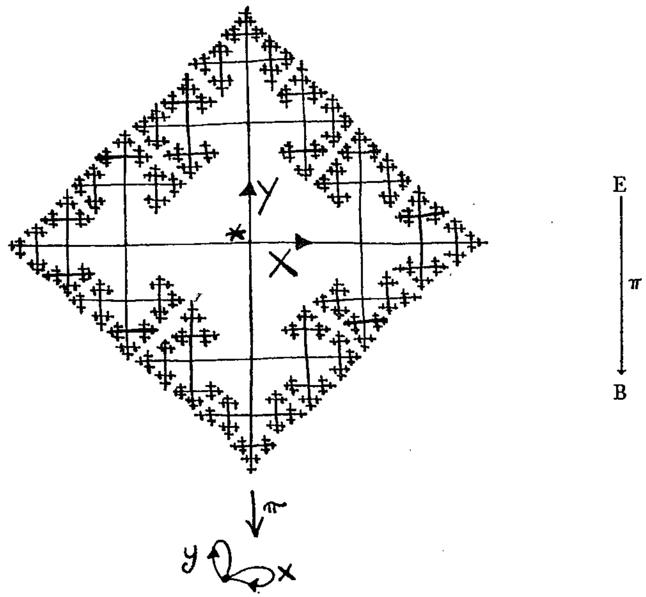
From "On Knots" by L. Kauffman
Princeton Univ Press (1987)

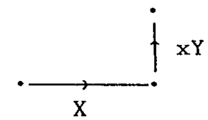
XI

FREE DIFFERENTIAL CALCULUS



Here is a picture of the figure eight (B) and its universal covering space E. Now $\pi_1(B) = (x,y\,|\;\;)$, the free group on two generators. Let $G = (x,y\,|\;\;)$ and note that G is the group of automorphisms of E over B. Thus E has "generating" 1-simplices X and Y as depicted. X is the lift of x as an element in π_1 starting at $*$. Y is the lift of y. By regarding E as the set of translates of X and Y under the action of G, we can write the lift of any word $\omega \in \pi_1(B)$ as a formal sum of simplices with coefficients in G. Thus (letting ω denote the lift),

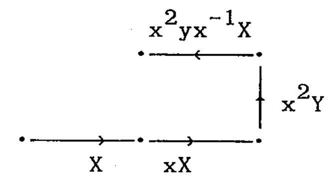
$$\widetilde{xy} = X + xY$$



Note that we lift in lexicographic order. <u>Therefore</u>

$$\widetilde{\omega_1\omega_2} = \widetilde{\omega}_1 + \omega_1\widetilde{\omega}_2.$$

Example: $x^2yx^{-1} = \omega$.



$$\tilde{\omega} = \tilde{x} + x(\widetilde{xyx^{-1}})$$
$$= X + x(\tilde{x} + x\widetilde{yx^{-1}}) \qquad (\widetilde{x^{-1}} = x^{-1}X)$$
$$= X + xX + x^2(\tilde{y} + y\widetilde{x^{-1}})$$
$$\tilde{\omega} = X + xX + x^2Y + x^2yx^{-1}X.$$

Now collect terms.

$$\tilde{\omega} = (1+x+x^2yx^{-1})X + (x^2)Y.$$

The coefficients belong to $\Gamma = Z[G]$, the group ring o over the integers.

The coefficient of X is called $\partial\omega/\partial x$.

The coefficient of Y is called $\partial\omega/\partial y$.

$$\tilde{\omega} = \left[\frac{\partial\omega}{\partial x}\right]X + \left[\frac{\partial\omega}{\partial y}\right]Y.$$

These definitions extend to any number of variables, and form the beginning of Fox's free differential calculus. From this point of view it is easy to derive some rules for differentiation:

1°. $\frac{\partial(\omega\tau)}{\partial x} = \frac{\partial\omega}{\partial x} + \omega\frac{\partial\tau}{\partial x}$ (and same for $\partial/\partial y$). This is just a restatement of $\widetilde{\omega\tau} = \tilde{\omega}+\omega\tilde{\tau}$.

2°. Since, by definition, $\frac{\partial}{\partial x}(1) = 0$ and $1 = \omega^{-1}\omega$, we have

$$0 = \frac{\partial(1)}{\partial x} = \frac{\partial(\omega^{-1}\omega)}{\partial x} = \frac{\partial\omega^{-1}}{\partial x} + \omega^{-1}\frac{\partial\omega}{\partial x}.$$

Hence $\frac{\partial\omega^{-1}}{\partial x} = -\omega^{-1}\frac{\partial\omega}{\partial x}$.

