Entanglement Criteria - Quantum and Topological

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ABSTRACT

This paper gives a criterion for detecting the entanglement of a quantum state, and uses it to study the relationship between topological and quantum entanglement. It is fundamental to view topological entanglements such as braids as *entanglement operators* and to associate to them unitary operators that are capable of creating quantum entanglement. The entanglement criterion is used to explore this connection.

Keywords: quantum entanglement,topological entanglement, braiding, linking, Yang-Baxter operator, non-locality

1. INTRODUCTION

This paper discusses relationships between topological entanglement and quantum entanglement. The present paper is a sequel to our previous work[5] and it explores more deeply the ideas in that earlier paper. We propose that it is fundamental to view topological entanglements such as braids as *entanglement operators* and to associate to them unitary operators that perform quantum entanglement. One can compare the way the unitary operator corresponding to an elementary braid has (or has not) the capacity to entangle quantum states. One can examine the capacity of the same operator to detect linking. The detection of linking involves working with closed braids or with link diagrams. In both cases, the algorithms for computing link invariants are very interesting to examine in the light of quantum computing. These algorithms can usually be decomposed into one part that is a straight composition of unitary operators, and hence can be seen as a sequence of quantum computer instructions, and another part that can be seen either as preparation/detection, or as a quantum network with cycles in the underlying graph. For the background on knotting, linking, Yang-Baxter equation and state sum invariants of links, we refer the reader to our paper.[5]

The paper is organized as follows. Section 2 introduces our result giving a set of equations that characterize entanglement of a quantum state. This section then shows how this criterion can be used to analyze a general class of solutions to the Yang-Baxter equation and their corresponding link invariants. The section ends with a discussion of invariants of local unitary transformations in relation to this algebraic criterion for entanglement. Section 3 is a discussion of the structure of entanglement in relation to measurement. In particular, we discuss the EPR thought experiment and discuss the Bell inequalities in the $CHSH$ formulation. We use our criterion for entanglement to show how unentangled states cannot violate the Bell inequalties and we give an example of an entangled state that also does not violate the Bell inequalities for a given choice of operators. The upshot of this discussion is that just as there is a complex relationship between quantum entanglement and topological entanglement, there is also a complex relationship between quantum entanglement and non-locality. One would hope for a deeper connection between topology and non-locality. It is our hope that this study will help in that goal.

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2. ENTANGLEMENT CRITERIA

Let

$$\phi = \Sigma a_\alpha |\alpha>$$

where α runs over all binary strings of length n. Thus we regard ϕ as a general element of $V^{\otimes n}$ where V is a complex vector space of dimension two with basis $\{|0>, |1>\}$.

With α a binary string as above, let $|\alpha|$ denote the number of ones in the string. Thus $|1101| = 3$. Let e_i denote the string of length n with all zeroes except for a 1 in the i-th place. We shall write $i \in \alpha$ to mean that the i-th place in the string α is occupied by a 1. Thus $i \in e_i$ and $2 \in 11$.

Theorem. The state $\phi = \Sigma a_\alpha |\alpha>$ is unentangled if and only if the following equations are satisfied for each coefficient in ϕ.:

$$a_{0...0}^{|\alpha|-1} a_\alpha = \Pi_{i \in \alpha} a_{e_i}.$$

Proof. If ϕ is unentangled then ϕ has the form of an n-fold tensor product as shown below, with k a complex constant

$$\phi = k(|0\cdots0> +A_{1...0}|1\cdots0>)(|0\cdots0> +A_{01...0}|01\cdots0>)$$

$$\cdots(|0\cdots0> +A_{0...1}|0\cdots1>)$$

$$= k\Pi_{i=1}^n(|e_0> +A_{e_i}|e_i>)$$

$$= k\Sigma_\alpha A_\alpha |\alpha>$$

where α runs over all binary strings of length n, $A_{0...0} = A_{e_0} = 1$, and

$$A_\alpha = \Pi_{i \in \alpha} A_{e_i}.$$

Since here

$$a_\alpha = kA_\alpha,$$

it follows at once from this decomposition that

$$a_{0...0}^{|\alpha|-1} a_\alpha = \Pi_{i \in \alpha} a_{e_i}.$$

Conversely, if the coefficients of ϕ satisfy this formula, then it is easy to see that ϕ factorizes in the above form. This completes the proof of the Theorem.

