Geometry of Simplexes

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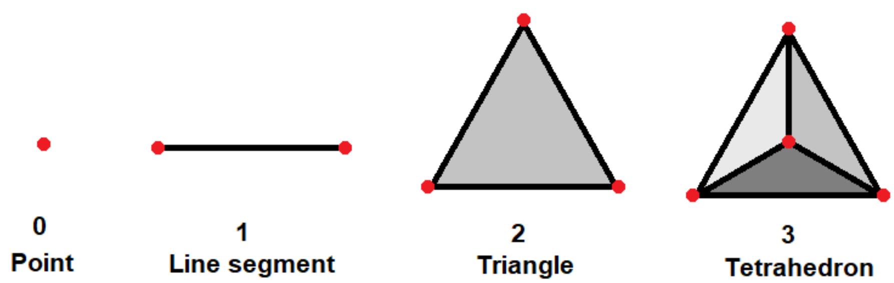
April 1, 2019

Note to MATH 300 students

Hey guys. I just hacked out this thing in about 2 1/2 hours to give you an idea of how I want your essay to look. Please don't plagiarize me too heavily, this is only a model. —Jon

1 Introduction

Consider the following sequence of geometric objects, known as *simplexes*[1].



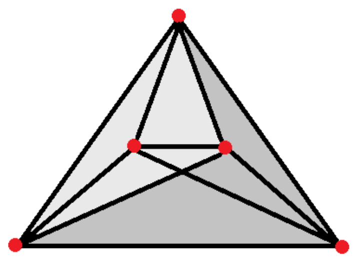
The numbers indicate the dimension of each object. For example, the tetrahedron is a 3-dimensional simplex, or *3-simplex*. Each N-simplex is generated by taking the convex hull of the preceding simplex plus one additional point, positioned in N-space so that it is affine-independent with the preceding simplex. The sequence continues into higher dimensions, so N-simplexes can be defined for arbitary $N \in \mathbb{N}$.

Polytopes in general, and simplexes in particular, can be broken down into constituent n-dimensional *elements*. These are the object's vertices, edges, faces,

[1]The plural *simplices* is also used.

and cells, corresponding to $n = 0, 1, 2,$ & 3, respectively; n-elements are also defined for higher n. For example, inspection of the figure reveals the tetrahedron to have four vertices, six edges, four faces, and one cell.

Although physical models of N-simplexes exist only for $N = 0, 1, 2,$ & 3, much insight can be gleaned into the structure of higher-dimensional simplexes by enumerating their n-elements. For example, the 4-simplex, or *pentachoron*, has five vertices, ten edges, ten faces, five cells, and one 4-element. The image below shows a plane projection of the pentachoron's edge-lattice.



4
Pentachoron

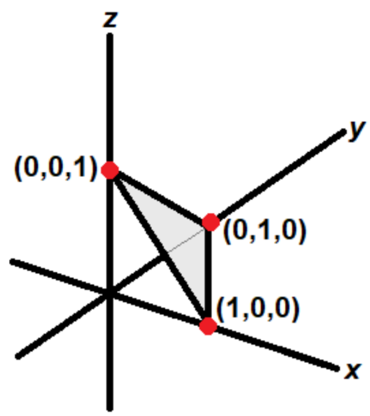
The goal of this paper is to develop formulas which enumerate the n-elements of an N-simplex. In section 2, basic terms are defined. In section 3, two formulas are proven, one explicit, the other recursive. Additionally, a table of values is provided for N and n up through 5. Section 4 concludes with some reflections on related polytopes.

2 Definitions

We begin by defining a specific instance of the N-simplex, to serve as a model for the general case.

DEFINITION 1. The *standard N-simplex* is the convex hull in $\mathbb{R}^{N+1}$ of all points with all coordinates zero, except for a single coordinate with value 1. These $(N+1)$ points are the *vertices* of the simplex.

EXAMPLE. The equilateral triangle in $\mathbb{R}^3$ with corners at $(1,0,0)$, $(0,1,0)$, & $(0,0,1)$, is the standard 2-simplex. See figure.



DEFINITION 2. An *N-simplex* is a figure in $\mathbb{R}^k$ $(k \geq N)$ that is geometrically similar to the standard N-simplex.

EXAMPLE. Any equilateral triangle in the plane, or in 3-space, or in any higher space, is a 2-simplex.

Next we give a useful definition for the n-elements of an N-simplex. Note that this definition does not apply to polytopes other than simplexes. A more general definition may be found in the literature.

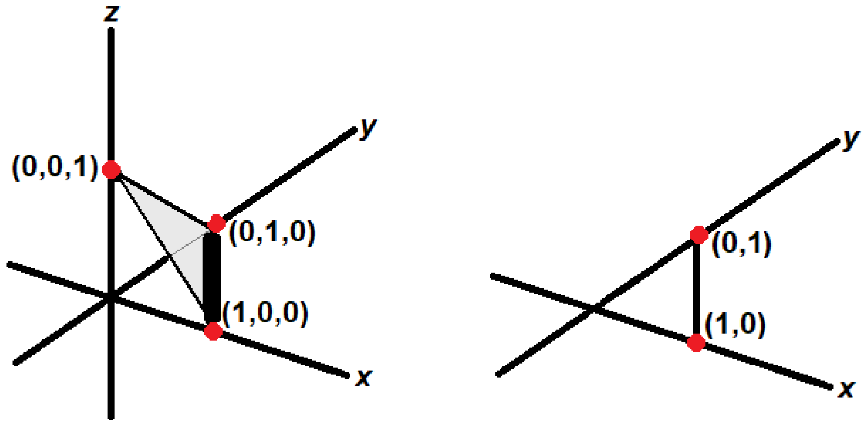
DEFINITION 3. The convex hull of any $(n+1)$ vertices of a simplex is a *n-element* of the simplex. For $n = 0, 1, 2, 3$, the n-elements are called, respectively, *vertices, edges, faces*, and *cells*.

