On the Links–Gould Invariant of Links

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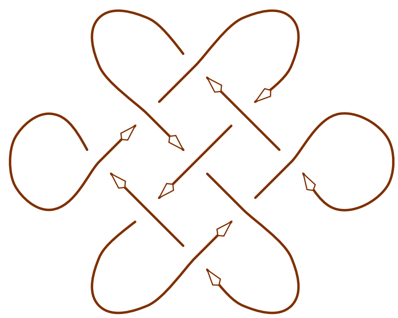

Abstract

We introduce and study in detail an invariant of $(1,1)$ tangles. This invariant, derived from a family of four dimensional representations of the quantum superalgebra $U_q[gl(2|1)]$, will be referred to as the Links–Gould invariant. We find that our invariant is distinct from the Jones, HOMFLY and Kauffman polynomials (detecting chirality of some links where these invariants fail), and that it does not distinguish mutants or inverses. The method of evaluation is based on an abstract tensor state model for the invariant that is quite useful for computation as well as theoretical exploration.

1 Introduction

Since the discovery of the Jones polynomial [14], several new invariants of knots, links and tangles have become available due to the development of sophisticated mathematical techniques. Among these, the quantum algebras as defined by Drinfeld [9] and Jimbo [13], being examples of quasi-triangular Hopf algebras, provide a systematic means of solving the Yang–Baxter equation and in turn may be employed to construct representations of the braid group. From each of these representations, a prescription exists to compute invariants of oriented knots and links [34, 39, 41], from which the Jones polynomial is recoverable using the simplest quantum algebra $U_q\,[sl(2)]$ in its minimal (2-dimensional) representation.

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From such a large class of available invariants, it is natural to ask if generalisations exist, with the view to gaining a classification. One possibility is to look to multiparametric extensions in order to see which invariants occur as special cases. A notable example is the HOMFLY invariant [10] which includes both the Jones and Alexander–Conway invariants [2, 6] as particular cases as well as the invariants arising from minimal representations of $U_q\left[sl(n)\right]$ [39]. Another is the Kauffman polynomial which includes the Jones invariant as well as those obtained from the quantum algebras $U_q\left[o(n)\right]$ and $U_q\left[sp(2n)\right]$ in the q-deformations of the defining representations [39].

The work of Turaev and Reshetikhin [35] shows that the algebraic properties of quantum algebras are such that an extension of this method to produce invariants of oriented tangles is permissible. A tangle diagram is a link diagram with free ends. An associated invariant takes the form of a tensor operator acting on a product of vector spaces. Zhang [40] has extended this formalism to the case of quantum superalgebras which are $\mathbb{Z}_2$-graded generalisations of quantum algebras.

Since quantum superalgebras give rise to nontrivial one-parameter families of irreducible representations, it is possible to utilise them for the construction of two variable invariants. This was first shown by Links and Gould [26] for the simplest case using the family of four dimensional representations of $U_q\left[gl(2|1)\right]$. It was also made known that a one variable reduction of this invariant coincides with a one variable reduction of the Kauffman polynomial by the use of the Birman–Wenzl–Murakami algebra. Extensions to more general representations of quantum superalgebras are discussed in [12].

Thus far, little has been investigated with regard to the Links–Gould invariant. Here we report on some properties and behaviour. The method of evaluating the invariant involves a prior construction of the quantum R-matrix associated with a family of four dimensional representations. Having obtained this matrix, the construction of the invariant follows from properties of ribbon Hopf (super)algebras and their representations. Here we consider the invariants of $(1, 1)$ tangles for the following reason: for invariants derived from representations of quantum superalgebras with zero q-superdimension, the corresponding invariant is also zero. If the representation is irreducible, the quantum superalgebra symmetry of the procedure ensures that the invariant of $(1, 1)$ tangles takes the form of some scalar multiple of the identity matrix. (See [35] for a discussion of this symmetry.) We take this scalar to be the invariant.

In this paper, we prove that the Links–Gould invariant is not able to distinguish between mutant links (§4.8), nor is it able to distinguish a knot from its inverse (i.e. from the knot obtained by reversing the orientation) (Proposition 3.2). However it is good at distinguishing some knots and links from their mirror images (see Propositions 2 and 4.1), and it is distinguished from the HOMFLY and Jones polynomials by this behaviour (see §4.5 for specific examples). We provide many examples and a complete description of the state model for the invariant in abstract tensor form. This description of the invariant directly facilitates the construction of a computer program in MATHEMATICA for calculation of the invariant.

2 Construction of the R Matrix

We consider the family of four dimensional representations of the quantum superalgebra $U_q\left[gl\left(2|1\right)\right]$, which depend on a free complex parameter α. This superalgebra has 7 simple generators $\{E^1{}_1, E^2{}_2, E^3{}_3, E^1{}_2, E^2{}_1, E^2{}_3, E^3{}_2\}$ on which we define a $\mathbb{Z}_2$ grading in terms of the natural grading on the indices $[1]=[2]=0$, $[3]=1$ by:

$$\left[E^i{}_j\right] = [i]+[j] \qquad (\mathrm{mod}\ 2)\,.$$

The $U_q\left[gl\left(2|1\right)\right]$ generators satisfy the commutation relations:

$$\begin{aligned}
\left[E^1{}_2, E^2{}_1\right] &= \left[E^1{}_1 - E^2{}_2\right]_q \\
\left\{E^2{}_3, E^3{}_2\right\} &= \left[E^2{}_2 + E^3{}_3\right]_q \\
\left[E^i{}_i, E^j{}_k\right] &= \delta^j{}_i E^i{}_k - \delta^i{}_k E^j{}_i, \qquad i,j,k = 1,2,3,
\end{aligned}$$

where [,] and { , } denote the usual commutator and anticommutator, respectively and we have employed the q *bracket*, defined for a wide class of objects x by:

$$[x]_q \triangleq \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Let $\{|i\rangle\}_{i=1}^4$ denote a basis for the four dimensional $U_q\left[gl\left(2|1\right)\right]$ module V. Consistent with the $\mathbb{Z}_2$ grading on $U_q\left[gl\left(2|1\right)\right]$, we grade the basis states by:

$$[|1\rangle] = [|4\rangle] = 0, \qquad [|2\rangle] = [|3\rangle] = 1.$$

We define a dual basis $\{\langle i|\}_{i=1}^4$; in component form, these are represented by the transpose complex conjugates of the original basis: $\langle i| = \overline{|i\rangle}^t \equiv |i\rangle^\dagger$. Then: $\langle i||j\rangle \equiv \langle i|j\rangle = \delta_{ij}$. In terms of these dual bases, we define a representation π of the $U_q\left[gl\left(2|1\right)\right]$ generators; their action on the basis vectors is given by:

$$\begin{aligned}
\pi\left(E^1{}_1\right) &= -|2\rangle\langle 2| - |4\rangle\langle 4| \\
\pi\left(E^2{}_2\right) &= -|3\rangle\langle 3| - |4\rangle\langle 4| \\
\pi\left(E^3{}_3\right) &= \alpha|1\rangle\langle 1| + (\alpha+1)\left(|2\rangle\langle 2| + |3\rangle\langle 3|\right) + (\alpha+2)|4\rangle\langle 4| \\
\pi\left(E^1{}_2\right) &= -|3\rangle\langle 2| \\
\pi\left(E^2{}_1\right) &= -|2\rangle\langle 3| \\
\pi\left(E^2{}_3\right) &= [\alpha]_q^{1/2}|1\rangle\langle 3| - [\alpha+1]_q^{1/2}|2\rangle\langle 4| \\
\pi\left(E^3{}_2\right) &= [\alpha]_q^{1/2}|3\rangle\langle 1| - [\alpha+1]_q^{1/2}|4\rangle\langle 2|.
\end{aligned}$$

Associated with $U_q\left[gl(2|1)\right]$ there is a co-product structure ($\mathbb{Z}_2$-graded algebra homomorphism) $\Delta : U_q\left[gl(2|1)\right] \rightarrow U_q\left[gl(2|1)\right] \otimes U_q\left[gl(2|1)\right]$ given by:

$$\begin{aligned}
\Delta({E^i}_i) &= I \otimes {E^i}_i + {E^i}_i \otimes I, \qquad i = 1,2,3, \\
\Delta({E^1}_2) &= {E^1}_2 \otimes q^{-\frac{1}{2}({E^1}_1 - {E^2}_2)} + q^{\frac{1}{2}({E^1}_1 - {E^2}_2)} \otimes {E^1}_2 \\
\Delta({E^2}_1) &= {E^2}_1 \otimes q^{-\frac{1}{2}({E^1}_1 - {E^2}_2)} + q^{\frac{1}{2}({E^1}_1 - {E^2}_2)} \otimes {E^2}_1 \\
\Delta({E^2}_3) &= {E^2}_3 \otimes q^{-\frac{1}{2}({E^2}_2 + {E^3}_3)} + q^{\frac{1}{2}({E^2}_2 + {E^3}_3)} \otimes {E^2}_3 \\
\Delta({E^3}_2) &= {E^3}_2 \otimes q^{-\frac{1}{2}({E^2}_2 + {E^3}_3)} + q^{\frac{1}{2}({E^2}_2 + {E^3}_3)} \otimes {E^3}_2 .
\end{aligned}$$

There exists another possible co-product structure: $\overline{\Delta}$, defined by $\overline{\Delta} = T \cdot \Delta$, where $T : U_q\left[gl\left(2|1\right)\right] \otimes U_q\left[gl\left(2|1\right)\right] \rightarrow U_q\left[gl\left(2|1\right)\right] \otimes U_q\left[gl\left(2|1\right)\right]$ is the twist map, defined for homogeneous elements $a, b \in U_q\left[gl\left(2|1\right)\right]$:

$$T\left(a \otimes b\right) = (-)^{[a][b]}\left(b \otimes a\right).$$

The tensor product module has the following decomposition with respect to the co-product for generic values of α:

$$V \otimes V = V_1 \oplus V_2 \oplus V_3. \tag{1}$$

We construct symmetry adapted bases $\{|\Psi_1^k\rangle\}_{k=1}^4$ and $\{|\Psi_3^k\rangle\}_{k=1}^4$, for the spaces V_1 and V_3 respectively in terms of the basis elements of V:

$$\begin{aligned}
|\Psi_1^1\rangle &= |1\rangle \otimes |1\rangle \\
|\Psi_2^1\rangle &= (q^\alpha + q^{-\alpha})^{-\frac{1}{2}}\left(q^{\alpha/2}|1\rangle \otimes |2\rangle + q^{-\alpha/2}|2\rangle \otimes |1\rangle\right) \\
|\Psi_3^1\rangle &= (q^\alpha + q^{-\alpha})^{-\frac{1}{2}}\left(q^{\alpha/2}|1\rangle \otimes |3\rangle + q^{-\alpha/2}|3\rangle \otimes |1\rangle\right) \\
|\Psi_4^1\rangle &= \left(q^\alpha + q^{-\alpha}\right)^{-\frac{1}{2}}[2\alpha+1]_q^{-\frac{1}{2}} \times \\
&\quad \left[[\alpha+1]_q^{\frac{1}{2}}\left(q^\alpha|1\rangle \otimes |4\rangle + q^{-\alpha}|4\rangle \otimes |1\rangle\right) + [\alpha]_q^{\frac{1}{2}}\left(q^{\frac{1}{2}}|2\rangle \otimes |3\rangle - q^{-\frac{1}{2}}|3\rangle \otimes |2\rangle\right)\right] \\
|\Psi_1^3\rangle &= \left(q^{\alpha+1} + q^{-\alpha-1}\right)^{-\frac{1}{2}}[2\alpha+1]_q^{-\frac{1}{2}} \times \\
&\quad \left[[\alpha]_q^{\frac{1}{2}}\left(q^{\alpha+1}|4\rangle \otimes |1\rangle + q^{-\alpha-1}|1\rangle \otimes |4\rangle\right) + [\alpha+1]_q^{\frac{1}{2}}\left(q^{-\frac{1}{2}}|3\rangle \otimes |2\rangle - q^{\frac{1}{2}}|2\rangle \otimes |3\rangle\right)\right] \\
|\Psi_2^3\rangle &= \left(q^{\alpha+1} + q^{-\alpha-1}\right)^{-\frac{1}{2}}\left(q^{(\alpha+1)/2}|4\rangle \otimes |2\rangle + q^{-(\alpha+1)/2}|2\rangle \otimes |4\rangle\right) \\
|\Psi_3^3\rangle &= \left(q^{\alpha+1} + q^{-\alpha-1}\right)^{-\frac{1}{2}}\left(q^{(\alpha+1)/2}|4\rangle \otimes |3\rangle + q^{-(\alpha+1)/2}|3\rangle \otimes |4\rangle\right) \\
|\Psi_4^3\rangle &= |4\rangle \otimes |4\rangle .
\end{aligned}$$

Dual bases $\{\langle\Psi_1^k|\}_{k=1}^4$ and $\{\langle\Psi_3^k|\}_{k=1}^4$, are found from the definitions:

$$\langle\Psi_j^k| = |\Psi_j^k\rangle^\dagger, \qquad k = 1,3, \qquad j = 1,\ldots,4, \tag{2}$$

$$\left(|i\rangle \otimes |j\rangle\right)^\dagger = (-)^{[|i\rangle][|j\rangle]}\left(\langle i| \otimes \langle j|\right), \qquad i,j = 1,\ldots,4. \tag{3}$$

