Show on the Wirtinger generators,

$c = b^{-1}ab \Longrightarrow \langle c\rangle = (1-t^{-1})\langle a\rangle + t^{-1}\langle b\rangle$

$c = bab^{-1} \Longrightarrow \langle c\rangle = (1-t)\langle a\rangle + t\langle b\rangle$.

(iii) Show that a homomorphism such as in (ii) exists exactly for those t that are <u>roots</u> of $\Delta_K(t) = 0$.

(iv) Compute the affine representations of the trefoil and of the figure-eight knot.

XII
CYCLIC BRANCHED COVERINGS

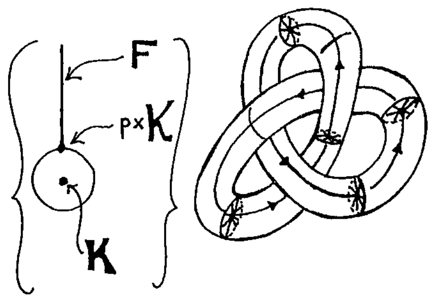
In Chapter IX we illustrated Seifert's approach to branched covering spaces. In this chapter we turn to th spaces in a more systematic, and partially four-dimensi manner. Let K be an oriented knot or link in the oriented sphere S^3. Then there is a homomorphism $\phi : \pi_1(S^3-K) \longrightarrow Z$ defined by the equation $\phi(\alpha) = \ell k(\alpha,$ where this denotes the sum of the linking numbers with individual components of K. Let $\phi_b : \pi_1(S^3-K) \longrightarrow Z/nZ$ denote the composition of ϕ with the surjection $Z \longrightarrow Z/nZ$. The <u>n-fold cyclic covering of K</u> is by definition the covering space of S^3-K that corresponds this representation.

These coverings can be described, as we have done i Chapter IX, by first cutting S^3 along a spanning surfa for F, and then pasting n copies together cyclically end-to-end. Another useful description is as follows: There exists a mapping $\psi : S^3-K \longrightarrow S^1$ that induces the map ϕ on the fundamental groups. (The proof involves some obstruction theory.) This map ψ is unique up to homotopy, and it can be adjusted so that

(1) For a small tubular neighborhood of K, $N(K)$,

there is a product structure $N(K) \cong K \times D^2$ so that $\psi|\partial N(K)$ is equivalent to projection on $S^1 = \partial D^2$.

(2) There is a point $p \in S^1$ such that $\psi^{-1}(p) \cap \partial N(K)$ is $\cong$ to $K \times p$ for this $p \in \partial D^2 = S^1$. That is, $\psi^{-1}(p)$ is a parallel copy of K. And $F = \psi^{-1}(p) \cap E_K$ is a connected, oriented, spanning surface for this (parallel copy of) K. Here E_K denotes the closure of the exterior of this tubular neighborhood.



We shall call a $\psi : S^3\text{-}K \longrightarrow S^1$ satisfying (1) and (2) above a <u>good representation for K</u>.

Some knots have especially good representation in that ψ can be a <u>fibering</u>. That is, $\psi^{-1}(p)$ gives a spanning surface for all p, and ψ^{-1}(small neighborhood(p)) $\cong$ Surface × neighborhood(p) for all p. Such a knot is called a <u>fibered knot</u>. Some examples of fiberings will appear shortly.

To return to coverings, let $\psi : S^3\text{-}K \longrightarrow S^1$ be a
representation for K. Contemplate the following diagr

$$\begin{array}{ccc} \mu_n(K) = \widetilde{S^3\text{-}K} & \longrightarrow & S^1 \\ \downarrow \pi & & \downarrow \lambda_n \\ S^3\text{-}K & \xrightarrow{\ \psi\ } & S^1 \end{array}$$

where $\lambda_n(z) = z^n$ (complex multiplication on the circl

The meaning of the diagram is that $\mu_n(K)$ is defi
to be the <u>pull-back of</u> λ_n <u>by</u> ψ. In general,

$$\begin{array}{ccc} P & \xrightarrow{\ \pi_Y\ } & Y \\ \downarrow \pi_X & & \downarrow b \\ X & \xrightarrow{\ a\ } & Z \end{array} \qquad P = \{(x,y) \in X \times Y | a(x) = b(y)\}$$

This diagram defines the pull-back (or equalizer) of tw
mappings.

$S^3\text{-}K$ is the n-fold cyclic cover of K. Notice t
in general

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & S^1 \\ \downarrow \pi & & \downarrow \\ X & \xrightarrow{\ \psi\ } & S^1 \end{array} \qquad \begin{array}{c} z \\ \downarrow \\ z^n \end{array}$$

will produce an n-fold covering of X:

$$\tilde{X} = \{(x,z) \mid \psi(x) = z^n\},$$

$$\pi^{-1}(X) = \{(x,z) \mid z^n = \psi(x)\}$$

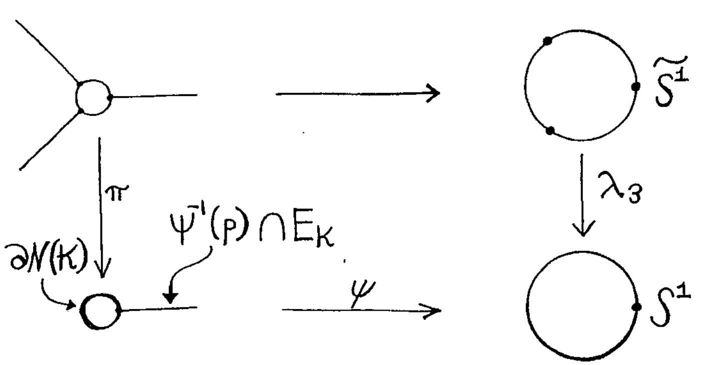
Thus we produce the n-fold covers by solving equations of the form $z^n = k$.

Branched covers follow similarly. Let E_K be the exterior of K as described above. Let $\partial E_K = K \times S^1$ via the given identification of $N(K)$ with $K \times D^2$. Then $\pi : \tilde{E}_K \longrightarrow E_K$ restricts on the boundary to $K \times S^1 \longrightarrow K \times S^1$, $(x,z) \longrightarrow (x,z^n)$. Since this extends over $K \times D^2$ (same formula), we get

$$M(K) = \tilde{E}_K \cup \widetilde{(K{\times}D^2)} \xrightarrow{\pi} E_K \cup (K{\times}D^2) = S^3.$$

This is the n-fold cyclic branched covering space, branched along K.

All we have done so far, is to redescribe our previous construction. But it is important to keep making the comparison. Thus in the covering construction, splitting along F is included in that $F = \psi^{-1}(p)$ and $S^1 \xrightarrow{\lambda_n} S^1$ can be described by splitting S^1 on a point



Of course, the infinite cyclic cover is obtained using $\lambda_\infty : \mathbb{R} \longrightarrow S^1$, $\lambda_\infty(r) = e^{2\pi i r}$.

