CHAPTER 2

CONSTRUCTION OF THE STEENROD SQUARES

The object of this chapter is to construct the Steenrod squares. These are cohomology operations of type $(Z_2, n; Z_2, n + i)$.

We remark now, once and for all, that there are analogous operations for Z_p coefficients, where p is an odd prime; but they will not be treated in this book.

THE COMPLEX $K(Z_2,1)$

In Chapter 1 we constructed a CW-complex $K(\pi,n)$ for any abelian group π where $n \geq 2$. We now have need for an explicit complex $K(Z_2,1)$.

Proposition 1

Let $P = P(\infty) = \bigcup_n P(n)$ denote infinite-dimensional real projective space, the limit of $P(n)$ under the natural injections $P(n) \to P(n+1)$. Then P is a $K(Z_2,1)$ space.

PROOF: The n-sphere S^n is a covering space for $P(n)$ with covering group Z_2. From the exact homotopy sequence of this covering, it follows that $\pi_1(P(n)) = Z_2$ and that $\pi_i(P(n))$ is trivial for $1 < i < n$. The result follows.

Let S^∞ denote the infinite-dimensional sphere, i.e., the union of all S^n under the natural injections $S^n \to S^{n+1}$. We can easily give a cell structure for S^∞ as a CW-complex. In each dimension $i \geq 0$, we have two cells, which will be denoted d_i and Td_i. The action of the homology boundary ∂ is given by $\partial d_i = d_{i-1} + (-1)^i Td_{i-1}$, with $\partial T = T\partial$, $TT = 1$. (A sketch of S^2 will make this plausible.)

We can compute the homology of S^∞ from these formulas, verifying that S^∞ is acyclic. Indeed, in even dimensions the only non-zero cycles are generated by $d_{2j} - Td_{2j} = \partial d_{2j+1}$; in odd dimensions, only the sign is changed in the argument.

The homology of P is more interesting. We obtain a cell structure for $P = P(\infty)$ by collapsing S^∞ under the action of Z_2 considered as generated by T; in other words, identify d_i with Td_i for every $i \geq 0$. Thus P has exactly one cell in each dimension, denoted by e_i; and the boundary formula is $\partial e_{2j} = 2e_{2j-1}$, $\partial e_{2j-1} = 0$. Therefore $\tilde{H}_i(P;Z)$ must be Z_2 for i odd, 0 for i even.

By the universal coefficient theorems, since $H_*(P;Z)$ is Z_2 in odd dimensions, we have also the following: $H^*(P;Z)$ is Z_2 in positive even dimensions; $H_*(P;Z_2)$ and likewise $H^*(P;Z_2)$ are Z_2 in all dimensions.

Thus we know the homology and cohomology groups of $K(Z_2,1)$ in all dimensions and for any coefficients and in particular for coefficients Z or Z_2. We turn to the calculation of the cohomology ring.

Let W denote the chain complex of S^∞. Then W is a Z_2-free acyclic chain complex with two generators in each dimension $i \geq 0$.

To compute the ring structure of $H^*(P)$, we must give a diagonal map for W. The action of T on W gives an action of T on $W \otimes W$ by $T(u \otimes v) = T(u) \otimes T(v)$.

Define $r\colon W \to W \otimes W$ by

$$r(d_i) = \textstyle\sum_{0 \leq j \leq i} (-1)^{j(i-j)} d_j \otimes T^j d_{i-j}$$
$$r(Td_i) = T(r(d_i))$$

where T^j of course denotes T or 1 according to whether j is odd or even. Then r is a chain map with respect to the usual boundary in $W \otimes W$, namely, $\partial(u \otimes v) = (\partial u) \otimes v + (-1)^{\deg u} u \otimes (\partial v)$. The verification is direct but tedious, and we omit it.

If h denotes the diagonal map of Z_2, then it is clear from the definitions that r is *h-equivariant*, that is, $r(gw) = h(g)r(w)$ for $g \in Z_2$ and $w \in W$. Therefore r induces a chain map $s\colon W/T \to W/T \otimes W/T$, where $W/T = W/Z_2$ is the chain complex of $P = P(\infty)$. Explicitly,

$$s(e_i) = \textstyle\sum_{0 \leq j \leq i} (-1)^{j(i-j)} e_j \otimes e_{i-j}$$

This map s is a chain approximation to the diagonal map Δ of P, and so we may use it to find cup products in $H^*(P;Z_2)$. Let α_i denote the nontrivial element of $H^i(P;Z_2)=Z_2$. The summation for $s(e_{j+k})$ contains a term $e_j \otimes e_k$ with coefficient 1 mod 2. Therefore,

$$\begin{aligned}\langle \alpha_j \cup \alpha_k, e_{j+k}\rangle &= \langle \Delta^*(\alpha_j \times \alpha_k), e_{j+k}\rangle \\ &= \langle \alpha_j \times \alpha_k, s(e_{j+k})\rangle \\ &\equiv 1 \pmod 2\end{aligned}$$

so that $\alpha_j \cup \alpha_k = \alpha_{j+k}$. We have indicated the proof of the following result.

Proposition 2

$H^*(Z_2,1;Z_2)=H^*(P;Z_2)$, as a ring, is the polynomial ring $Z_2[\alpha_1]$ on one generator, the non-zero one-dimensional class α_1.

THE ACYCLIC CARRIER THEOREM

To state the fundamental theorem on acyclic carriers, we need some terminology. Let π and G be groups (not necessarily abelian) and let $Z[\pi]$ denote the group ring of π. Let K be a π-free chain complex with a $Z[\pi]$-basis B of homogeneous elements, called "cells." For two cells $\sigma,\tau \in B$, let $[\tau:\sigma]$ denote the coefficient of σ in $\partial\tau$; this is an element in $Z[\pi]$. Let L be a chain complex on which G acts, and let h be a homomorphism $\pi \to G$.

Definition

An *h-equivariant carrier* $\mathcal{C}$ from K to L is a function $\mathcal{C}$ from B to the subcomplexes of L such that:

1. if $[\tau:\sigma] \neq 0$ then $\mathcal{C}\sigma \subset \mathcal{C}\tau$;

2. for $x \in \pi$ and $\sigma \in B$, $h(x)\mathcal{C}\sigma \subset \mathcal{C}\sigma$.

The carrier $\mathcal{C}$ is said to be *acyclic* if the subcomplex $\mathcal{C}\sigma$ is acyclic for every cell $\sigma \in B$. The h-chain map $f\colon K \to L$ is said to be *carried by* $\mathcal{C}$ if $f\sigma \in \sigma\mathcal{C}$ for every $\sigma \in B$.