Now, the general form of the basis vectors $|\Psi_j^k\rangle$ is:

$$|\Psi_j^k\rangle = \sum_m \theta_m^{kj} \left(|x_m^{kj}\rangle \otimes |y_m^{kj}\rangle\right),$$

where the θ_m^{kj} are in general complex scalar functions of q and α. From (2) and (3), and choosing the parameters q and α to be real and positive, the duals of these vectors are given by:

$$\langle\Psi_j^k| = \sum_m (-)^{[|x_m^{kj}\rangle][|y_m^{kj}\rangle]} \theta_m^{kj} \left(\langle x_m^{kj}| \otimes \langle y_m^{kj}|\right).$$

As the R matrix is unique, analytic continuation makes our final results valid for any complex q and α. For the duals, we obtain:

$$\begin{aligned}
\langle\Psi_1^1| &= \langle 1| \otimes \langle 1| \\
\langle\Psi_2^1| &= (q^\alpha + q^{-\alpha})^{-\frac{1}{2}} \left(q^{\frac{1}{2}\alpha}\langle 1| \otimes \langle 2| + q^{-\frac{1}{2}\alpha}\langle 2| \otimes \langle 1|\right) \\
\langle\Psi_3^1| &= (q^\alpha + q^{-\alpha})^{-\frac{1}{2}} \left(q^{\frac{1}{2}\alpha}\langle 1| \otimes \langle 3| + q^{-\frac{1}{2}\alpha}\langle 3| \otimes \langle 1|\right) \\
\langle\Psi_4^1| &= \left(q^\alpha + q^{-\alpha}\right)^{-\frac{1}{2}} [2\alpha+1]_q^{-\frac{1}{2}} \times \\
&\quad \left[[\alpha+1]_q^{\frac{1}{2}} \left(q^\alpha\langle 1| \otimes \langle 4| + q^{-\alpha}\langle 4| \otimes \langle 1|\right) - [\alpha]_q^{\frac{1}{2}} \left(q^{\frac{1}{2}}\langle 2| \otimes \langle 3| - q^{-\frac{1}{2}}\langle 3| \otimes \langle 2|\right)\right] \\
\langle\Psi_1^3| &= (q^{\alpha+1} + q^{-\alpha-1})^{-\frac{1}{2}} [2\alpha+1]_q^{-\frac{1}{2}} \times \\
&\quad \left[[\alpha]_q^{\frac{1}{2}} \left(q^{\alpha+1}\langle 4| \otimes \langle 1| + q^{-\alpha-1}\langle 1| \otimes \langle 4|\right) - [\alpha+1]_q^{\frac{1}{2}} \left(q^{-\frac{1}{2}}\langle 3| \otimes \langle 2| - q^{\frac{1}{2}}\langle 2| \otimes \langle 3|\right)\right] \\
\langle\Psi_2^3| &= \left(q^{\alpha+1} + q^{-\alpha-1}\right)^{-\frac{1}{2}} \left(q^{\frac{1}{2}(\alpha+1)}\langle 4| \otimes \langle 2| + q^{-\frac{1}{2}(\alpha+1)}\langle 2| \otimes \langle 4|\right) \\
\langle\Psi_3^3| &= (q^{\alpha+1} + q^{-\alpha-1})^{-\frac{1}{2}} \left(q^{\frac{1}{2}(\alpha+1)}\langle 4| \otimes \langle 3| + q^{-\frac{1}{2}(\alpha+1)}\langle 3| \otimes \langle 4|\right) \\
\langle\Psi_4^3| &= \langle 4| \otimes \langle 4|.
\end{aligned}$$

From the basis vectors $|\Psi_j^k\rangle$ and their duals $\langle\Psi_j^k|$ for V_1 and V_3, we construct projectors P_1 and P_3, defined by:

$$P_1 = \sum_{k=1}^4 |\Psi_1^k\rangle\langle\Psi_1^k|, \qquad P_3 = \sum_{k=1}^4 |\Psi_3^k\rangle\langle\Psi_3^k|.$$

Note that the multiplication operation on the graded space $V \otimes V$ is given by:

$$(|i\rangle \otimes |j\rangle)(\langle k| \otimes \langle l|) = (-)^{[|j\rangle][\langle k|]} (|i\rangle\langle k| \otimes |j\rangle\langle l|), \qquad i,j,k,l = 1,2,3,4. \tag{4}$$

Now let I be the identity operator on $V \otimes V$, viz: $I = \sum_{ij=1}^4 e^i{}_i \otimes e^j{}_j$, where $e^k{}_l = |k\rangle\langle l|$ is an elementary rank 2 tensor. As we have $P_1 + P_2 + P_3 = I$, we thus do not need to explicitly construct P_2 (or even a basis for V_2); we simply set:

$$P_2 = I - P_1 - P_3. \tag{5}$$

Where g is a classical Lie superalgebra, the corresponding quantum superalgebra $U_q[g]$ admits a universal R matrix $R \in U_q[g] \otimes U_q[g]$ satisfying (among other relations):

$$\begin{aligned} R\Delta(a) &= \overline{\Delta}(a) R, && \forall a \in U_q[g], \\ R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, && \text{in } U_q[g] \otimes U_q[g] \otimes U_q[g], \end{aligned} \tag{6}$$

where the subscripts refer to the embedding of R acting on the triple tensor product space. From any representation of $U_q[g]$, one may obtain a tensor solution of (6) by replacing the superalgebra elements with their matrix representatives. Similarly to (4), multiplication of tensor products of matrices a, b, c, d is governed by:

$$(a \otimes b)(c \otimes d) = (-)^{[b][c]}(ac \otimes bd), \qquad \text{homogeneous } b, c.$$

We introduce the *graded permutation operator* P on the tensor product space $V \otimes V$, defined for graded basis vectors $v^k, v^l \in V$ by:

$$P(v^k \otimes v^l) = (-)^{[k][l]}(v^l \otimes v^k),$$

and extended by linearity. (We use the shorthand $\left[v^k\right] \equiv [k]$.) With this, we define:

$$\sigma = PR,$$

which can be shown to satisfy the equation:

$$(\sigma \otimes I)(I \otimes \sigma)(\sigma \otimes I) = (I \otimes \sigma)(\sigma \otimes I)(I \otimes \sigma). \tag{7}$$

From [26], we have (with a slight change of notation and a convenient choice of normalisation):

$$\sigma = q^{-2\alpha}P_1 - P_2 + q^{2\alpha+2}P_3.$$

Using (5), this simplifies to:

$$\sigma = \left(1 + q^{-2\alpha}\right)P_1 + \left(1 + q^{2\alpha+2}\right)P_3 - I.$$

From the above form of σ, it is straightforward to deduce that σ satisfies the polynomial identity:

$$q^{-1}\sigma^3 + \left(q^{-1} - q^{-2\alpha-1} - q^{2\alpha+1}\right)\sigma^2 + \left(q - q^{-2\alpha-1} - q^{2\alpha+1}\right)\sigma + qI = 0.$$

The above skein relation may be used to evaluate the invariant in some cases, but not all since it is of third order. The invariant may also be directly evaluated for a class of links using quantum superalgebra theoretic results [12].

We will represent rank 2 tensors as matrices, that is, the elementary rank 2 tensor ${e^i}_k$ is represented by the elementary (4×4) matrix $e_{i,k}$. We adopt the (standard) convention that the elementary rank 4 tensor ${e^{ij}}_{kl} = {e^i}_k \otimes {e^j}_l$ is constructed by insertion of a copy of the elementary rank 2 tensor ${e^j}_l$ at each location of ${e^i}_k$ (i.e. each element of ${e^i}_k$ is multiplied by the whole of ${e^j}_l$). This means that ${e^{ij}}_{kl}$ is represented by the elementary (16×16) matrix $e_{4(i-1)+j,4(k-1)+l}$.

Let A be an arbitrary graded rank 4 tensor acting on $V \otimes V$, then for scalar coefficients ${A^{ij}}_{kl}$:

$$A = \sum_{ijkl} {A^{ij}}_{kl} \left({e^i}_k \otimes {e^j}_l \right).$$

Our convention then tells us that the coefficient ${A^{ij}}_{kl}$ is the $(4(i-1)+j, 4(k-1)+l)$ entry of A, written explicitly:

$${A^{ij}}_{kl} \mapsto A_{4(i-1)+j,4(k-1)+l}.$$

We wish to remove the grading on V, and convert the matrix representing σ to its ungraded counterpart. Recall that basis vectors v^k satisfy ${e^i}_j v^k = {\delta^k}_j v^i$, hence the action of A on $V \otimes V$ is:

$$\begin{aligned}
A(v^k \otimes v^l) &= \sum_{ijmn} {A^{ij}}_{mn} \left({e^i}_m \otimes {e^j}_n \right) (v^k \otimes v^l) \\
&= \sum_{ijmn} {A^{ij}}_{mn} (-)^{[k]([j]+[n])} \left({e^i}_m v^k \otimes {e^j}_n v^l \right) \\
&= \sum_{ijmn} {A^{ij}}_{mn} (-)^{[k]([j]+[n])} \left({\delta^k}_m v^i \otimes {\delta^l}_n v^j \right) \\
&= \sum_{ijmn} {A^{ij}}_{mn} (-)^{[k]([j]+[n])} {\delta^k}_m {\delta^l}_n \left(v^i \otimes v^j \right) \\
&= \sum_{ij} {A^{ij}}_{kl} (-)^{[k]([j]+[l])} \left(v^i \otimes v^j \right).
\end{aligned}$$

Now, in this sum, the parity factor is constructed from the degrees of vectors; in the ungraded case, there would be no such factor, indeed we would have:

$$\overline{A}(v^k \otimes v^l) = \sum_{ij} {\overline{A}^{ij}}_{kl} \left(v^i \otimes v^j \right).$$

This motivates us to set:

$${\overline{A}^{ij}}_{kl} = (-)^{[k]([j]+[l])} {A^{ij}}_{kl}.$$

Under these conventions, the explicit form of σ is presented (as a matrix!) in §4.

3 Knot Theory

3.1 Link Examples

In Table 1, we list the links to be studied. (By the term 'knot', we intend a link of one component.) We use the well-known notation of Alexander and Briggs (1926) [3], the data being abstracted from [1], itself citing [36] and [8] (beware that the tables in this latter article are presented in microfiche form only).

K	$w(K)$	Chiral?	Invertible?
0_1 (Unknot)	0	*No* (trivial)	*Yes* (trivial)
2_1^2 (Hopf Link)	2	*No* (trivial)	*Yes* (trivial)
3_1 (Trefoil)	3	*Yes* [1, p 176]	*Yes* (trivial)
4_1 (Figure Eight)	0	*No* ([1, p 14]; see [17, p 198] for an elegant graphical proof)	*Yes* (as 8_{17} is the smallest non-invertible knot)
5_1^2 (Whitehead Link)	1	*Yes* [16, pp 49-50]	*Yes*
8_{17}	0	*No* [16, p 455]	*No* [19, p 162]
9_{42}	1	*Yes* [18, p 218]	*Yes*
10_{48}	0	*Yes* [18, p 218]	*Yes*

Table 1: Data for the links to be investigated, including their Alexander–Briggs (and common) names, their writhes $w(K)$, and whether they are chiral and invertible. Diagrams of the links are presented in Figures 9 to 13.

3.2 Reflection and Inversion – Chirality and Invertibility

Throughout, we shall write "=" to denote *ambient isotopy* of link diagrams, meaning that they are equivalent under the Reidemeister moves (original: [33], but see, e.g. [18]). We shall use the following definitions, but the reader must be aware that conflicting terminology appears in the literature.

Reflection: We shall denote by K^* the mirror image (or reflection) of a knot K. A knot is *chiral* if it is distinct from its mirror image; i.e. there are actually two distinct knots with the same name, $K^* \neq K$, e.g. the trefoil knot is chiral: $(3_1)^* \neq 3_1$. Note that this definition doesn't require an orientation. A knot is *amphichiral* if it is ambient isotopic to its mirror image, i.e. $K^* = K$.

The HOMFLY[1] (and hence the Jones) polynomial and the Kauffman polynomial can distinguish many (but not all) knots from their reflections. The first chiral knot that neither the HOMFLY nor the Kauffman polynomial can distinguish is 9_{42}, i.e. $9_{42}^* \neq 9_{42}$, but the polynomials are equal. Similarly, the knot 10_{48} is chiral, but the HOMFLY polynomial fails to detect this, although the Kauffman does detect it [18, p 218] (wrongly labeled 10_{79}).

Inversion: Assign an orientation to a knot. Denote the *inverse* of a knot K by K^{-1}, obtained by reversing the orientation. Whilst this is a simple concept for a knot, there are of course many possibilities for the reversal of only some components of oriented, multi-component links; we shall not go into these here.

Commonly, $K = K^{-1}$, and we say that K is *invertible*. For example, the trefoil knot is invertible $(3_1)^{-1} = 3_1$. Less commonly, $K \neq K^{-1}$, and we say that K is *noninvertible*. The first example of a noninvertible (prime) knot is 8_{17}.

Both the reflection and the inverse are automorphisms of order two, i.e. $(K^*)^* = K$ and $(K^{-1})^{-1} = K$. The notions may of course be combined, we obtain: $(K^*)^{-1} = (K^{-1})^*$.

To illustrate, using the Trefoil Knot 3_1 (see Figure 9). We have two equivalence classes: $3_1 = (3_1)^{-1}$ and $(3_1)^* = ((3_1)^{-1})^* = ((3_1)^*)^{-1}$.

3.3 Abstract Tensor Conventions

By a 'positive oriented' or 'right-handed' crossing, we shall intend a crossing such that if the thumb of the right hand points in the direction of one of the arrows, the fingers of the right hand will point in the direction of the other arrow. The opposite situation is naturally called a 'negative oriented' or 'left-handed' crossing.

