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# Rational Tangles

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This paper gives an elementary and self-contained proof of Conway's Basic Theorem on rational tangles. This theorem states that two rational tangles are topologically equivalent if and only if they have the same associated rational fraction. Our proof divides into a geometric half that relates the arithmetic of continued fractions to the topology of tangles and an algebraic part that defines the fraction of any tangle via the bracket model for the Jones polynomial. We present an application to molecular biology. 

## 1. INTRODUCTION

We give an elementary and self-contained proof of J. H. Conway's Basic Theorem on Rational Tangles. Conway associated a finite continued fraction to each rational tangle. The sum of this continued fraction is the fraction of the tangle. Conway's Theorem states that *two rational tangles are ambient isotopic if and only if their fractions are equal*. The meanings of these terms will be explained in the body of the paper. Conway's Theorem is first stated (without proof) in his paper [3]. Proofs of the theorem using a fair amount of mathematical machinery have appeared in the literature of knot theory (see [2]). In the meantime, the subject of tangles has become of wider interest due to its applications to the topology of DNA. We are pleased with the elementary nature of our proof and we believe that it provides a good place to begin learning these aspects of knot theory and its applications.



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The paper is organized as follows. Section 1 defines the notions of tangle, rational tangle, and the continued fraction and fraction associated with a rational tangle. Then, by using a few simple lemmas about the topology of rational tangles coupled with an elementary algebraic identity of Lagrange for continued fractions, we show that *tangles having the same fraction are ambient isotopic*. Our proof actually provides an algorithm for transforming a rational tangle into an equivalent one. This is one half of the ‘‘if and only if’’ of the Conway Theorem. What is unique in our approach to this half is the fact that a topological interpretation for Lagrange’s identity provides the simple key to the proof. This completes the ‘‘hard’’ part of the Theorem and Section 3 turns to the ‘‘easier’’ part of showing that the fraction of a rational tangle is a topological invariant of that tangle. That is, we must show that if two tangles are ambient isotopic, then they have the same fraction. This task is accomplished by showing that a topologically invariant ‘‘fraction’’ can be defined for any tangle whatsoever, and that this fraction agrees with the rational tangle fraction that we have defined in Section 2. We define the general tangle fraction by using the bracket model of the Jones polynomial [8, 9] at a special value. The advantage of this approach is that the bracket properties can be verified by entirely elementary means, and the bracket formalism is just suited to handling the tangles.

We also remark that the definition of tangle fraction given here coincides with the definition of conductivity for tangles explained in our earlier paper [7]. In fact, we can use the bracket definition of conductivity to deduce the formula for the conductivity of a dual graph in the plane (Remark after Theorem 3.3). The discussion in Section 3 completes our proof of the Conway Theorem. Section 4 is a quick discussion of the applications of rational tangles to the analysis of DNA recombination.

## RATIONAL TANGLES AND THEIR ALGEBRA— CONWAY’S THEOREM

The key concept for this section is the notion of a *tangle*. A tangle is analogous to a link except that it has free ends. These ends are depicted as strands that enter a box (the tangle box) within which there are no free ends. Inside the box one may find closed loops that are knotted and linked with the tangle strands. The strands of the tangle may themselves be knotted and linked. Note that if you begin at one of the free ends emanating from the tangle box and walk along it, you will enter the box, and eventually leave the box to meet another end of strand. Thus a tangle box must have an even number of ends. We will be concerned in this section with tangles with four ends. Such tangles have two strands, each strand participating in two ends. In considering the topology of tangles, we allow the strands to move inside the box by ambient isotopy, but the ends of the strands must remain fixed, and the strand movements are confined to the box.

Rational tangles were introduced by John Conway as basic building blocks for the construction of knots. They are the simplest tangles in the sense that they can be "unwound." More precisely, we visualize the four endpoints of the strands of a tangle as restricted to the surface of a sphere and the rest of the tangle inside the sphere. If we restrict the endpoints of the tangle to move on the surface and the rest of the tangle to move inside the sphere, then a rational tangle is one that can be deformed into two straight lines. In Fig. 2.1A, the tangle can be unwound by grabbing the ends b and c and untwisting while holding a and d fixed. In Fig. 2.1B, we unwind by first using the ends c and d to unwind the bottom and then using b and d to unwind the remaining twist.