We can, if we like, use a mapping $\bar{\psi} : S^3 \longrightarrow D^2$ that $\bar{\psi}$ is differentiably transverse to $0 \in D^2$ and $\bar{\psi}^{-1}(0) = K$. Then branched covers are directly formed the pull-backs:

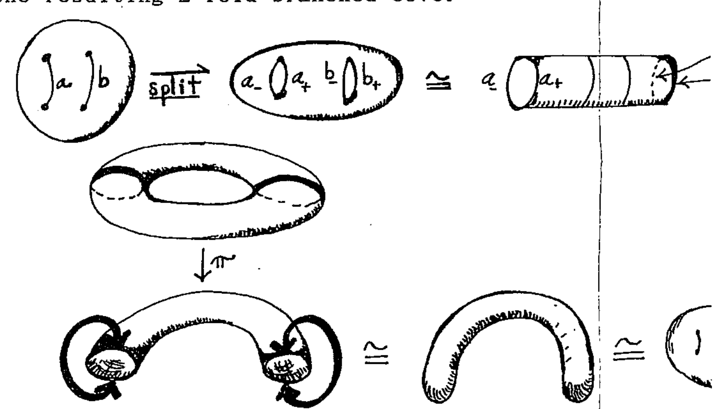
$$\begin{array}{ccccc} \tilde{S}^3 & \longrightarrow & D^2 & & z \\ \downarrow & & \downarrow & & \downarrow \\ S^3 & \xrightarrow{\bar{\psi}} & D^2 & & z^n \end{array}$$

RIEMANN SURFACE DIGRESSION

Take

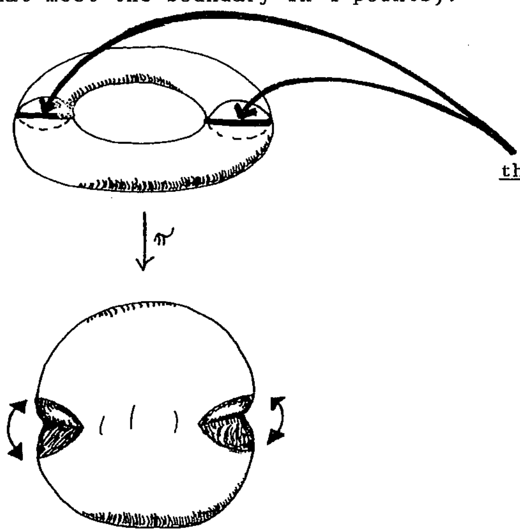
$$\begin{cases} K = 4 \text{ points in } S^2. \\ F = 2 \text{ intervals joining 2 pairs of poin} \end{cases}$$

Draw a picture of S^2 split along F, and a picture the resulting 2-fold branched cover:

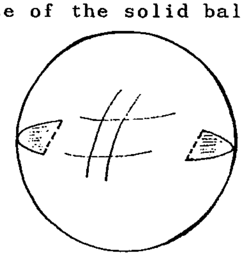


Thus the torus is the 2-fold branched cover of S^2 along 2 points.

And by the same construction, we see that the solid torus $D^2 \times S^1$ is the 2-fold branched cover of D^3 along two arcs (that meet the boundary in 4 points).



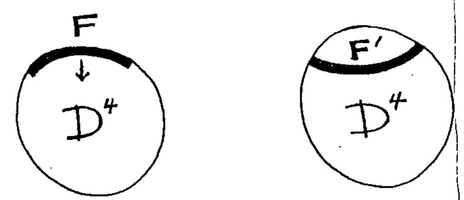
Here you see the 3-ball split along two disks. Each disk bounds one of the interior arcs that form the branch set. (Another part of the boundary of each disk is an arc on the surface of the solid ball.)



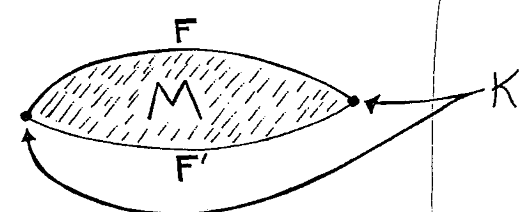
branch set in the 3-ball

This bit of geometry will be useful to us for a number constructions, but first we generalize it! The solid as 2-fold branched cover of D^3 is our first example branched covering of a manifold with boundary.

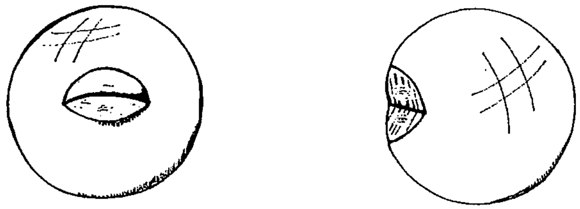
Moving back up one dimension, consider a connecte oriented surface $F \subset S^3$. Push this surface into D^4, keeping its boundary fixed in S^3.



In the diagram, F' denotes the pushed-in surfac As the surface is pushed in, it traces out a 3-manifol that is homeomorphic to $F \times I$ with $K \times I$ ($K = \partial F$) collapsed to a single copy of K.



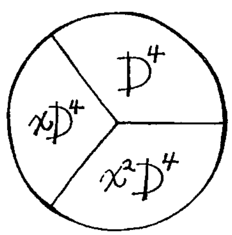
Now imagine splitting D^4 along M. The result 4-ball again with the surface 3-sphere now containing and M_- with $M_+ \cap M_- = F'$.



Here we have drawn a dimensionally-reduced diagram. Notice that this diagram is isomorphic to our pictures for branched covers of spheres and balls in dimensions two and three.

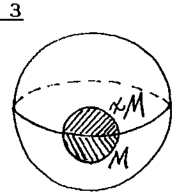
With this model in mind, we can describe cyclic branched covers of D^4 branched along F' (the push-in of F).

Thus, if $N_a(F) = N_a$ denotes the a-fold cyclic cover of D^4, branched along F', then we make N_a from a-copies of D^4, labelled $D^4, xD^4, x^2D^4, \cdots, x^{a-1}D^4$. Here the symbol x has order $a : x^a = 1$.



On ∂D^4 label $\begin{cases} M_+ & \text{as } xM \\ M_- & \text{as } M. \end{cases}$

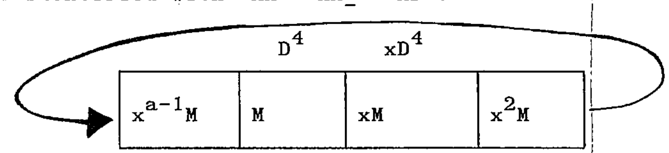
a = 3



$M \cup xM \subset D^4$

$\Longrightarrow x^kM \cup x^{k+1}M \subset x^kD^4.$

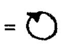
Identify appropriately labelled pieces. Thus $xM = M_+$ is identified with $xM = xM_- \subset xD^4$.

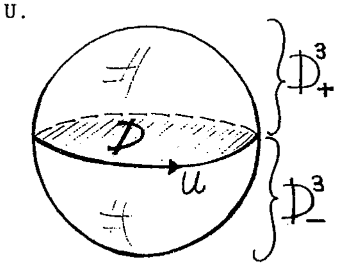


$x^aM = M$

It is then appropriate to identify $x: N_a \longrightarrow N_a$ the covering translation. We have $Na/x \equiv N_a/Z_a \cong D^4$.

Of course, the boundary $M_a = \partial N_a$ is the a-fold cyclic cover of S^3, branched along K.

Example 1: Let $U \subset S^3$ be the unknot of one componen $U =$  . Then the spanning surface F is a disk D ; the subspace $W = M \cup xM$ consists of the lower and up hemiballs D^3_- and D^3_+ of a 3-ball D^3 whose equator is U.

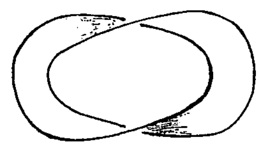


$xM = D^3_+$

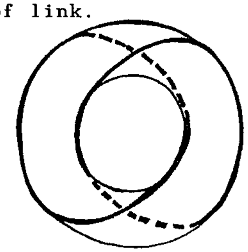
$M = D^3_-$

Of course, in this case $N_a \cong D^4$ for all a.

Example 2: Let $L =$  be the Hopf Link. Then tl spanning surface F is an annulus:



And $W = M \cup xM$ is a solid torus τ on whose boundary is embedded the Hopf link.

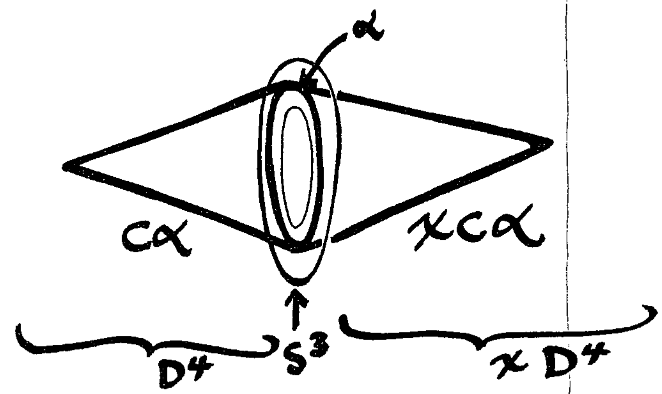


The surface of this torus is divided by L into two annuli: F and xF. M and xM meet along the central annular slice of this torus (as the page appears to cut it).