If the two outward-pointing arrows of a positive oriented crossing are pointed upwards, then we shall label the components of the crossing with indices a in top left, b in bottom left, c in top right, and d in bottom right, and associate with the crossing the (rank 4) tensor $\sigma^{a\ c}_{b\ d}$, where the position of the indices in the tensor corresponds with the positioning of the labels in the crossing. The inverse of σ will represent a negative oriented crossing, with the convention on the indices being the same as that of σ. A diagram of σ and σ^{-1} is provided in Figure 1.

[1] The HOMFLY polynomial is named by the conjunction of the initials of six of its discoverers [10], omitting those ("P" and "T") of two independent discoverers [32]. Przytycki, the omitted "P", has furthered the entymological spirit with the suggestion "FLYPMOTH" [31, p 256], which includes all the discoverers and has a muted reference to the "flyping" operation of the Tait, Kirkwood and Little – the original compilers of knot tables. (Another possibility is the letter sequence "HOMFLYPT".) Bar-Natan (Prasolov and Sossinsky [30, p 36] cite Bar-Natan [4], who cites "L Rudulph") goes further, adding a "U" for good measure, to account for any unknown discoverers, yielding the unpalatable "LYMPHTOFU"!

Figure 1: Definition of the tensors σ and σ^{-1} representing positive oriented and negative oriented crossings with upward pointing arrows, respectively.

We shall also require four (rank 2) tensors (i.e. genuine matrices) to represent all possible horizontally-oriented half-loops. We shall call these 'cap' and 'cup' matrices, and label them with the suggestive $\Omega^{\pm}$ and $\mho^{\pm}$, e.g. Ω^{+} is the upper loop with arrow pointing right. A diagram is provided in Figure 2.

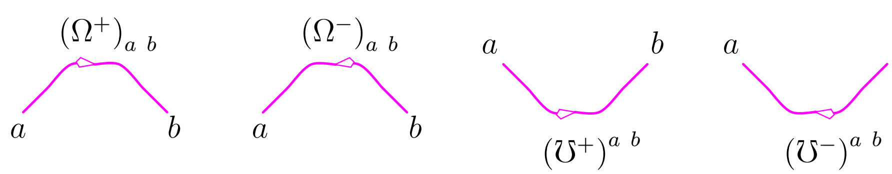


Figure 2: Definition of the tensors (matrices) $\Omega^{\pm}$ and $\mho^{\pm}$, representing all possibilities of horizontally-aligned half-loops.

With these basic tensors σ, σ^{-1}, $\Omega^{\pm}$ and $\mho^{\pm}$, we may evaluate an invariant for any particular link. However, this direct procedure tends to be computationally expensive, and parts of the computation are often repeated, so in practice, we define auxiliary symbols. We shall use the notation X to represent a rank 4 tensor such as σ or σ^{-1} with parallel pointing arrows (i.e. a 'channel' crossing in the terminology of Kauffman [18, p 76].)

The primary auxiliary tensors used are listed below; secondary ones will be mentioned where necessary. The Einstein summation convention is used throughout.

- The first auxiliary symbols are those of crossings that have been 'twisted' relative to σ and σ^{-1}. The left, right, and upside-down-twisted versions of X will be called X_l, X_r and X_d respectively. They are defined in the following manner:

$$\begin{aligned}
(X_l)^{a\ c}_{b\ d} &\triangleq X^{e\ a}_{d\ h} \cdot \left(\Omega^{-}\right)_{b\ e} \cdot \left(\mho^{-}\right)^{h\ c} \\
(X_r)^{a\ c}_{b\ d} &\triangleq X^{c\ g}_{f\ b} \cdot \left(\mho^{+}\right)^{a\ f} \cdot \left(\Omega^{+}\right)_{g\ d} \\
(X_d)^{a\ c}_{b\ d} &\triangleq X^{e\ g}_{f\ h} \cdot \left(\mho^{+}\right)^{a\ h} \cdot \left(\Omega^{+}\right)_{g\ b} \cdot \left(\mho^{+}\right)^{c\ f} \cdot \left(\Omega^{+}\right)_{e\ d}.
\end{aligned} \tag{8}$$

  Observe that X_d is a 'channel' crossing, whilst X_l and X_r are 'cross-channel' crossings. Diagrams are found in Figures 3 and 4.

- The next set of auxiliary symbols represent p copies of the *same* crossing X (for any channel crossing X) atop one another (see Figure 5). They are defined recursively in the following manner:

$$\left(X^{p+1}\right)^{a\ c}_{b\ d} \triangleq X^{a\ c}_{e\ f} \cdot (X^{p})^{e\ f}_{b\ d}, \qquad p = 1, 2, \ldots.$$

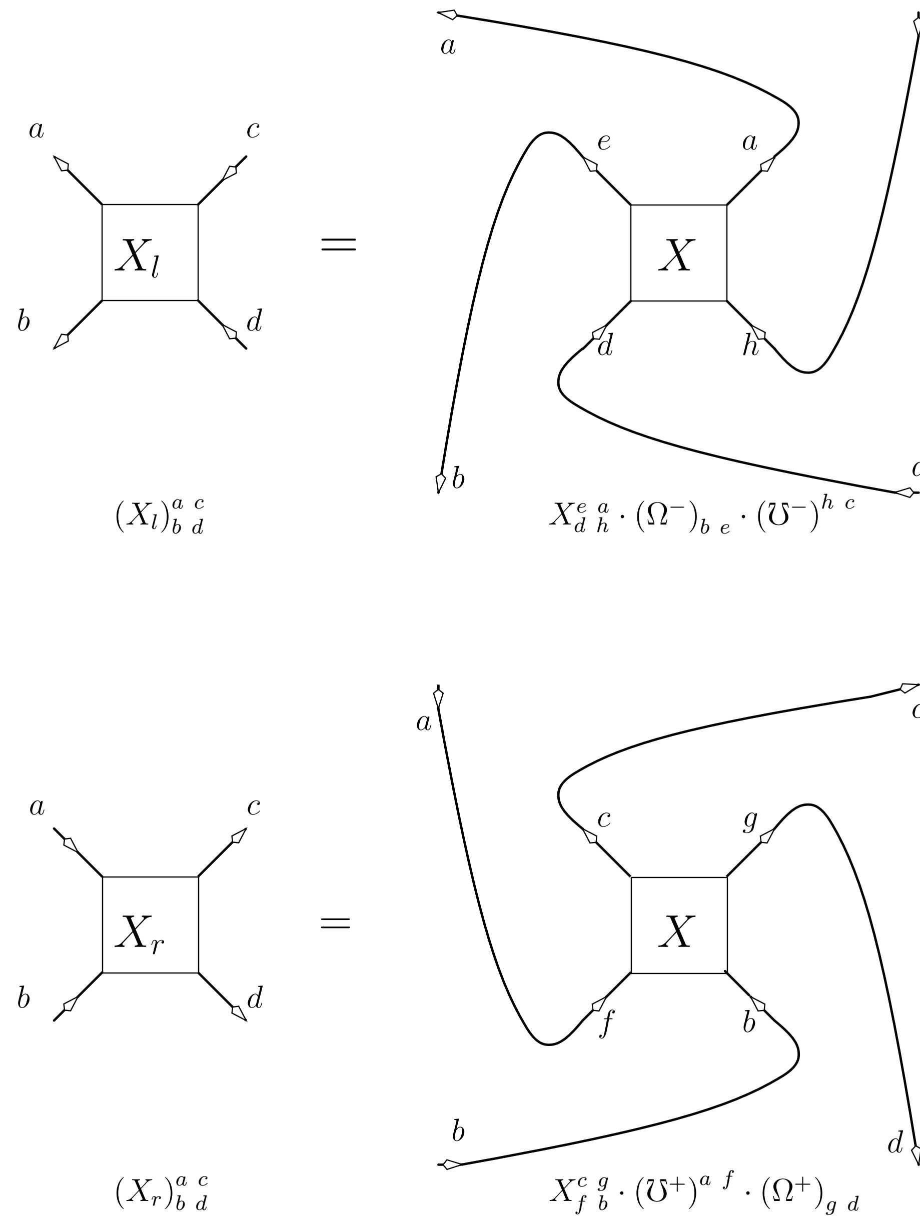
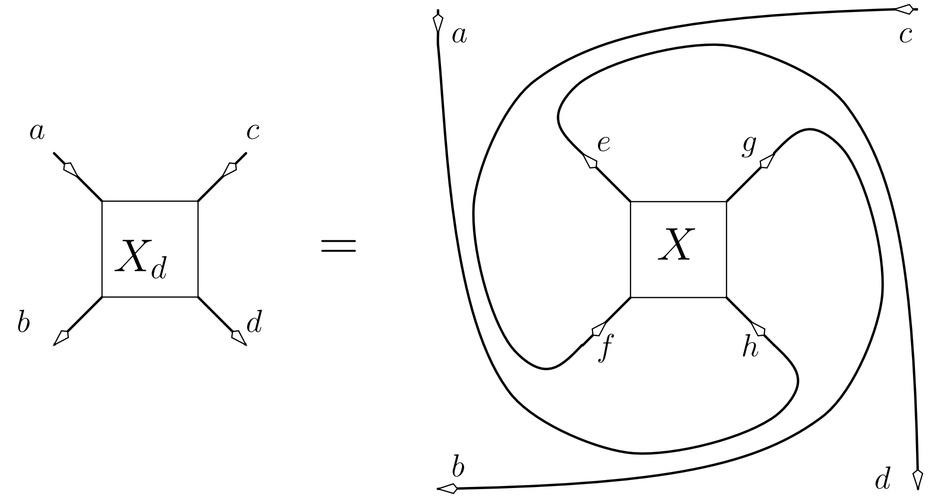
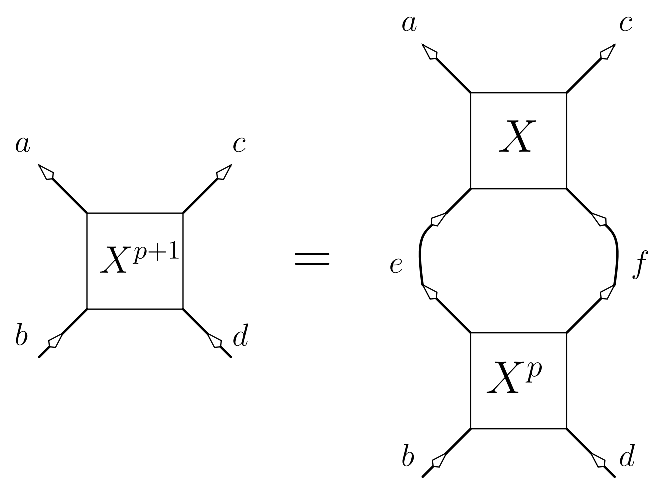


Figure 3: The primary auxiliary tensors X_l and X_r, where X is one of σ or σ^{-1}.



$(X_d)^{a\ c}_{b\ d}$ $\quad$ $X^{e\ g}_{f\ h} \cdot (\mho^+)^{a\ h} \cdot (\Omega^+)_{g\ b} \cdot (\mho^+)^{c\ f} \cdot (\Omega^+)_{e\ d}$

Figure 4: The primary auxiliary tensor X_d, where X is one of σ or σ^{-1}.



$(X^{p+1})^{a\ c}_{b\ d}$ $\quad$ $X^{a\ c}_{e\ f} \cdot (X^p)^{e\ f}_{b\ d}$

Figure 5: The primary auxiliary tensors X^{p+1} in terms of X and X^p; X is one of σ or σ^{-1}. If all arrows are reversed, then the definition also holds for X being σ_d or σ_d^{-1}; that is, any channel crossing.

- The third set of frequently-encountered patterns are where a crossing X is to the left or right of its own 'upside-downness' X_d. That is, fix an X as either σ or σ^{-1}, and examine the patterns formed from juxtaposing X and X_d. They are defined in the following manner:

$$\begin{aligned}(X_dX)^{a\ c}_{b\ d} &\triangleq (X_d)^{a\ e}_{b\ f} \cdot X^{g\ c}_{h\ d} \cdot \left(\Omega^-\right)_{e\ g} \cdot \left(\mho^+\right)^{f\ h} \\ (XX_d)^{a\ c}_{b\ d} &\triangleq X^{a\ e}_{b\ f} \cdot (X_d)^{g\ c}_{h\ d} \cdot \left(\Omega^+\right)_{e\ g} \cdot \left(\mho^-\right)^{f\ h}.\end{aligned}$$

  A diagram is found in Figure 6.

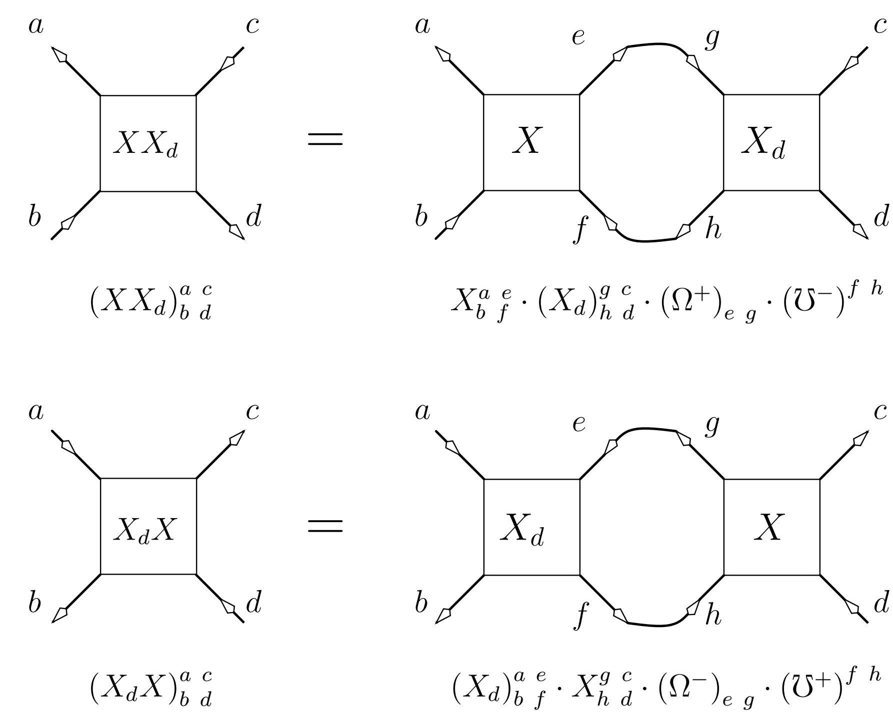


Figure 6: The primary auxiliary tensors X_dX and XX_d; X is one of σ or σ^{-1}.