Conversely, we can construct a rational tangle by starting with two horizontal or two vertical strands, picking two endpoints and twisting them, then picking another pair and twisting them, and so on, for a finite number of twists. We see an example in Fig. 2.1B by reading the figures from the bottom up. A twist of two diagonally opposite strands can be obtained by two successive twists of adjacent strands (Fig. 2.2). Thus we only need twists of adjacent structure to construct rational tangles.

We need a few definitions before defining a rational tangle formally.

A *horizontal* (resp. *vertical*) *integer tangle* $t_a$ (resp., $t'_a$) is a twist of two horizontal (resp. vertical) strands $|a|$ times in the positive or negative direction according to the sign of $a$. The directions are shown in Fig. 2.3.

The *horizontal sum* "$+$" and *vertical sum* "$+'$" of two tangles $A$ and $B$ are defined by the diagrams in Fig. 2.4. Thus twisting two adjacent strands of a tangle $A$ is equivalent to adding an integer $t_a$ on the right or the left (with $+$) or adding $t'_a$ on the top or bottom (with $+'$) (Fig. 2.5).

It is easy to see that $t_a + t_b \sim t_{a+b}$ and $t'_a +' t'_b \sim t'_{a+b}$ (Fig. 2.6). Essentially, a negative twist cancels a positive twist topologically so that cancellation of positive and negative integers is paralleled in tangle topology. This particular cancellation is an instance of the second Reidemeister move in knot theory. See Section 3 of this paper for a discussion and illustration of the Reidemeister moves.

At this point it is worth drawing attention to the notion of ambient isotopy of tangles. Two tangles $A$ and $B$ are said to be *ambient isotopic* if it is possible to *deform* $A$ into $B$ *without moving the endpoints and without moving the strands outside the tangle box* (we will usually just say *isotopic*). When we say *deform*, we mean topological deformation, a movement of the strands that is continuous and does not allow any strand to penetrate