Theorem 1

Let $\mathcal{C}$ be an acyclic carrier from K to L. Let K' be a subcomplex of K which is a $Z(\pi)$-free complex on a subset of B. Let $f\colon K' \to L$ be an h-equivariant chain map carried by $\mathcal{C}$. Then f extends over all of K to an h-equivariant chain map carried by $\mathcal{C}$. Moreover the extension is unique up to an h-equivariant chain homotopy carried by $\mathcal{C}$.

Note the important special case where K' is empty.

The proof proceeds by induction on the dimension; suppose that f has been extended over all of K^q and consider a $(q+1)$-cell $\tau \in B$. Then $\partial\tau = \sum a_i\sigma_i$ where $a_i = [\tau : \sigma_i] \in Z(\pi)$. Thus $f(\partial\tau) = \sum f(a_i\sigma_i) = \sum h(a_i)f(\sigma_i)$, which is in $\mathcal{C}\tau$ by properties (1) and (2). Since f is a chain map, $f(\partial\tau)$ is a cycle, but then, since $\mathcal{C}\tau$ is acyclic, there must exist x in $\mathcal{C}\sigma$ such that $\partial x = f(\partial\tau)$. Choose any such x, and put $f(\tau) = x$. This is the essential step in the construction; f is extended over K^{q+1} by requiring it to be h-invariant. Uniqueness is proved by applying the construction to the complex $K \times I$ and its subcomplex $K' \times I \cup K \times \dot{I}$.

CONSTRUCTION OF THE CUP-i PRODUCTS

Now let K be the chain complex of a simplicial complex, and let W be the Z_2-free complex discussed above. Define the action of Z_2 (generated by T) on $W \otimes K$ by $T(w \otimes k) = (Tw) \otimes k$, and on $K \otimes K$ by $T(x \otimes y) = (-1)^{dg\,x\,dg\,y}(y \otimes x)$. Define a carrier $\mathcal{C}$ from $W \otimes K$ to $K \otimes K$ by

$$\mathcal{C}: d_i \otimes \sigma \longrightarrow C(\sigma \times \sigma)$$

where by $C(\sigma \times \sigma)$ we mean the following: $K = C(X)$, the chain complex of the simplicial complex X. By the Eilenberg-Zilber theorem, there is a canonical chain-homotopy equivalence $\Psi: C(X \times X) \longrightarrow C(X) \otimes C(X)$. Then for σ a generator of K, i.e., a simplex of X, by $C(\sigma \times \sigma)$ we mean the subcomplex $\Psi C(\sigma \times \sigma)$ of $C(X) \otimes C(X)$. Then $\mathcal{C}$ is clearly acyclic and h-equivariant, where h is the identity map of Z_2. Therefore there exists an h-equivariant chain map

$$\varphi: W \otimes K \longrightarrow K \otimes K$$

carried by $\mathcal{C}$.

We should examine this φ because it plays a principal role in the definition of the squaring operations. Consider the restriction $\varphi_0 = \varphi \mid d_0 \otimes K$. This can be viewed as a map $K \longrightarrow K \otimes K$, and as such it is carried by the diagonal carrier. Thus it is a chain approximation to the diagonal, suitable for computing cup products in K. The same remarks apply to $T\varphi_0: \sigma \longrightarrow \varphi(Td_0 \otimes \sigma)$. Since both φ_0 and $T\varphi_0$ are carried by $\mathcal{C}$, they must be equivariantly homotopic. In fact it is not hard to verify that the chain homotopy is given by $\varphi_1: K \longrightarrow K \otimes K: \sigma \longrightarrow \varphi(d_1 \otimes \sigma)$. Further, φ_1 and $T\varphi_1$ are equivariantly homotopic homotopies; a homotopy is given by φ_2; and so forth.

We now use φ to construct cochain products.

Definition

For each integer $i \geq 0$, define a "cup-i product"

$$C^p(K) \otimes C^q(K) \to C^{p+q-i}(K): (u,v) \to u \cup_i v$$

by the formula

$$(u \cup_i v)(c) = (u \otimes v)\varphi(d_i \otimes c) \qquad c \in C_{p+q-i}(K)$$

For the definition we must make an explicit choice of φ, but it will be seen that this choice is not essential.

Coboundary formula

$$\delta(u \cup_i v) = (-1)^i \, \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v - (-1)^i u \cup_{i-1} v - (-1)^{pq} v \cup_{i-1} u$$

(It is understood that $u \cup_{-1} v = 0$.)

The proof is a routine computation from the definitions: let c be a chain of $C_{p+q-i+1}(K)$; then

$$(\delta(u \cup_i v))(c) = (u \cup_i v)(\partial c) = (u \otimes v)\varphi(d_i \otimes \partial c)$$

By definition, $\partial(d_i \otimes c) = \partial d_i \otimes c + (-1)^i d_i \otimes \partial c$; so this becomes

$$\begin{aligned}
&(-1)^i(u \otimes v)\, \varphi\partial(d_i \otimes c) - (-1)^i(u \otimes v)\varphi(\partial d_i \otimes c) \\
&\quad = (-1)^i(u \otimes v)\, \partial\varphi(d_i \otimes c) - (-1)^i(u \otimes v)\varphi(d_{i-1} \otimes c) \\
&\qquad\qquad\qquad\qquad - (-1)^{2i}(u \otimes v)\varphi(Td_{i-1} \otimes c) \\
&\quad = (-1)^i \, \delta(u \otimes v)\varphi(d_i \otimes c) - (-1)^i(u \otimes v)\varphi(d_{i-1} \otimes c) \\
&\qquad\qquad\qquad\qquad - (u \otimes v)T\varphi(d_{i-1} \otimes c) \\
&\quad = (-1)^i \, \delta(u \otimes v)\varphi(d_i \otimes c) - (-1)^i(u \otimes v)\varphi(d_{i-1} \otimes c) \\
&\qquad\qquad\qquad\qquad - (-1)^{pq}(v \otimes u)\varphi(d_{i-1} \otimes c)
\end{aligned}$$

By definition of $\delta(u \otimes v)$, the first term may be written

$$(-1)^i(\delta u \otimes v)\varphi(d_i \otimes c) + (-1)^{i+p}(u \otimes \delta v)\varphi(d_i \otimes c)$$

and so we finally have, from the definitions,

$$\begin{aligned}
(\delta(u \cup_i v))(c) = (-1)^i \, \delta u \cup_i v + (-1)^{i+p} u \cup_i \delta v - (-1)^i u \cup_{i-1} v \\
- (-1)^{pq} v \cup_{i-1} u
\end{aligned}$$

which is the required formula.