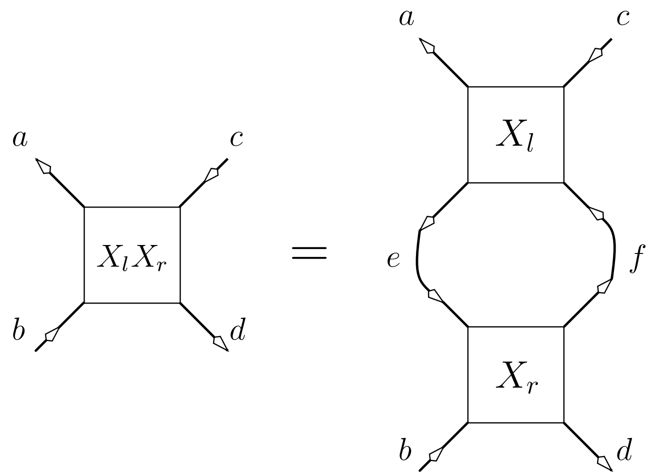
- The final set of frequently-encountered patterns are where a crossing X_l is placed atop above a crossing X_r (or vice-versa). We obtain:

$$\begin{aligned}(X_lX_r)^{a\ c}_{b\ d} &\triangleq (X_l)^{a\ c}_{e\ f} \cdot (X_r)^{e\ f}_{b\ d} \\ (X_rX_l)^{a\ c}_{b\ d} &\triangleq (X_r)^{a\ c}_{e\ f} \cdot (X_l)^{e\ f}_{b\ d}.\end{aligned}$$

  A diagram is found in Figure 7. A moment's thought demonstrates that the diagram for X_lX_r is a right rotation of the diagram for X_dX. In fact, we have the identity:

$$(X_lX_r)^{a\ c}_{b\ d} = (X_dX)^{e\ a}_{d\ h} \cdot \left(\mho^+\right)^{h\ c} \cdot \left(\Omega^+\right)_{b\ e},$$

  although in practice we shall not use it. (A diagram parallel to Figure 3 would demonstrate this.)



$(X_l X_r)^{a\ c}_{b\ d}$ $\qquad$ $(X_l)^{a\ c}_{e\ f} \cdot (X_r)^{e\ f}_{b\ d}$

Figure 7: The primary auxiliary tensor $X_l X_r$; X is one of σ or σ^{-1}. $X_r X_l$ is obtained by swapping every r and l in this diagram.

3.4 The Effects of Reflection and Inversion on the Tensors

Reflection: Let K^* be the reflection of a tangle K; and say that we have constructed a tensor representing K. Every positive (respectively negative) crossing in K will have been replaced by the equivalent negative (respectively positive) crossing in K^*. Thus, the tensor corresponding to K^* will be that of K with every σ replaced by σ^{-1}, and every σ^{-1} replaced by σ. This carries through to the auxiliary tensors; i.e. $\sigma_d \sigma$ will be replaced with $\sigma_d^{-1}\sigma^{-1}$, etc. The caps $\Omega^\pm$ and cups $\mho^\pm$ will remain unchanged.

From the uniqueness [20] of the universal R matrix for any quantum superalgebra the following relation holds (for appropriate normalisation):

$$R^{-1}(q) = R(q^{-1}),$$

which in turn leads to the relation

$$\sigma^{-1}(q) = P\sigma(q^{-1})P.$$

Thus, up to a basis transformation, σ and σ^{-1} are interchangeable by the change of variable $q \mapsto q^{-1}$. It then follows that the invariant for K^* is obtainable from that of K by the same change of variable, which leads to the following:

Proposition 3.1
If K is amphichiral then the invariant LG_K is palindromic.[2]

[2] We intend “palindromic” to mean that the polynomial is invariant under the mapping $q \mapsto q^{-1}$.

Inversion: Again, if K^{-1} is the inverse of K, then every arrow in K will have been replaced with an arrow in the opposite direction. The tensor corresponding to K^{-1} will thus have the following changes: For the crossings, where X is either σ or σ^{-1}, interchange $X \iff X_d$ and $X_l \iff X_r$; and for the caps and cups, interchange only the signs, i.e. $\Omega^{\pm} \iff \Omega^{\mp}$ and $\mho^{\pm} \iff \mho^{\mp}$.

This has the effect that the tensor representing K is replaced by the dual tensor acting on the dual space [35]. Recalling that the tensors representing $(1,1)$ tangles act as scalar multiples of the identity on V, then the dual tensor has exactly the same form, from which we conclude:

Proposition 3.2

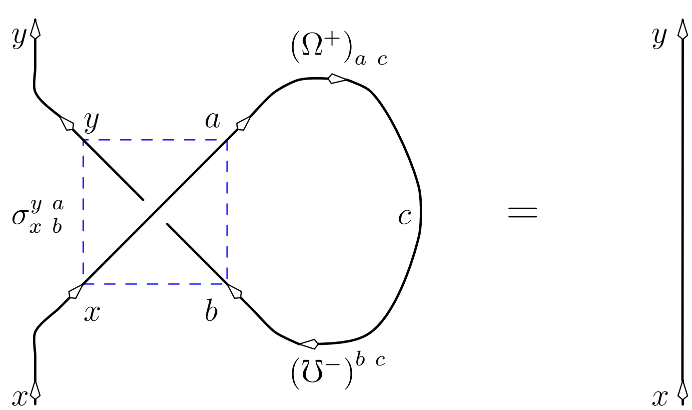
A knot invariant derived from an irreducible representation of a quantum (super)algebra is unable to detect inversion.

3.5 Abstract Tensor Expressions for the Example Links

We list the abstract tensors $(T_K)^y_x$ that represent the $(1,1)$-tangle (open diagram) forms of the example links. In each case, the indices x and y are the lower and upper loose ends of the tangle in question. The Links–Gould invariant is then formed by setting x and y to be the same, i.e.

$$LG_K\,(q, p = q^{\alpha}) \;\triangleq\; (T_K)^i_i$$

(no sum on i), for any allowable index i. We typically choose $i = 1$. Our invariant does not need to be *writhe-normalised*, due to the choice of normalisation of σ and the cap and cup matrices $\Omega^{\pm}$ and $\mho^{\pm}$. Figure 8 depicts removal of a loop from a diagram.



$\sigma^{y\ a}_{x\ b} \cdot (\Omega^+)_{a\ c} \cdot (\mho^-)^{b\ c}$ $\qquad$ δ^y_x

Figure 8: Removal of a Single, Positive Loop.

Braid presentations for the example knots are taken from [14, pp 109-110] and [15, pp 381-386].

$\mathbf{0_1}$ (Unknot): A braid presentation is the trivial $e \in B_1$. As the unclosed tangle representing the Unknot is rather meaningless, we use simply $(T_{0_1})_x^y \triangleq \delta_x^y$.

$\mathbf{2_1^2}$ (Hopf Link): A braid presentation is ${\sigma_1}^2 \in B_2$. Diagrams pertaining to the Hopf Link and Trefoil are found in Figure 9.

$$(T_{2_1^2})_x^y \triangleq \left(\sigma^2\right)_{x\ b}^{y\ a} \cdot \left(\Omega^+\right)_{a\ c} \cdot \left(\mho^-\right)^{b\ c}.$$

$\mathbf{3_1}$ (Trefoil): A braid presentation is ${\sigma_1}^3 \in B_2$. This knot has also been called the *overhand knot* (as that is how it is tied) and the *cloverleaf knot* [7, pp 3-4].

$$(T_{3_1})_x^y \triangleq \left(\sigma^3\right)_{x\ b}^{y\ a} \cdot \left(\Omega^+\right)_{a\ c} \cdot \left(\mho^-\right)^{b\ c}.$$

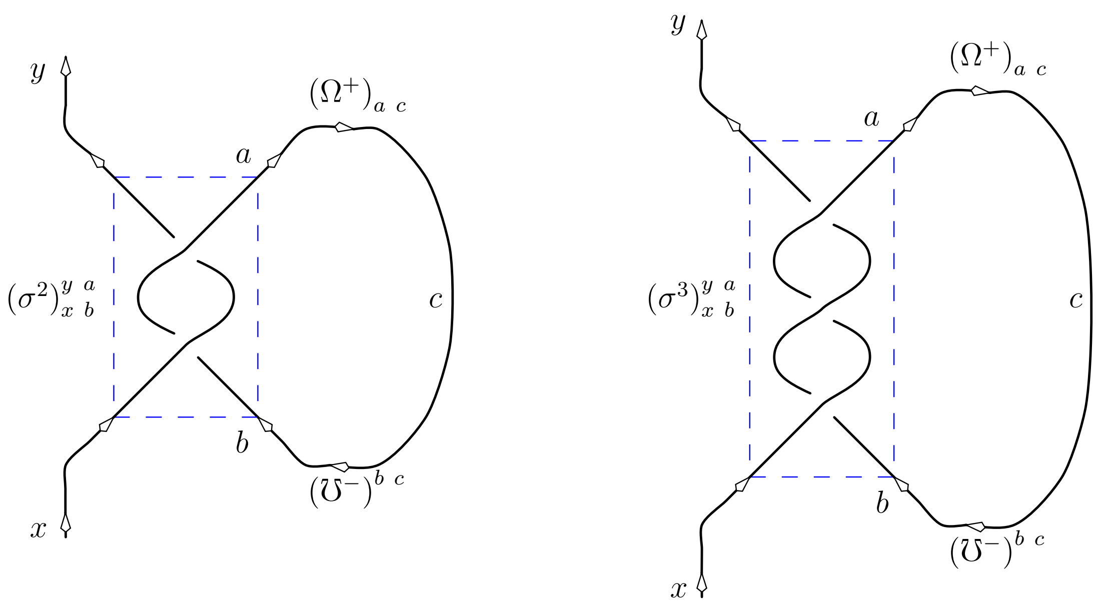


Figure 9: Tangle form of 2_1^2 (the Hopf Link) and 3_1 (the positive Trefoil).

4_1 (Figure Eight): A braid presentation is $\left(\sigma_1\sigma_2^{-1}\right)^2 \in B_3$, and a diagram is found in Figure 10. This knot has also been called the *Four-Knot* (as it is the only 4 crossing knot) and *Listing's Knot* [7, p 4].

$$(T_{4_1})^y_x \triangleq \left(\sigma_l^{-1}\sigma_r^{-1}\right)^{y\ b}_{a\ c} \cdot (\sigma_r)^{c\ e}_{d\ f} \cdot \sigma^{a\ d}_{x\ g} \cdot \left(\Omega^-\right)_{b\ e} \cdot \left(\mho^-\right)^{g\ f}.$$

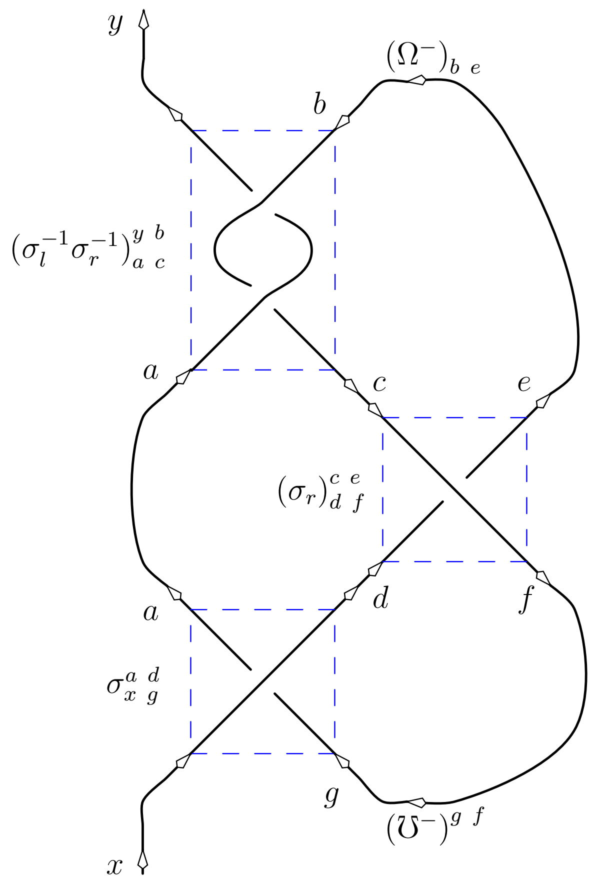


Figure 10: Tangle form of 4_1 (the Figure Eight Knot).

5_1^2 (Whitehead Link): A braid presentation is $\left(\sigma_1\sigma_2^{-1}\right)^2\sigma_2^{-1} \in B_3$ and a diagram is found in Figure 11. (This link is named after the topologist J H C Whitehead, not the logician Alfred North Whitehead [17, p 200].)