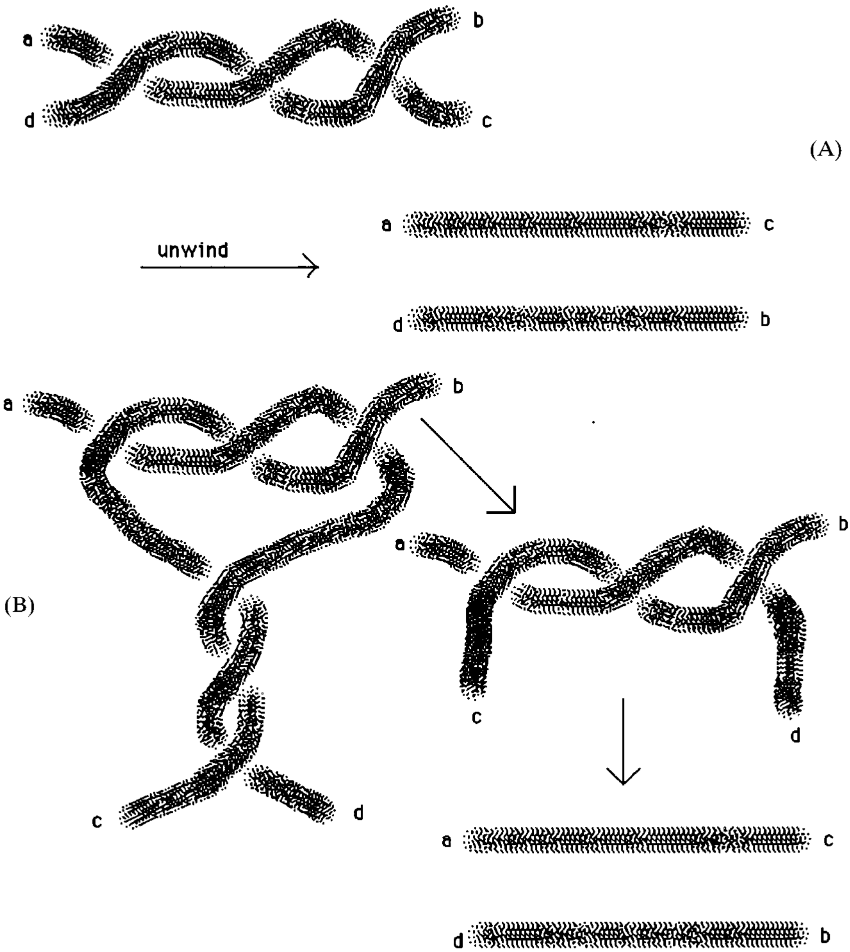



FIGURE 2.1

either itself or another strand. Of course, we usually visualize this by taking a representation of $A$ and a representation of $B$, and trying to change $A$ until it looks like $B$. In the end, if $A$ and $B$ are isotopic, the changed version of $A$ should be identical to $B$. In Section 3 we will explain how isotopy can be expressed by the combinatorial Reidemeister moves on diagrams. It will then be possible to give ways to calculate invariants of isotopy. In this section, we use some elementary isotopies to simplify the presentation of rational tangles.

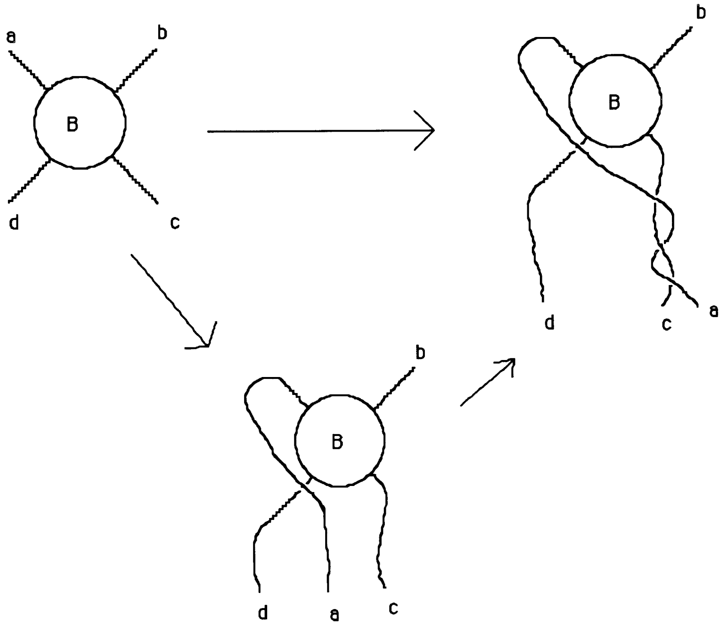



FIG. 2.2. Twining diagonally opposite strands via twists of adjacent strands.

Now we define a rational tangle (with $n$ integer tangles) inductively.

DEFINITION. For any sequence of integers $a_1, \dots, a_n$, choose a sequence of integer tangles $T_{a_1}, \dots, T_{a_n}$, where $T_{a_i} = t_{a_i}$ or $t'_{a_i}$.

(1) Let $B_1 = T_{a_1}$.

(2) For $k < n$.

(i) if $T_{a_{k+1}} = t_{a_{k+1}}$, let $B_{k+1} = t_{a_{k+1}} + B_k$ or $B_k + t_{a_{k+1}}$,

(ii) if $T_{a_{k+1}} = t'_{a_{k+1}}$, let $B_{k+1} = t'_{a_{k+1}} +' B_k$ or $B_k +' t'_{a_{k+1}}$.

Any tangle constructed in this way is a *rational tangle with n integer tangles*.

*Note*. The number of integer tangles in a rational tangle is *not unique*. For example, $t_5 \sim t_3 + t_2$, and, by our definition, $t_5$ has one integer tangle and $t_3 + t_2$ has two.

*Note*. This definition of "rational tangle with $n$ integer tangles" is simply a formalization of our description of a rational tangle as obtained from two horizontal or two vertical strands by successive twisting of pairs of adjacent endpoints. These twistings can be construed as the (horizontal