THE SQUARING OPERATIONS

We emphasize that the cup-i products are defined on integral cochains and take values in integral cochains. But suppose $u \in C^p(K)$ is a cocycle mod 2, that is, $\delta u = 2a$, $a \in C^{p+1}(K)$. It follows from the coboundary formula that $u \cup_i u$ is also a cocycle mod 2. We can therefore define operations

$$Sq_i\colon Z^p(K;Z_2) \to Z^{2p-i}(K;Z_2)\colon u \to u \cup_i u$$

in the obvious way. Moreover, we can compose this with the natural projection of cocycles onto cohomology classes.

Lemma 1

The resulting function $Sq_i\colon Z^p(K;Z_2) \to H^{2p-i}(K;Z_2)$ is a homomorphism.

PROOF: Let c be a cochain of the appropriate dimension, and write down $Sq_i(u+v)(c)$. As expected, one obtains the terms $Sq_i(u)(c)$ and $Sq_i(v)(c)$ plus two cross terms, but the sum of the cross terms is a coboundary mod 2:

$$(u \cup_i v)(c) + (v \cup_i u)(c) = \delta(u \cup_{i+1} v)(c) \qquad (\text{mod } 2)$$

Lemma 2

If u is a coboundary, so also is $Sq_i(u)$.

PROOF: If $u = \delta a$, $Sq_i(u) = \delta(a \cup_i \delta a + a \cup_{i-1} a) \qquad (\text{mod } 2)$

Proposition 3

The above operation passes to a homomorphism:

$$Sq_i\colon H^p(K;Z_2) \to H^{2p-i}(K;Z_2)$$

This follows from the preceding lemmas. Of course "homomorphism" is understood in the sense of additive groups, not of rings.

Proposition 4

Let f be a continuous map $K \to L$. Then f^* commutes with Sq_i as in the diagram below.

$$\begin{array}{ccc} H^p(L;Z_2) & \xrightarrow{Sq_i} & H^{2p-i}(L;Z_2) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ H^p(K;Z_2) & \xrightarrow{Sq_i} & H^{2p-i}(K;Z_2) \end{array}$$

PROOF: By the simplicial approximation theorem, we may assume f is simplicial. Let u be a p-cochain of L. From the definitions, we have the formulas

$$f^*(Sq_i(u))\colon c \to (u \otimes u)\varphi_L(d_i \otimes f(c)) = (u \otimes u)\varphi_L(1 \otimes f)(d_i \otimes c)$$
$$Sq_i(f^*(u))\colon c \to (f^*u \otimes f^*u)\varphi_K(d_i \otimes c) = (u \otimes u)(f \otimes f)\varphi_K(d_i \otimes c)$$

where c is a $(2p-i)$-chain of K. But the two chain maps $\varphi_L(1 \otimes f)$ and $(f \otimes f)\varphi_K$ are both carried by the acyclic carrier $\mathcal{C}$ from $W \otimes K$ to $L \otimes L$

given by $\mathcal{C}(d_i \otimes \sigma) = C(f\sigma \times f\sigma)$. Thus they are equivariantly chain-homotopic, and hence the two images displayed above are cohomologous. In fact, if h is the homotopy, the difference between the above cochains is δg where $g(e) = (u \otimes u)h(d_i \otimes e)$.

The definition of the squaring operations is now complete, in the sense that we can draw the following inference.

Corollary 1

The operation Sq_i is independent of the choice of φ.

The corollary is proved by putting $K = L$ in Proposition 4, letting φ_K, φ_L denote two different choices of φ, and taking for f the identity map.

Proposition 5

If u is a cochain of dimension p, then $Sq_0(u)$ is the cup-product square u^2.

This follows from the remarks given after the definition of φ, since $(u \cup_0 u)(c) = (u \otimes u)\varphi_0(c)$ and φ_0 is suitable for computing cup products.

We have already begun to assemble the basic properties of the squaring operations, and henceforward it will be more convenient to modify the notation as follows.

Definition

Denote by Sq^i (with upper index) the natural homomorphisms

$$Sq^i\colon H^p(K;Z_2) \to H^{p+i}(K;Z_2) \qquad i = 0, 1, \ldots, p$$

given by $Sq^i = Sq_{p-i}$. For values of i outside the range $0 \leq i \leq p$, Sq^i is understood to be the zero homomorphism.

Thus Sq^i raises dimension by i in the cohomology of K.

COMPATIBILITY WITH COBOUNDARY AND SUSPENSION

We now wish to define squaring operations in relative cohomology. Let L be a subcomplex of K; we have an exact sequence, at the cochain level,

$$0 \to C^*(K,L) \xrightarrow{q^*} C^*(K) \xrightarrow{j^*} C^*(L) \to 0$$

We may assume that $\varphi_L = \varphi_K \mid W \otimes L$, since $\varphi_K(d_i \otimes \sigma) \in C(\sigma \times \sigma) \subset L \otimes L$ for $\sigma \in L$. This implies that for $u,v \in C^*(K)$, $j^*(u \cup_i v) = j^*u \cup_i j^*v$. Define relative cup-$i$ products as follows. Let $u,v \in C^*(K,L)$; then $j^*(q^*u \cup_i q^*v) = 0$, so that $(q^*u \cup_i q^*v)$ is in the image of q^*, by exactness; but q^* is one-to-one, and hence we may define $u \cup_i v$ as the unique cochain in $C^*(K,L)$ such that $q^*(u \cup_i v) = q^*u \cup_i q^*v$. It is trivial to verify that the coboundary formula for cup-i products carries over into this context. Therefore we obtain homomorphisms

$$Sq^i\colon H^p(K,L;Z_2) \to H^{p+i}(K,L;Z_2)$$

by the same process as in the absolute case. It is obvious from the definition that $q^*Sq^i = Sq^i q^*$.

In what follows, all coefficients are in Z_2 and we drop the Z_2 from the notation.

We recall the definition of the coboundary homomorphism $\delta^*\colon H^*(L) \to H^*(K,L)$. Let a be a cocycle and $\bar{a}$ its cohomology class, $a \in \bar{a} \in H^p(L)$. Then $a = j^*b$ for some $b \in C^p(K)$. Then $j^*(\delta b) = (\delta a) = 0$, and so $\delta b = q^*c$ for some $c \in C^{p+1}(K,L)$. Since q^* is one-to-one, c must be a cocycle, and by definition $\delta^*(a) = \bar{c}$, the cohomology class of c.