Remark. The simplest example of this theorem is the case of two entangled qubits. By the theorem,

$$\phi = a_{00}|00> +a_{01}|01> +a_{10}|10> +a_{11}|11>$$

is unentangled exactly when

$$a_{00}a_{11} = a_{10}a_{01}.$$

This is the criterion that we checked by hand earlier in the paper. For three qubits we have the equations

$$a_{000}a_{110} = a_{100}a_{010}$$

$$a_{000}a_{101} = a_{100}a_{001}$$

$$a_{000}a_{011} = a_{010}a_{001}$$

$$a_{000}^2 a_{111} = a_{100}a_{010}a_{001}.$$

We are now in a position to compare topological and quantum entanglement for the larger class of solutions to the Yang-Baxter equation that we mentioned in our previous paper.[5] Recall that if $M_{\alpha,\beta}$ is a matrix with entries on the unit circle (α and β range over all binary strings of length n), then we can define

$$R|\alpha,\beta> = M_{\alpha,\beta}|\beta,\alpha>,$$

and R is a unitary solution to the Yang-Baxter equation. See[5] for our previous discussion of this solution.

Corresponding to a link diagram K, we define[5] a state summation S_K by summing over all assignments of binary strings α to each component of the link K, (colorings of the diagram K) and taking the product of the matrix entries $M_{\alpha,\beta}$ associated via R to each crossing in the colored diagram. It then easy to see that if K is a link of two components K_1 and K_2, then

$$S_K = \Sigma_{\alpha \neq \beta} M_{\alpha,\alpha}^{w(K_1)} M_{\beta,\beta}^{w(K_2)} M_{\alpha,\beta}^{2lk(K_1,K_2)} + \Sigma_\alpha M_{\alpha,\alpha}^{w(K_1)} M_{\alpha,\alpha}^{w(K_2)} M_{\alpha,\alpha}^{2lk(K_1,K_2)}.$$

Here $w(K_i)$ denotes the writhe of the component K_i. That is, $w(K_i)$ denotes the sum of the signs of all the self-crossings of K_i. Recall that the linking number of K, denoted $lk(K) = lk(K_1,K_2)$, is one-half the sum of the signs of the crossings shared by K_1 and K_2. Finally, the writhe of K, denoted $w(K)$, stands for the sum of all the signs in the diagram K whether they are between two components, or with a component and itself. The following formula is an immediate consequence of these definitions

$$w(K) = w(K_1) + w(K_2) + 2lk(K_1,K_2).$$

In order to separate out the topological dependence so that we can see how this state summation can detect the linking number of the link, it is useful to assume that $M_{\alpha,\alpha} = \lambda$ is a constant independent of the binary string α. We shall make this assumption from now on. We can then write the formula for the state sum in the form

$$S_K = \Sigma_{\alpha \neq \beta} \lambda^{w(K_1)} \lambda^{w(K_2)} M_{\alpha,\beta}^{2lk(K_1,K_2)} + \Sigma_\alpha \lambda^{w(K_1)} \lambda^{w(K_2)} \lambda^{2lk(K_1,K_2)}$$

$$= \Sigma_{\alpha \neq \beta} \lambda^{w(K)} (M_{\alpha,\beta}^2/\lambda^2)^{lk(K_1,K_2)} + \Sigma_\alpha \lambda^{w(K)}$$

$$= \lambda^{w(K)} \Sigma_{\alpha \neq \beta} (M_{\alpha,\beta}^2/\lambda^2)^{lk(K_1,K_2)} + 2^n.$$

Thus we obtain the topological invariant Z_K defined by the equation

$$Z_K = \lambda^{-w(K)} S_K = \Sigma_{\alpha \neq \beta} (M_{\alpha,\beta}^2/\lambda^2)^{lk(K_1,K_2)} + 2^n.$$

We conclude, as in the case of two qubits, that Z_K can detect linking number so long as $M_{\alpha,\beta}^2 \neq \lambda^2$.