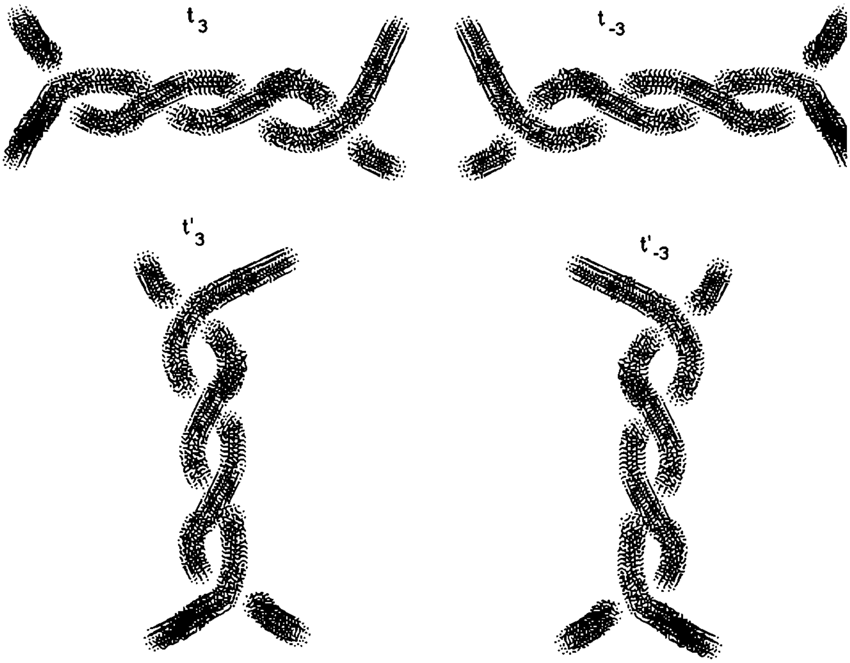
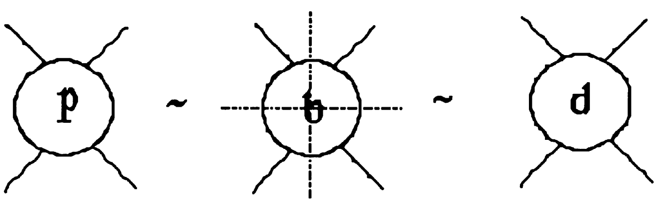



FIGURE 2.3

or vertical) addition of horizontal or vertical tangles on the right, left, top, or bottom. Thus *any rational tangle is a rational tangle with n integer tangles for some n*. The next theorem gives a fundamental simplification in the description of rational tangles.

FLIP THEOREM 1. *A* 180° *rotation* (*flip*) *of a rational tangle b in the horizontal or vertical axis is ambient isotopic to b*.



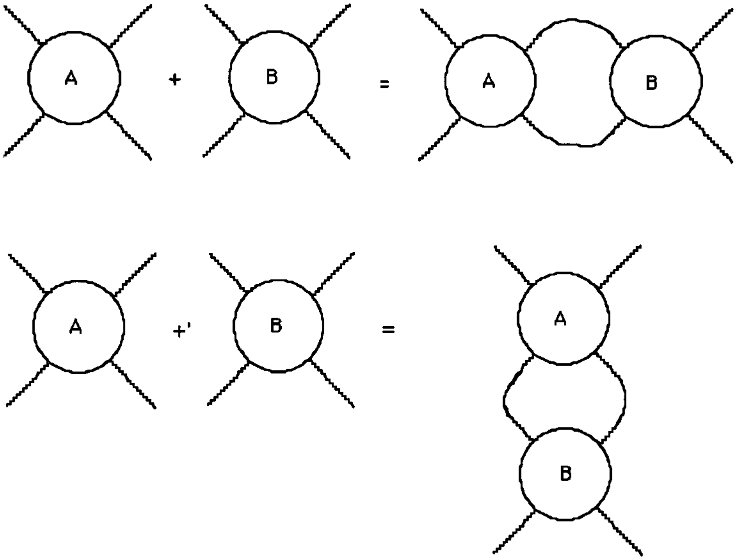



FIGURE 2.4

*Note.* This theorem is, in general, not true for nonrational tangles. For example, consider the tangle illustrated in Figure 2.6.1. It is not rational. One strand has a knot in it and this strand is linked with the other strand. This tangle is not equivalent to the tangle obtained by performing a vertical flip because there is no way to deform the knot on one strand to the other one, since the endpoints are fixed.

*Proof of Flip Theorem* 1. We proceed by induction on the number $n$ of integer tangles in $b$.

If $n = 1$, then $b = t_a$ or $t'_a$ for some $a$. In either case a horizontal or vertical flip leaves the tangle invariant.