Proposition 6

Sq^i commutes with δ^* as in the diagram.

$$\begin{array}{ccc} H^p(L) & \xrightarrow{Sq^i} & H^{p+i}(L) \\ {\scriptstyle\delta^*}\downarrow & & \downarrow{\scriptstyle\delta^*} \\ H^{p+1}(K,L) & \xrightarrow{Sq^i} & H^{p+i+1}(K,L) \end{array} \qquad \text{(coefficients } Z_2)$$

PROOF: Using the notation of the last paragraph, $Sq^i\bar{a}$ and $Sq^i(\delta^*\bar{a})$ are represented by $a \cup_{p-i} a$ and $c \cup_{p+1-i} c$, respectively. Now

$$q^*(c \cup_{p+1-i} c) = q^*c \cup_{p+1-i} q^*c = \delta b \cup_{p+1-i} \delta b = \delta b' \qquad (\text{mod } 2)$$

where $b' = (b \cup_{p+1-i} \delta b) + (b \cup_{p-i} b)$. Moreover, $j^*(b') = 0 + (a \cup_{p-i} a)$. Therefore, by definition of δ^*,

$$\delta^*(Sq^i(\bar{a})) = \delta^*[a \cup_{p-i} a] = [c \cup_{p+1-i} c]$$

and the class on the right is $Sq^i(\delta^*(a))$.

Recall that, given any space X, we may define the *cone* over X and the *suspension* of X from the product space $X \times I$ by collapsing $X \times 0$ or $X \times \dot{I}$, respectively, to a point. In reduced cohomology, we have the *suspension isomorphism* $S^*\colon \tilde{H}^*(X) \to \tilde{H}^*(SX)$ defined by the composition

$$\tilde{H}^p(X) \xrightarrow[\approx]{\delta^*} H^{p+1}(CX,X) \xrightarrow[\approx]{} \tilde{H}^{p+1}(SX)$$

which raises dimension by one. The second isomorphism is proved by an excision argument and is based on a map. This, together with the naturality of the squaring operations and Proposition 6, yields the following fundamental property.

Proposition 7

The squares commute with suspension:

$$Sq^i \cdot S^* = S^* \cdot Sq^i \colon \tilde{H}^p(X) \to \tilde{H}^{p+i+1}(SX)$$

We can apply Proposition 7 to obtain an example of a non-trivial squaring operation which is not just a cup product. Let K denote the real projective plane; its cohomology ring with Z_2 coefficients is just the polynomial ring over Z_2 generated by the non-trivial one-dimensional cohomology class α and truncated by the relation $\alpha^3 = 0$. From Proposition 5, $Sq^1(\alpha) = Sq_0(\alpha) = \alpha^2$, so that $S^*Sq^1(\alpha)$ is non-zero. Hence $Sq^1S^*(\alpha)$ is also non-zero, showing that the operation Sq^1 is non-trivial in $H^2(SK; Z_2)$.

DISCUSSION

The squaring operations constructed in this chapter are a special case of the reduced power operations of Steenrod. These operations have been very important in the development of algebraic topology; most of this book is devoted to their properties and applications. And there are many more.

The specific construction we have given is neither the simplest possible nor the most subtle. Steenrod's original definition is more direct; his most recent definition is far more elegant. The construction we have given is adopted as a middle ground—one from which the algebraic properties are easily deduced, and yet the geometric genesis is not totally obliterated.

The reader will observe that we have defined the squaring operations in the simplicial cohomology theory, yet we will use these operations in singular theory. We justify this usage as follows. In their book (pp. 123–124), Steenrod and Epstein show that the squares, if defined for finite regular cell complexes, have unique extension to both singular and Čech theory for arbitrary pairs. But a finite regular cell complex has the homotopy type of a simplicial complex, on which we have defined the operations.

EXERCISE

1. Suppose the cocycle $u \in C^{2p}(X; Z)$ satisfies $\delta u = 2a$ for some a.

i. Show that $u \cup_0 u + u \cup_1 u$ is a cocycle mod 4.

ii. Define a natural operation, the *Pontrjagin square*,

$$P_2 \colon H^{2p}(\ ; Z_2) \to H^{4p}(\ ; Z_4)$$

iii. Show that $\rho P_2(u) = u \cup u$, where $\rho: H^*(\ ;Z_4) \to H^*(\ ;Z_2)$ denotes reduction mod 2.

iv. Show that $P_2(u+v) = P_2(u) + P_2(v) + u \cup v$, where $u \cup v$ is computed with the non-trivial pairing $Z_2 \otimes Z_2 \to Z_4$.

REFERENCES

General

1. N. E. Steenrod [2].

Other definitions of the squares

1. N. E. Steenrod [3,5,6].
2. ——— and D. B. A. Epstein [1].

CHAPTER 3

PROPERTIES OF THE SQUARES

We assemble the fundamental properties of the squaring operations in an omnibus theorem.

Theorem 1

The operations Sq^i, defined (for $i \geq 0$) in the previous chapter, have the following properties:

0. Sq^i is a natural homomorphism $H^p(K,L; Z_2) \to H^{p+i}(K,L; Z_2)$
1. If $i > p$, $Sq^i(x) = 0$ for all $x \in H^p(K,L; Z_2)$
2. $Sq^i(x) = x^2$ for all $x \in H^i(K,L; Z_2)$
3. Sq^0 is the identity homomorphism
4. Sq^1 is the Bockstein homomorphism
5. $\delta^* Sq^i = Sq^i \delta^*$ where $\delta^*: H^*(L; Z_2) \to H^*(K,L; Z_2)$
6. *Cartan formula*: $Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y)$
7. *Adem relations*: For $a < 2b$, $Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ where the binomial coefficient is taken mod 2

We remark that the above properties completely characterize the squaring operations and may be taken as axioms, as is done in the book of Steenrod and Epstein.

Properties (0), (1), (2) and (5) have been proved in the last chapter. This chapter will be devoted to the proof of (3), (4), (6), and (7).

Sq^1 AND Sq^0

Let β denote the Bockstein homomorphism attached to the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$. Then β is a homomorphism $H^*(K,L;Z_2) \to H^*(K,L;Z)$, which raises dimension by one. It is defined on $x \in H^p(K,L;Z_2)$ as follows: represent the class x by a cocycle c; choose an integral cochain c' which maps to c under reduction mod 2; then $\delta c'$ is divisible by 2 and $\frac{1}{2}(\delta c')$ represents βx.