Now lets return the the matrix R and see about its entanglement capabilities. We are assuming that all the $M_{\alpha,\alpha}$ are equal to λ. Then if $\phi = \Sigma_{\alpha,\beta}|\alpha,\beta>$, then

$$R\phi = \Sigma_{\alpha,\beta} M_{\beta,\alpha}|\alpha,\beta>.$$

Using our entanglement criterion (and writing 0 for the zero string $0\cdots0$), we conclude that the state $R\phi$ is unentangled exactly when the following equations are satisfied for all α and β.

$$\lambda^{|\alpha|+|\beta|-1} M_{\alpha,\beta} = \Pi_{i \in \alpha} M_{e_i,0} \Pi_{j \in \beta} M_{0,e_j}.$$

In the case $\alpha = \beta$ this equation becomes

$$\lambda^{|\alpha|+|\alpha|-1} M_{\alpha,\alpha} = \Pi_{i \in \alpha} M_{e_i,0} \Pi_{j \in \alpha} M_{0,e_j}$$

$$\lambda^{|\alpha|+|\alpha|-1} \lambda = \Pi_{i \in \alpha} M_{e_i,0} \Pi_{j \in \alpha} M_{0,e_j}$$

$$\lambda^{2|\alpha|} = \Pi_{i \in \alpha} M_{e_i,0} \Pi_{j \in \alpha} M_{0,e_j}$$

Thus, letting

$$m_{\alpha,0} = \Pi_{i \in \alpha} M_{e_i,0}$$

and

$$m_{0,\alpha} = \Pi_{j \in \alpha} M_{0,e_j}$$

we have

$$\lambda^{2|\alpha|} = m_{\alpha,0} m_{0,\alpha}$$

and

$$\lambda^{|\alpha|+|\beta|-1} M_{\alpha,\beta} = m_{\alpha,0} m_{0,\beta}.$$

From these formulas we find that

$$m_{0,\alpha} m_{\alpha,0} m_{0,\beta} m_{\beta,0} \lambda^{-2} M_{\alpha,\beta}^2 = m_{\alpha,0} m_{0,\beta} m_{\alpha,0} m_{0,\beta}.$$

Hence

$$m_{0,\alpha} m_{\beta,0} \lambda^{-2} M_{\alpha,\beta}^2 = m_{0,\beta} m_{\alpha,0}.$$

Therefore

$$M_{\alpha,\beta}/\lambda^2 = (m_{\alpha,0}/m_{0,\alpha})(m_{0,\beta}/m_{\beta,0}).$$

The state $R\phi$ is unentangled exactly when this last equation is satisfied. We see from this that *if the matrix M is symmetric, i.e. if $M_{\alpha,\beta} = M_{\beta,\alpha}$ for all α and β then the invariant Z_K detects linking exactly when $R\phi$ is an entangled state.* On the other hand, if M is not symmetric, then the invariant can detect linking even when the state $R\phi$ is unentangled. This is the generalization of our previous results, using the entanglement criteria proved here. The generalization shows that while there is no necessary relation between quantum entanglement and the ability to detect topological linking, there are cases of invariants where the two properties are directly related to one another.