Firstly, we define a temporary tensor to reduce computation:

$$(W)^{c\ i}_{x\ d} \triangleq \left(\sigma^{-2}\right)^{c\ e}_{x\ f} \cdot (\sigma_d^2)^{g\ i}_{h\ d} \cdot \left(\Omega^+\right)_{e\ g} \cdot \left(\mho^-\right)^{f\ h},$$

where we have written $\sigma^{-2} \triangleq \left(\sigma^{-1}\right)^2$. With this, we have:

$$(T_{5_1^2})^y_x \triangleq (W)^{c\ i}_{x\ d} \cdot (\sigma_r\sigma_l)^{a\ y}_{i\ b} \cdot \left(\Omega^+\right)_{c\ a} \cdot \left(\mho^+\right)^{d\ b}.$$

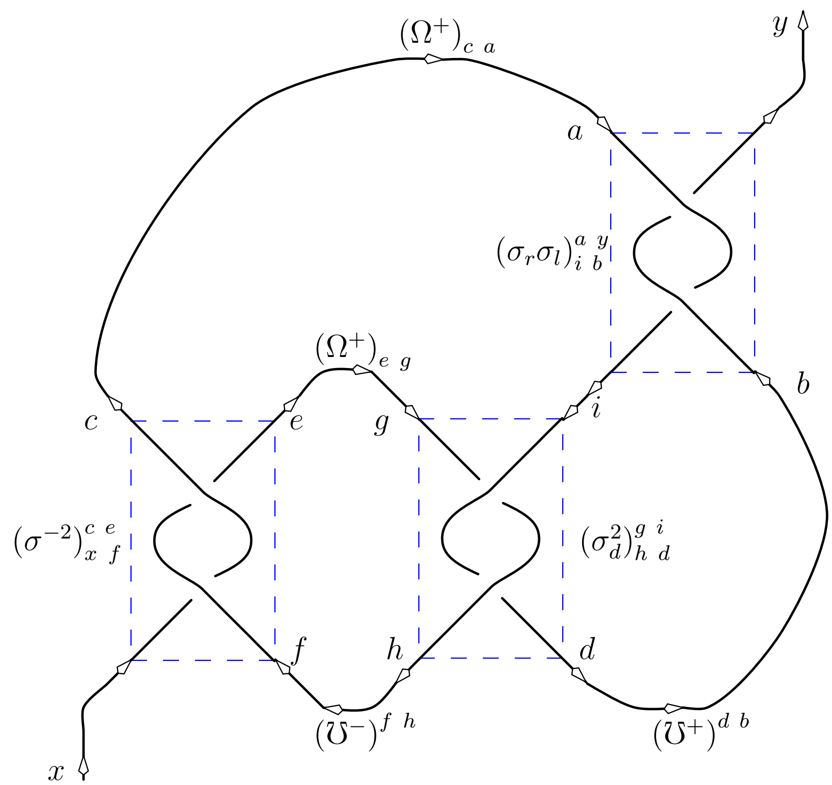


Figure 11: Tangle form of 5_1^2 (the Whitehead Link).

8_{17}: A braid presentation is $\left(\sigma_1^{-1}\sigma_2\right)^2\sigma_2^2\sigma_1^{-2}\sigma_2 \in B_3$, and a diagram is found in Figure 12. Again, we define some temporary tensors to reduce computation:

$$\begin{aligned}(EA)_{b\ d\ f}^{y\ c\ e} &\triangleq \left(\sigma^{-2}\right)_{g\ f}^{c\ e}\cdot\left(\sigma^{2}\right)_{b\ d}^{y\ g}\\ (EB)_{x\ i\ j}^{b\ d\ f} &\triangleq \left(\sigma^{-1}\right)_{k\ l}^{d\ f}\cdot\sigma_{m\ n}^{b\ k}\cdot\left(\sigma^{-1}\right)_{o\ j}^{n\ l}\cdot\sigma_{x\ i}^{m\ o}.\end{aligned}$$

With these, we have:

$$(T_{8_{17}})_x^y \triangleq (EA)_{b\ d\ f}^{y\ c\ e}\cdot(EB)_{x\ i\ j}^{b\ d\ f}\cdot\left(\Omega^+\right)_{c\ r}\cdot\left(\mho^-\right)^{i\ r}\cdot\left(\Omega^+\right)_{e\ q}\cdot\left(\mho^-\right)^{j\ q}.$$

To reduce computation, we may define even more auxiliary tensors:

$$(EB)_{x\ i\ j}^{b\ d\ f} = (EC)_{m\ n\ l}^{b\ d\ f}\cdot(ED)_{x\ i\ j}^{m\ n\ l},$$

where:

$$\begin{aligned}(EC)_{m\ n\ l}^{b\ d\ f} &\triangleq \left(\sigma^{-1}\right)_{k\ l}^{d\ f}\cdot\sigma_{m\ n}^{b\ k}\\ (ED)_{x\ i\ j}^{m\ n\ l} &\triangleq \left(\sigma^{-1}\right)_{o\ j}^{n\ l}\cdot\sigma_{x\ i}^{m\ o}.\end{aligned}$$

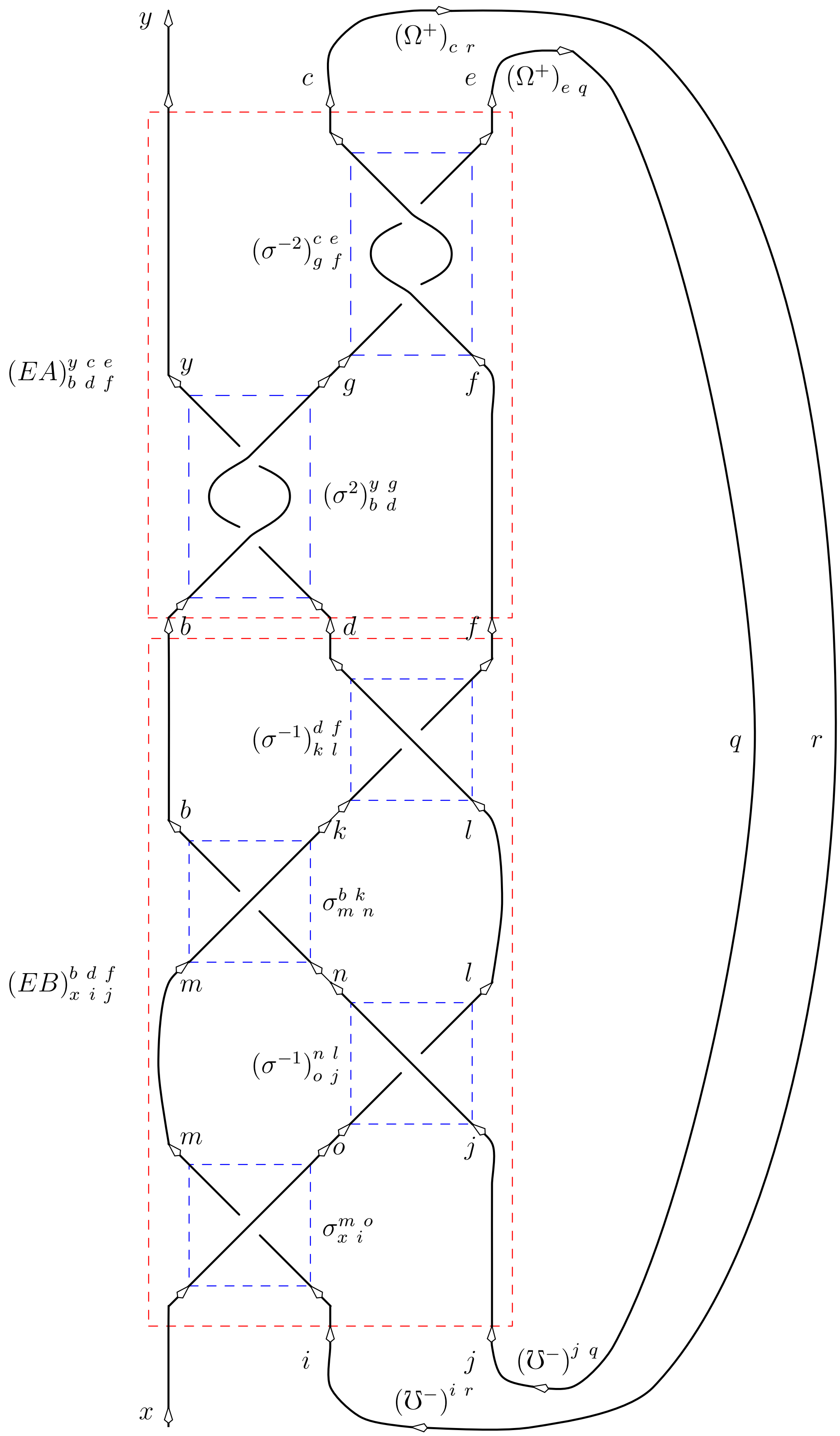


Figure 12: Tangle form of 8_{17}.

9_{42}: A braid presentation is $\sigma_1^3\sigma_3\sigma_2^{-1}\sigma_3\sigma_1^{-2}\sigma_2^{-1} \in B_4$, and a diagram is found in Figure 14. Again, we define a temporary tensor to reduce computation:

$$
\begin{aligned}
(N)_{b\ h}^{a\ y} &\triangleq (\sigma_d^2)_{b\ d}^{a\ c} \cdot \left(\sigma^{-3}\right)_{f\ h}^{e\ y} \cdot \left(\Omega^-\right)_{c\ e} \cdot \left(\mho^+\right)^{d\ f}. \\
(T_{9_{42}})_x^y &\triangleq (N)_{b\ h}^{a\ y} \cdot \left(\sigma_d^{-1}\sigma^{-1}\right)_{i\ j}^{b\ h} \cdot (\sigma\sigma_d)_{x\ m}^{k\ i} \cdot \left(\mho^+\right)^{m\ j} \cdot \left(\Omega^+\right)_{k\ a}.
\end{aligned}
$$

10_{48}: A braid presentation is $\sigma_1^{-2}\sigma_2^4\sigma_1^{-3}\sigma_2 \in B_3$, and a diagram is found in Figure 13.

$$
\begin{aligned}
(T_{10_{48}})_x^y \triangleq\ & \left(\sigma^{-2}\right)_{b\ f}^{a\ y} \cdot \left(\sigma^4\right)_{d\ h}^{f\ g} \cdot \left(\sigma^{-3}\right)_{c\ e}^{b\ d} \cdot (\sigma)_{x\ i}^{e\ h} \cdot \\
& \left(\Omega^-\right)_{j\ a} \cdot \left(\mho^+\right)^{j\ c} \cdot \left(\Omega^+\right)_{g\ k} \cdot \left(\mho^-\right)^{i\ k}.
\end{aligned}
$$

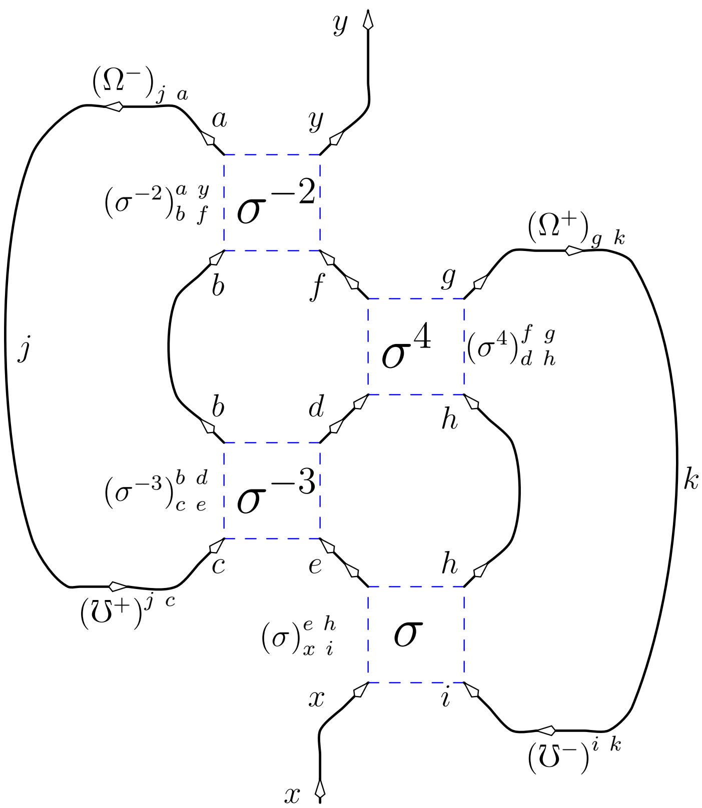


Figure 13: Tangle form of 10_{48}.