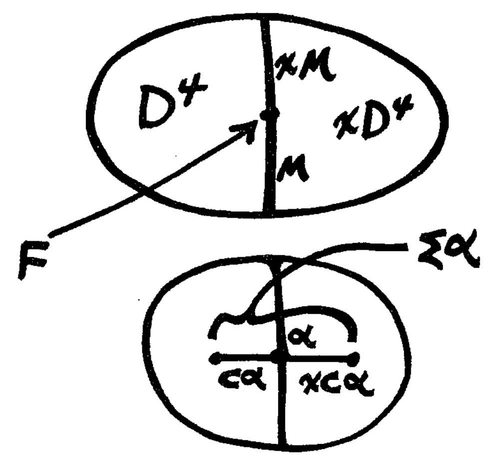
In this case we see that $M_2(L)$ can be described by taking two solid tori (the result of drilling out W from S^3) and identifying them along their boundaries via the involution $x : S^1 \times S^1 \circlearrowleft$ obtained from this construction. This shows that $M_2(L)$ is a Lens space ([ST]). In fact, it is homeomorphic to $\mathbb{RP}^3$ (= D^3 with antipodal identifications on the boundary). Do this as an exercise.

We can also see clearly the structure of $N_2(L)$. For this consists in $N_2(L) = D^4 \cup xD^4$ where the two 4-balls are pasted along W. Pasting two 4-balls along solid tori in their 3-sphere boundaries obviously produces a space

with $H_2(N_2(L)) \cong Z$ generated by a suspension $\Sigma\alpha$ of generator of the solid torus: $\Sigma\alpha = c\alpha - xc\alpha$ where $c\alpha$ denotes the result of coning α into the center of D



We can, if we like, choose $\alpha \subset F$ so that it generate $H_1(F)$.

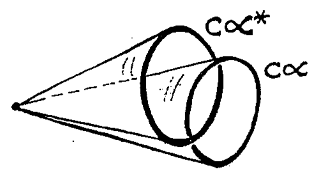


This is the sort of schematic diagram we use for this situation.

We can also see the self-intersection number of the 2-cycle $\Sigma\alpha \subset N_2(L)$. To find $(\Sigma\alpha)\cdot(\Sigma\alpha)$ deform $\Sigma\alpha$ to $d(\Sigma\alpha)$ via

$$d(\Sigma\alpha) = d(c\alpha - xc\alpha) = c\alpha_* - xc\alpha^*$$

where α_* denotes pushing α in the positive normal direction to F in S^3. The involution x exchanges pushing up and pushing down, so this is identified with $x\alpha^*$. On each side the result is



$$(c\alpha)\cdot(c\alpha_*) = \ell k(\alpha,\alpha_*) = +1$$

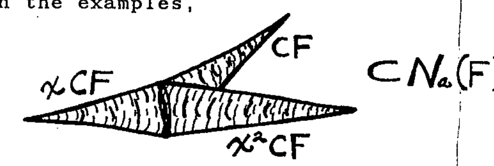
Thus $(\Sigma\alpha)\cdot(\Sigma\alpha) = 2$. (See [GA].)

We now generalize the geometry of this example.

LEMMA 12.1. $N_a(F)$ *has the homotopy type of* $\cup_{i=0}^{a-1} x^i CF$ *where* $CF \subset D^4$ *denotes the cone over* $F \subset S^3$ *with the apex of the cone at the center of* D^4. *Given a cycle* α *on* F, *let* $\Sigma\alpha$ *denote the cycle* $\Sigma\alpha = c\alpha - xc\alpha$. *Then* $H_1(N_a(F)) = 0$ *and* $H_3(N_a(F)) = 0$ *while*

$$H_2(N_a(F)) \cong \oplus_{k=0}^{a-2} x^k \Sigma H_1(F).$$

Proof: As in the examples,



Now we want to determine the intersection form $f : H_2(N_a) \times H_2(N_a) \longrightarrow Z$. (See [K4].)

THEOREM 12.2. *The intersection form* f *(above) is giv.. by the formulas:*

$$f(x^i\Sigma\alpha, x^j\Sigma\beta) = \begin{cases} \theta(\alpha,\beta) + \theta(\beta,\alpha), & i \equiv j \pmod a \\ -\theta(\alpha,\beta), & i \equiv j-1 \pmod a \\ -\theta(\beta,\alpha), & i \equiv j+1 \pmod a \end{cases}$$

[f is zero in the other cases].

Hence f *has matrix*
$$\begin{bmatrix} \tilde\theta & -\theta & & & \\ \theta' & \tilde\theta & -\theta & & \\ & -\theta' & \tilde\theta & & \\ & & & \ddots & -\theta \\ & & & -\theta' & \tilde\theta \end{bmatrix}$$
wh..

θ *is the Seifert form for* $F \subset S^3$, *and there are* $(a-1) \times (a-1)$ *blocks.*

Proof:

$$f(\Sigma\alpha,\Sigma\beta) = f(\Sigma\alpha, d\Sigma\beta)$$
$$= f(c\alpha - xc\alpha,\ c\beta_* - xc\beta^*)$$
$$= f(c\alpha, c\beta_*) + f(c\alpha, c\beta^*)$$
$$= \ell k(\alpha,\beta_*) + \ell k(\alpha,\beta^*)$$
$$\therefore\ f(\Sigma\alpha,\Sigma\beta) = \theta(\alpha,\beta) + \theta(\beta,\alpha).$$

Here the deformation $d\Sigma\beta$ is produced just as in the Hopf Link example. We also need $f(\Sigma\alpha, x\Sigma\beta)$ and $f(x\Sigma\alpha,\Sigma\beta)$.

$$f(\Sigma\alpha, x\Sigma\beta) = f(d\Sigma\alpha, x\Sigma\beta)$$
$$= f(c\alpha_* - xc\alpha^*,\ xc\beta - x^2 c\beta)$$
$$= -f(xc\alpha^*,\ xc\beta)$$
$$= -f(c\alpha^*, c\beta)$$
$$= -\ell k(\alpha^*,\beta)$$
$$\therefore\quad f(\Sigma\alpha, x\Sigma\beta) = -\theta(\alpha,\beta).$$

A similar calculation shows that $f(x\Sigma\alpha,\Sigma\beta) = -\theta(\beta,\alpha)$.

These calculations, and the fact that x is an isometry of f suffice to prove the theorem.

Remark: Everything can be generalized to other dimensions. We shall make use of this in Chapter XIX.

We now need some discussion of the algebraic topology of N_a and ∂N_a. Let $j : H_2(N_a) \longrightarrow H_2(N_a,\partial N_a)$ denote

the mapping induced by the inclusion $N_a \subset (N_a,\partial N_a)$. T is part of the exact sequence of the pair:

$$0 \longrightarrow H_3(\partial N_a) \longrightarrow H_2(N_a) \xrightarrow{\ j\ } H_2(N_a,\partial N_a) \longrightarrow H_1(\partial N_a) \longrightarrow 0$$

Thus a matrix for j with respect to appropriate bases will be a relation matrix for $H_1(\partial N_a)$. [Don't forget ∂N_a is the a-fold cyclic cover of S^3 branched along

LEMMA 12.3. *Let* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *be bases for* $H_2(N_a)$ *and* $H_2(N_a,\partial N_a)$ *that are dual in the sense of Poincare-Lefschetz duality. [That is, if* $\mathfrak{B} = \{b_1,\cdots,b_n\}$ *and* $\mathfrak{B}' = \{b_1',\cdots,b_n'\}$ *and* $\langle\ ,\ \rangle : H_2(N_a) \times H_2(N_a,\partial N_a) \longrightarrow Z$ *the (nonsingular by Poincare-Lefschetz) intersection pa ing, then* $\langle b_i, b_k'\rangle = \delta_{ik}$*.] Then the* <u>*matrix of*</u> j, <u>*wit respect to*</u> $\mathfrak{B},\mathfrak{B}'$ <u>*is the intersection matrix for*</u> f <u>*wi respect to*</u> $\mathfrak{B}$.