The composition of β and the reduction homomorphism gives a homomorphism

$$\delta_2 \colon H^p(K,L;Z_2) \to H^{p+1}(K,L;Z_2)$$

which we also call "the Bockstein homomorphism"; in fact, it is the Bockstein of the sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$. To show that this is Sq^1, we will use the following lemma, which in the light of (3) and (4) will be seen to be a special case of the Adem relations (7).

Lemma 1

$\delta_2 Sq^j = 0$ if j is odd; $\delta_2 Sq^j = Sq^{j+1}$ if j is even.

PROOF: Given $u \in H^p(K,L;Z_2)$, let c be an integral cochain such that the reduction mod 2 of c is in the class u. Then $Sq^j u$ is the class of $(c \cup_{p-j} c)$. Now $\delta c = 2a$ for some integral cochain $a \in C^{p+1}(K,L)$. Writing i for $(p-j)$, we have, by the coboundary formula,

$$\delta(c \cup_i c) = (-1)^i 2a \cup_i c + (-1)^j c \cup_i 2a - (-1)^i c \cup_{i-1} c - (-1)^p c \cup_{i-1} c$$

Thus $\delta_2(Sq^j u)$ is represented by the mod 2 cocycle

$$a \cup_i c + c \cup_i a + (s)(c \cup_{i-1} c)$$

where the coefficient s is 0 or 1 according to whether j is even or odd, respectively. But the sum of the first two terms is a coboundary, namely, $\delta(c \cup_{i+1} a)$ (mod 2), and the last term represents $(s)(Sq^{j+1}u)$. This proves the lemma.

As a special case of the lemma, $\delta_2 Sq^0 = Sq^1$. This shows that (4) follows from (3). It remains to prove (3).

Property (3) must be true in the real projective plane $P(2)$, for in that case $\delta_2 Sq^0(\alpha) = Sq^1(\alpha) = (\alpha^2) \neq 0$, and so $Sq^0(\alpha) \neq 0$, which proves $Sq^0(\alpha) = \alpha$ since α is the only non-zero element of $H^1(P(2);Z_2) = Z_2$. We can deduce that (3) is true in the circle S^1 by taking a map $f \colon S^1 \to P(2)$ such that $f^* \colon \alpha \to \sigma$, where σ denotes the generator of $H^*(S^1;Z_2)$, and applying the naturality condition: $Sq^0\sigma = Sq^0 f^*\alpha = f^*Sq^0\alpha = f^*\alpha = \sigma$. Then (3) holds in every sphere S^n by suspension (Proposition 6 of Chapter 2).

Let K be a complex of dimension n; map K to S^n so that σ pulls back to a given class in $H^n(K;Z_2)$ (use the Hopf-Whitney theorem, Theorem 5 of Chapter 1). Then (3) holds for that class. Now let K be any complex, of unrestricted dimension; (3) must hold on any n-dimensional cohomology class in K because the injection j of the n-skeleton K^n into K induces a monomorphism $j^*: H^n(K;Z_2) \to H^n(K^n;Z_2)$ and the result follows as before by naturality. Thus (3) has been shown to hold in absolute cohomology.

Now let K,L be a pair and $K \cup CL$ the space obtained by attaching to K the cone over L, attached at the common subspace L. We then have isomorphisms, commuting with Sq^0,

$$H^*(K,L) \approx H^*(K \cup CL, CL) \approx \tilde{H}^*(K \cup CL)$$

The first isomorphism is by excision; the second, by contractibility of CL. This completes the proof of (3) and hence also of (4).

THE CARTAN FORMULA AND THE HOMOMORPHISM Sq

The Cartan formula (6) has two forms, one where we interpret the multiplication as the external cross product and another in which it is interpreted as the cup product. We will prove the first form and deduce the second as a corollary.

Consider the composition

$$W \otimes K \otimes L \xrightarrow{r \otimes 1} W \otimes W \otimes K \otimes L \xrightarrow{T} W \otimes K \otimes W \otimes L$$
$$\xrightarrow{\varphi_K \otimes \varphi_L} K \otimes K \otimes L \otimes L \xrightarrow{T} K \otimes L \otimes K \otimes L$$

where $r: W \to W \otimes W$ was defined in Chapter 2 and T permutes the second and third factors (we are not concerned with sign changes because we want conclusions in Z_2 coefficients). This composition, which we denote by $\varphi_{K \otimes L}$, is easily seen to be suitable for computations of cup-i products and hence Sq^i, in $K \otimes L$.

Using the same letters to denote (co-)homology classes or their representatives, and writing p,q,n for dim u, dim v, and $p+q-i$, respectively, we compute as follows:

$$\begin{aligned}
Sq^i(u \times v)(a \otimes b) &= ((u \otimes v) \cup_n (u \otimes v))(a \otimes b) \\
&= (u \otimes v \otimes u \otimes v)\varphi_{K \otimes L}(d_n \otimes a \otimes b) \\
&= (u \otimes u \otimes v \otimes v)\textstyle\sum \varphi_K(d_j \otimes a) \otimes T^j\varphi_L(d_{n-j} \otimes b)
\end{aligned}$$

the last step using the definition of r, and the summation being over j, $0 \leq j \leq n$;

$$\begin{aligned} Sq^i(u \times v)(a \otimes b) &= \sum (u \cup_j u)(a) \otimes (v \cup_{n-j} v)(b) \\ &= \sum (Sq^{p-j}u)(a) \otimes (Sq^{q-n+j}v)(b) \\ &= \sum (Sq^{p-j}u \times Sq^{q-n+j}v)(a \otimes b) \end{aligned}$$

Hence, since $Sq^i x$ is zero for i outside the range $0 \leq i \leq \dim x$, we have

$$\begin{aligned} Sq^i(u \times v) &= \textstyle\sum_{j=0}^{n} Sq^{p-j}u \times Sq^{q-n+j}v \\ &= \textstyle\sum_{s=i-q}^{p} Sq^s u \times Sq^{i-s}v \qquad s = p - j \\ &= \textstyle\sum_{s=0}^{i} Sq^s u \times Sq^{i-s}v \end{aligned}$$

which completes the proof of the first form of (6).

In order to prove the cup-product form of (6), let Δ denote the diagonal map of K; if $x, y \in H^*(K; Z_2)$, $x \cup y = \Delta^*(x \times y)$, and so

$$\begin{aligned} Sq^i(x \cup y) &= Sq^i \Delta^*(x \times y) \\ &= \Delta^* Sq^i(x \times y) \\ &= \Delta^* \textstyle\sum_{j=0}^{i} Sq^j x \times Sq^{i-j}y \\ &= \textstyle\sum Sq^i x \cup Sq^{i-j}y \end{aligned}$$

This completes the proof of the Cartan formula (6) in both interpretations.