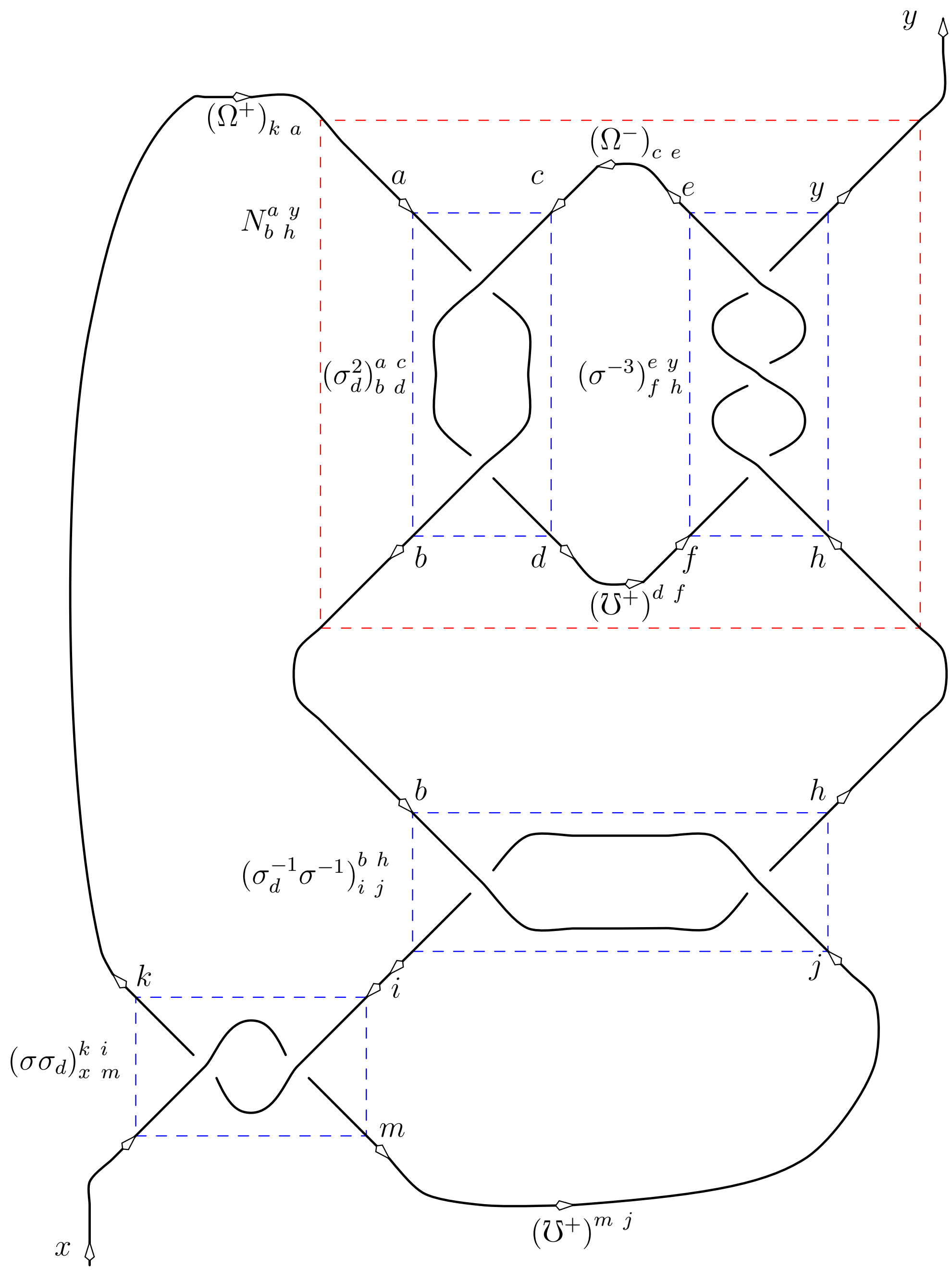


Figure 14: Tangle form of 9_{42}.

4 The Links–Gould Tangle Invariant

4.1 Crossing Matrices σ and σ^{-1}

From the results of §2, we have the matrices σ and σ^{-1}, using the substitution $p \triangleq q^{\alpha}$:

$$
\sigma = \left[\begin{array}{cccc|cccc|cccc|cccc}
p^{-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & p^{-1} & \cdot & \cdot & p^{-2}-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q^2-1 & \cdot & \cdot & -q & \cdot & \cdot & -qY & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq & \cdot & \cdot \\
\hline
\cdot & \cdot & p^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \cdot & \cdot & \cdot & Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & -qY & \cdot & \cdot & Y & \cdot & \cdot & Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq & \cdot & \cdot & \cdot & \cdot & \cdot & p^2q^2-1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & pq & \cdot & \cdot & p^2q^2-1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^2q^2
\end{array}\right],
$$

$$
\sigma^{-1} = \left[\begin{array}{cccc|cccc|cccc|cccc}
p^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & p^2-1 & \cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & p^2-1 & \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & Y^2q^{-2} & \cdot & \cdot & Yq^{-1} & \cdot & \cdot & -Yq^{-2} & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & Yq^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & -q^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}q^{-2}-1 & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1}q^{-1} & \cdot & \cdot \\
\hline
\cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -Yq^{-2} & \cdot & \cdot & -q^{-1} & \cdot & \cdot & q^{-2}-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}q^{-2}-1 & \cdot & \cdot & p^{-1}q^{-1} & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1}q^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-1}q^{-1} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^{-2}q^{-2}
\end{array}\right],
$$

where $Y = (p^{-2} - q^2 + p^2q^2 - 1)^{1/2}$.

4.2 Caps and Cups $\Omega^{\pm}$ and $\mho^{\pm}$

Where $\mho^{\pm} = (\Omega^{\pm})^{-1}$, we will use:

$$
\begin{aligned}
\Omega^{-} &= \begin{bmatrix} q^{-2\alpha} & \cdot & \cdot & \cdot \\ \cdot & -q^{-2(\alpha+1)} & \cdot & \cdot \\ \cdot & \cdot & -q^{-2\alpha} & \cdot \\ \cdot & \cdot & \cdot & q^{-2(\alpha+1)} \end{bmatrix} \equiv \begin{bmatrix} p^{-2} & \cdot & \cdot & \cdot \\ \cdot & -p^{-2}q^{-2} & \cdot & \cdot \\ \cdot & \cdot & -p^{-2} & \cdot \\ \cdot & \cdot & \cdot & p^{-2}q^{-2} \end{bmatrix}, \\
\mho^{-} &= \begin{bmatrix} q^{2\alpha} & \cdot & \cdot & \cdot \\ \cdot & -q^{2(\alpha+1)} & \cdot & \cdot \\ \cdot & \cdot & -q^{2\alpha} & \cdot \\ \cdot & \cdot & \cdot & q^{2(\alpha+1)} \end{bmatrix} \equiv \begin{bmatrix} p^{2} & \cdot & \cdot & \cdot \\ \cdot & -p^{2}q^{2} & \cdot & \cdot \\ \cdot & \cdot & -p^{2} & \cdot \\ \cdot & \cdot & \cdot & p^{2}q^{2} \end{bmatrix}, \\
\Omega^{+} &= \mho^{+} = I_4.
\end{aligned}
$$

The choices for $\Omega^{\pm}$ and $\mho^{\pm}$ are not unique.

- Ω^{+} and $\mho^{+}$ may be chosen from consistency considerations in Figure 3. The simple choices:

  $$\left(\Omega^{+}\right)_{a\ b} = \left(\mho^{+}\right)^{a\ b} = \delta_{a\ b}$$

  (i.e. $\Omega^{+} = \mho^{+} = I_4$), ensure that the definition (8), i.e.

  $$(X_r)^{a\ c}_{b\ d} \triangleq X^{c\ g}_{f\ b} \cdot \left(\mho^{+}\right)^{a\ f} \cdot \left(\Omega^{+}\right)_{g\ d}$$

  (where X is either σ or σ^{-1}), simplifies to the elegant form:

  $$(X_r)^{a\ c}_{b\ d} = X^{c\ d}_{a\ b}.$$

- For the choice of Ω^{-} and $\mho^{-}$, we invoke the following result from [27, Lemma 2, p 354] (see also [26]):

  $$(I \otimes \mathrm{str})\left[(I \otimes q^{-2h_\rho})\sigma\right] = kI,$$

  for some constant k depending on the normalisation of σ. Note that str denotes the supertrace, and that in this case:

  $$\pi(q^{-2h_\rho}) = \begin{bmatrix} q^{-2\alpha} & \cdot & \cdot & \cdot \\ \cdot & q^{-2\alpha-2} & \cdot & \cdot \\ \cdot & \cdot & q^{-2\alpha} & \cdot \\ \cdot & \cdot & \cdot & q^{-2\alpha-2} \end{bmatrix}.$$

From Figure 8, we require:

$$\sigma_{xb}^{ya} \cdot \left(\Omega^{+}\right)_{ac} \cdot \left(\mho^{-}\right)^{bc} = \delta_{x}^{y},$$

which, along with the condition:

$$\left(\Omega^{-}\right)_{ab} \cdot \left(\mho^{-}\right)^{bc} = \delta_{a}^{c},$$

imposes the choice:

$$\begin{aligned} \left(\mho^{-}\right)^{bc} &= (-)^{[b]} \pi(q^{-2h_\rho})_{c}^{b}, \\ \left(\Omega^{-}\right)_{bc} &= (-)^{[b]} \pi(q^{2h_\rho})_{c}^{b}. \end{aligned}$$

For other references on the construction of the cap and cup matrices, see the papers by Reshetikhin and Turaev [35], and particularly Zhang [40] for the superalgebra case.

4.3 Results

Some evaluations of the invariant are presented in Table 2. This $U_q\left[gl\left(2|1\right)\right]$ oriented invariant *is* an invariant of ambient isotopy.

K	$LG_K\left(q,p\right)$
0_1	1
2_1^2	$-1 + p^{-2} - q^2 + p^2 q^2$
3_1	$1 + p^{-4} - p^{-2} + 2q^2 - p^{-2}q^2 - p^2 q^2 - p^2 q^4 + p^4 q^4$
4_1	$7 + \left(p^{-4}q^{-2} + p^4 q^2\right) - 3\left(p^{-2} + p^2\right) - 3\left(p^{-2}q^{-2} + p^2 q^2\right) + 2\left(q^{-2} + q^2\right)$
5_1^2	$-10 + p^{-6}q^{-2} - 3p^{-4} - 3p^{-4}q^{-2} + 4p^{-2}q^{-2} + 9p^{-2} - 2q^{-2}$ $-8q^2 + 2p^{-2}q^2 + 9p^2q^2 + 4p^2 + 2p^2q^4 - 3p^4q^2 - 3p^4q^4 + p^6q^4$
8_{17}	see §4.6
$9_{42}, 10_{48}$	see §4.5

Table 2: The Links–Gould $U_q\left[gl\left(2|1\right)\right]$ oriented polynomial invariant $LG_K\left(q,p\right)$, evaluated using the open diagram form of various links K.

4.4 Behaviour of the Invariant

Fix a knot K, and denote by K^* the reflection of K and by K^{-1} the inverse of K. From the polynomial for K, we may immediately write down the polynomials for K^* and K^{-1}. For the reflection, we have:

$$LG_{K^*}(q,p) = LG_K\left(q^{-1},p^{-1}\right). \tag{9}$$

For the inverse, we have:

$$LG_{K^{-1}}(q,p) = LG_K\left(q,q^{-1}p^{-1}\right).$$

(this follows from $\alpha \mapsto -(\alpha+1)$).

Chirality: As we have:

$$K = K^* \quad \rightarrow \quad LG_K(q,p) = LG_{K^*}(q,p),$$

then we have, conversely, that:

$$LG_K(q,p) \neq LG_{K^*}(q,p) \quad \rightarrow \quad K \neq K^*, \tag{10}$$

i.e. if the polynomials corresponding to K and K^* are distinct, then K must be chiral. Using the identity (9), the test of (10) becomes:

$$LG_K(q,p) \neq LG_K\left(q^{-1},p^{-1}\right) \quad \rightarrow \quad K \neq K^*,$$

i.e. if $LG_K(q,p)$ is *not* palindromic, then K is chiral.

Invertibility: We make the observation that the representation of $U_q[gl(2|1)]$ acting on the dual module V^* is given by the replacement $\alpha \mapsto -(\alpha+1)$ (with an appropriate redefinition of the Cartan elements). Thus for a given $(1,1)$ tangle K, with invariant $LG_K(q,p)$, the tangle invariant $LG_{K^{-1}}$ of its inverse K^{-1} is obtained as $LG_{K^{-1}}(q,p) = LG_K(q,q^{-1}p^{-1})$. However, in view of Proposition 3.2, such an invariant is unable to detect inversion.

We summarise these results in a proposition:

Proposition 4.1
Let $LG_K(q,p)$ be the Links–Gould polynomial invariant for the knot K.