Proof: Let $m_{ij} = f(b_i,b_j)$. Then

$$m_{ij} = f(b_i,b_j) = f(b_i, j(b_j))$$
$$= f\left[b_i,\ \sum_{k=1}^{n} J_{kj} b_k\right] \quad \text{where J is the matrix of the ma}$$
$$= \sum_{k=1}^{n} J_{kj} f(b_i,b_k)$$
$$= \sum_{k=1}^{n} J_{kj}\delta_{ik} = J_{ij}.$$

$\therefore \quad M_{ij} = J_{ij}$.

Thus we now know that the matrix $\theta \otimes \Lambda_a + \theta' \otimes \Lambda_a'$ is a relation matrix for $H_1(\partial N_a)$. The matrix Λ_a is shown below.

$$\Lambda_a = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix} \quad (a-1) \times (a-1).$$

For example, if $a = 2$ then $\theta+\theta'$ is a relation matrix for $H_1(M_2(K))$. In the Hopf Link example, $\theta = (1)$, so (2) is the relation matrix, whence $H_1(M_2(\bigcirc\!\!\!\bigcirc)) = Z_2$. To continue the Hopf Link example, we have that

$$\Lambda_a + \Lambda_a' = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \quad (a-1) \times (a-1)$$

is a relation matrix for $H_1(M_a(\bigcirc\!\!\!\bigcirc))$. Thus

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \Rightarrow H_1(M_3(\bigcirc\!\!\!\bigcirc)) \cong Z/3Z.$$

<u>Exercise</u>. Compute $H_1(M_a(\bigcirc\!\!\!\bigcirc))$ for $a = 2,3,4,5,6,\cdots$.

As a next task, let's see how to compute the signature of N_a. By definition, the <u>signature of a manifold of dimension 4k is the signature of its 2k-dimensional intersection form</u>. Thus

$$\sigma(N_a) = \sigma(f) = \sigma(\theta \otimes \Lambda_a + \theta' \otimes \Lambda_a').$$

We shall proceed with a mixture of algebra and geo etry. Remember that $x : H_2(N_a) \longrightarrow H_2(N_a)$ is an isome of the form f $(f(x\alpha, x\beta) = f(\alpha,\beta)$ for all $\alpha, \beta \in H_2(N$ Thus it would be helpful to decompose $H_2(N_a)$ into a s of eigenspaces for x. Eigenspaces for different eigen values of an isometry of a form f are perpendicular w resepct to f.

In order to construct the eigenspaces we extend coefficients to the complex numbers $\mathbb{C}$. Then f becom <u>hermitian form over</u> $\mathbb{C}$. This means that $f(\lambda a, \mu b) = \lambda\bar{\mu} f(a,b)$ when $\lambda, \mu \in \mathbb{C}$; and $f(a,b) = \overline{f(b,a)}$ for $a,b \in H_2(N_a;\mathbb{C})$. A hermitian form has real eigenvalues hence a well-defined signature. The hermitian form obtained by extending scalars from a real form has the signature as the real form (exercise).

Thinking of f over $\mathbb{C}$, construct eigenvectors a follows: Let $\omega \in \mathbb{C}$ and $e \in H_2(N_a)$. Define, <u>for</u> $\underline{\omega^a}$

$$V(\omega,\varepsilon) = e + x\bar{\omega}e + x^2\bar{\omega}^2 e + \cdots + x^{a-1}\bar{\omega}^{a-1}e.$$

Thus $V(\omega,e) \in H_2(N_a;\mathbb{C})$. And $V(\omega,e)$ is an eigenvecto for x with eigenvalue ω.

$$\begin{aligned} xV(\omega,e) &= x(1 + x\bar{\omega} + x^2\bar{\omega}^2 + \cdots + x^{a-1}\bar{\omega}^{a-1})e \\ &= (x + x^2\bar{\omega} + x^3\bar{\omega}^2 + \cdots + x^{a-1}\bar{\omega}^{a-2} + x^a\bar{\omega}^{a-1})e \\ &= (\bar{\omega}^{a-1} + x + x^2\bar{\omega} + x^3\bar{\omega}^2 + \cdots + x^{a-1}\bar{\omega}^{a-2})e \\ &= \omega(1 + x\bar{\omega} + x^2\bar{\omega}^2 + \cdots + x^{a-1}\bar{\omega}^{a-1})e \end{aligned}$$

$\therefore xV(\omega,e) = \omega V(\omega,e). \qquad (\omega^a = 1)$

Furthermore, you can easily check that if $\{e_1, e_2, \cdots, e_n\}$ is a basis for $\Sigma H_1(F) \subset H_2(N_a(F))$, then $\{V(\omega,e_1),\cdots,V(\omega,e_n)\}$ is linearly independent whenever $\omega^a = 1$, $\omega \neq 1$. Therefore, let $V(\omega) \subset H_2(N_a;\mathbb{C})$ be the subspace $V(\omega,\Sigma H_1(F))$ for each a^{th} root of unity:

$$V(\omega) = \{(1+x\bar{\omega}+\cdots+x^{a-1}\bar{\omega}^{a-1})e \mid e \in \Sigma H_1(F)\}.$$

Then (by dimension count) we have that

$$H_2(N_a;\mathbb{C}) \cong \oplus_{k=1}^{a-1} V(\omega^k)$$

where $\omega = \exp(2\pi i/a)$. This is an eigenspace decomposition.

LEMMA 12.4. *The form* $f|V(\omega^k)$ has matrix

$$a((1-\omega^k)\theta+(1-\omega^k)\theta')$$

where θ *is the Seifert matrix for* $F \subset S^3$.

Proof: Let $\alpha = \Sigma X$, $\beta = \Sigma Y \in \Sigma H_1(F)$. Let ω be any a^{th} root of unity, and $f = f|V(\omega)$

$$f(V(\omega,\alpha),V(\omega,\beta)) = f\left[\sum_{i=0}^{a-1} x^i\bar{\omega}^i\alpha,\ \sum_{j=0}^{a-1} x^j\bar{\omega}^j\beta\right]$$
$$= \sum_{i=0}^{a-1} f(x\omega\alpha,x\omega\beta) + \sum_{i=0}^{a-1} f(x^i\bar{\omega}^i\alpha,x^{i+1}\bar{\omega}^{i+1}\beta)$$
$$+ \sum_{i=0}^{a-1} f(x^{i+1}\bar{\omega}^{i+1}\alpha,x^i\bar{\omega}^i\beta).$$

These are the only possible nonzero terms (using Theor 12.2). Consequently,

$$f(V(\omega,\alpha),V(\omega,\beta)) = \sum_{i=0}^{a-1} f(\alpha,\beta) + \sum_{i=0}^{a-1} f(\alpha,x\bar{\omega}\beta) + \sum_{i=0}^{a-1} f(x$$
$$= a[f(\alpha,\beta) + \omega f(\alpha,x\beta) + \bar{\omega}f(x\alpha,\beta)]$$
$$= a[\theta(x,y) + \theta(y,x) - \omega\theta(x,y) - \bar{\omega}\theta(y,$$
$$= a[(1-\omega)\theta(x,y) + (1-\bar{\omega})\theta(y,x)].$$

This completes the proof. ■

DEFINITION 12.5. *Let* $K \subset S^3$ *be an oriented knot or* l $F \subset S^3$ *a connected oriented spanning surface for* K. θ *be the Seifert form for* F *and let* $\omega \neq 1$ *be a com* *number. Define the* *<u>ω-signature of K</u>* *by the formula*

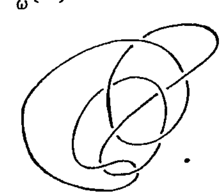
$$\sigma_\omega(K) = \sigma((1-\omega)\theta + (1-\bar{\omega})\theta').$$

(Compare [T].)