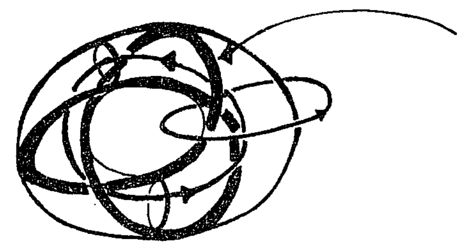
3°. For $n > 0$, $\frac{\partial x^n}{\partial x} = 1 + x + x^2 + \cdots + x^{n-1}$. (Since $x^n = (1+x+x^2+\cdots+x^{n-1})X$.) Hence for $n < 0$

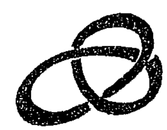
$$\frac{\partial x^n}{\partial x} = -x^n(1+x+x^2+\cdots+x^{|n|-1}).$$

The Jacobian Matrix

Here is Fox's algorithm for computing the Alexander polynomial from a presentation of $\pi_1(S^3-K)$. Let $\pi_1(S^3-K) = (x_1,\cdots,x_n|r_1,\cdots,r_m)$ be a presentation. Regard $r_1,\cdots,r_m$ as elements in the free group generated by $x_1,\cdots,x_n$. Form the Jacobian matrix $J = \left[\frac{\partial r_i}{\partial x_j}\right]$. Let $\phi : \pi_1(S^3-K) \longrightarrow Z = (t|\ )$ be the Abelianizing map. Let $J^\phi = \left[\frac{\partial r_i}{\partial x_j}\right]^\phi$ be the image of the Jacobian matrix under the map. Its entries are now in $Z[t,t^{-1}]$. $\Delta_K(t)$ is, up to balance, any generator of the ideal generated by largest minors of J^ϕ. (This is a principal ideal.)

Let $K_{a,b}$ be a torus knot of type a,b. Here $\gcd(a,b) = 1$. We can see that $\pi_1(S^3-K_{a,b}) = (\alpha,\beta|\alpha^a = \beta^b)$ by looking at $S^3-K_{a,b}$ as the union of pieces interior and exterior to the torus where $K_{a,b}$ lives, and using the Seifert-VanKampen Theorem [MA].





$$G = \pi_1(S^3-K_{a,b}) = (\alpha,\beta|\alpha^a = \beta^b).$$

It turns out that we can write a relation $A = B$ in the form $A-B$ and put $\left[\frac{\partial(A-B)}{\partial x_i}\right]^\phi$ into the Jacobian.

In this case the map $\phi : G \longrightarrow (t|\ )$ is $\phi(\alpha) = t^b$, $\phi(\beta) = t^a$ (remember $\gcd(a,b) = 1$). Thus

$$J = \left[\frac{\partial(\alpha^a-\beta^b)}{\partial\alpha}, \frac{\partial(\alpha^a-\beta^b)}{\partial\beta}\right]$$

$$J = [1+\alpha+\alpha^2+\cdots+\alpha^{a-1}, -(1+\beta+\beta^2+\cdots+\beta^{b-1})]$$

$\therefore\ J^{\phi} = [1+t^b+t^{2b}+\cdots+t^{(a-1)b},\ -(1+t^a+t^{2a}+\cdots+t^{a(b-1)})]$.

Now

$$1 + t^b + t^{2b} + \cdots + t^{(a-1)b} = \left[\frac{t^{ab}-1}{t^b-1}\right]$$

$$1 + t^a + t^{2a} + \cdots + t^{a(b-1)} = \left[\frac{t^{ab}-1}{t^a-1}\right]$$

$$\Delta(t) \doteq \gcd\left[\frac{t^{ab}-1}{t^b-1},\ \frac{t^{ab}-1}{t^a-1}\right]$$

$$\Delta(t) \doteq \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)}.$$

This is the Alexander polynomial for the torus knot of type (a,b).

$$\Delta(t) \doteq \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)}.$$

If <u>a</u> and <u>b</u> are <u>both</u> <u>odd</u> then

$$\Delta(-1) \doteq \frac{(-2)(-2)}{(-2)(-2)} = 1.$$

Hence these knots have vanishing Arf invariant.