Assume the theorem is true for any rational tangle with $\leq n$ integer tangles, and let $b$ be a tangle with $n + 1$ integer tangles. By the definition of tangles $b$ is equal to one of the tangles $t_a + B$, $B + t_a$, $t'_a +' B$, or $B +' t'_a$ for some $a$ and some rational tangle $B$ with $n$ integer tangles.

Suppose that $b = t_a + B$. Then a horizontal flip leaves the $t_a$ part invariant and, by the induction hypothesis, takes $B$ into a tangle $B^\wedge$ isotopic to $B$. (Note that, as discussed above, tangle isotopy leaves the endpoints of the tangle fixed, so that the isotopy of $B^\wedge$ to $B$ induces an isotopy of $t_a + B^\wedge$ to $t_a + B$.) Hence $t_a + B$ is isotopic to its horizontal flip $t_a + B^\wedge$.

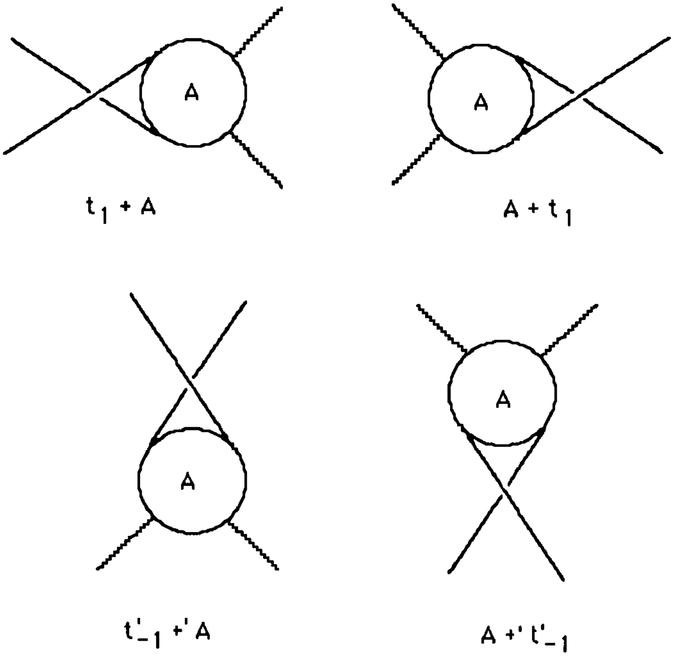



FIGURE 2.5

Vertical flips take a little more work. Consider specifically $b = t_3 + B$. Its vertical flip is $B' + t_3$, where $B'$ is the vertical flip of $B$. By the induction hypothesis $B \sim B'$, so $B' + t_3 \sim B + t_3$. Hence we must prove that $t_3 + B \sim B + t_3$. To see this, we twist $B$ through three 180° twists (holding the endpoints of $t_3 + B$ fixed), as shown in Fig. 2.7. Such a flip of a subtangle is called a *flype*. Since we performed an odd number of flypes, the resulting subtangle becomes $B^\wedge$ (the horizontal flip of $B$). Again, by the induction hypothesis, $B^\wedge \sim B$ and we are done.

The other cases are handled in exactly the same way, using horizontal or vertical flypes as needed. *The key idea is that flyping allows us to move an integer tangle from the right to the left, from the left to the right, from the bottom to the top, and from the top to the bottom.*

We state an essential part of the proof separately.

