From "Lehrbuch der Topologie" by Seifert + Threlfall

$p = 3$). We construct a new lens $\mathfrak{L}'$ having screw angle $2\pi q'/p$ from these pieces. The tetrahedron $\mathfrak{T}_i$ has a triangle Δ_i in common with the lower lens cap and has a triangle $\bar{\Delta}_i$ in common with the upper lens cap. The lens axis b is common to all $\mathfrak{T}_i$. The lens edges lying opposite to b are all equivalent to one another and form one and the same edge a in the lens space. The triangles are equivalent in pairs to one another, as a consequence of the association of the lens caps. In particular, $\underline{\Delta}_i$ is equivalent to $\bar{\Delta}_{i+q}$, where $i + q$ is to be reduced mod p if necessary. We now construct the new lens $\mathfrak{L}'$ from the p tetrahedra. To do this we first join tetrahedron $\mathfrak{T}_{1+q}$ to tetrahedron $\mathfrak{T}_1$ by identifying the equivalent triangles $\bar{\Delta}_{1+q}$ and $\underline{\Delta}_1$. We then join $\mathfrak{T}_{1+2q}$ to $\mathfrak{T}_{1+q}$ by identifying triangles $\bar{\Delta}_{1+2q}$ and $\underline{\Delta}_{1+q}$, and so forth. We ultimately arrive at a lens $\mathfrak{L}'$ which differs from the original lens only in the fact that the edges a and b have interchanged their roles. If $2\pi q'/p$ is the screw angle of $\mathfrak{L}'$, then in the cyclic sequence $\mathfrak{T}_1, \mathfrak{T}_{1+q}, \ldots, \mathfrak{T}_2, \ldots$ in which the tetrahedra are arranged about the axis a of $\mathfrak{L}'$, the tetrahedra $\mathfrak{T}_1$ and $\mathfrak{T}_2$, of which the latter coincides with $\mathfrak{T}_{2+xp}$, will differ by q' places on one side and will differ by $p - q'$ places on the other side. Consequently, the difference of indexes of $\mathfrak{T}_{2+xp}$ and $\mathfrak{T}_1$ in the cyclic sequence will be equal to qq' or $q(p - q')$, respectively. But this difference is also equal to $(2 + xp) - 1$. Thus,

$$(2 + xp) - 1 = qq' \quad \text{or} \quad (2 + xp) - 1 = q(p - q'),$$

which gives

$$qq' \equiv \pm 1 \pmod{p}.$$

Thus the lens spaces (7, 2) and (7, 3) are homeomorphic for example because $2 \cdot 3 \equiv -1$ (mod 7). But one cannot decide from Theorem II whether the lens spaces $(p, 1)$ and $(p, 2)$ are homeomorphic. We shall later (§77) introduce an invariant which is not associated to the fundamental group and which permits certain lens spaces to be distinguished. It can show, for example, that (5, 1) and (5, 2) are distinct spaces. On the other hand, Theorem II does not distinguish between (7, 1) and (7, 2).

It should be noted that every lens space can be decomposed into two full rings having a torus as their common boundary. Bore a full cylinder $\mathfrak{B}$ out of the lens by boring along (and concentric with) the lens axis b. After identification of the lens caps, $\mathfrak{B}$ will close to form a full ring. The same is true for the complementary space $\mathfrak{A}$ which remains after boring out $\mathfrak{B}$. To see this we need only to decompose the lens $\mathfrak{L}$ into p tetrahedra as previously and to assemble the lens $\mathfrak{L}'$ from them. The complementary space $\mathfrak{A}$ then becomes a cylinder in $\mathfrak{L}'$ which surrounds the axis a of $\mathfrak{L}'$. When equivalent points are identified, $\mathfrak{A}$ will close to form a full ring (cf. §63).

Start here.

As another example we shall investigate the *spherical dodecahedron space* (Kneser [8, p. 256]). This space arises from a dodecahedron when one twists opposite lying pentagons by $\pi/5$ radians relative to one another and then identifies them. The edge network of the dodecahedron, which completely determines the space, is drawn in Fig. 112. There exist $\alpha^0 = 5$ nonequivalent vertices O, P, Q, R, S. There are ten nonequivalent edges, each formed by identifying three equivalent edges. The Euler characteristic is $N = -5 + 10 - 6 + 1 = 0$; thus we are dealing with a manifold. We select O as the initial point of the closed paths and we select the paths $a, h, f^{-1}, f^{-1}d$ as auxiliary paths leading to the vertices P, Q, R, S, respectively. The generating path classes of the fundamental group will then be represented by the closed paths

$$\begin{aligned} A &= aa^{-1}, & B &= abh^{-1}, \\ C &= hcf, & D &= f^{-1}d(d^{-1}f), \\ E &= (f^{-1}d)e, & F &= f^{-1}f, \\ G &= (f^{-1}d)ga^{-1}, & H &= hh^{-1}, \\ J &= aif, & K &= hk(d^{-1}f). \end{aligned}$$

The relations of type (I) follow after one writes the right-hand sides of the above equations in capital letters instead of lower case letters. We then get $A = D = F = H = 1$ and the remaining relations become trivial.

