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THE PALLET GRAPH OF A FOX COLORING

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Abstract. We introduce the notion of a graph associated with a Fox p-coloring of a knot, and show that any non-trivial p-coloring requires at least $\lfloor \log_2 p \rfloor + 2$ colors. This lower bound is best possible in the sense that there is a p-colorable virtual knot which attains the bound.

1. Introduction

A *p-coloring* of a diagram D of a knot K, introduced by Fox [1] in 1961, is a map from the set of the arcs of D to $\mathbb{Z}/p\mathbb{Z}$,

$$\gamma : \{\text{the arcs of } D\} \to \mathbb{Z}/p\mathbb{Z},$$

such that at each crossing the sums of the images (called the *colors*) of the under-crossing arcs is equal to twice the color of the over-crossing arc. We say that a p-coloring γ is *trivial* if it is a constant map.

Harary and Kauffman [2] study the number of distinct colors appeared in a non-trivially p-colored knot diagram (D, γ). Let $N(D, \gamma) = \#\mathrm{Im}(\gamma) > 1$ be the cardinality of the image of γ. For a p-colorable knot K in $\mathbb{R}^3$, we denote by $C_p(K)$ the minimal number of $N(D, \gamma)$ for all the non-trivially p-colored diagrams (D, γ) of K. We remark that the notation $C_p(K)$ is used in the original paper [2], and also written as $\mathrm{mincol}_p(K)$ in some papers.

There are several studies on this number found in [4, 5, 6, 7]. In particular, it is known in [6, 7] that

- $C_3(K) = 3$ for any 3-colorable knot K,
- $C_5(K) = 4$ for any 5-colorable knot K,
- $C_7(K) = 4$ for any 7-colorable knot K, and
- $C_{11}(K) \geq 5$ for any 11-colorable knot K.

The first aim of this paper is to generalize these results as follows:

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Theorem 1.1. *Let p be an odd prime. Any p-colorable knot K satisfies*

$$C_p(K) \geq \lfloor \log_2 p \rfloor + 2,$$

where $\lfloor x \rfloor$ is the maximal integer less than or equal to x.

All of this can be done as well for virtual knots, with virtual crossings imposing no conditions on the colors: An arc of a virtual knot diagram is a curve that begins and ends at under-crossings, possibly passing through several virtual crossings, and the coloring conditions are derived from real crossings only [3].

For a p-colorable virtual knot K, we denote by $C_p^v(K)$ the minimal number of $N(D,\gamma)$ for all the non-trivially p-colored diagrams (D,γ) of K in virtual knot category. The second aim of this paper is to prove that the inequality is best possible for virtual knots as follows.

Theorem 1.2. *Let p be an odd prime. There is a p-colorable virtual knot K with*

$$C_p^v(K) = \lfloor \log_2 p \rfloor + 2.$$

This paper is organized as follows: In Section 2, we introduce a graph associated with a p-coloring which we call the pallet graph. We prove Theorem 1.1 by calculating the determinant of a matrix associated with the pallet graph. In Section 3, we prove Theorem 1.2 by constructing a tree with $\lfloor \log_2 p \rfloor + 2$ vertices for each p which is the pallet graph of some p-colored virtual knot diagram.

2. Determinant of a matrix

We will start this section with a calculation of a matrix. Let $\mathcal{M}_n$ be the set of $n \times n$ matrices with integer entries such that

- each row contains at most two 1's and at most one -2, and
- all the entries other than 1 and -2 are 0.

We denote by $\det(X)$ the determinant of X.

Lemma 2.1. *Any matrix X in $\mathcal{M}_n$ satisfies $|\det(X)| \leq 2^n$.*

Proof. We prove the lemma by induction on n. For $n = 1$, we have $X = (0), (1)$, or (-2) and the inequality holds. For $n > 1$, we divide the proof into three cases.

(i) If X has a row which contains no -2, then the cofactor expansion along the row induces

$$|\det(X)| \leq 1 \cdot 2^{n-1} + 1 \cdot 2^{n-1} = 2^n.$$

(ii) If X has a row which contains no 1 but one -2, then the cofactor expansion along the row induces

$$|\det(X)| \leq 2 \cdot 2^{n-1} = 2^n.$$

(iii) Consider the case other than (i) and (ii); that is, every row contains exactly

- one 1 and one -2, or
- two 1's and one -2.