COROLLARY. *If B is a rational tangle and a is an integer, then*

$$t_a + B \sim B + t_a \qquad \text{and} \qquad t'_a +' B \sim B +' t'_a.$$

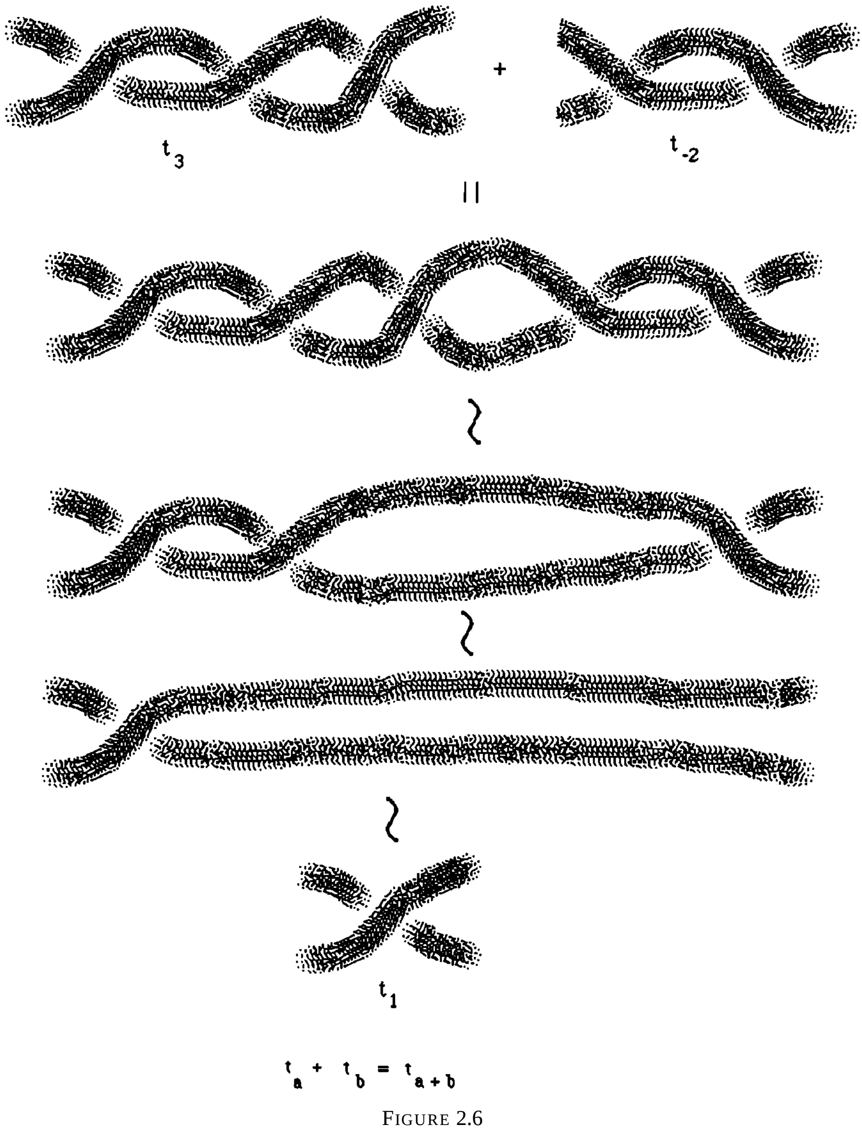



FIGURE 2.6

FIGURE 2.6.1

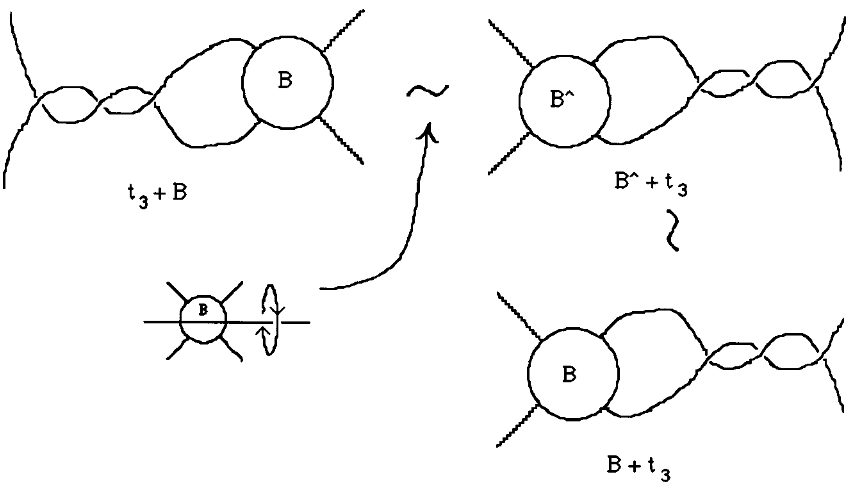



FIGURE 2.7

Now we define a subclass of rational tangles, the basic tangles. A *basic horizontal* (resp., *vertical*) *tangle* is a rational tangle built as follows:

(1) Start with a horizontal tangle $t_a$ (resp., a vertical tangle $t'_a$).