We remarked before that the Sq^i are homomorphisms only in the sense of groups; the Cartan formula makes it clear that they are not ring homomorphisms, but they can be combined into a ring homomorphism in the following sense.

Definition

Define $Sq\colon H^*(K; Z_2) \to H^*(K; Z_2)$ by

$$Sq(u) = \textstyle\sum_i Sq^i u$$

The sum is essentially finite; the image $Sq(u)$ is not in general homogeneous, i.e., it need not be contained in H^p for some p. (It is understood that each Sq^i is defined on nonhomogeneous elements $u \in H^*$ by requiring it to be additive.)

Proposition 1

Sq is a ring homomorphism.

PROOF: Clearly $Sq(u) \cup Sq(v) = (\sum_i Sq^i u) \cup (\sum_j Sq^j v)$ has $Sq^i(u \cup v)$ as its $(p+q+i)$-dimensional term, by the Cartan formula. Thus $Sq(u) \cup Sq(v) = Sq(u \cup v)$.

As an application of this homomorphism, we compute the Sq^i on any power of any one-dimensional cohomology class of any complex, as follows.

Proposition 2

For $u \in H^1(K; Z_2)$, $Sq^i(u^j) = \binom{j}{i} u^{j+i}$.

PROOF: $Sq(u) = Sq^0 u + Sq^1 u = u + u^2$ by properties (1) to (3). But Sq is a ring homomorphism. Therefore $Sq(u^j) = (u + u^2)^j = u^j \sum_k \binom{j}{k} u^k$, and the proposition follows by comparing coefficients.

In particular this gives us the action of all the Sq^i in $H^*(Z_2,1; Z_2)$, since this is generated as a ring by a one-dimensional class.

We pursue this line not only for its intrinsic interest but because it will serve us in the proof of the Adem relations.

SQUARES IN THE n-FOLD CARTESIAN PRODUCT OF $K(Z_2,1)$

Definition

Let K_n be the topological product of n copies of $K(Z_2,1)$. Here we may take for $K(Z_2,1)$ the complex $P(\infty)$ discussed in Chapter 2.

Since $H^*(Z_2,1; Z_2)$ is the polynomial ring on the one-dimensional class, it follows by the Künneth theorem that the ring $H^*(K_n; Z_2)$ is the polynomial ring over Z_2 on generators $x_1, \ldots, x_n$, where x_i is the non-trivial one-dimensional class of the ith copy of $K(Z_2,1)$. In this polynomial ring $Z_2[x_1,\ldots,x_n]$, we have the subring S of symmetric polynomials, which (by the fundamental theorem of symmetric algebra) may be written as $Z_2[\sigma_1,\ldots,\sigma_n]$ where σ_j is the elementary symmetric function of degree j (for example, $\sigma_1 = x_1 + \cdots + x_n$).

Proposition 3

In $H^*(K_n; Z_2)$, $Sq^i(\sigma_n) = \sigma_n \sigma_i$ $(1 \leq i \leq n)$.

PROOF: $Sq(\sigma_n) = Sq(\prod x_i) = \prod Sq(x_i) = \prod (x_i + x_i^2) = \sigma_n(\prod (1 + x_i)) = \sigma_n \sum_{i=0}^n \sigma_i$. The result follows.

Corollary 1

In $H^*(Z_2,n; Z_2)$, $Sq^i \iota_n \neq 0$ for $0 \leq i \leq n$.

PROOF: By Theorem 1 of Chapter 1, we can find a map f of K_n into $K(Z_2,n)$ such that $f^*(\iota_n) = \sigma_n$. Then $f^* Sq^i(\iota_n) = Sq^i f^*(\iota_n) = Sq^i(\sigma_n) = \sigma_n \sigma_i \neq 0$, which proves the corollary.

We can show more; we can find a host of linearly independent elements in $H^*(Z_2,n;Z_2)$ by using compositions of the squares.

Notation

Given a sequence $I = \{i_1,\ldots,i_r\}$ of (strictly) positive integers, denote by Sq^I the composition $Sq^{i_1}\cdots Sq^{i_r}$. By convention, $Sq^I = Sq^0$, the identity, when I is the empty sequence (the unique sequence with $r = 0$).

For example, $Sq^{\{2,1\}}(x) = Sq^2(Sq^1(x))$.

Definitions

A sequence I as above is *admissible* if $i_j \geq 2(i_{j+1})$ for every $j < r$. (This condition is vacuously satisfied if $r \leq 1$.) In this case we may also refer to Sq^I as admissible. The *length* of any sequence I is the number of terms, r in the above notation. The *degree* $d(I)$ is the sum of the terms, $\sum_j i_j$. (Thus Sq^I raises dimension by $d(I)$.) For an admissible sequence I, the *excess* $e(I)$ is $2i_1 - d(I)$.

For the excess, we have

$$\begin{aligned} e(I) &= 2i_1 - d(I) \\ &= i_1 - i_2 - \cdots - i_r \\ &= (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_r) \end{aligned}$$

The last expression justifies the name, but the first two are more convenient in practice.

Recall that S denotes the symmetric polynomial subring of the polynomial ring $H^*(K_n;Z_2) = Z_2[x_1,\ldots,x_n]$. We define an ordering on the monomials of S as follows: given any such monomial, write it as $m = \sigma_{j_1}^{e_1}\sigma_{j_2}^{e_2}\cdots\sigma_{j_s}^{e_s}$ with the j_k in decreasing order, $j_1 > j_2 > \cdots$. Then put $m < m'$ if $j_1 < j_1'$ or if $j_1 = j_1'$ and $(m/\sigma_{j_1}) < (m'/\sigma_{j_1})$.

Theorem 2

If $d(I) \leq n$, then $Sq^I(\sigma_n)$ can be written $\sigma_n \cdot Q_I$ where $Q_I = \sigma_{i_1}\cdots\sigma_{i_r} +$ (a sum of monomials of lower order).

