Math 423, Spring 2010, Final Exam — Solutions

1. $v-e+f=2$, $3v=2e$, $f=\sum_{i=2}^{\infty} f_i$

$2e=\sum_{i=2}^{\infty} i f_i \Rightarrow 6v-6e+6f=12$

$\Rightarrow 6f-2e=12$

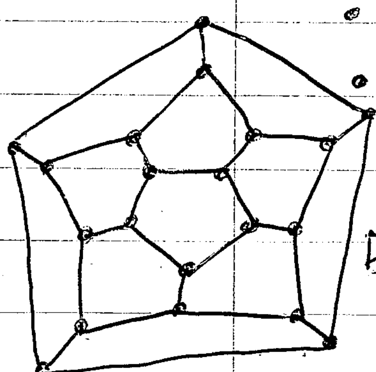
$\Rightarrow \sum_i 6f_i - \sum_i i f_i = 12$

$\Rightarrow \sum_i (6-i) f_i = 12$

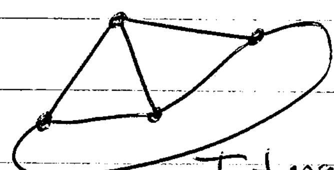
$\Rightarrow \boxed{4f_2+3f_3+2f_4+f_5=12+f_7+2f_8+3f_9+\cdots}$

$\Rightarrow$ there must be a small (2, 3, 4 or 5 sided) region.

- hexagonal plane lattice (honey comb) has only 6 sided regions.
- dodecahedral graph has 12 5 sided regions.
- tetrahedral graph has only 3 sided regions.

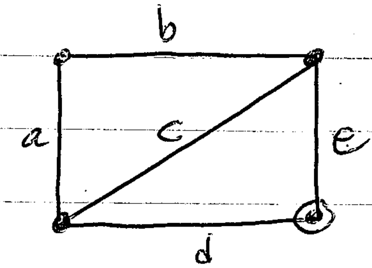


Dodec



Tetra

2.



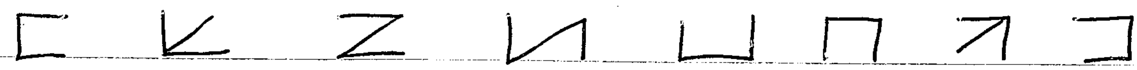
Wang algebra:

$\mathcal{W}=(a+b)(a+c+d)(b+c+e)$

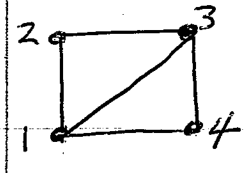
$=(\cancel{a^2}+ac+ad+ab+bc+bd)(b+c+e)$

$=\underline{abc}+abd+acd+\underline{abc}+bcd+ace+ade+abe+bce+bde$

$=abd+acd+bcd+ace+ade+abe+bce+bde$



Eight spanning trees.



Kirchhoff Matrix $\mathcal{K} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}$

$\mathcal{M}$

$$\det(\mathcal{M}) = \begin{vmatrix} -3 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -3 \end{vmatrix}$$

$$= -3\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= -3(6-1) - (-3-1) + (1+2)$$

$$= -15 + 4 + 3 = -8$$

Thus $|\det(\mathcal{M})| = 8$ = # spanning trees.

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3. $A = A(G)$

$$(A^k)_{ij} = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in \{1, 2, \ldots, n\}} A_{i\alpha_1} A_{\alpha_1 \alpha_2} \cdots A_{\alpha_{k-1} j}$$

$\Updownarrow$

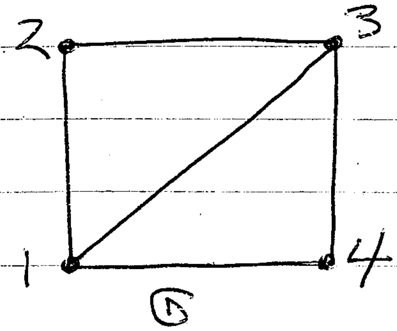
Nodes(G)

each term $= 1$ iff $[i \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{k-1} \to j]$ is a walk on G of length k.

$\therefore$ $(A^k)_{ij}$ = # walks of length k from i to j.

$P(t) = C_A(t) = \mathrm{Det}(A - tI)$ char poly.

Then $P(A) = 0$ & this gives recursion relation for $(A^k)_{ij}$.