(2) Add a vertical tangle $t'_b$ on the *bottom* (resp., a horizontal tangle $t_a$ on the *left*), then a horizontal tangle $t_c$ on the *right* (resp., a vertical tangle $t'_c$ on the *bottom*), then a $t'_d$ on the bottom (resp., $t_d$ on the left), and so on, stopping after a finite number of such steps.

If we start with $t_a$, the resulting tangle is a basic horizontal tangle, and if we start with $t'_a$, it is a basic vertical angle.

For example, the construction $t_3 \to t_3 +' t'_2 \to (t_3 +' t'_2) + t_{-2}$ ends in a basic horizontal tangle, whereas $t'_3 \to t_2 + t'_3 \to (t_2 + t'_3) +' t_{-2}$ ends in a basic vertical tangle (Fig. 2.8).

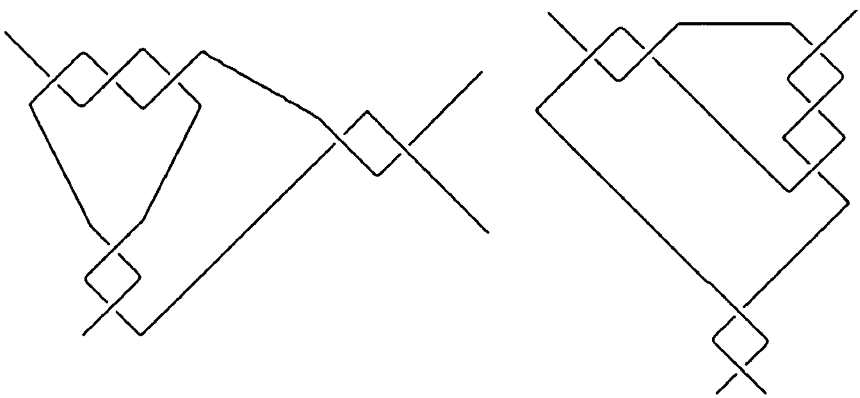
When constructing a rational tangle, we can use the last corollary to convert horizontal twists on the left to those on the right and vertical twists on the top to those on the bottom.

Moreover, the corollary plus the fact that $t_a + t_b \sim t_{a+b}$ and $t'_a +' t'_b \sim t'_{a+b}$ allows us to avoid adding two horizontal or two vertical twists in a row. Hence we have

THEOREM 2. *Every rational tangle is isotopic to a basic tangle.*

Note that an algorithm is implicit in the proof of Theorem 2.

In order to classify the basic tangles, we introduce three more operations.

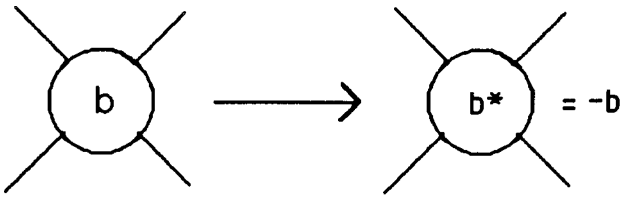


$(t_3 +' t'_2) + t_{-2}$ $\qquad$ $(t_2 + t'_3) +' t'_{-2}$

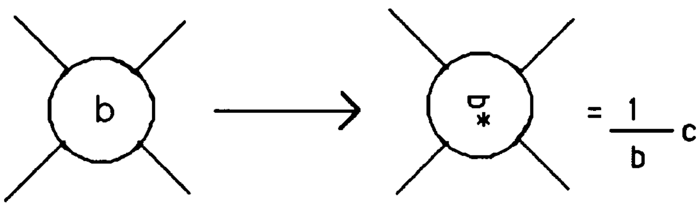
FIGURE 2.8

DEFINITION. If $b$ is a rational tangle, then

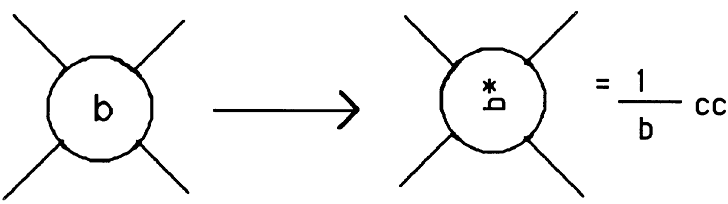
(1) $-b = b^*$ is the mirror image of $b$ (reverse all crossings)




(2) The *c-inverse* (clockwise inverse) $1/_{c}b$ of $b$ is the tangle obtained by rotating $b$ 90° clockwise and taking the mirror image.




