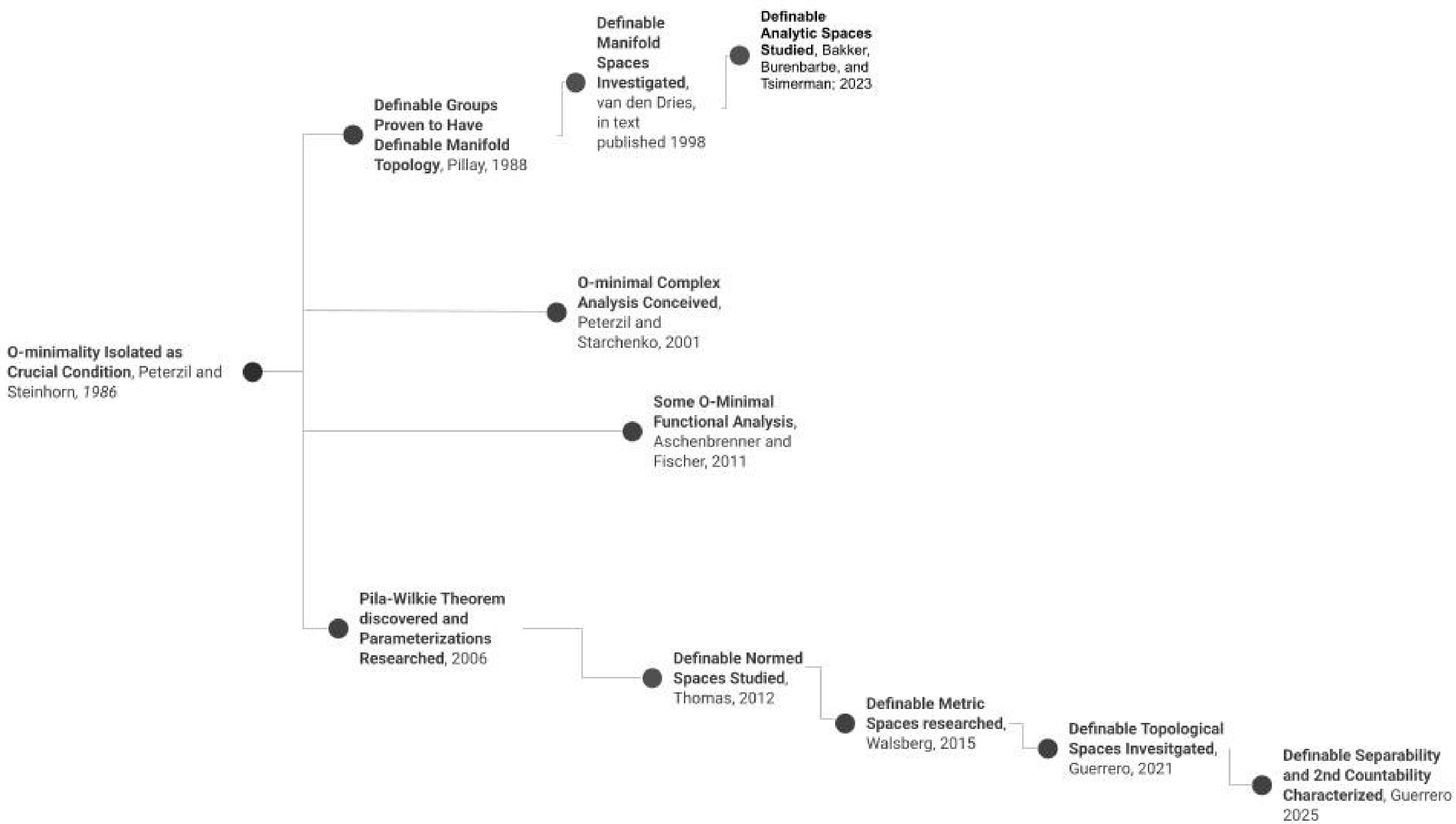
A Minimal Introduction to O-Minimality

By John Solak

Overview

1. O-Minimality and the Monotonicity Theorem
2. Cells, Dimension, and Cell Decomposition
3. Definable Choice, Curve Selection
4. Recent work of Pablo Guerrero

Some History of O-minimality



Basic Definitions

Definition- A structure $\mathcal{M}$ is called O-minimal iff every definable set $X \subseteq M$ can be composed into a finite union of points and intervals, i.e. $X = \bigcup_{i \leq m} a_i \cup \bigcup_{j \leq n} I_j$

Monotonicity Theorem: Let $f: (a, b) \to M$ be a definable (not necessarily continuous) function. Then there are breakpoints $a_0 \coloneqq a < a_1 < \cdots < a_k < a_{k+1} =: b$ in the interval (a, b) such that on each subinterval (a_j, a_{j+1}), the function f is continuous and is either constant or strictly monotonic (increasing or decreasing, not both).