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

adjacency matrix

$$p(t) = \begin{vmatrix} -t & 1 & 1 & 1 \\ 1 & -t & 1 & 0 \\ 1 & 1 & -t & 1 \\ 1 & 0 & 1 & -t \end{vmatrix} \quad (= \det(A - tI))$$

$$= \begin{vmatrix} 0 & 1-t^2 & 1+t & 1 \\ 1 & -t & 1 & 0 \\ 0 & 1+t & -t-1 & 1 \\ 0 & t & 0 & -t \end{vmatrix} = - \begin{vmatrix} 1-t^2 & 1+t & 1 \\ 1+t & -1-t & 1 \\ t & 0 & -t \end{vmatrix}$$

$$= -t \begin{vmatrix} 1+t & 1 \\ -1-t & 1 \end{vmatrix} + t \begin{vmatrix} 1-t^2 & 1+t \\ 1+t & -1-t \end{vmatrix}$$

$$= -t(1+t+1+t) + t((1-t^2)(-1-t) - (1+t)^2)$$

$$= -2t(1+t) + t(1+t)((1-t^2)(-1) - (1+t))$$

$$= -2t(1+t) + t(1+t)^2((1-t)(-1) - 1)$$

$$= -2t(1+t) + t(1+t)^2(t-2)$$

$$= (1+t)[-2t + t(1+t)(t-2)]$$

$$= (1+t)[-2t + t(t-2+t^2-2t)]$$

$$= (1+t)[-2t + t^2 - 2t + t^3 - 2t^2]$$

$$= (1+t)[-4t - t^2 + t^3]$$

Thus $p(t) = (t+1)\left[t^3 - t^2 - 4t\right]$

$$= t^4 - t^3 - 4t^2$$
$$+ t^3 - t^2 - 4t$$
$$p(t) = t^4 - 5t^2 - 4t$$

So we have $\boxed{A^4 = 5A^2 + 4A}$

by Cayley Hamilton Theorem.

<u>Check it</u> :

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 5 & 2 \\ 5 & 5 & 4 & 5 \\ 5 & 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 14 & 9 \\ 9 & 10 & 9 & 10 \\ 14 & 9 & 15 & 9 \\ 9 & 10 & 9 & 10 \end{bmatrix}$$

$$5A^2 + 4A = \begin{bmatrix} 15 & 5 & 10 & 5 \\ 5 & 10 & 5 & 10 \\ 10 & 5 & 15 & 5 \\ 5 & 10 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 0 \\ 4 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 14 & 9 \\ 9 & 10 & 9 & 10 \\ 14 & 9 & 15 & 9 \\ 9 & 10 & 9 & 10 \end{bmatrix}$$

$$= A^4$$

Thus we have $A^4 = 5A^2 + 4A$

So $A^{K+4} = 5A^{K+2} + 4A^K$

and $(A^{K+4})_{ij} = 5(A^{K+2})_{ij} + 4(A^K)_{ij}$

giving recursion with initial conds A, A^2, A^3, A^4 as computed.

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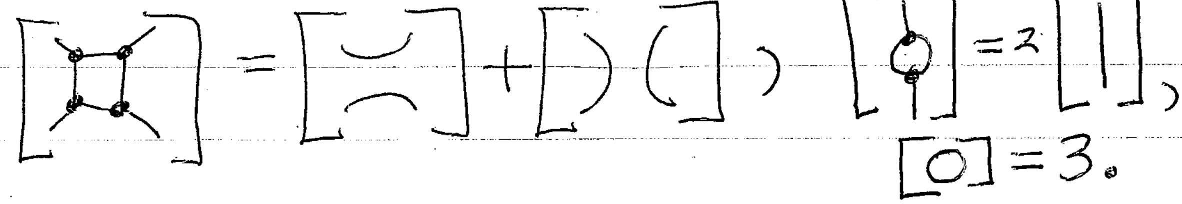
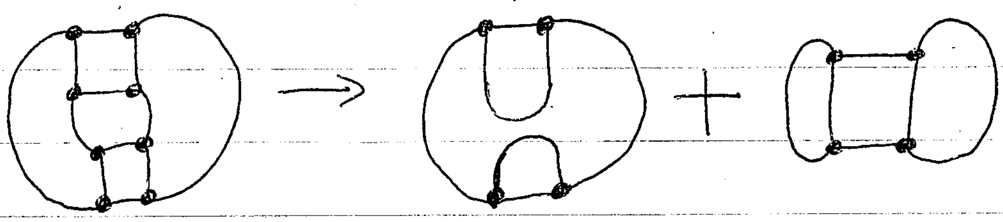
4. 