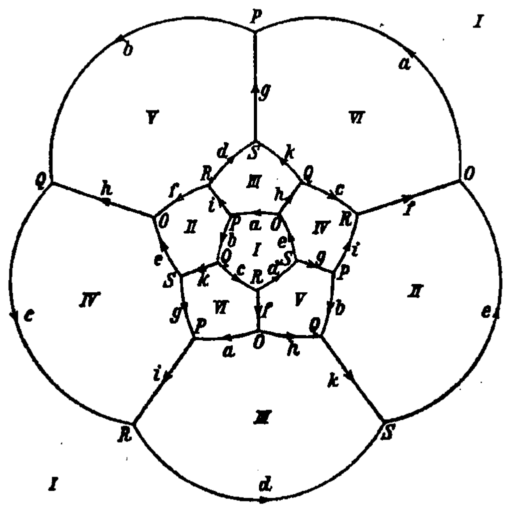

FIG. 112

By running around the pentagons we get the following six relations of type (II):

$$\left.\begin{array}{r} ABCDE = 1 \\ BKEF^{-1}J^{-1} = 1 \\ AJDK^{-1}H^{-1} = 1 \\ CJ^{-1}G^{-1}EH = 1 \\ BH^{-1}F^{-1}DG = 1 \\ AG^{-1}K^{-1}CF = 1 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{r} BCE = 1 \\ BKEJ^{-1} = 1 \\ J = K \\ CJ^{-1}G^{-1}E = 1 \\ B = G^{-1} \\ G^{-1}K^{-1}C = 1 \end{array}\right.$$

Elimination of G and K gives

$$BCE = 1,$$
$$BJEJ^{-1} = 1,$$
$$CJ^{-1}BE = 1,$$
$$BJ^{-1}C = 1.$$

From the first and fourth of these relations we get

$$E = C^{-1}B^{-1}, \qquad J = CB.$$

Using these to eliminate E and J from the second and third relations, we get

$$BCBC^{-1} \cdot B^{-2}C^{-1} = 1 \tag{I}$$

and

$$CB^{-1}C^{-1}BC^{-1}B^{-1} = 1. \tag{II}$$

We determine the first homology group from these two relations by making relations (I) and (II) Abelian! As always, we use additive notation for Abelian groups, and we denote the elements of the homology group by means of symbols with bars. We get

$$\bar{C} = 0, \tag{$\bar{\text{I}}$}$$

$$-\bar{C} - \bar{B} = 0; \tag{$\overline{\text{II}}$}$$

thus $\bar{B} = \bar{C} = 0$. That is, the first homology group consists of the null element alone. Since the dodecahedron space is orientable, we have the following values for the Betti numbers:

$$p^0 = 1, \quad p^1 = p^2 = 0, \quad p^3 = 1.$$

There are no torsion coefficients.

The numbers above are just the numerical invariants of the 3-sphere. Thus the homology groups are not in themselves sufficient to distinguish whether the 3-sphere does or does not coincide with the dodecahedron space. To decide this we examine whether the fundamental groups of these spaces differ. To do so we transform the somewhat untransparent relations (I) and (II) further. We set (II) into (I) at the position indicated by the dot in (I). In place of (I) we get the relation

$$BCBC^{-1} \cdot CB^{-1}C^{-1}BC^{-1}B^{-1} \cdot B^{-2}C^{-1} = 1,$$

or, after shortening this,

$$B^2C^{-1}B^{-3}C^{-1} = 1. \qquad \text{(I')}$$

By introducing a new generator U into (I) and (II) where U is defined by $C = U^{-1}B$, we get

$$B^2 \cdot B^{-1}U \cdot B^{-3} \cdot B^{-1}U = 1,$$

$$U^{-1}B \cdot B^{-1} \cdot B^{-1}U \cdot B \cdot B^{-1}U \cdot B^{-1} = 1$$

or

$$B^4 = UBU, \qquad U^2 = BUB$$

or also

$$B^5 = (BU)^2 = U^3. \qquad \text{(III)}$$

We recognize from the relations (III) that the dodecahedron space is not homeomorphic to the 3-sphere. That is because the fundamental group does not consist of the unit element alone. Instead, the relations (III) are satisfied by the icosahedral group, if one interprets B as a rotation of $2\pi/5$ radians about a vertex of the icosahedron and interprets U as a rotation of $2\pi/3$ radians, having the same sense of rotation, about the midpoint of a triangle adjoining that vertex. The icosahedral group is therefore either the group (III) itself or a factor group of (III). In either case the fundamental group does not consist of just the unit element alone. It is possible to show, by the way, that (III) is of order 120 and is the "binary icosahedral group."*

The spherical dodecahedron space is a manifold which has the same homology groups as a 3-sphere without, however, being homeomorphic to it. Such a manifold is called a *Poincaré space*. Infinitely many Poincaré spaces are known. But the spherical dodecahedron space is the only one known which has a finite fundamental group.[33]

The homology groups are not sufficient to characterize the 3-sphere. Whether the 3-sphere is characterized by its fundamental group is the content of the "Poincaré conjecture," which remains unproven to this day. Since the fundamental group of the 3-sphere consists of the unit element alone, we can also state the problem as follows: Aside from the 3-sphere do there exist other 3-dimensional closed manifolds such that each closed path can be contracted to a point (is null homotopic)?**

* *Jahresber. Deutsch. Math.-Verein.* **42** (1932), problem 84, p. 3.