2.1. More About Entanglement Criteria

An element of the unitary group $U(2)$ can be represented by a matrix U of the type shown below, with λ and μ complex numbers such that $\lambda\bar{\lambda} + \mu\bar{\mu} = e^{i\theta}$ so that $Det(U) = e^{i\theta}$.

$$U = \left(\begin{array}{cc} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{array} \right)$$

We wish to consider the relationship between our algebraic criteria for entanglement and the results of performing local unitary transformations on a state. To this end note that

$$U|0> = \lambda|0> -\bar{\mu}|1>$$

$$U|1> = \mu|0> +\bar{\lambda}|1>,$$

and that if

$$\psi = \Sigma a_{\alpha 0 \beta}|\alpha 0 \beta> +a_{\alpha 1 \beta}|\alpha 1 \beta>,$$

then

$$(I^k \otimes U \otimes I^l)\psi = \Sigma a_{\alpha 0 \beta}|\alpha> (\lambda|0> -\bar{\mu}|1>)|\beta> +a_{\alpha 1 \beta}|\alpha> (\mu|0> +\bar{\lambda}|1>)|\beta>$$

$$= \Sigma(\lambda a_{\alpha 0 \beta} + \mu a_{\alpha 1 \beta})|\alpha 0 \beta> +(-\bar{\mu} a_{\alpha 0 \beta} + \bar{\lambda} a_{\alpha 1 \beta})|\alpha 1 \beta>$$

$$= \Sigma a'_{\alpha 0\beta}|\alpha 0\beta> + a'_{\alpha 1\beta}|\alpha 1\beta> .$$

Thus

$$a'_{\alpha 0\beta} = \lambda a_{\alpha 0\beta} + \mu a_{\alpha 1\beta}$$

$$a'_{\alpha 1\beta} = -\bar{\mu} a_{\alpha 0\beta} + \bar{\lambda} a_{\alpha 1\beta}.$$

Let

$$v_{\alpha i\beta} = \begin{pmatrix} a_{\alpha 0\beta} \\ a_{\alpha 1\beta} \end{pmatrix}$$

$$v'_{\alpha i\beta} = \begin{pmatrix} a'_{\alpha 0\beta} \\ a'_{\alpha 1\beta} \end{pmatrix}$$

Then

$$v'_{\alpha i\beta} = U v_{\alpha i\beta}$$

Hence if

$$M = \begin{pmatrix} a_{\alpha 0\beta} & a_{\gamma 0\delta} \\ a_{\alpha 1\beta} & a_{\gamma 1\delta} \end{pmatrix}$$

then $|Det(M)|^2$ is invariant under the local coordinate transformation U since with

$$M' = \begin{pmatrix} a'_{\alpha 0\beta} & a'_{\gamma 0\delta} \\ a'_{\alpha 1\beta} & a'_{\gamma 1\delta} \end{pmatrix}$$

we have that $M' = UM$.

There are many choices of these two by two determinants that are invariant under local coordinate transformations. It is easy to show that they all vanish for an unentangled state. We conjecture that if all of them vanish, then the state is unentangled. This point will be taken up in a sequel to this paper. *Note that the value of $|Det(M)|^2$ is only an invariant for the specific local transformation with which it is associated,* but that in the case of two qubits the non-zero values of $|Det(M)|^2$ are exactly $|a_{00}a_{11} - a_{01}a_{10}|^2$ which we know to determine entanglement in this case. We see here that for two qubits the value of $|Det(M)|^2$ *is* invariant under all local unitary transformations of the state. This can also be verified by a density matrix calculation.

3. A REMARK ABOUT EPR, ENTANGLEMENT AND BELL'S INEQUALITY

It is remarkable that the simple algebraic situation of an element in a tensor product that is not itself a a tensor product of elements of the factors corresponds to subtle nonlocality in physics. It helps to place this algebraic structure in the context of a gedanken experiment to see where the physics comes in. Consider

$$S = (|0 > |1 > + |1 > |0 >)/\sqrt{2}.$$

In an EPR thought experiment, we think of two "parts" of this state that are separated in space. We want a notation for these parts and suggest the following:

$$L = (\{|0 >\}|1 > + \{|1 >\}|0 >)/\sqrt{2},$$

$$R = (|0 > \{|1 >\} + |1 > \{|0 >\})/\sqrt{2}.$$

In the left state L, an observer can only observe the left hand factor. In the right state R, an observer can only observe the right hand factor. These "states" L and R together comprise the EPR state S, but they are accessible individually just as are the two photons in the usual thought experiement. One can transport L and R individually and we shall write

$$S = L * R$$

to denote that they are the "parts" (but not tensor factors) of S.

