking in a stable first order theory T.

Thus, a property of the entire theory rather than of the acl-geometry of particular strongly minimal sets.

Non-trivial flat strongly minimal sets

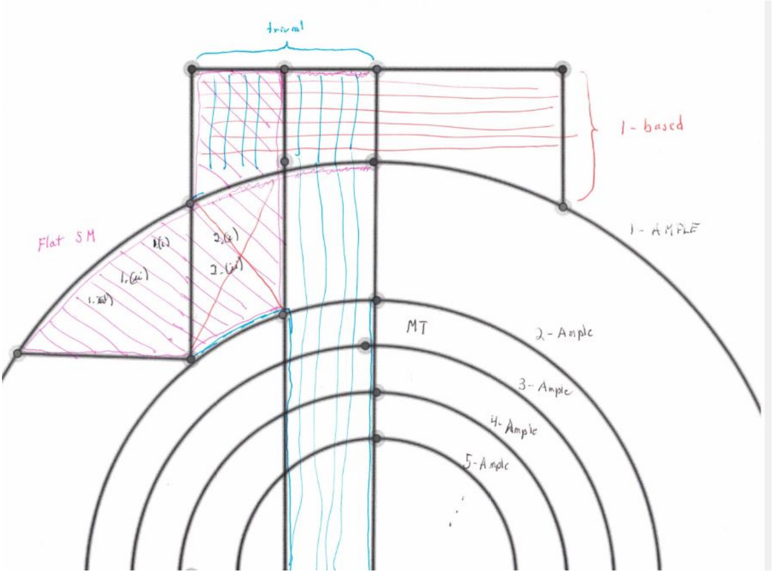
Flat: dimension computed by inclusion-exclusion.

non-trivial flat and so 1-ample but not 2-ample

1. Strongly minimal theories
   - (i) no binary function (generically 2-discrete)
   - (ii) no commutative binary function (no elim imag)
   - (iii) definable binary functions exist
     - (a) strongly minimal quasigroups: $(M, R, *)$ [Bal20] and an example of Hrushovski [Hru93, Proposition 18]
     - (b) strongly minimal theory that eliminates imaginaries (flat) INFINITE vocabulary) (Verbovskiy)
2. almost strongly minimal theories
   - (i) Non-Desarguesian projective planes definably coordinatized by ternary fields [Bal95]
   - (ii) almost strongly minimal buildings [Ten00a, Ten00b]

2-ample but not 3-ample sm sets (not flat) [MT19]

The Ample Hierarchy of stable theories



Further Questions

1. generic n-discrete and arbitrary finite relational language
2. explore the consequences in combinatorics and universal algebra
3. How do the examples of Andrews in computable model theory fit into the scheme? How does the Muller-Tent example fit in the Zilber classification?

References I

John T. Baldwin.
A field guide to Hrushovski constructions.
Report: http://www.math.uic.edu/~jbaldwin/pub/hrutrav.pdf.

John T. Baldwin.
Some projective planes of Lenz Barlotti class I.
Proceedings of the A.M.S., 123:251–256, 1995.

John T. Baldwin.
Strongly minimal Steiner Systems II: Coordinatizaton and Strongly Minimal Quasigroups.
in preparation, 2020.

References II

Silvia Barbina and Enrique Casanovas.
Model theory of Steiner triple systems.
Journal of Mathematical Logic, 201x.
https://doi.org/10.1142/S0219061320500105.

Andreas Baudisch, Amador Martin-Pizarro, and Martin Ziegler.
Ample hierarchy.
Fund. Math., 224(2):97–153, 2014.

John T. Baldwin and G. Paolini.
Strongly Minimal Steiner Systems I.
Journal of Symbolic Logic, 2020?
arXiv:1903.03541.

W. Burnside.
Groups of Finite Order.
Cambridge, 1897.

References III

John T. Baldwin and V. Verbovskiy.
Dcl in Hrushovski Constructions.
in preparation, 2020.

Trevor Evans.
Universal Algebra and Euler's Officer Problem.
The American Mathematical Monthly,, 86(6):466–473, 1976.

Trevor Evans.
Finite representations of two-variable identities or why are finite fields important in combinatorics?
In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 135–141. North-Holland, Amsterdam, 1982.

References IV

D. Evans.
An introduction to ampleness.
talks given at British Postgraduate Model Theory Conference, Oxford, January 2015; http://wwwf.imperial.ac.uk/~dmevans/OxfordPGMT.pdf, 2011.

Kitty Holland.
Model completeness of the new strongly minimal sets.
The Journal of Symbolic Logic, 64:946–962, 1999.

E. Hrushovski.
A new strongly minimal set.
Annals of Pure and Applied Logic, 62:147–166, 1993.

References V

M. Mermelstein.
Geometry preserving reducts of hrushovskis non-collapsed construction.
Masters thesis, 2013.

I. Muller and K. Tent.
Building-like geometries of finite morley rank.
J. Eur. Math. Soc., 21:3739–3757, 2019.
DOI: 10.4171/JEMS/912.

Gianluca Paolini.
New ω-stable planes.
Reports on Mathematical Logic, 2021.
to appear.

D.J.S. Robinson.
A Course in the Theory of Groups.
Springer-Verlag, 1982.

References VI

S. Świerczkowski.
Algebras which are independently generated by every n elements.
Fund. Math., 49:93–104, 1960/1961.

Sherman K Stein.
Homogeneous quasigroups.
Pacific Journal of Mathematics, 14:1091–1102, 1964.

Katrin Tent.
A note on the model theory of generalized polygons.
Journal of Symbolic Logic, 65:692–702, 2000.

Katrin Tent.
Very homogeneous generalized ngons of finite Morley rank.
Journal of London Math. Soc., 62:1–15, 2000.