Cells, Dimension, and Cell Decomposition

Definition Cells are defined inductively. First, some notation. Let the definable, continuous functions be denoted

$C(X) := \{f: M^n \to M: f \text{ is continuous and definable}\}.$

Let $C_\infty(X) := C(X) \cup \{-\infty, \infty\}$

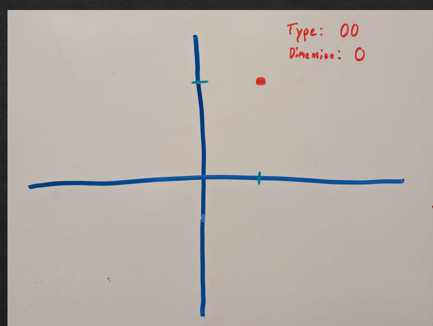
- In $M^0 = \{()\}$, containing just the empty tuple, the only cell is the singleton $\{()\}$, which has $dim(\{()\}) = 0$.
- In M^{n+1}, we define for each previous cell two new ones. Let $X \subseteq M^n$ be a cell of dimension k. Then, for any $f \in C(X)$, the graph $\Gamma(f)$ is a cell of dimension k. Second, for any $g, h \in C_\infty(X)$, the space between their graphs,

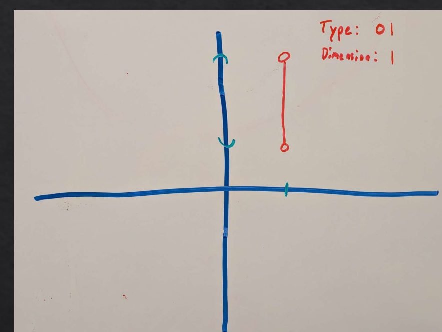
$$\{(\overline{x}, y) \in M^{n+1}: \overline{x} \in X, g(\overline{x}) < y < h(\overline{x})\}$$

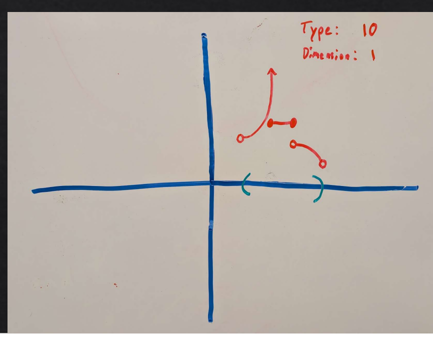
is a cell with dimension $k + 1$.

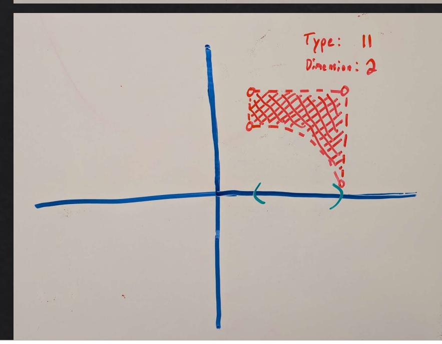
Examples of Cells

In M, the only *types* of cell are a point or an interval (with $\pm\infty$ allowed in the interval). In M^2, there are four different *types* of cells, and we classify them by whether each coordinate increases the dimension (given by a 1 in that coordinate) or maintains the dimension (given by a 0 in that coordinate). Here are examples of each type.









Cell Decomposition Theorem

Theorem:

1. Given any definable sets $A_1, \dots, A_k \subseteq M^n$, there is a finite decomposition of M^n into cells partitioning each of the A_i.
2. For each definable set $A \subseteq M^n$ and every definable function $f: A \to M$, there is a finite decomposition of M^n into cells B_i such that for every i, $f|_{B_i}$ is continuous.

Remark: point 2) can be strengthened to say that $f|_{B_i}$ is, in each coordinate, either strictly increasing, strictly decreasing, or independent, of that coordinate.

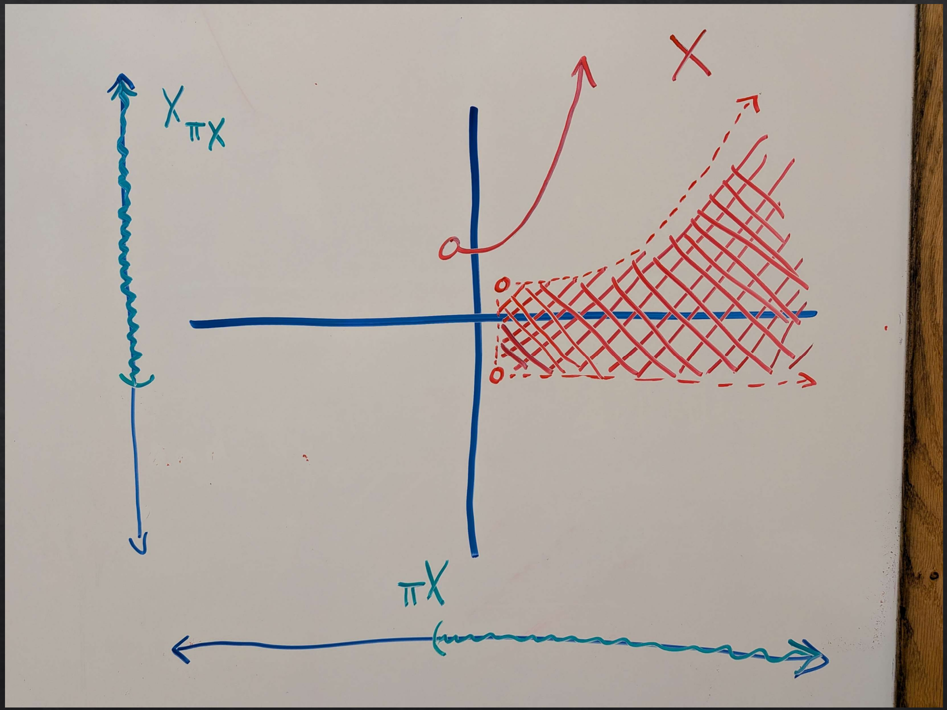
Definable Choice and Curve Selection pt. 1

