Essay 2: Simplexes, Cubes, Orthoplexes

Math 300, Fall 2018, Schneider

1 Content

In this essay you will systematically construct a coherent mathematical theory of N-dimensional polytopes. You will formally define a number of geometric objects in $\mathbb{R}^n$ and prove some enumerative formulas about their structure. Be sure to give concrete examples in dimensions 1, 2, & 3 via graphical illustrations on which the reader will base their higher-dimensional intuition.

Please define, explain, and illustrate the following terms:

- N-dimensional convex polytope
- n-dimensional face of a convex polytope
- N-simplex
- N-cube
- N-orthoplex.

Definitions should be presented as their need in the essay arises. Rather than simply declaring a new definition "out of the blue", first introduce the concept in the context of the discussion already underway.

Be sure to state the shapes of the faces of a simplex, cube, and orthoplex. Justify these statements using your definitions. For each of these three shapes, give both recursive and explicit formulas for the number of n-faces. Give a table of values for the face-counts, and prove your formulas.

2 Style and expectations

Your audience is at least as mathematically sophisticated as yourself. Write in the style of an advanced college math textbook or a research article. I will be looking for well-organized exposition of a coherent mathematical theory. Motivate the theory, define your terms, and prove your theorems. Your proofs should explicitly use your definitions.

Assume your reader is familiar with general properties of $\mathbb{R}^n$ (such as its coordinate system, the notion of convexity, the dimension of convex sets, etc.), and with common shapes of dimension ≤ 3 (polygons, the platonic solids), but not polytopes of dimension ≥ 4.

You may reference our class lectures, any credible book, article, or website, or the notes below. Cite published sources in a bibliography section.

3 Optional topics

Tangential to your main development, you may wish to include coverage of related topics such as

- the five regular polyhedra
- the six regular polychora
- vertex figures and Schläfi symbols
- duality
- another topic of your own choosing or invention.

Include this coverage only if it does not derail the main discussion.

4 Math

These notes are intentionally condensed, abridged, and abbreviated. Expand them into human-readable form in your essay.

4.1 Polytopes and faces

There are various ways of defining *N-polytope* and *n-face*. Definitions can be found in various sources, such as Mathworld; or, you may compose your own definitions (so long as they are equivalent to the standard meanings). Choose definitions that are useful to your proofs later in the paper.

4.2 Regular polytopes

You do not need to give rigorous coverage of *symmetry* and *regularity*, but you may wish to mention why cubes, simplexes, and orthoplexes are significant in light of this notion, giving motivation to this study.

An N-polytope X is *regular* if, for every $n \leq N$ and every pair of n-faces P and P', the polytope has a symmetry taking P to P'.

In dimension 1, a line segment is the only possible polytope. In dimensions 2, 3, & 4, there are many regular polytopes, including but not limited to the simplex, cube, and orthoplex. In dimension 5 and up, these three are the *only* regular polytopes. Thus it is worthwhile to study these three types of polytope specifically.

4.3 Definitions

The **standard N-cube** is the subset of $\mathbb{R}^N$ whose vertices are of the form $(x_1, \ldots, x_N)$ with each $x_i = \pm 1$. Any figure in $\mathbb{R}^m$ $(m \geq N)$ that is geometrically similar to the standard N-cube is an **N-cube**.

The standard N-**simplex** is the subset of $\mathbb{R}^{N+1}$ whose vertices are of the form $(x_1, \ldots, x_{N+1})$ with one coordinate equal to 1 and N coordinates equal to 0.

The standard N-**orthoplex** is the subset of $\mathbb{R}^N$ whose vertices are of the form $(x_1, \ldots, x_N)$ with one coordinate equal to ± 1 and $N-1$ coordinates equal to 0.

Explain why these definitions satisfy your definitions of polytopes and vertices. Pictorially illustrate each definition for dimensions $N = 0$, 1, 2, 3, & 4. State, and later prove, what is the shape of an n-face on a cube, simplex, and orthoplex.

4.4 Formulas

$$\begin{aligned} C(N,n) &= \text{number of } n\text{-faces in an } N\text{-cube.} \\ \text{Recursive: } C(N,n) &= 2 \cdot C(N-1,n) + C(N-1,n-1) \\ \text{Explicit: } C(N,n) &= 2^{N-n} \cdot \binom{N}{n} \end{aligned}$$

$$\begin{aligned} S(N,n) &= \text{number of } n\text{-faces in an } N\text{-simplex.} \\ \text{Recursive: } S(N,n) &= S(N-1,n) + S(N-1,n-1) \\ \text{Explicit: } S(N,n) &= \binom{N}{n} \end{aligned}$$

$$\begin{aligned} O(N,n) &= \text{number of } n\text{-faces in an } N\text{-orthoplex.} \\ \text{Recursive: } O(N,n) &= O(N-1,n) + 2 \cdot O(N-1,n-1) \\ \text{Explicit: } O(N,n) &= 2^{N} \cdot \binom{N}{n} \end{aligned}$$

Additionally, you should "initialize" the recursive formulas by stating (and justifying) values for $C(N,0)$ and $C(0,n)$, and likewise for S and O.

4.5 Tables

You should include three tables of values. Here's a template.

		Little cube (n)					
		0	1	2	3	4	5
Big cube (N)	0						
	1						
	2						
	3						
	4						
	5						

4.6 Proofs

You have several theorems to state and prove. Here are a few to get you started.

Lemma:
In the standard N-cube, an n-face's vertices are precisely those that match each other in exactly $N - n$ of their coordinates $(x_1, \ldots, x_N)$.
Proof:
The vertices so specified all lie on a common n-hyperplane, and no other vertex of the N-cube lies on that hyperplane. $\square$

Theorem:
The faces of the standard N-cube are themselves cubes.
Proof:
The vertices of an n-face agree in $N-n$ coordinates. In each of these coordinate directions, translate $\mathbb{R}^N$ so that these values go to 0. This sends the n-face to the standard n-cube in $\mathbb{R}^n$ (identified as a coordinate hyperplane of $\mathbb{R}^N$). $\square$

Theorem:
The number of n-faces on an N-cube is given by

$$C(N,n) \;=\; 2^{N-n} \cdot \binom{N}{n}$$

Proof:
(Presented in lecture)

Theorem:
The number of n-faces on an N-cube satisfies

$$\begin{aligned} C(N,0) &= 2^N \\ C(0,n) &= 0 \quad (n \geq 1) \\ C(N,n) &= 2 \cdot C(N-1,n) + C(N-1,n-1) \end{aligned}$$

Proof:
(Presented in lecture)