

Boolean Algebra

Louis H. Kauffman

1 Introduction

The purpose of these notes is to introduce Boolean notation for elementary logic. In this version of things we use 0 for F (False) and 1 for T (True). Negation is represented by placing a bar (or overline) across an expression. Thus we write

$$\sim A = \overline{A}.$$

The overline can go across a complex expression. Thus we have

$$\sim (A \vee B) = \overline{A \vee B}.$$

In Boolean notation, we use multiplication for “and” and addition for “or”. Thus, we write

$$A \vee B = A + B$$

and we write

$$A \wedge B = AB.$$

Note, for example, how DeMorgan’s Law transcribes in the Boolean notation:

$$\sim (A \vee B) = \overline{A + B},$$

$$\sim A \wedge \sim B = \overline{A} \overline{B},$$

Thus

$$\overline{A + B} = \overline{A} \overline{B}.$$

Remark on Boolean Arithmetic. The Boolean values of 0 and 1 form a very simple arithmetic with the following rules.

1. $\overline{0} = 1$
2. $\overline{1} = 0$
3. $0 + 0 = 0$
4. $0 + 1 = 1 + 0 = 1$
5. $1 + 1 = 1$ (watch out for that one!)
6. $1 \times 1 = 1$
7. $0 \times 0 = 0 \times 1 = 1 \times 0 = 0$

It is a remarkable fact that *all* the identities in basic logic and Boolean algebra are simply the identities that are true about this arithmetic. For example, the identity $\overline{\overline{A}} = A$ can be interpreted as saying that "For any element A of the Boolean Arithmetic $\overline{\overline{A}} = A$ ". And you only have to check that this is true for $A = 0$ and for $A = 1$ to prove it.

Note that in Boolean notation we have

$$(A \Rightarrow B) = \overline{A}B.$$

This makes a compact notation for implication.

Here is a list of identities that you are familiar with, written in Boolean notation. You can make these into exercises by either translating them into logic or set notation and seeing that they are true via truth tables or Venn diagrams *or* you can verify that they are true in Boolean arithmetic.

1. $\overline{\overline{A}} = A$
2. $A + B = B + A$
3. $(A + B) + C = A + (B + C)$
4. $AB = BA$
5. $(AB)C = A(BC)$
6. $A + A = A$
7. $1 + A = A$
8. $0 + A = A$
9. $A + \overline{A} = 1$
10. $A \overline{A} = 0$
11. $A(B + C) = AB + AC$
12. $A + BC = (A + B)(A + C)$
13. $A + B = \overline{\overline{A} \overline{B}}$

2 Exclusive Or

Here we define $A\Delta B$ to mean "A or B but not both A and B". Specifically we define operation by the equation

$$A\Delta B = A\overline{B} + \overline{A}B.$$

Here is an example of a derivation of the formula

$$\overline{A\Delta B} = \overline{A} \overline{B} + AB.$$

$$\begin{aligned}
\overline{A\Delta B} &= \overline{\overline{AB} + \overline{AB}} \\
&= \overline{\overline{AB}} \overline{\overline{AB}} \\
&= (\overline{A} + B)(A + \overline{B}) \\
&= \overline{A} \overline{B} + AB
\end{aligned}$$

We do this by using the DeMorgan Law twice. With this fact in hand, we can prove other things about exclusive or. For example, we can show that $(A\Delta B)\Delta B = A$ as follows.

$$\begin{aligned}
&(A\Delta B)\Delta B \\
&= (A\Delta B) \overline{B} + \overline{A\Delta B} B \\
&= (\overline{AB} + \overline{AB}) \overline{B} + (\overline{A} \overline{B} + AB) B \\
&= (\overline{AB} \overline{B} + \overline{AB} \overline{B}) + (\overline{A} \overline{B} B + AB B) \\
&= (\overline{AB} + 0) + (0 + AB) \\
&= \overline{AB} + AB \\
&= A(\overline{B} + B) \\
&= A(1) \\
&= A.
\end{aligned}$$

We'll stop here for now.