Context: o-minimal expansion of an ordered, divisible, Abelian group.

Definable Representative Let $X \subseteq M^n$ be a definable set. Let 1 denote a fixed, positive element.

1. If $n = 1$, let $e(X)$ be $inf(X)$, if $inf(X) \in X$. Else, let $a := inf(X), b := sup\{x \in X: (a, x) \subseteq X\}$. There are four cases:
   1. $a = -\infty, b = +\infty$, then $e(X) := 0$
   2. $a = -\infty, b < \infty$, then $e(X) := b - 1$
   3. $-\infty < \text{a},\ b = +\infty$, then $e(X) := a + 1$
   4. $-\infty < a, b < \infty$, then $e(X) := \frac{a+b}{2}$
2. For $X \subseteq M^{n+1}$, let πX denote the projection of X onto the first n coordinates. For any $b \in \pi X$, let X_{b} denote the fiber $\{c \in M: (b, c) \in X\}$. Let $a := e(\pi X)$. Then, define $e(X) := \left(\text{a}, e(X_{\mathrm{a}})\right)$.

Example of Definable Choice



Definable Choice and Curve Selection pt. 2

Proposition

1. (Definable Skolem Functions) Let $X \subseteq M^{m+n}$ be a definable set (parameters in M^n) and let $\pi: M^{m+n} \to M^m$ be the usual projection. Then, there is a definable map $f: \pi(X) \to M^n$ with $\Gamma(f) \subseteq X$

Proof: For $\overline{x} \in \pi X$, *define* $f(\overline{x}) := e(X_{\overline{x}})$.

Definable Choice and Curve Selection pt. 3

Proposition (Curve Selection) Let $X \subseteq M^n$ be a definable set. Then, for any $a \in Cl(X) \setminus X$, there is some $\epsilon \in M$ with $\epsilon > 0$ and a definable, continuous, injective function $\gamma: (0, \epsilon) \rightarrow X$ with $lim_{t\rightarrow 0}\gamma(t) = a$.

No definable, injective curve for an isolated point

One such non-self-intersecting curve.

Recent Work of Pablo Guerrero

Definition Let $X \subseteq M^{\mathrm{n}}$ be a definable set and let $\mathcal{T}$ be a topology on X. We say that $(X, \mathcal{T})$ is a definable topological space iff there exists a definable basis i.e. a uniformly definable family of definable, open sets, $\{U_b : b \in B\} \subseteq M^{n+\mathrm{m}}$, where $B \subseteq M^m$ is also a definable set.

Definition A definable topological space is called definably separable/has the definable chain condition iff there does not exist an infinite, definable family of pairwise disjoint, open sets.

Definition A definable topological space is called definably second countable iff there exists a definable basis $\{U_b : b \in B\}$ such that for all $x \in X$ and all $b \in B$, if U_b is a neighborhood of x, then,

$$dim\{c \in B : x \in U_c \subseteq U_b\} = dim(B)$$

His Results

Theorem Let $\mathcal{M}$ be an o-minimal expansion of $(\mathbb{R},<)$. Let $(X,\mathcal{T})$ be a definable topological space. Then, the following are equivalent:

1. $(X,\mathcal{T})$ is definably separable
2. $(X,\mathcal{T})$ is separable
3. $(X,\mathcal{T})$ has the countable chain condition

Theorem Let $\mathcal{M}$ be an o-minimal expansion of $(\mathbb{R},<)$. Let $(X,\mathcal{T})$ be a definable topological space. Then, $(X,\mathcal{T})$ is definably second-countable iff it is second-countable.

Thank you!

Bibliography

- *Tame Topology and O-minimal Structures,* by Lou van den Dries
- *Definable Separability and Second-Countability,* by Pablo Andújar Guerrero
- *Math 509 Lecture Notes: Model Theory of the Real Numbers*, by Alex Kruckman