Let $\vec{v}_j$ be the jth column of X. We may assume that the $(1,1)$-entry of X is -2. Consider the matrix

$$Y = \left(-\sum_{j=1}^{n} \vec{v}_j, \vec{v}_2, \ldots, \vec{v}_n \right).$$

Then we see that $Y \in \mathcal{M}_n$ and the first row of Y satisfies the case (i). Therefore, we have $|\det(X)| = |\det(Y)| \leq 2^n$. □

Definition 2.2. Let (D, γ) be a non-trivially p-colored diagram. The *pallet graph* G of (D, γ) is a simple graph such that

(i) the vertices of G correspond to the colors on the arcs of (D, γ), that is, the elements of the image $\mathrm{Im}(\gamma)$, and
(ii) two different vertices c and c' of G are connected by an edge labeled $c'' = (c + c')/2$ if and only if there is a crossing of (D, γ) whose lower arcs admit the colors c and c' and the upper admits c''.

We take a maximal tree of the pallet graph G. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges of T, and $c_1, c_2, \ldots, c_n$ the vertices of T, where $n = N(D, \gamma)$. We define the $(n-1) \times n$ matrix $A = (a_{ij})$ with integer entries such that

- $a_{ij} = 1$ if the edge e_i is incident to the vertex c_j,
- $a_{ij} = -2$ if the edge e_i is labeled by c_j, and
- $a_{ij} = 0$ otherwise.

Lemma 2.3. *Let A be the $(n-1) \times n$ matrix as above, and A_j the $(n-1) \times (n-1)$ submatrix obtained from A by deleting the jth column $(1 \leq j \leq n)$.*

(i) $\det(A_j)$ *is divisible by p.*
(ii) $\det(A_j)$ *is odd.*

Proof. (i) The simultaneous equation $A\vec{x} = \vec{0}$ over the field $\mathbb{Z}/p\mathbb{Z}$ has two independent solutions $\vec{x} = {}^{\mathrm{t}}(1, 1, \ldots, 1), {}^{\mathrm{t}}(c_1, c_2, \ldots, c_n)$. Since the rank of A is at most $n - 2$, we have $\det(A_j) \equiv 0 \pmod{p}$.

(ii) The matrix A over $\mathbb{Z}_2$ is coincident with the incident matrix of T. For each $1 \leq i \leq n-1$, let $c_{\sigma(i)}$ be the vertex between two endpoints of the edge e_i which is farther than the other away from the vertex c_j. Since T is a tree, we see that

$$\det(A_j) \equiv a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n-1,\sigma(n-1)} \equiv 1 \pmod 2.$$

□

Proof of Theorem 1.1. Let (D,γ) be a non-trivially p-colored diagram of a knot K, and A the $(n-1)\times n$ matrix constructed as above, where $n = N(D,\gamma)$. By Lemmas 2.1 and 2.3, it holds that $p \leq |\det(A_j)| < 2^{n-1}$, that is, $n > \log_2 p + 1$. □

Remark 2.4. By definition, the proof of Theorem 1.1 can be also applied for a virtual knot; any p-colorable virtual virtual knot K satisfies

$$C_p^v(K) \geq \lfloor \log_2 p \rfloor + 2.$$

3. Construction of a graph

Recall that a pallet graph G over $\mathbb{Z}/p\mathbb{Z}$ satisfies the following properties:

(P1) G is a connected simple graph with two or more vertices.
(P2) If two different vertices c and $c' \in \mathbb{Z}/p\mathbb{Z}$ are connected by an edge, then the label $c'' = (c+c')/2$ of the edge also appears as a vertex of G.

We remark that G has at least $\lfloor \log_2 p \rfloor + 2$ vertices by Theorem 1.1.

Lemma 3.1. *There is a graph G with exactly $\lfloor \log_2 p \rfloor + 2$ vertices satisfying* (P1) *and* (P2).

Proof. Put $k = \lfloor \log_2 p \rfloor$; that is, k is the integer satisfying $2^k < p < 2^{k+1}$. There are integers $m_1, m_2, \dots, m_s$ uniquely satisfying

$$2^{k+1} - p = 2^{m_s} + \cdots + 2^{m_1} + 1$$

with $1 \leq m_1 < m_2 < \cdots < m_s < k$. Since $m_{j+1} \geq m_j + 1$ and $m_1 \geq 1$, it holds that $m_j \geq j$ for each j. Similarly, since $m_{j-1} \leq m_j - 1$ and $m_s \leq k-1$, it holds that $m_j \leq k - 1 - (s-j)$ for each j. Therefore, we obtain

$$0 \leq m_j - j \leq k - s - 1.$$

We take $1+(k-s+1)+s = k+2$ elements a, $b(0), b(1), \ldots, b(k-s)$, $c(1), c(2), \ldots,$ $c(s)$ in $\mathbb{Z}/p\mathbb{Z}$ such that

$$\begin{cases} a = 0, & \\ b(i) = 2^i & \text{for } i = 0, 1, \ldots, k-s, \text{ and} \\ c(j) = 2^{k-j+1} - (2^{m_s - j} + \cdots + 2^{m_j - j}) & \text{for } j = 1, 2, \ldots, s. \end{cases}$$