We prove this theorem by induction on the length of I; it reduces to the last proposition if $r = 1$. Let r be the length of I, and write J for $i_2, \ldots, i_r$. We assume the result for sequences of length $< r$ (in particular for J); then, using the Cartan formula,

$$\begin{aligned} Sq^I(\sigma_n) = Sq^{i_1}Sq^J(\sigma_n) &= Sq^{i_1}(\sigma_n Q_J) \\ &= \textstyle\sum_{m=0}^{i_1} Sq^m(\sigma_n)Sq^{i_1-m}(Q_J) \\ &= \sigma_n\sigma_{i_1}Q_J + \textstyle\sum_{m=0}^{i_1-1} \sigma_n\sigma_m Sq^{i_1-m}(Q_J) \end{aligned}$$

By the induction hypothesis on Q_J, $Sq^I(\sigma_n)$ may be written

$$\sigma_n\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r} + \sigma_n\sigma_{i_1}\text{ (lower terms of } Q_J) + \textstyle\sum_{m=0}^{i_1-1}\sigma_n\sigma_m Sq^{i_1-m}(Q_J)$$

Observe that $Sq^j(\sigma_i) \leq \sigma_{i+j}$ and also that $Sq^i(\sigma_i) = \sigma_i^2 < \sigma_{2i}$. Therefore the largest possible term after the first term in the above expression is obtained from the last group of terms by taking $m = i_1 - i_2 + 1$, so that $i_1 - m = i_2 - 1$; thus this term is

$$x = \sigma_n\sigma_{i_1-i_2+1}\sigma_{2i_2-1}\sigma_{i_3}\cdots\sigma_{i_r}$$

Now I is admissible, and so $2i_2 - 1 < 2i_2 \leq i_1$, but this implies that $x < \sigma_n\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$, and the proof is complete.

As I runs through all admissible sequences of degree $\leq n$, the monomials $\sigma_I = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ are linearly independent in S and hence in $H^*(K_n;Z_2)$. From the above theorem, the $Sq^I(\sigma_n)$ are also linearly independent. We can draw the following inference.

Corollary 2

As I runs through the admissible sequences of degree $\leq n$, the elements $Sq^I(\iota_n)$ are linearly independent in $H^*(Z_2,n;Z_2)$.

Choose a map $f\colon K_n \to K(Z_2,n)$ such that $f^*(\iota_n) = \sigma_n$. Then the corollary follows from the preceding remarks.

Proposition 4

If $u \in H^n(K;Z_2)$ for any space K, and I has excess $e(I) > n$, then $Sq^I u = 0$. If $e(I) = n$, then $Sq^I u = (Sq^J u)^2$, where J denotes the sequence obtained from I by dropping i_1.

This proposition may be considered as a generalization of properties (1) and (2) of Theorem 1. To prove the first statement, note that if $e(I) = i_1 - i_2 - \cdots - i_r > n$, then $i_1 > n + i_2 + \cdots + i_r = \dim(Sq^J u)$, so that $Sq^I u = Sq^{i_1}(Sq^J u) = 0$ by (1). The second statement of the proposition follows in a similar way from (2).

These results are included in a theorem of Serre, which states that $H^*(Z_2,n;Z_2)$ is exactly the polynomial ring over Z_2 with generators $\{Sq^I(\iota_n)\}$, as I runs through all admissible sequences of excess less than n. We will establish Serre's theorem in Chapter 9, using spectral-sequence methods.

As a corollary to Serre's theorem, we mention that the map $f\colon K_n \to K(Z_2,n)$ such that $f^*(\iota_n) = \sigma_n$ clearly has the property that f^* is a monomorphism through dimension $2n$. We will use this fact in the proof of the Adem relations, and so we call attention to the fact that our proof of Theorem 1 will not be complete until we have proved Serre's theorem.

THE ADEM RELATIONS

We now discuss the Adem relations (7).

An Adem relation has the form

$$R = Sq^a Sq^b + \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \equiv 0 \pmod 2$$

where $a < 2b$, and $[a/2]$ denotes the greatest integer $\leq a/2$. We usually drop the limits of summation from the expression, since the lower limit is implicit in the term Sq^c while the upper limit is implicit in the convention that the binomial coefficient $\binom{x}{y}$ is zero if $y < 0$. We also use the standard convention that $\binom{x}{y} = 0$ if $x < y$. (As an exercise in the use of these conventions, the reader may note that the Adem relations give $Sq^{2n-1}Sq^n = 0$ for every n.)

We will establish the Adem relations through a series of lemmas.

Lemma 2

Let y be a fixed cohomology class such that $R(y) = 0$ for every Adem relation R. Then $R(xy) = 0$ for every one-dimensional cohomology class x (and every R).

We will defer the proof of Lemma 2 to the end, since it is elementary but long and complicated.

Lemma 3

For every R and for every $n \geq 1$, $R(\sigma_n) = 0$ where $\sigma_n \in H^n(K_n; Z_2)$ as before.

PROOF: Let 1 denote the unit in the ring $H^*(K_n; Z_2)$; then $R(1) = 0$ for every R, by dimensional arguments. Then $R(x_1) = R(1x_1) = 0$ by Lemma 2; and finally $R(\sigma_n) = R(x_1 \cdots x_n) = 0$ by induction on n using Lemma 2.

Lemma 4

Let y be any cohomology class of dimension n of any space K, with Z_2 coefficients, and let $R = R(a,b)$ be the Adem relation for $Sq^a Sq^b$ where $a + b \leq n$. Then $R(y) = 0$.

PROOF: By Serre's theorem we have a map $f^*: H^*(Z_2,n; Z_2) \to H^*(X^n; Z_2)$ which takes ι_n to σ_n and is a monomorphism through dimension $2n$. We have $R(\sigma_n) = 0$ by Lemma 2, and so $R(\iota_n) = 0$, since these elements have dimension $n + a + b \leq 2n$. The result for y follows by naturality, using a map $g: K \to K(Z_2,n)$ such that $g^*(\iota_n) = y$.

Lemma 5

Let R be an Adem relation. If $R(y) = 0$ for every class y of dimension p, then $R(z) = 0$ for every class z of dimension $(p-1)$.

PROOF: Let u denote the generator of $H^1(S^1; Z_2)$. Clearly $Sq^i u = 0$ for all $i > 0$. Therefore, by the Cartan formula, $R(u \times z) = u \times R(z)$. But $u \times z$ has dimension p; hence $R(u \times z) = 0$, and so $R(z) = 0$.

The Adem relations follow easily from Lemma 4 and Lemma 5 by induction on dimension.

It remains to prove Lemma 2.

We begin by recalling the formula $\binom{p}{q} = \binom{p-1}{q-1} + \binom{p-1}{q}$, which holds for all p,q except for the case $p = q = 0$.