<u>Exercise</u>. (a) Use S-equivalence to show that $\sigma_\omega(K)$ an invariant of the knot K.

(b) Show that $\sigma_\omega(K)$ vanishes on slice knots.

(c) Compute $\sigma_\omega(K)$ for all ω for the knot



(d) Prove that if $Te = (1+\bar{\omega}x+\cdots+\bar{\omega}^{a-1}x^{a-1})e$ then for $e \in \Sigma H_1(F)$, $Te = 0 \Longrightarrow e = 0$.

We can now summarize our calculations in the

THEOREM 12.6. *Let* $N_a(F)$ *be the a-fold cyclic cover of* D^4 *branched along* $F' \subset D^4$ *where* F' *is the push-in of* $F \subset S^3$. F *is a connected, oriented spanning surface for a knot or link* $K \subset S^3$. *Then the signature of* $N_a(F)$ *is given by the formula*

$$\sigma(N_a(F)) = \sum_{i=0}^{a-1} \sigma_{\omega^i}(K)$$

where $\omega = \exp(2\pi i/a)$.

Proof: This follows at once from the preceding discussion.

Example: Let K be the trefoil  with Seifert form $\theta = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Then

$$M = (1-\omega)\theta + (1-\bar{\omega})\theta' = \begin{bmatrix} \omega+\bar{\omega}-2 & 1-\omega \\ 1-\bar{\omega} & \omega+\bar{\omega}-2 \end{bmatrix}.$$

Hence

$$|M_\omega| = (\omega+\bar{\omega}-2)^2 - (1-\omega)(1-\bar{\omega})$$
$$= \omega^2 + \bar{\omega}^2 + 2 - 4\omega - 4\bar{\omega} + 4 - (1-\omega-\bar{\omega}+1)$$
$$\therefore\ |M_\omega| = \omega^2 + \bar{\omega}^2 - 3\omega - 3\bar{\omega} + 4.$$

Actually, it is better to observe that

$$|M_\omega| = |(1-\omega)(\theta-\omega\theta')| = (1-\omega)^2\,\Delta_K(\bar{\omega})$$
$$= (1-\omega)^2(\bar{\omega}^2-\bar{\omega}+1).$$

Note that M_ω will be singular at roots of $\Delta_K(t) = 0$. From this we will be able to calculate $\sigma_\omega(K)$ for all

Exercise: Do the calculation in this direct form.

Let's do this exercise:

$$M_\omega = \begin{bmatrix} \omega+\bar{\omega}-2 & 1-\omega \\ 1-\bar{\omega} & \omega+\bar{\omega}-2 \end{bmatrix}.$$

Given a hermitian matrix $H = \begin{bmatrix} a & z \\ \bar{z} & a \end{bmatrix}$ with $a \neq 0$ we c find a matrix H' congruent to it by (1) multiply the first row by $-\bar{z}/a$ and add it to the second row; (2) tiply the first column by $-z/a$ and add it to the secc column. The result is

$$H' = \begin{bmatrix} a & 0 \\ 0 & a-z\bar{z}/a \end{bmatrix}.$$

In our case, $a = \omega+\bar{\omega}-2$ and hence $a = 0 \Longrightarrow \omega+\bar{\omega} =$ $\Longrightarrow \mathrm{Re}(\omega) = 1 \Longrightarrow \omega = 1$ (since $\omega \in S^1$). Since we're or interested in σ_ω for $\omega \neq 1$, our H' applies and we have

$$a - z\bar{z}/a = (\omega+\bar{\omega}-2) - \frac{(1-\omega)(1-\bar{\omega})}{(\omega+\bar{\omega}-2)}$$

$$= (\omega+\bar{\omega}-2) - \frac{(1-\omega-\bar{\omega}+1)}{(\omega-\bar{\omega}-2)}$$

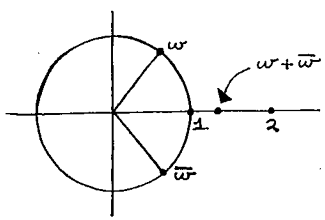
$$= (\omega-\bar{\omega}-2) + 1$$

$$\therefore \quad a - z\bar{z}/a = (\omega+\bar{\omega}-1).$$

Hence

$$H' = \begin{bmatrix} \omega+\bar{\omega}-2 & 0 \\ 0 & \omega+\bar{\omega}-1 \end{bmatrix}.$$

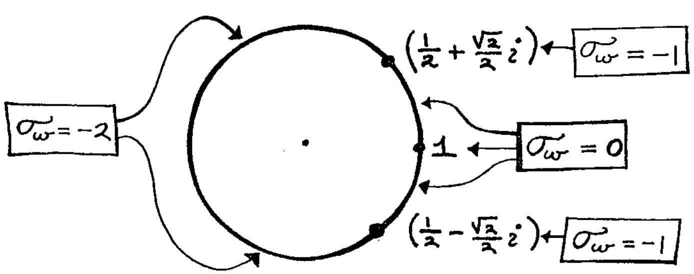
Examine the unit circle:



$$\omega \neq 1 \Rightarrow \omega+\bar{\omega}-2 < 0.$$

$$\therefore \quad \sigma_\omega = -1 + \text{sgn}(\omega+\bar{\omega}-1)$$

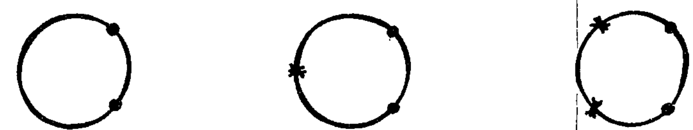
Thus we need to know when $\omega+\bar{\omega} = 1$. $\left\{\begin{matrix}\omega = a+bi \\ \omega+\bar{\omega} = 1\end{matrix}\right\} \Rightarrow a = 1/2$ $\Rightarrow b = \pm\sqrt{3}/2$. This is no surprise, since we know that the form goes singular at the roots of the Alexander polynomial. We can now draw a diagram to indicate the values of σ_ω( ) $= \sigma_\omega(K)$ for different $\omega \in S^1$:



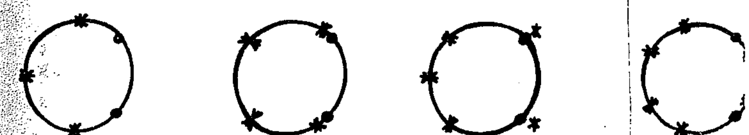
It is easy to use this diagram to calculate

$$\sigma(N_\ell(K)) \underset{\text{def.}}{\equiv} \sigma(K,\ell)$$

for $\ell = 1,2,3,4,\cdots$. Just remember (Theorem 12.6) tha $\sigma(K,\ell) = \sum_{i=1}^{\ell-1} \sigma_{\omega^i}(K)$ where $\omega = e^{2\pi i/\ell}$, and plot these points on the diagram: Let $\sigma[\ell] = \sigma(K,\ell)$.



$\sigma[1] = 0$ (empty plot!) $\sigma[2] = -2$ $\sigma[3] = -$



$\sigma[4] = -6$ $\sigma[5] = -8$ $\sigma[6] = -8$ $\sigma[7] =$

Exercise. $\sigma[\ell+6] = \sigma[\ell]-8 \quad \forall_\ell \geq 1$.