We connect the vertices corresponding to these numbers to obtain a graph G as follows:

(i) $b(0)$ is connected to a by an edge labeled

$$\frac{a + b(0)}{2} = \frac{p+1}{2} = 2^k - (2^{m_s - 1} + \cdots + 2^{m_1 - 1}) = c(1).$$

(ii) For each $1 \leq i \leq k-s$, $b(i)$ is connected to a by an edge labeled

$$\frac{a + b(i)}{2} = 2^{i-1} = b(i-1).$$

(iii) For each $1 \leq j \leq s-1$, $c(j)$ is connected to $b(m_j - j)$ by an edge labeled

$$\begin{aligned} \frac{b(m_j - j) + c(j)}{2} &= 2^{k-j} - (2^{m_s - j - 1} + \cdots + 2^{m_{j+1} - j - 1}) \\ &= c(j+1). \end{aligned}$$

(iv) $c(s)$ is connected to $b(m_s - s)$ by an edge labeled

$$\frac{b(m_s - s) + c(s)}{2} = 2^{k-s} = b(k-s).$$

Since the graph G is connected, we have the conclusion. $\square$

Figure 1 shows an example of the graph constructed in Lemma 3.1 for $p = 601$.

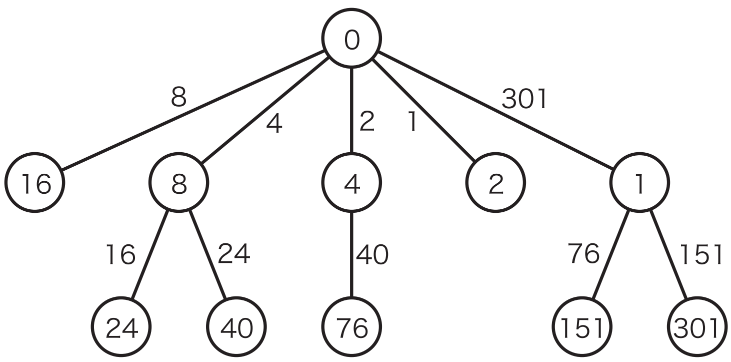

Figure 1

LEMMA 3.2. *Let G be a graph satisfying the properties* (P1) *and* (P2). *Then there is a non-trivially p-colored virtual knot diagram whose pallet graph is G.*

Proof. It is sufficient to construct a Gauss diagram instead of a virtual knot diagram (cf. [3]). We take a closed path of G which passes all the edges of G. Let $c_1, c_2, \ldots, c_n$ be the vertices of G, and $c_{k(1)}, c_{k(2)}, \ldots, c_{k(m)}$ the sequence of vertices on the path in this order.

To construct a Gauss diagram, we divide a circle into m arcs by m points $P_1, P_2, \ldots, P_m = P_0$, and assign the color $c_{k(i)}$ to each arc between P_{i-1} and P_i $(i = 1, 2, \ldots, m)$. We take m points $Q_1, Q_2, \ldots, Q_m$ on the circle such that Q_i is in the interior of an arc labled $(c_{k(i)} + c_{k(i+1)})/2$, where $c_{k(m+1)} = c_{k(1)}$.

We consider a Gauss diagram equipped with the oriented chords $\overrightarrow{Q_iP_i}$ $(i = 1, 2, \ldots, m)$ and any signs on them. The Gauss diagram presents a non-trivially p-colored diagram such that P_i and Q_i correspond to lower and upper crossings, respectively. Then we see that G is the pallet graph of the p-colored diagram. □

Proof of Theorem 1.2. By Lemmas 3.1 and 3.2, there is a non-trivially p-colored virtual knot diagram (D, γ) such that its pallet graph G has exactly $\lfloor \log_2 p \rfloor + 2$ vertices. The virtual knot K presented by D satisfies $C_p^v(K) \leq N(D, \gamma) = \lfloor \log_2 p \rfloor + 2$. The opposite inequality follows by Theorem 1.1. □

REMARK 3.3. (i) Several statements proved in this paper hold even for any odd composite p.

(ii) It is an open question whether any p-colorable knot K satisfies

$$C_p(K) = \lfloor \log_2 p \rfloor + 2.$$

The equality holds for $p = 3, 5, 7$ (cf. [6, 7]).

(iii) Let $\mathrm{c}(K)$ denote the crossing number of K. Since $\mathrm{c}(K) \geq C_p(K)$, any p-colorable knot K satisfies

$$\mathrm{c}(K) \geq \lfloor \log_2 p \rfloor + 2$$

by Theorem 1.1. It is an open question whether the equality does not hold for other than the trefoil knot $(p = 3)$ and the figure-eight knot $(p = 5)$.

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