- If $LG_K(q,p)$ is *not* invariant under the transformation $q \mapsto q^{-1}$ (which implies $p \mapsto p^{-1}$), then LG detects the chirality of K.
- $LG_K(q,p)$ enjoys the symmetry property:

$$LG_K(q,p) = LG_K\left(q,q^{-1}p^{-1}\right). \tag{11}$$

4.5 The Chirality of 9_{42} and 10_{48}

The polynomials for 9_{42} and 10_{48} are:

$$
\begin{aligned}
&LG_{9_{42}}(q,p) = \\
&\quad 3 + p^{-8}q^{-6} - 2p^{-6}q^{-6} - 2p^{-6}q^{-4} + p^{-4}q^{-6} + 3p^{-4}q^{-4} + p^{-4}q^{-2} + p^{-4} - p^{-2}q^{-4} \\
&\quad -p^{-2}q^{-2} - 3p^{-2} - 3p^{-2}q^{2} + 6q^{2} + 2q^{4} - p^{2}q^{-2} - p^{2} - 3p^{2}q^{2} - 3p^{2}q^{4} + p^{4}q^{-2} \\
&\quad +3p^{4} + p^{4}q^{2} + p^{4}q^{4} - 2p^{6} - 2p^{6}q^{2} + p^{8}q^{2} \\
&LG_{10_{48}}(q,p) = \\
&\quad 165 + 5p^{-8} - 25p^{-6} + 68p^{-4} - 129p^{-2} - 132p^{2} + 67p^{4} - 22p^{6} + 4p^{8} + p^{-16}q^{-8} \\
&\quad -3p^{-14}q^{-8} + 4p^{-12}q^{-8} - 4p^{-10}q^{-8} + 4p^{-8}q^{-8} - 2p^{-6}q^{-8} - 3p^{-14}q^{-6} + 12p^{-12}q^{-6} \\
&\quad -21p^{-10}q^{-6} + 24p^{-8}q^{-6} - 22p^{-6}q^{-6} + 13p^{-4}q^{-6} - 3p^{-2}q^{-6} + 16q^{-4} + 5p^{-12}q^{-4} \\
&\quad -23p^{-10}q^{-4} + 50p^{-8}q^{-4} - 69p^{-6}q^{-4} + 67p^{-4}q^{-4} - 43p^{-2}q^{-4} - 3p^{2}q^{-4} + 94q^{-2} \\
&\quad -6p^{-10}q^{-2} + 29p^{-8}q^{-2} - 72p^{-6}q^{-2} + 119p^{-4}q^{-2} - 132p^{-2}q^{-2} - 43p^{2}q^{-2} + 13p^{4}q^{-2} \\
&\quad -2p^{6}q^{-2} + 88q^{2} - 2p^{-6}q^{2} + 12p^{-4}q^{2} - 39p^{-2}q^{2} - 129p^{2}q^{2} + 119p^{4}q^{2} - 69p^{6}q^{2} \\
&\quad +24p^{8}q^{2} - 4p^{10}q^{2} + 12q^{4} - 2p^{-2}q^{4} - 39p^{2}q^{4} + 68p^{4}q^{4} - 72p^{6}q^{4} + 50p^{8}q^{4} - 21p^{10}q^{4} \\
&\quad +4p^{12}q^{4} - 2p^{2}q^{6} + 12p^{4}q^{6} - 25p^{6}q^{6} + 29p^{8}q^{6} - 23p^{10}q^{6} + 12p^{12}q^{6} - 3p^{14}q^{6} - 2p^{6}q^{8} \\
&\quad +5p^{8}q^{8} - 6p^{10}q^{8} + 5p^{12}q^{8} - 3p^{14}q^{8} + p^{16}q^{8}.
\end{aligned}
$$

Neither of these polynomials are palindromic, hence LG *does* distinguish the chirality of these knots.

4.6 The Noninvertibility of 8_{17} is not Detected

Recall that 8_{17} is the smallest noninvertible knot. We find its polynomial invariant to be given by:

$$
\begin{aligned}
&LG_{8_{17}}(q,p) = \\
&\quad 139 + (p^{-12}q^{-6} + p^{12}q^{6}) - 4\,(p^{-10}q^{-6} + p^{10}q^{6}) - 4\,(p^{-10}q^{-4} + p^{10}q^{4}) \\
&\quad +7\,(p^{-8}q^{-6} + p^{8}q^{6}) + 18\,(p^{-8}q^{-4} + p^{8}q^{4}) + 7\,(p^{-8}q^{-2} + p^{8}q^{2}) - 7\,(p^{-6}q^{-6} + p^{6}q^{6}) \\
&\quad -36\,(p^{-6}q^{-4} + p^{6}q^{4}) - 36\,(p^{-6}q^{-2} + p^{6}q^{2}) - 7\,(p^{-6} + p^{6}) + 3\,(p^{-4}q^{-6} + p^{4}q^{6}) \\
&\quad +40\,(p^{-4}q^{-4} + p^{4}q^{4}) + 82\,(p^{-4}q^{-2} + p^{4}q^{2}) + 40\,(p^{-4} + p^{4}) + 3\,(p^{-4}q^{2} + p^{4}q^{-2}) \\
&\quad -22\,(p^{-2}q^{-4} + p^{2}q^{4}) - 102\,(p^{-2}q^{-2} + p^{2}q^{2}) - 102\,(p^{-2} + p^{2}) - 22\,(p^{-2}q^{2} + p^{2}q^{-2}) \\
&\quad +4\,(q^{-4} + q^{4}) + 68\,(q^{-2} + q^{2}).
\end{aligned}
$$

As 8_{17} is amphichiral, the polynomial invariant is palindromic, as predicted by Proposition 4.1. Furthermore, we may observe the invariance:

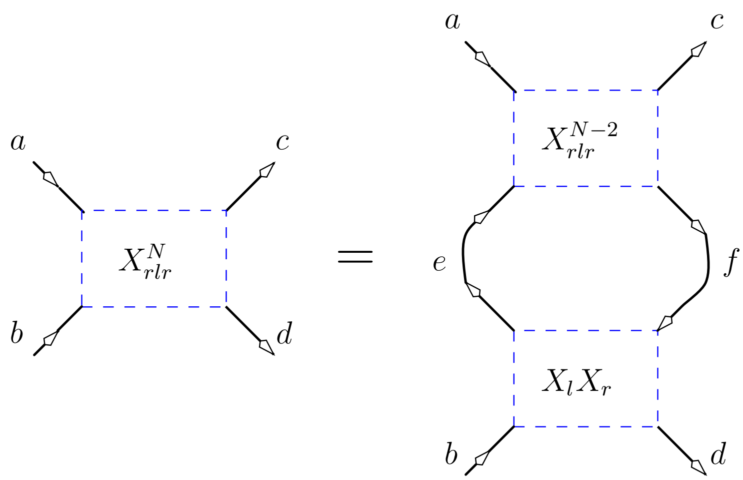
$$LG_{8_{17}}(q,p) = LG_{8_{17}}\left(q, q^{-1}p^{-1}\right),$$

which is consistent with our assertion that our polynomial invariant cannot detect the noninvertibility of *any* knot. More experiments to illustrate this claim are supplied in §4.7.

4.7 A Class of Noninvertible Pretzel Knots

A class of noninvertible knots has been presented by Trotter [38]; this class provides an easily-programmable set of examples to see if a knot invariant detects noninvertibility. Trotter is of the opinion that the knots are chiral. These pretzel knots were in fact the *first* noninvertible knots to be described [28, p 25].

The structure of the knots (p, q, r) in this family is depicted by its simplest example in Figure 16. Note that p, q, and r must all be distinct, odd, and greater than 1. In Figure 16, the notation X^N_{rlr} refers to the 2-braid of N crossings formed by the placing of X_r atop X_l, with X_r as the top and bottom crossings, for X being either σ or σ^{-1}. The recursive definition for such knots is provided in Figure 15.



$(X^N_{rlr})^{a\ c}_{b\ d}$ $\qquad$ $(X^{N-2}_{rlr})^{a\ c}_{e\ f} \cdot (X_l X_r)^{e\ f}_{b\ d}$

Figure 15: Recursive definition of the towers X_{rlr} used in the evaluation of the Links–Gould link invariant for the Trotter pretzel knots; X is either σ or σ^{-1}. The minimum is the case $N = 1$, which corresponds to X_r, i.e. $X^1_{rlr} \triangleq X_r$. A parallel definition of X_{lrl} might be given.

The tensor associated with the pretzel is:

$$T_{TP(p,q,r)}{}^{y}_{x} \triangleq (\sigma^{-p}_{rlr})^{a\ c}_{x\ d} \cdot (\sigma^{-q}_{rlr})^{e\ g}_{f\ h} \cdot (\sigma^{-r}_{rlr})^{i\ k}_{j\ l} \cdot (\Omega^-)_{a\ k} \cdot (\Omega^+)_{c\ e} \cdot (\Omega^+)_{g\ i} \cdot (\mho^+)^{d\ f} \cdot (\mho^+)^{h\ j} \cdot (\mho^+)^{l\ y}.$$

Experiments show that the Links–Gould invariant for this class of noninvertible knots always displays the symmetry of (11), for all $p, q, r \leqslant 67$. This amounts to 5456 knots, the smallest being the $(3, 5, 7)$ pretzel, a knot of $3 + 5 + 7 = 15$ crossings, and the largest being the $(63, 65, 67)$ pretzel, a knot of $63 + 65 + 67 = 195$ crossings. Incidentally, we find that the invariant demonstrates that all those pretzels are chiral.

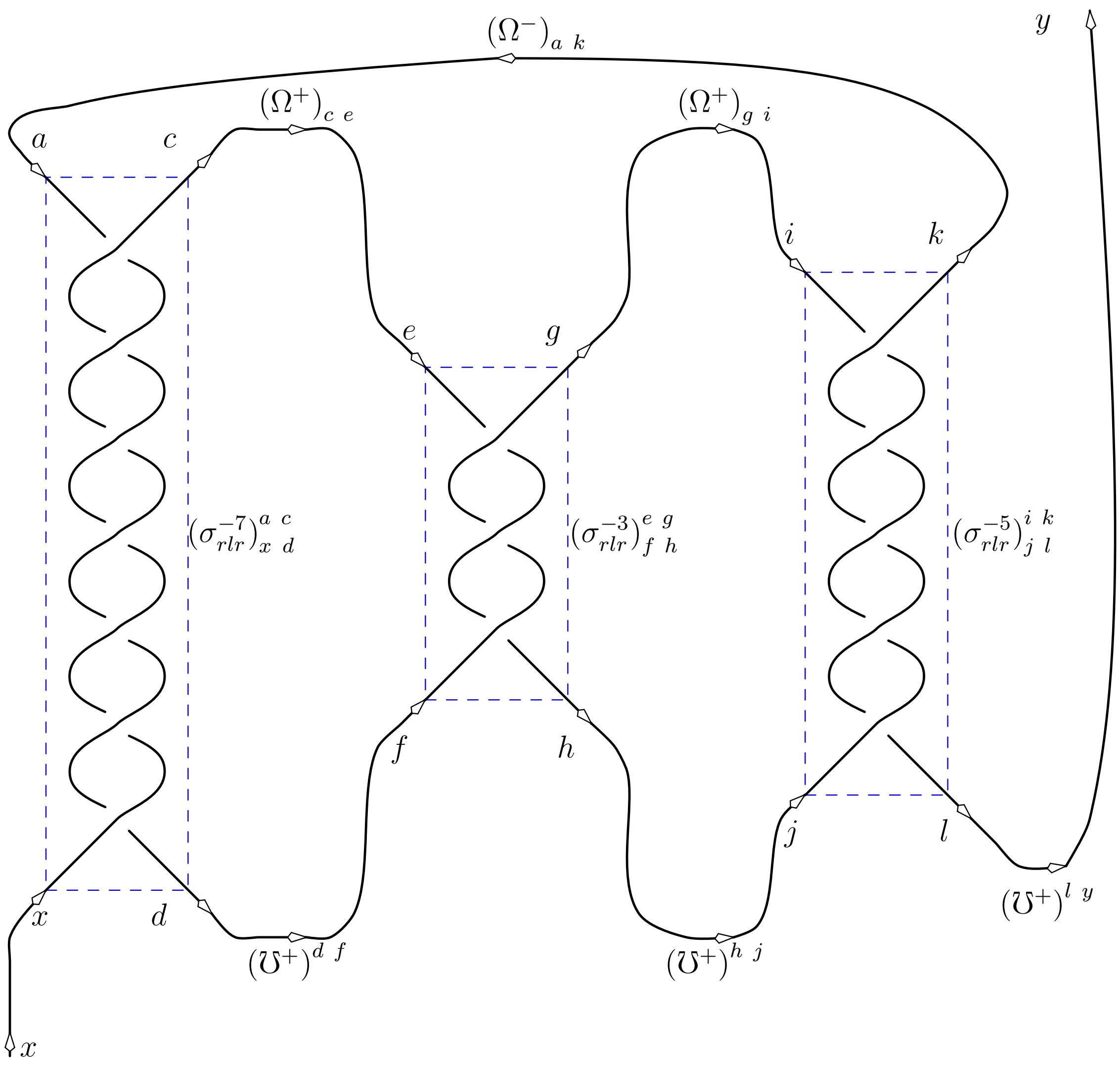


Figure 16: The (noninvertible) pretzel knots of Trotter, in tangle form. This illustration is of the smallest possible one, with $p = 7$, $q = 3$, $r = 5$.

4.8 The Kinoshita–Terasaka Pair of Mutant Knots

The Kinoshita–Terasaka pair is an example of a pair of 11 crossing mutant knots that are known to be distinct. To be precise, more commonly, the first of the pair is usually known as the "Kinoshita–Terasaka Knot", and the second has been called the "Conway Knot". In the original source by Kinoshita and Terasaka [21, p 151], the knot involved is the one labelled $\kappa\,(2,2)$ (reproduced in [28, p 53]). They had constructed this knot as an example of a nontrivial 11 crossing knot with Alexander polynomial equal to 1. The source used to draw our example is from [1, p 174]; note that these diagrams have 12 crossings, so they are not minimal.

A number of proofs of their distinctness are at hand:

- Adams [1, p 106] states that Francis Bonahon and Lawrence Siebenmann first showed this in 1981. Adams [1, p 174] goes on to state that in 1986 David Gabai [11] showed that their *minimal genus Seifert surfaces* have different *genera*[3].

- More recently, Morton and Cromwell [29] have constructed a Vassiliev invariant of *type*[4] 11 which distinguishes them. This Vassiliev invariant is based on the HOMFLY polynomial for framed links, and the authors compare it with another invariant, itself coming from $SU_q\,(3)$, which does *not* distinguish them.

  More specifically, they show that the '$SU_q\,(N)$ invariant' for the module with Young diagram $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ will distinguish at least *some* mutant pairs (in particular the KT pair), for all $N \geqslant 4$, but will definitely *not* distinguish any for $N = 2, 3$.

More generally, it is known that neither the HOMFLY nor the Kauffman polynomial can distinguish *any* pair of mutants [1, p 174]. In fact Lickorish [24] used skein theoretical arguments to show this; and furthermore, Lickorish and Lipson [25] and, independently Przytycki [31] again used skein theoretical arguments to show that two equally twisted 2-*cables* (definition in [1, p 118]) of a mutant pair would have the same HOMFLY polynomial. Perhaps the strongest statement that can be made in this direction was provided in 1994 by Chmutov, Duzhin and Lando [5], who proved that *all* Vassiliev invariants of type less than 9 will agree on *any* pair of mutants.

