Final Exam - Math 215 - Fall 2009

Do problems $1, 2, 3, 4, 5, 6, 7$. Write all your proofs with care, using full sentences and correct reasoning.

1. Prove $1^3 + 2^3 + \cdots n^3 = n^2(n+1)^2/4$ for all $n = 1, 2, 3, \cdots$.

2. Prove that the following two statements are equivalent:

$$(B \Rightarrow A) \wedge (C \Rightarrow A)$$

and

$$(B \vee C) \Rightarrow A.$$

In your proof, do *not* use truth tables. Use facts of the type

$$A \Rightarrow B \; = (\sim A) \vee B$$

and

$$\sim (A \wedge B) = \; (\sim A) \vee (\sim B),$$

and give a completely algebraic proof.

3. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $h = g \circ f : X \longrightarrow Z$ with $h(x) = g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is surjective and g is surjective, then h is surjective. Given an example where h is surjective but not both f and g are surjective.

4. The sets in this problem are all finite.
(i) Given sets A and B prove the inclusion-exclusion formula

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(ii) Given sets A , B and C prove the inclusion-exclusion formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(iii) Let D_3 denote the number of bijections

$$f : \{1, 2, 3\} \longrightarrow \{1, 2, 3\}$$

such that $f(k)$ is not equal to k for $k = 1, 2, 3$. Determine the value of D_3 by using the principle of inclusion-exclusion. Check your answer by directly constructing all such bijections.

5. **(a)** Let X be any set. Let $P(X)$ denote the set of subsets of X. Let $S : X \longrightarrow P(X)$ be any well-defined mapping from X to its power set $P(X)$. Show that S is not surjective. Your proof should apply to both finite and infinite sets.

(b) Give an example of a proper subset X of the real numbers R that is in $1-1$ correspondence with all of R. Explain why your subset has this property.

(c) Prove that the set $N \times N$ of ordered pairs of natural numbers is countable.

6. Let C^n_r denote the binomial choice coefficient. Thus C^n_r is equal to the number of r-element subsets of a set with n-elements. This is sometimes phrased as the number of ways to choose r things from n things.

(a) State the binomial theorem for $(x + y)^n$ in terms of the coefficients C^n_r. Give the shortest correct proof of the binomial theorem that you know.

(b) Find a general formula for $(x+y+z)^n$ by applying the binomial theorem.

7. Let $S \subset \{1, 2, \ldots, 2n\}$ where S has $n + 1$ elements. Then S contains two numbers such that one divides the other. [Any number can be written uniquely as an odd number times a power of 2, $m = (2k - 1)2^j$. Show that there are exactly n odd numbers in the list $\{1, 2, \ldots, 2n\}$, and use this to conclude that in a selection of $n + 1$ numbers there must be an occurence of at least two numbers of the form $(2k - 1)2^j$ with the *same* k and *different* values of j.]