Lemma 6

$\binom{p}{q} + \binom{p}{q+1} + \binom{p-1}{q-1} + \binom{p-1}{q+1} \equiv 0 \pmod 2$ except in the cases $(p = q = 0)$ and $(p = 0, q = -1)$.

This lemma follows from the formula just cited. (The easiest way to see the sense of these two formulas is to consider Pascal's triangle.)

To prove Lemma 2 is to show $R(xy) = 0$ where x is any one-dimensional class and y has the property that $R(y) = 0$ for every R. We begin by applying the Cartan formula to $Sq^b(xy)$; since $\dim x = 1$, $Sq^b(xy) = xSq^b y + x^2 Sq^{b-1} y$. Again by the Cartan formula,

$$\begin{aligned} Sq^a Sq^b(xy) &= Sq^a(xSq^b y + x^2 Sq^{b-1} y) \\ &= xSq^a Sq^b y + x^2 Sq^{a-1} Sq^b y + x^2 Sq^a Sq^{b-1} y + 0 \\ &\quad + x^4 Sq^{a-2} Sq^{b-1} y \end{aligned}$$

the zero coming from $Sq^1(x^2)$, which is zero mod 2. In a similar manner we find that

$$\begin{aligned} \sum (s) Sq^{a+b-c} Sq^c(xy) &= x \sum (s) Sq^{a+b-c} Sq^c y + x^2 \sum (s) Sq^{a+b-c-1} Sq^c y \\ &\quad + x^2 \sum (s) Sq^{a+b-c} Sq^{c-1} y + x^4 \sum (s) Sq^{a+b-c-2} Sq^{c-1} y \end{aligned}$$

where $s = s(c) = \binom{b-c-1}{a-2c}$. In these two formulas, the first terms match, since

$$xSq^a Sq^b y + x \sum (s) Sq^{a+b-c} Sq^c y = xR(y) = 0$$

Now $a < 2b$ implies $(a-2) < 2(b-1)$, and so the fourth terms also match: since $R(y) = 0$ for all R, in particular, for $R(a-2, b-1)$,

$$\begin{aligned} Sq^{a-2} Sq^{b-1} y &= \textstyle\sum_c \binom{b-c-2}{a-2c-2} Sq^{a+b-c-3} Sq^c y \\ &= \textstyle\sum_{c'} \binom{b-c'-1}{a-2c'} Sq^{a+b-c'-2} Sq^{c'-1} y \end{aligned}$$

where $c' = c + 1$. Thus it remains to show that

$$Sq^a Sq^{b-1} y + \sum \binom{b-c-1}{a-2c-1} Sq^{a+b-c-1} Sq^c y$$
$$= \sum (s) Sq^{a+b-c-1} Sq^c y + \sum (s) Sq^{a+b-c} Sq^{c-1} y$$

where the second term on the left-hand side (LHS) replaces $Sq^{a-1}Sq^b y$, using $R(a-1, b)$. We consider three cases.

CASE 1: $a = 2b - 2$. Then $a - 2c = 2(b - c - 1)$, and so $(s) = \binom{k}{2k} = 0$ unless $k = 0$, that is, unless $c = b - 1$; so RHS $= Sq^a Sq^{b-1} y + Sq^{a+1} Sq^{b-2} y$. Similarly, $\binom{b-c-1}{a-2c-1} = \binom{k}{2k-1} = 0$ unless $k = 1$, that is, unless $c = b - 2$; so LHS $= Sq^a Sq^{b-1} y + Sq^{a+1} Sq^{b-2} y$, and the two sides are equal.

CASE 2: $a = 2b - 1$. Proved by a similar argument.

CASE 3: $a < 2b - 2$. Then, by $R(a, b-1)$,

$$Sq^a Sq^{b-1} y = \sum_c \binom{b-c-2}{a-2c} Sq^{a+b-c-1} Sq^c y$$

Also,

$$\sum (s) Sq^{a+b-c} Sq^{c-1} y = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^{c-1} y$$
$$= \sum_{c'} \binom{b-c'-2}{a-2c'-2} Sq^{a+b-c'-1} Sq^{c'} y$$

where $c' = c - 1$, so we are reduced to verifying that

$$\binom{b-c-2}{a-2c} + \binom{b-c-1}{a-2c-1} \equiv \binom{b-c-1}{a-2c} + \binom{b-c-2}{a-2c-2} \qquad (\text{mod } 2)$$

But this follows from Lemma 5, with $p = b - c - 1$, $q = a - 2c - 1$. The exceptional cases are excluded automatically, because $p = 0$, $q = 0$ or -1 means $b = c + 1$, $a = 2c + 1$ or $2c$, respectively, contradicting the Case 3 hypothesis.

This completes the proof of Lemma 2.

We attach a short table of representative Adem relations.

$Sq^1 Sq^1 = 0,\ Sq^1 Sq^3 = 0, \ldots;\ Sq^1 Sq^{2n+1} = 0$

$Sq^1 Sq^2 = Sq^3,\ Sq^1 Sq^4 = Sq^5, \ldots;\ Sq^1 Sq^{2n} = Sq^{2n+1}$

$Sq^2 Sq^2 = Sq^3 Sq^1,\ Sq^2 Sq^6 = Sq^7 Sq^1, \ldots;\ Sq^2 Sq^{4n-2} = Sq^{4n-1} Sq^1$

$Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1, \ldots;\ Sq^2 Sq^{4n-1} = Sq^{4n+1} + Sq^{4n} Sq^1$

$Sq^2 Sq^4 = Sq^6 + Sq^5 Sq^1, \ldots;\ Sq^2 Sq^{4n} = Sq^{4n+2} + Sq^{4n+1} Sq^1$

$Sq^2 Sq^5 = Sq^6 Sq^1, \ldots;\ Sq^2 Sq^{4n+1} = Sq^{4n+2} Sq^1$

$Sq^3 Sq^2 = 0, \ldots;\ Sq^3 Sq^{4n+2} = 0$

$Sq^3 Sq^3 = Sq^5 Sq^1; \ldots$

$Sq^{2n-1} Sq^n = 0$

DISCUSSION

Theorem 1 lists the major properties of the squaring operations, and, as we remarked, these properties characterize the squares uniquely. Parts (0) to (5) are due to Steenrod. The Cartan formula (6) was indeed discovered by Cartan; the Adem relations (7) were proved independently, and by very different methods, by Adem and Cartan.

Theorem 2, the surrounding material, and the proof we have given of the Adem relations include work of Cartan, Serre, and Thom. From our point of view this material will be amplified and completed by the calculations of Serre given in Chapter 9.

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