$= 2^2 \cdot 3 + 2^2 \cdot 3 = \underline{24}$.

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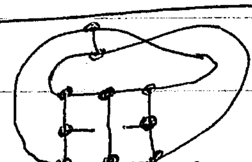
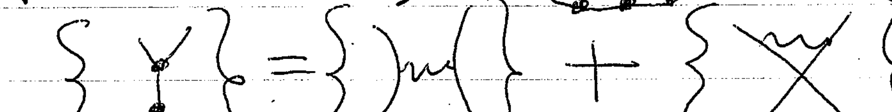
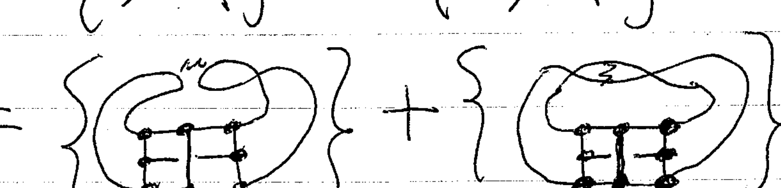
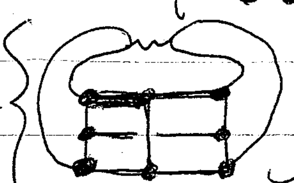
5. Petersen uncol.  $= P$

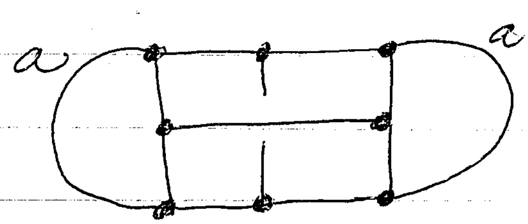
(Here)m(≡) ≠ (i.e. colored with unequal colors)

({G} = all colorings of G) Use 

$\Rightarrow \{P\} =$ 

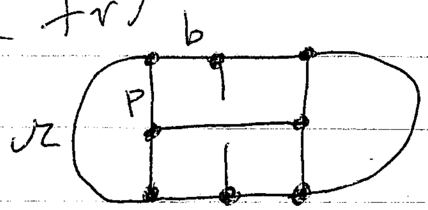
But  $= \phi$ empty set

That is $\nexists$ coloring of

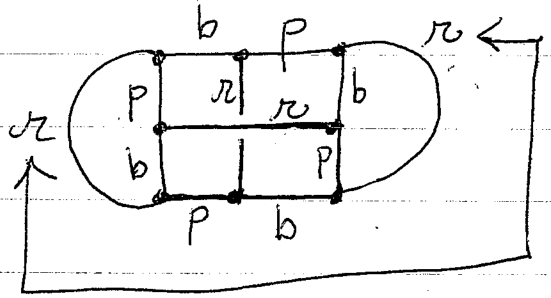


with, $a \neq a'$.

<u>Proof.</u> If try



then



forced same.

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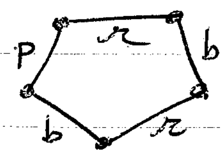
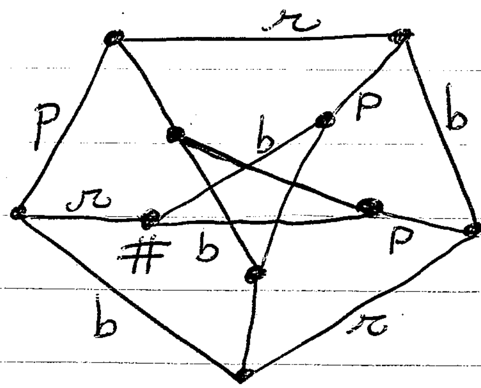
Since colorings of $\mathcal{P}$ would break up into two sets of solutions to the above with $a \neq a'$, $\mathcal{P}$ is uncolorable. //

<u>Ali's solution</u>: Generic coloring of a 5-cycle in a cubic graph is



(<u>one</u> extra color). But this leads to a contradiction in Petersen:



Hence Petersen is uncolorable.