The curious thing about this formalism is that it includes a little bit of macroscopic physics implicitly, and so it makes it a bit more apparent what EPR were concerned about. After all, lots of things that we can do to L or R do not affect S. For example, transporting L from one place to another, as in the original experiment where the photons separate. On the other hand, if Alice has L and Bob has R and Alice performs a local unitary transformation on "her" tensor factor, this applies to both L and R since the transformation is actually being applied to the state S. This is also a "spooky action at a distance" whose consequence does not appear until a measurement is made.

To go a bit deeper it is worthwhile seeing what entanglement, in the sense of tensor indecomposability, has to do with the structure of the EPR thought experiment. To this end, we look at the structure of the Bell inequalities using the $CHSH$ formalism as explained in book by Nielsen and Chuang.[10] For this we use the following observables with eigenvalues ± 1.

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_1,$$

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_1,$$

$$S = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}_2 / \sqrt{2},$$

$$T = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}_2 / \sqrt{2}.$$

The subscripts 1 and 2 on these matrices indicate that they are to operate on the first and second tensor factors, repsectively, of a quantum state of the form

$$\phi = a|00 > + b|01 > + c|10 > + d|11 > .$$

To simplify the results of this calculation we shall here assume that the coefficients a, b, c, d are real numbers. We calculate the quantity

$$\Delta = <\phi|QS|\phi> + <\phi|RS|\phi> + <\phi|RT|\phi> - <\phi|QT|\phi>,$$

finding that

$$\Delta = (2 - 4(a+d)^2 + 4(ad - bc))/\sqrt{2}.$$

Classical probability calculation with random variables of value ± 1 gives the value of $QS + RS + RT - QT = \pm 2$ (with each of Q, R, S and T equal to ± 1). Hence the classical expectation satisfies the Bell inequality

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2.$$

That quantum expectation is not classical is embodied in the fact that Δ can be greater than 2. The classic case is that of the Bell state

$$\phi = (|01> - |10>)/\sqrt{2}.$$

Here

$$\Delta = 6/\sqrt{2} > 2.$$

In general we see that the following inequality is needed in order to violate the Bell inequality

$$(2 - 4(a+d)^2 + 4(ad - bc))/\sqrt{2} > 2.$$

This is equivalent to

$$(\sqrt{2} - 1)/2 < (ad - bc) - (a+d)^2.$$

Since we know that ϕ is entangled exactly when $ad - bc$ is non-zero, this shows that an unentangled state cannot violate the Bell inequality. This formula *also* shows that it is possible for a state to be entangled and yet not violate the Bell inequality. For example, if

$$\phi = (|00> - |01> + |10> + |11>)/2,$$

then $\Delta(\phi)$ satisfies Bell's inequality, but ϕ is an entangled state. We see from this calculation that entanglement in the sense of tensor indecomposability, and entanglement in the sense of Bell inequality violation for a given choice of Bell operators are not equivalent concepts. On the other hand, Benjamin Schumacher has pointed out[11] that any entangled two-qubit state will violate Bell inequalities for an appropriate choice of operators. We will expand the discussion of this point in a joint paper[3] under preparation. This deepens the context for our question of the relationship between topological entanglement and quantum entanglement. The Bell inequality violation is an indication of true quantum mechanical entanglement. One's intuition suggests that it is *this* sort of entanglement that should have a topological context. We will continue in the search for that context.

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