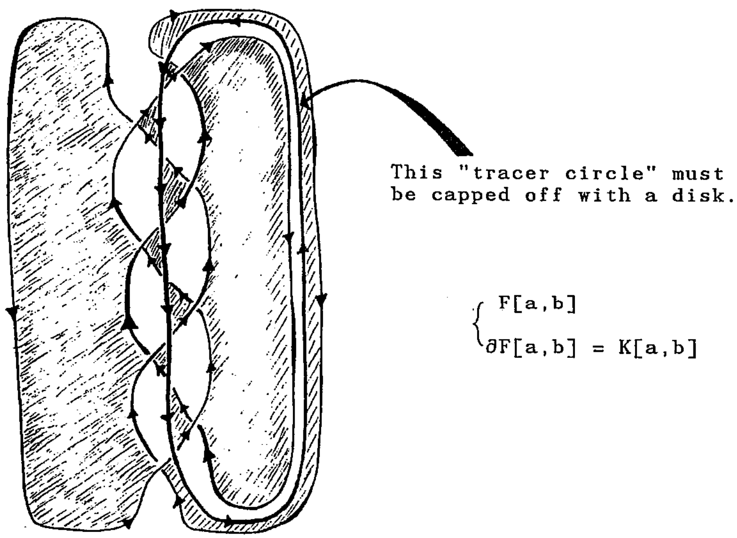
The last exercise can be generalized:

Exercise 12.7. Let $K \subset S^3$ be a knot such that the ro of the Alexander polynomial $\Delta_K(t)$ are contained in th d^{th} roots of unity. Let $\sigma[\ell] = \sigma[\ell](K)$ denote the s nature of the ℓ-fold branched cover of D^4, branched along a pushed-in spanning surface for K. Prove the

PERIODICITY THEOREM. *There exists a constant* c *such that* $\sigma[\ell+d] = c+\sigma[\ell]$ *for all* $\ell \geq 1$. (See [DK], [N].)

TORUS KNOTS

A torus knot of type (a,b), denoted K[a,b], has Seifert pairing (with respect to a Seifert surface for the usual drawing) $\theta = -\Lambda_a \otimes \Lambda_b$. There are many ways to see this. For the moment, take it as an exercise and contemplate the form of spanning surface for K[3,4]:



In three-dimensional space, the surface F[a,b] has an action of $(Z/aZ) \times (Z/bZ)$ corresponding exactly to our algebraic actions for Λ_a and Λ_b. Thus we have $x : x^a = 1$, $y : y^b = 1$ acting as isometries of θ. Consequently, <u>we can apply the same algebraic technique</u>

<u>that we used for cyclic branched covers and 4-manifolds diagonalize</u>:

$$-\Lambda_a \otimes \Lambda_b \underset{\mathbb{C}}{\sim} -\Omega_a \otimes \Omega_b$$

and conclude that the signature of K[a,b] is the signature of

$$(-\Omega_a \otimes \Omega_b) + (-\Omega_a \otimes \Omega_b)^* = M_{a,b}$$

where Ω_a is as shown below.

$$\Omega_a = \begin{bmatrix} 1-\omega & & & & \\ & 1-\omega^2 & & & \\ & & 1-\omega^3 & & \\ & & & \ddots & \\ & & & & 1-\omega^{a-1} \end{bmatrix}$$

with $\omega = e^{2\pi i/a}$. Our formalism shows that $\Lambda_a \underset{\mathbb{C}}{\sim} \Omega_a$ wh $\underset{\mathbb{C}}{\sim}$ means congruence of matrices. that is, there is an invertible complex matrix P such that $P^*\Lambda_a P = \Omega_a$. Th change of basis respects the isometry x so that we obt $-\Omega_a \otimes \Omega_b$ as isomorphic to the Seifert form over the com plex numbers.

The signature of the matrix $M_{a,b}$ is given by the formula

$$\sigma[a,b] = \sum_{\substack{1 \leq i \leq a-1 \\ 1 \leq j \leq b-1}} \operatorname{sgn} \operatorname{Re}[-(1-\omega^i)(1-\tau^j)]. \qquad (\tau = e^{2\pi i/b})$$

Here

$$\mathrm{sgn}(\alpha) = \begin{cases} +1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0 \\ 0 & \text{otherwise.} \end{cases}$$

To determine these signs more explicitly, note that

$$1-e^{i\theta} = (e^{-i\theta/2}-e^{i\theta/2})e^{i\theta/2}$$
$$= -2i\ \sin(\theta/2)e^{i\theta/2}$$

$$\therefore\quad -(1-\omega^k)(1-\tau^\ell) = 4\ \sin(\pi k/a)\sin(\pi \ell/b)\cdot e^{i\pi(k/a+\ell/b)}$$

$$\therefore\quad \mathrm{sgn}(\mathrm{Re}[-(1-\omega^k)(1-\tau^\ell)]) = \mathrm{sgn}(\mathrm{Re}(e^{i\pi(k/a+\ell/b)}))$$
$$= \mathrm{sgn}(\cos(i\pi(k/a+\ell/b)))$$
$$= \begin{cases} +1 & \text{if } -\frac{1}{2} < k/a+\ell/b < +\frac{1}{2} \pmod 2 \\ -1 & \text{if } \frac{1}{2} < k/a+\ell/b < 3/2 \pmod 2 \end{cases}$$

$$\therefore\quad \epsilon[k,\ell] \underset{\text{def}}{\equiv} \begin{cases} +1 & \text{if } 0 < \frac{k}{a} + \frac{\ell}{b} + \frac{1}{2} < 1 \pmod 2 \\ -1 & \text{if } 1 < \frac{k}{a} + \frac{\ell}{b} + \frac{1}{2} < 2 \pmod 2 \end{cases}$$

We have the explicit formula:

$$\sigma(K[a,b]) = \sum_{\substack{1 \le k \le a-1 \\ 1 \le \ell \le b-1}} \epsilon[k,\ell]$$

for the signature of the torus knot $K[a,b]$.

For example, if $a = 2$, $b = 3$ then

$$k = 1,\quad \ell = 1,2 : \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = 1 + \frac{1}{3} \Longrightarrow \epsilon[1,1] = -1,$$
$$\frac{1}{2} + \frac{2}{3} + \frac{1}{2} = 1 + \frac{2}{3} \Longrightarrow \epsilon[1,2] = -1.$$

Hence $\sigma(K[2,3]) = -2$, as we know.

Since these computations work just as well for linl

as for knots we can set up a table of signatures $\sigma[a,b]$

a \ b	2	3	4	5	6	7	8	9	10	11	12	13
2	-1											
3	-2	-4										
4	-3	-6	-7									
5	-4	-8	-8	-12								
6	-5	-8	-11	-16	-17							
7	-6	-8	-14	-16	-18	-24						
8	-7	-10	-15	-20	-23	-30	-31					
9	-8	-12	-16	-24	-26	-32	-32	-40				
10	-9	-14	-19	-24	-29	-34	-39	-48	-49			
11	-10	-16	-22	-24	-34	-40	-42	-48	-50	-60		
12	-11	-16	-23	-28	-35	-42	-47	-54	-59	-70	-71	
13	-12	-16	-24	-32	-36	-48	-52	-56	-64	-72	-72	-84

(These signatures are brought to you courtesy of the Rac
Shack Model 100 Portable Computer.)

You are strongly urged to try your hand at proving some of the patterns that leap to the eye. It is true t for fixed a, the signatures $\sigma[a,k] = f(k)$ have a qua periodicity to the effect: d = least common multiple $(a,2)$ then $f(k+d) = f(k)+c$ for a fixed constant c. This is actually a case of the periodicity theorem [Exercise 12.7].

But there are other patterns. Look at the signatur of the links of type (a,a):

-1,	-4,	-7,	-12,	-17,	-24,	-31,	-40,	-49,	-60,	-71,	-84
3	3	5	5	7	7	9	9	11	11	13	

The successive differences indicate the pattern.

These facts imply that $\sigma[a,b] \neq 0$ for all a,b. <u>Hence no torus knot is a slice knot.</u>

XIII

SIGNATURE THEOREMS

We will return to knots and cyclic branched coveri in the next section. Here we prove general results abou the signature of a manifold. (Unless otherwise specifi homology is taken with real coefficients.)