** *Editor's Note*: As of January 1979, this famous problem is still open! However, new (unpublished) results of W. Thurston have established the following weak version: If a simply connected 3-manifold M is a cyclic branched covering space of S^3, then M is in fact homeomorphic to S^3.

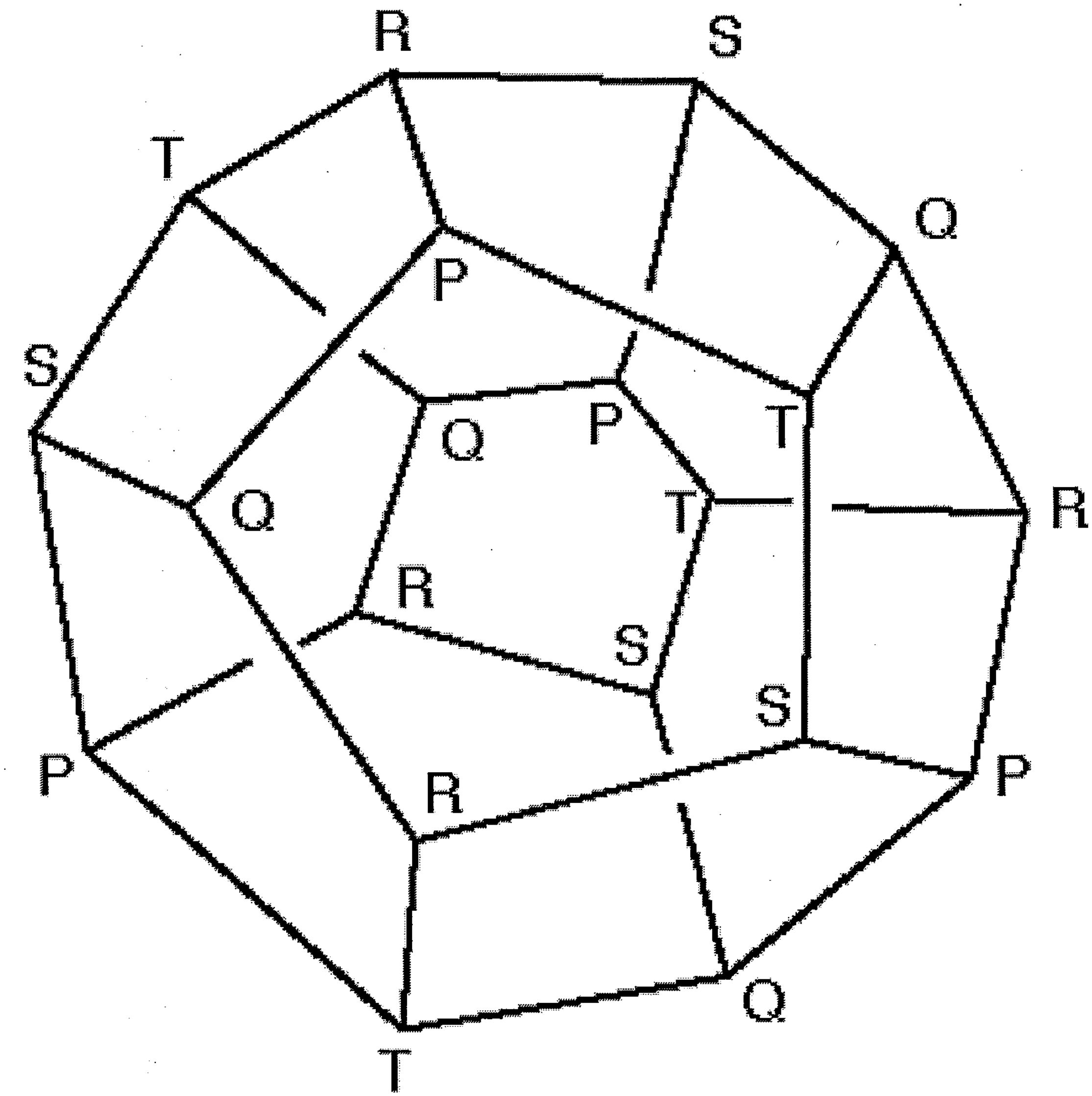

Figure 97 - The Dodecahedral Space as Identification space from a Solid Dodecahedron