The question of whether the Links–Gould invariant is able to distinguish mutants is immediately answerable in the negative. Theorem 5 of [29] states that if the modules occurring in the decomposition of $V \otimes V$ each have unit multiplicity, as indeed (1) shows in our case, then the invariant is unable to detect mutations. Whilst this was proved in [29] for the case of quantum algebras, the extension to the case of quantum superalgebras is quite straightforward. As an example, we have explicitly evaluated the Links–Gould invariant for the aforementioned pair of mutants.

[3] *If L is an oriented link in S^3* (i.e. the 3-sphere), *then a Seifert surface for L is an oriented surface R embedded in S^3 such that $\partial R = L$ and no component is closed.* [11, p 677]. That is, a Seifert surface is a 2-manifold with boundary being the link in question; the genus of such a surface being a topological classifying label [1, p 95-106]. The original reference for the Seifert algorithm is contained in [37].

[4] A Vassiliev invariant is defined [29, p 229] to be of *type* d if it is zero on any link diagram of $d+1$ nodes, and to be of *degree* d if it is of type d but not of type $d-1$.

We illustrate KT, the first of Kinoshita–Terasaka pair, in Figure 17, where the tensors KTA and KTB are defined below, in Figures 19 and 18. From KT, we may build the mutant KT' by replacing the component KTA with KTA' (depicted in Figure 20), which is formed by reflection of KTA about a horizontal line.

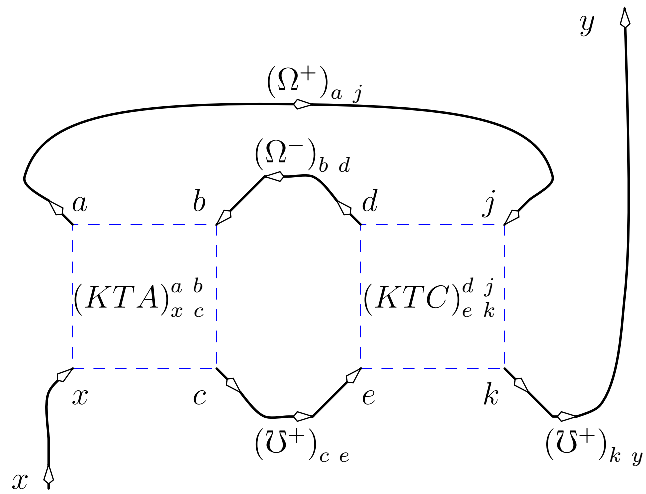


Figure 17: KT, the first of the Kinoshita–Terasaka pair of mutant knots, where the sub-diagrams KTA, KTA' and KTC are found in Figures 19, 20 and 18 respectively. (The mutant KT' of KT is obtained by exchanging KTA with KTA'.)

The tensors associated with KT and KT' are:

$$\begin{aligned}
(T_{KT})^{y}_{x} &\triangleq (KTA)^{a\ b}_{x\ c} \cdot (KTC)^{d\ j}_{e\ k} \cdot \left(\Omega^-\right)_{b\ d} \cdot \left(\mho^+\right)^{c\ e} \cdot \left(\Omega^+\right)_{a\ j} \cdot \left(\mho^+\right)^{k\ y}, \\
(T_{KT'})^{y}_{x} &\triangleq (KTA')^{a\ b}_{x\ c} \cdot (KTC)^{d\ j}_{e\ k} \cdot \left(\Omega^-\right)_{b\ d} \cdot \left(\mho^+\right)^{c\ e} \cdot \left(\Omega^+\right)_{a\ j} \cdot \left(\mho^+\right)^{k\ y},
\end{aligned}$$

where

$$\begin{aligned}
(KTA)^{a\ b}_{q\ c} &\triangleq (\sigma\sigma_d)^{a\ b}_{d\ e} \cdot \left(\sigma^{-2}\right)^{d\ f}_{q\ g} \cdot \left(\sigma_d^{-1}\right)^{h\ e}_{i\ c} \cdot \left(\Omega^+\right)_{f\ h} \cdot \left(\mho^-\right)^{g\ i} \\
(KTA')^{a\ b}_{q\ c} &\triangleq \left(\sigma^{-2}\right)^{a\ f}_{d\ g} \cdot \left(\sigma_d^{-1}\right)^{h\ b}_{i\ e} \cdot (\sigma\sigma_d)^{d\ e}_{q\ c} \cdot \left(\Omega^+\right)_{f\ h} \cdot \left(\mho^-\right)^{g\ i} \\
(KTC)^{d\ j}_{e\ k} &\triangleq (KTB)^{d\ f}_{e\ g} \cdot \left(\sigma_l^{-1}\sigma_r^{-1}\right)^{h\ j}_{i\ k} \cdot \left(\Omega^-\right)_{f\ h} \cdot \left(\mho^+\right)^{g\ i} \\
(KTB)^{d\ f}_{e\ g} &\triangleq \sigma^{d\ b}_{a\ c} \cdot (\sigma_d^2)^{l\ f}_{m\ n} \cdot \left(\sigma^{-1}\sigma_d^{-1}\right)^{a\ n}_{e\ g} \cdot \left(\Omega^+\right)_{b\ l} \cdot \left(\mho^-\right)^{c\ m}.
\end{aligned}$$

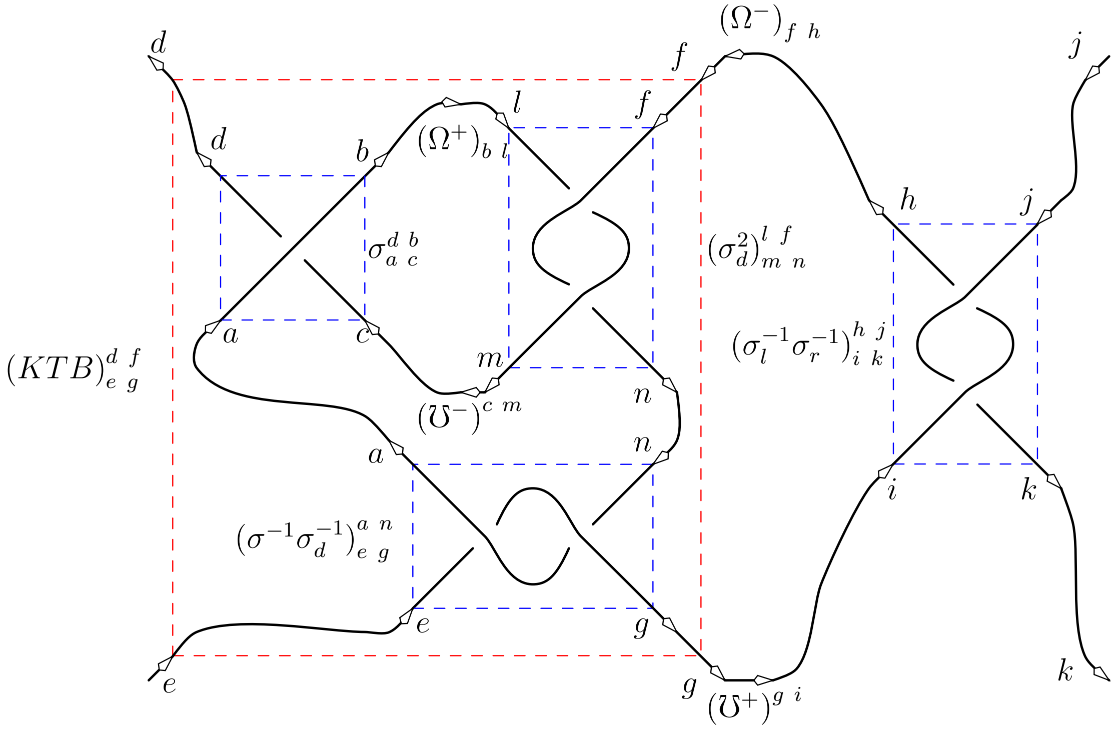


Figure 18: The component KTC of the Kinoshita–Terasaka pair of mutant knots KT and KT'.

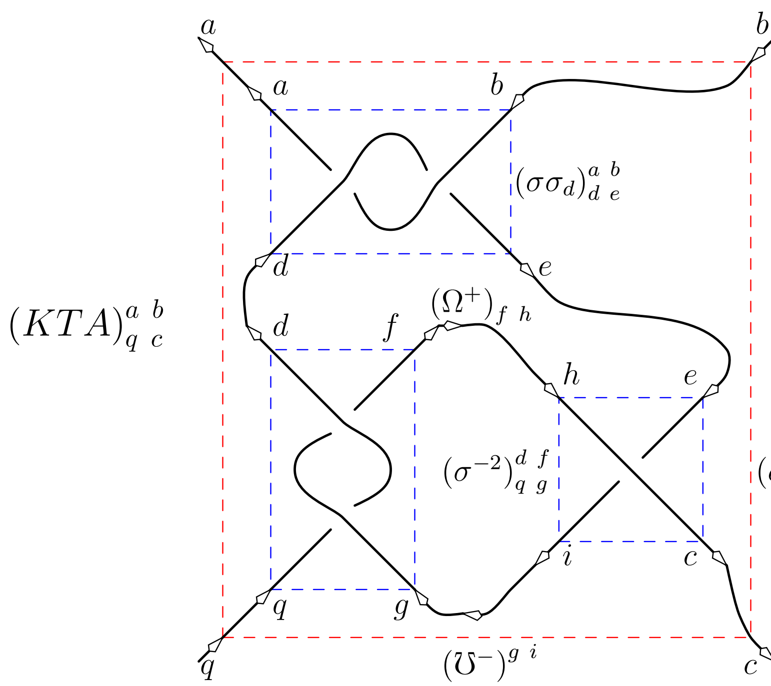


Figure 19: The component KTA of KT, the first of the Kinoshita–Terasaka pair.

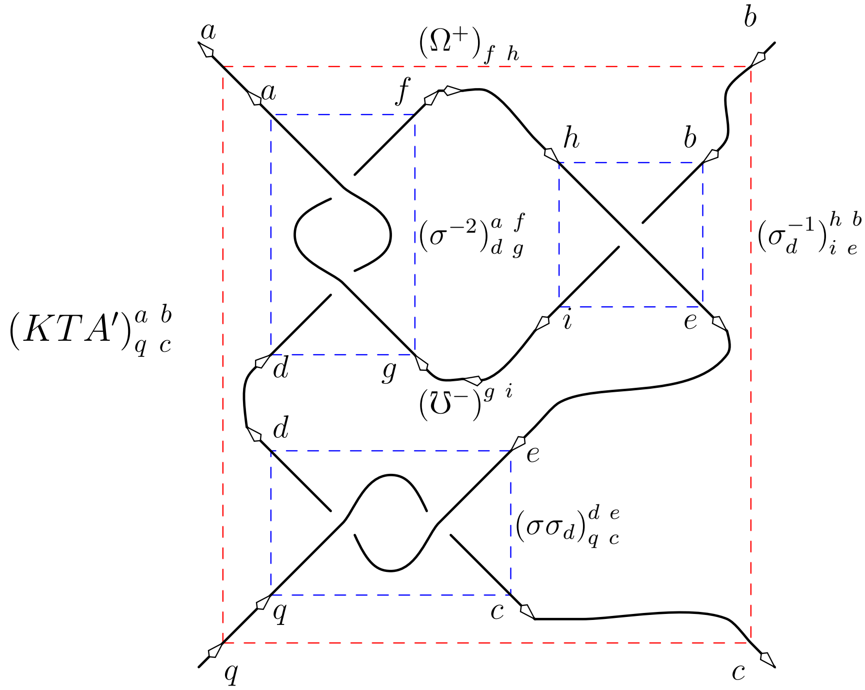


Figure 20: The component KTA' of KT', the second of the Kinoshita–Terasaka pair.

4.9 Links–Gould Polynomials of the K–T Mutants

We find the Links–Gould polynomials of both mutants to be:

$$
\begin{aligned}
&LG_{KT}(q,p) = \\
&\quad -23 - p^{-6}q^{-8} - p^{-6}q^{-6} + 2p^{-6}q^{-4} + p^{-6}q^{-2} - p^{-6} + p^{-4}q^{-8} + 6p^{-4}q^{-6} - 3p^{-4}q^{-4} \\
&\quad -9p^{-4}q^{-2} + 2p^{-4} + 3p^{-4}q^{2} - 7p^{-2}q^{-6} - 7p^{-2}q^{-4} + 18p^{-2}q^{-2} + 9p^{-2} - 11p^{-2}q^{2} \\
&\quad -2p^{-2}q^{4} + 2q^{-6} + 14q^{-4} - 8q^{-2} + 6q^{2} + 10q^{4} - 7p^{2}q^{-4} - 7p^{2}q^{-2} + 18p^{2} + 9p^{2}q^{2} \\
&\quad -11p^{2}q^{4} - 2p^{2}q^{6} + p^{4}q^{-4} + 6p^{4}q^{-2} - 3p^{4} - 9p^{4}q^{2} + 2p^{4}q^{4} + 3p^{4}q^{6} - p^{6}q^{-2} - p^{6} \\
&\quad +2p^{6}q^{2} + p^{6}q^{4} - p^{6}q^{6},
\end{aligned}
$$

hence the Links–Gould link invariant *does not* distinguish between these mutants.

As predicted by the theorem of [29], the tensors KTA and KTA' are in fact *identical*, which explains why the pair of mutants yield the same invariant.

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Louis Kauffman thanks the National Science Foundation for support of this research under grant number DMS-9205277 and the NSA for partial support under grant number MSPF-96G-179.

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