THEOREM 13.1 (Novikov Addition Theorem). *Let* M^{4n} *be* (*4n-dimensional manifold that is obtained by gluing two manifolds along a common boundary. Then the signature* (*M is the sum of the signatures of these manifolds. Thu if* $M = Y_+ \cup Y_-$ *where* Y_+ *and* Y_- *are 4n-manifolds,* $X = Y_+ \cap Y_-$ *is a 4n-1 manifold* ($X = \partial Y_+$ and $X = \partial$Y *then* $\sigma(M) = \sigma(Y_+) + \sigma(Y_-)$. *(Orientations compatible wi this pasting.)* (See [AS].)

Proof: Use the Mayer-Vietoris sequence to decompose $H_{2n}(M) \cong G_+ \oplus G_- \oplus A \oplus B$ where

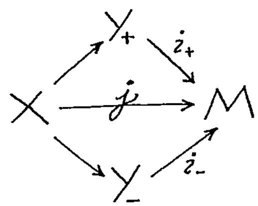
$$G_+ = \text{Image}(H_{2n}(Y_+) \longrightarrow H_{2n}(M))$$

$$G_- = \text{Image}(H_{2n}(Y_-) \longrightarrow H_{2n}(M))$$

$$A = \{x \in H_{2n-1}(X) \mid i_+x = i_-x = 0\}$$

$$B = H_{2n}(X)/\{x \in H_{2n}(X) \mid j(x) = 0 \text{ in } H_{2n}(M)\}.$$

Here:



the diagram of inclusions

Thus

G_+ = 2n-cycles on Y_+.

G_- = 2n-cycles on Y_-.

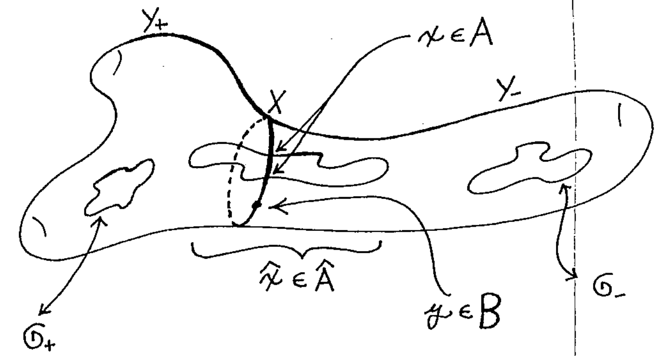
A = cycles in X of dim(2n−1) bounding in Y_+ and Y_-.

B = 2n-cycles in X that live in M.

We leave this decomposition as an exercise, but note: In the Mayer-Vietoris Sequence for $M = Y_+ \cup Y_-$ we have

$$\cdots \longrightarrow H_{2n}(M) \longrightarrow H_{2n-1}(X) \xrightarrow{f} H_{2n-1}(Y_+) \oplus H_{2n-1}(Y_-) \longrightarrow \cdots$$
$$H_{2n}(M) \downarrow A \nearrow H_{2n-1}(X)$$

since $A = \text{Kernel}(f)$. Therefore we have a surjection $H_{2n}(M) \longrightarrow A \longrightarrow 0$. A becomes a direct summand of $H_{2n}(M)$ by lifting back: Given $[x] \in H_{2n-1}(X)$ with $i_+[x] = 0 = i_-[x]$ there are chains α_+, α_- on Y_+ and Y_- respectively such that $\partial\alpha_+ = x$, $\partial\alpha_- = x$. Thus $\alpha_+ - \alpha_-$ is a 2n-cycle on M. Let $[\hat{x}] = [\alpha_+ - \alpha_-]$. Then $\hat{A} \subset H_{2n}(M)$ as the actual direct summand.

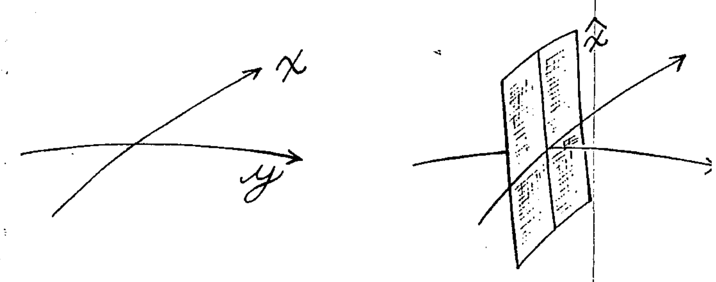


$M = Y_+ \cup Y_-$

Now observe the following basic fact about interse tions of cycles:

$$\boxed{x \in A,\ y \in H_{2n}(X) \Rightarrow x \cdot y = \hat{x} \cdot j(y)}$$

Here the left $\cdot$ denotes the intersection pairing $H_{2n-1}(X) \times H_{2n}(X) \longrightarrow Z$ and the right $.$ denotes the pairing $H_{2n}(M) \times H_{2n}(M) \longrightarrow Z$. One dimension down, the picture is:

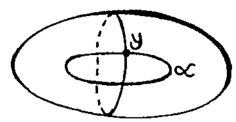


Claim: Given $y \in G_{2n}(X)$, then $j(y) = 0 \iff x \cdot y = 0$ for all $x \in A_{2n-1}(X)$.

Proof: $\Longrightarrow$: $j(y) = 0$ given.

$x \cdot y = \hat{x} \cdot j(y) = 0$

$\Longleftarrow$: $j(y) \neq 0$ given. Then by Poincaré duality on M $\exists\, \alpha \in H_2(M)$ with $\alpha \cdot j(y) \neq 0$



$\Rightarrow \alpha = \hat{x}$ some $x \in H_{2n-1}(M)$

$\therefore$ $0 \neq \alpha \cdot j(y) = \hat{x} \cdot j(y) = x \cdot y$. ■

We conclude, from this claim, that <u>A and B are Poincaré dual on M</u> (via duality of A and B on X). Therefore we can choose bases so that the intersection form on $H_{2n}(M)$ looks like:

•	G_+	G_-	$\hat{A}$	B
G_+	*	0	*	0
G_-	0	*	*	0
$\hat{A}$	*	*	*	I
B	0	0	I	0

where I is an identity matrix. This implies that $\sigma(M) = \sigma(\cdot|G_+) + \sigma(\cdot|G_-)$. Hence $\sigma(M) = \sigma(Y_+) + \sigma(Y_-)$. ■

THEOREM 13.2. *Let* M^{4n} *be a compact, oriented, 4n-dim*[…]*sional manifold. Suppose that* M^{4n} *forms the boundary* […] *a compact oriented manifold* N^{4n+1}. *Then* $\sigma(M^{4n}) = 0$.

Proof: Let $j : H_{2n}(M) \longrightarrow H_{2n}(N)$ denote the map induce[…] by inclusion. And let $A = \{x \in H_{2n}(M) \mid j(x) = 0\}$ den[…] the kernel of j. Note that $x \in A$ implies that there exists $X \in H_{2n+1}(N,M)$ with $\partial X = x$. Choose a lifting $\alpha : A \longrightarrow H_{2n+1}(N,M)$ such that $\partial\alpha(x) = x$ for all $x \in$[…] Then we have, for $x,y \in H_2(M)$, the formula $x \cdot y = j(x) \cdot \alpha$[…] where the first intersection denotes intersection number[…] in M, and the second denotes intersection numbers in[…]

If $a,b \in A$ then $a \cdot b = j(a) \cdot \alpha(b) = 0$.

If $b \in H_{2n}(M)$ and $j(b) \neq 0$, then by Poincaré–Lefschetz duality we have an $X \in H_{2n+1}(N,M)$ with $j(b) \cdot X \neq 0$. Hence $b \cdot (\partial X) \neq 0$ and $(\partial X) \in A$.