Suppose a is <u>odd</u> and b is <u>even</u>. Then $\Delta(-1)$ is indeterminate in this form. So apply L'Hospital's Rule.

$$\Delta(-1) \doteq \left[\frac{((ab)t^{ab-1})(t-1)+(t^{ab}-1)}{(at^{a-1})(t^b-1)+(t^a-1)(bt^{b-1})}\right]_{t=-1}$$

$$\doteq \frac{(-ab)(-2)+0}{(-2)(-b)}.$$

$$\therefore\ \Delta(-1) \doteq a.$$

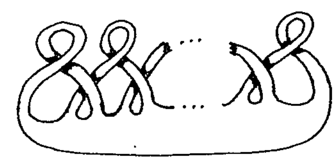
Thus in this case $\mathrm{ARF}(K_{a,b}) = 0$ or 1 according as

$a \equiv \pm 1$ or $\pm 3 \pmod 8$.

$$\mathrm{ARF}(K_{3,2}) = 1$$
$$\mathrm{ARF}(K_{5,2}) = 1$$
$$\mathrm{ARF}(K_{7,2}) = 0$$
$$\mathrm{ARF}(K_{9,2}) = 0$$
$$\mathrm{ARF}(K_{11,2}) = 1$$
$$\mathrm{ARF}(K_{13,2}) = 1$$
$$\cdots$$

A four-fold periodicity.

<u>Exercise 11.1</u>. $K_{n,2}$ has a spanning surface of form

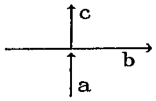


(1) Verify this periodicity (above) using topological script.

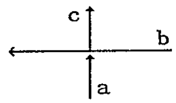
(2) Calculate $\nabla_K(t)$ for $K_{3,4}$ by using the Seifert pairing.

<u>One more remark</u> about Alexander Polynomial and Free Differential Calculus:

We can use the Wirtinger Presentation [F1] for $\pi_1(S^3-K)$. This associates one meridianal generator to eac arc in the knot diagram, and one relation to each crossing:



$c = b^{-1}ab,$



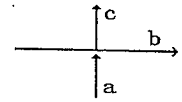
$c = bab^{-1}.$

Let $\phi : G = \pi_1(S^3-K) \longrightarrow Z = (t| \ )$. Then ϕ(any generator in Wirtinger) = t. Each relation $c = b^{\pm1} ab^{\mp1} = w$ gives rise to a relation in $H_1(X_\infty)$ of the form $[c] = \left[\frac{\partial\omega}{\partial a}\right]^{\phi}[a] + \left[\frac{\partial\omega}{\partial b}\right]^{\phi}[b]$. Since $\left[\frac{\partial\omega}{\partial c}\right]^{\phi} = 1$ this relation corresponds to a row in the Jacobian matrix for Fox's algorithm. In the case of this presentation, the determinant of any $(n-1) \times (n-1)$ minor will produce the Alexander polynomial. (The knot has n crossings.)

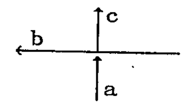
<u>Exercise 11.2</u>. a) Use the notation of the above remark and show that if $w = b^{-1}ab$ then $\left[\frac{\partial\omega}{\partial a}\right]^{\phi} = (1-t^{-1})$ and $\left[\frac{\partial\omega}{\partial b}\right]^{\phi} = t^{-1}$, while if $w = bab^{-1}$ then $\left[\frac{\partial\omega}{\partial a}\right]^{\phi} = (1-t)$ and $\left[\frac{\partial\omega}{\partial b}\right]^{\phi} = t$.

b) Choose a knot and calculate its Alexander polynomial using the Wirtinger presentation.

c) Part a) of this exercise shows that if [a], [b], [c] connote elements in $H_1(X_\infty)$ that correspond to lifts of the elements $a,b,c \in \pi_1(S^3-K)$, then the following relations ensue in $H_1(X_\infty)$ as a $Z[t,t^{-1}]$ module:



$\Longrightarrow [c] = (1-t^{-1})[a] + t^{-1}[b]$ (*



$\Longrightarrow [c] = (1-t)[a] + t[b].$

This part of the exercise asks you to compare these patterns with the patterns that arise from trying to represent the fundamental group as a group of affine transformations of the complex plane [DR]: Let

$$\mathcal{L} = \{T : \mathbb{C} \longrightarrow \mathbb{C} \mid T(z) = \alpha z+\beta, \text{ where } \alpha \text{ and } \beta \text{ are elements of } \mathbb{C}\}.$$

Call this the <u>affine group</u>. Let $G = \pi_1(S^3-K)$ with the Wirtinger presentation.

(i) Let $[\alpha,\beta]$ denote $T(z) = \alpha z+\beta$. Show that $[\alpha,\beta][\gamma,\delta] = [\alpha\gamma,\alpha\delta+\beta]$ where [ ][ ] denotes composition of maps.

(ii) Suppose $\phi : G \longrightarrow \mathcal{L}$ is a homomorphism of groups. Show that $\phi(a) = [t,\psi(a)]$, $t \in \mathbb{C}$, for a fixed t independent of the choice of a, given that $\ell k(a,K) = +1$. Hence this holds for all the Wirtinger generators. Given an element a with $\ell k(a,K) = +1$, let $\langle a\rangle = \psi(a)$. Thus $\phi(a) = [t,\langle a\rangle]$.