LEMMA. An n-element of the standard N-simplex is itself an n-simplex.

PROOF.

The n-element in question has $(n+1)$ vertices, which lie in an $(n+1)$-dimensional coordinate hyperplane $P \subset \mathbb{R}^{N+1}$. Regard P as $\mathbb{R}^{n+1}$ (by omitting those coordinates which take no value but 0 among the vertices). Within this hyperplane, the n-element is the standard n-simplex. $\square$

EXAMPLE. The bold edge in this standard 2-simplex is itself the standard 1-simplex within the xy-plane.



3 Main Theorems

The main focus of this study is the enumeration of elements in a simplex, denoted by the following function:

DEFINITION 4. For $N, n \in \mathbb{N}$, let $S(N, n)$ denote the number of n-elements in an N-simplex. (We may assume the simplex is the standard N-simplex.)

Visual inspection of the first few simplexes produces this table of values for $S(N, n)$. Data for dimensions 4 and 5 are also included. Each row corresponds to an N-simplex, while each column corresponds to an n-element.

		Little simplex (n)					
		0	1	2	3	4	5
	0	1	0	0	0	0	0
	1	2	1	0	0	0	0
Big	2	3	3	1	0	0	0
simplex	3	4	6	4	1	0	0
(N)	4	5	10	10	5	1	0
	5	6	15	20	15	6	1

Thus, for example, the "6" in row $N = 3$, column $n = 1$, indicates that a tetrahedron has six edges, $S(3, 1) = 6$.

The pattern of numbers in this table is known as *Pascal's triangle*, and is described by either of the two formulas presented below.

THEOREM 1.

$$S(N,0) = N+1 \tag{1}$$
$$S(0,n) = 0 \quad (n \geq 1) \tag{2}$$
$$S(N,n) = S(N-1,n) + S(N-1,n-1) \tag{3}$$

Equation 1 describes the leftmost column of the table. Equation 2 describes the top row of the table. Equation 3 states that each entry in the table is the sum of the entry above it and the entry above and to the left.

PROOF.

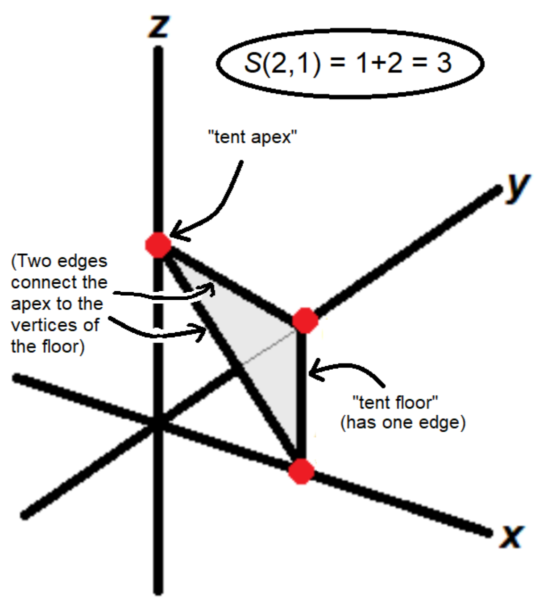
Equation 1 states that an N-simplex has $N+1$ vertices, as observed in Definition 1.

Equation 2 states that a 0-simplex has no elements of any dimension beside 0. This is evident, since the 0-simplex is itself a single point.

The standard N-simplex is composed of an $(N-1)$-simplex (a "tent floor" whose points have final coordinate 0), a vertex (a "tent apex" with final coordinate 1), and, for each $(n-1)$-element of the tent floor, an n-simplex connecting that element to the tent apex.

Equation 3 states that n-elements of the standard N-simplex $S(N,n)$ include the n-elements of the tent floor $S(N-1,n)$, and the n-elements connecting the tent apex to each $(n-1)$-element of the tent floor $S(N-1,n-1)$. □

The last sentence of the preceding proof is illustrated in the figure, which computes that a triangle has 3 edges.



Moving now from recursive to explicit:

THEOREM 2.

$$S(N,n) = \binom{N+1}{n+1}.$$

PROOF.

As noted in the proof of Theorem 1, an N-simplex has $N+1$ vertices. By Definition 3, an n-element is the convex hull of any $n+1$ of these vertices. There are $\binom{N+1}{n+1}$ ways of selecting these vertices. No two choices can have the same convex hull, since the points are affine-independent. Thus the correspondence is exact. □

4 Conclusion

The function $S(N,n)$ enumerating elements of a simplex is nothing other than the familiar binomial coefficients appearing in Pascal's triangle. This gives geometric significance to a function typically associated with algebra and combinatorics, and hopefully enhances the reader's intuition for higher-dimensional geometry.

The simplex is one of three types of regular polytope that exist in every dimension of space. The other two, cubes and orthoplexes, also have formulas enumerating their respective elements. A similar approach to that used here can be applied to those cases (although our definition of "element" needs to be revised to suit those cases).