Let $\{a_1, \cdots, a_r\}$ be a basis for A and $\{\hat{a}_1, \cdots, \hat{a}_r\} \subset H_{2n}(M)$ be dual (Poincaré dual) in the sen[…] that $a_i \cdot \hat{a}_j = \delta_{ij}$. Since A is ⊥ to A we know that $\{a_1, a_2, \cdots, a_r, \hat{a}_1, \cdots, \hat{a}_r\}$ is linearly independent.

Claim: This is a basis for $H_{2n}(M)$.

To see the claim, suppose $x \in H_{2n}(M)$ and let $\omega = \sum_i (x \cdot a_i)\hat{a}_i + \sum_j (x \cdot \hat{a}_j) a_j$. Then $j(x-\omega) \cdot H_{2n+1}(N,M) = (x-\omega) \cdot A = \{0\}$. Hence $j(x-\omega) = 0$ and therefore $(x-\omega) \in A$, whence $x = \omega$.

These remarks show that M^{4n} has intersection form $\begin{bmatrix} 0 & I \\ I & * \end{bmatrix}$ and hence $\sigma(M) = 0$. This completes the proof. ■

Remark: It follows from the argument that if $M^{2n} = \partial N^{2n+1}$ then

$$\dim(\mathrm{Ker}(H_n(M) \longrightarrow H_n(N)) = \frac{1}{2} \dim H_n(M).$$

Remark: Two closed, compact, oriented manifolds M_1^{4n}, M_2^{4n} are said to be <u>cobordant</u> if there exists a compact, oriented manifold N^{4n+1} with $\partial N = M_1 \cup (-M_2)$ (disjoint union). Theorem 13.2 implies that $0 = \sigma(\partial N) = \sigma(M_1) - \sigma(M_2)$. Hence $\sigma(M_1) = \sigma(M_2)$. <u>Signature is a cobordism invariant</u>.

Remark: We can now prove that the signatures associated with cyclic branched covers of knots are independent of the choice of spanning surface in the four-ball.

PROPOSITION 13.3. *Let* $K \subset S^3$ *be an oriented knot or link. Let* $F \subset D^4$ *be any properly embedded surface which is oriented with boundary* K. *Let* $N_a(F)$ *denote the a-fold cyclic covering of* D^4 *branched along* F. *Then* $\sigma(N_a(F))$ *depends only on the knot or link* $K \subset S^3$. *By our previous work this means that* $\sigma(N_a(F)) = \sigma_a(K)$.

Proof: Let $F' \subset D'^4$ be another surface bounding K. Then $\mathcal{S} = F \cup -F' \subset D^4 \cup -D'^4 = S^4$ is a compact oriented

surface embedded in S^4. We conclude that there exists 3-manifold $\mathcal{F} \subset D^5$ bounding $\mathcal{S} \subset S^4$. Hence $N_a^5(\mathcal{F})$ is 5-manifold with boundary $N_a^4(\mathcal{S}) = N_a^4(F) \cup -N_a^4(F')$. Henc $\sigma(N_a^4(\mathcal{S})) = 0$, and by Novikov, $\sigma(N_a^4(F)) = \sigma(N_a^4(F'))$.

Another fundamental property of the signature is th

PRODUCT THEOREM 13.4. *Let* M_1^{4n}, M_2^{4m} *be compact oriente manifolds, then* $\sigma(M_1 \times M_2) = \sigma(M_1)\sigma(M_2)$.

Proof: It suffices (by Kunneth Theorem) to prove that i the tensor product of bilinear forms on vector spaces V_1, V_2 over $\mathbb{R}$, $\sigma(V_1 \otimes V_2) = \sigma(V_1)\sigma(V_2)$. Let $\mathcal{A} = \{a_1, \cdots, a_k\}$ be a basis for V_1, $\mathcal{B} = \{b_1, \cdots, b_\ell\}$ l basis for V_2. Let $\langle\ ,\ \rangle : V_i \times V_i \longrightarrow \mathbb{R}$ represent th forms. We may assume that they are individually diagon ized. Hence $\{a_i \otimes b_j\}$ is a diagonalizing basis for $V_1 \otimes V_2$. If $\sigma(V_1) = P_1 - N_1$ and $\sigma(V_2) = P_2 - N_2$ where P_1 = number of i such that $\langle a_i, a_i \rangle > 0$ and N_1 = nu of i such that $\langle a_i, a_i \rangle < 0$. (Similarly for P_2 and N_2). Then

$$\begin{aligned} \sigma(V_1 \otimes V_2) &= (P_1P_2 + N_1N_2) - (P_1N_2 + P_2N_1) \\ &= (P_1 - N_1)(P_2 - N_2) \\ &= \sigma(V_1)\sigma(V_2). \end{aligned}$$ ■

Remark: If M^k is a compact oriented manifold and 4 does not divide $k = \dim(M)$, define the signature of M to be zero: $\sigma(M^k) = 0$ if $4|k$. Then $\sigma(M_1 \times M_2) = \sigma(M_1)\sigma(M_2)$ for manifolds of arbitrary dimension.

Example: Complex Projective Space. Complex projective space $\mathbb{CP}^2$ can be described in a number of ways. It is the set of complex lines in $\mathbb{C}^3$. Hence

$$\mathbb{CP}^2 = \{\langle z_0, z_1, z_2 \rangle \mid (z_0, z_1, z_2) \in \mathbb{C}^3 - \{0,0,0\}\}.$$

Here, $\langle\ ,\ ,\ \rangle$ denotes homogeneous coordinates so that $\langle z_0, z_1, z_2 \rangle = \langle \lambda z_0, \lambda z_1, \lambda z_2 \rangle$ whenever $\lambda \neq 0$. By reformulating this version you can show that $\mathbb{CP}^2 = D^4 \cup_H S^2$ where $H : S^3 \longrightarrow S^2$ is the Hopf map and $\cup_H$ denotes the mapping cylinder on $\partial D^4 = S^3 \longrightarrow S^2$. The Hopf map is the map $S^3 \longrightarrow S^3/S^1$ where S^1 = unit circle in $\mathbb{C}$ acts on $S^3 = \{(z_0, z_1) \,\big|\, |z_0|^2 + |z_1|^2 = 1\}$ by $\lambda(z_0, z_1) = (\lambda z_0, \lambda z_1)$. From this, one sees that $H_2(\mathbb{CP}^2) \cong Z$ and the generator has self-intersection $+1$. Hence $\sigma(\mathbb{CP}^2) = +1$.

G-SIGNATURE

In studying branched covering spaces we have been looking at manifolds with a cyclic group action. We found that signatures decomposed into sums of signatures of eigenspaces. These patterns fit into a more general context. We will outline this context and discuss how to

compute signatures of 4-dimensional manifolds admitting cyclic action. This is called the G-signature theoren (for 4-manifolds). We shall use it in Chapter XVII to study slice knots.

Let $G = C_d$ be a cyclic group of order d. Supp that G acts smoothly on N^{2n}, preserving orientatio N^{2n} is a compact, oriented manifold. Assume for now n is even. Let $B(\ ,\ ) : H_n(N;\mathbb{R}) \times H_n(N;\mathbb{R}) \longrightarrow \mathbb{R}$ be intersection form. (Symmetric since n is even.) Si the group acts on N, we have $B(gx, gy) = B(x,y)$ for $g \in G$.

Thus we have the following algebraic situation: symmetric form $B(\ ,\ ) : V \times V \longrightarrow \mathbb{R}$, V a vector spac over $\mathbb{R}$. Assume, for simplicity, that $B(\ ,\ )$ is nondegenerate. For the cyclic action, we are given a linear transformation $g : V \longrightarrow V$ with $g^d = 1$ and $B(gx, gy) = B(x,y)$ for all $x, y \in V$.

LEMMA 13.5. *Let* V *be as above. Then there exist su spaces* $V^+, V^- \subset V$ *such that*

(1) $V = V^+ \oplus V^-$.

(2) B *is positive definite on* V^+.

B *is negative definite on* V^-.

(3) $g(V^+) = V^+$, $g(V^-) = V^-$.

Proof: By averaging the standard inner product we can