(3) The *cc-inverse* (counterclockwise inverse) $1/_{cc}b$ of $b$ is the tangle obtained by rotating $b$ 90° degrees counterclockwise and taking the mirror image.



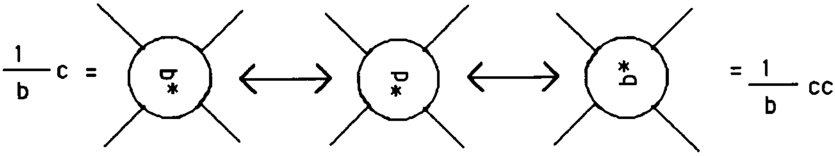



FIGURE 2.9

The reason for taking mirror images in the definition of inverses will become clear when we discuss numerical invariants.

THEOREM 3. *The c-inverse and cc-inverse of a rational tangle are ambient isotopic.*

*Proof.* We apply the flip theorem twice (a vertical followed by a horizontal flip as shown in Fig. 2.9.

Thus if a subtangle of a tangle is of the form $1/_c b$, $b$ rational, we can always replace it by $1/_{cc} b$ and, conversely, we can replace $1/_{cc} b$ by $1/_c b$. We will often write $1/b$ if the type of the inverse does not matter.

We need a few more properties of our operations in order to describe basic tangles algebraically.

PROPOSITION 4. (1) *If b is a basic horizontal tangle, then $1/_c b$ is a basic vertical tangle.*

(2) *If b is a basic vertical tangle, then $1/_{cc} b$ is a basic horizontal tangle.*

(3) $t'_a = 1/_c t_a = 1/_{cc} t_a$ *and* $t_a = 1/_c t'_a = 1/_{cc} t'_a$.

(4) $t_d + t'_e = t_d + 1/t_e$.

(5) $t_d +' t'_e = \cfrac{1}{t_e + \cfrac{1}{t_d}\,\mathrm{c}}\,\mathrm{cc}.$

(6) $-(b + c) = (-b) + (-c)$, $-(1/b) = 1/(-b)$, $-t_a = t_{-a}$, *and* $-t'_a = t'_{-a}$.

*Proof.* The results follow easily from the definitions (we leave it to the reader to draw these pictures).

Now we can build our basic tangles using the tangle operations $+$, $-$, $/_c$, $/_{cc}$. For example, consider the basic tangles of Fig. 2.8. The construction of the first tangle is given by

$$t_3 \to t_3 +' t_2 \to (t_3 +' t'_2) + t_{-2}$$

or

$$t_3 \to \cfrac{1}{t_2 + \cfrac{1}{t_3}\mathrm{c}}\mathrm{cc} \to \cfrac{1}{t_2 + \cfrac{1}{t_3}\mathrm{c}}\mathrm{cc} + t_{-2}$$

$$\sim t_{-2} + \cfrac{1}{t_2 + \cfrac{1}{t_3}\mathrm{c}}\mathrm{cc}$$

and the second is given by

$$t'_3 \to t_2 + t'_3 \to (t_2 + t'_3) + {}'t'_{-2}$$

or

$$\frac{1}{t_3} \to t_2 + \frac{1}{t_3} \to \cfrac{1}{t_{-2} + \cfrac{1}{t_2 + \cfrac{1}{t_3}}\mathrm{c}}\mathrm{cc}$$

In both cases we have a "continued fraction" in elementary tangles. Moreover, the second tangle is isotopic to the reciprocal of the first (recall Theorem 3).

Similarly, any basic horizontal tangle $b$ can be written in the form

$$b = t_{a_n} + \cfrac{1}{t_{a_{n-1}} + \cfrac{1}{\ddots + \cfrac{1}{t_{a_1}}}} \tag{1}$$

and any basic vertical tangle $b$ in the form

$$b = \cfrac{1}{t_{a_n} + \cfrac{1}{t_{a_{n-1}} + \cfrac{1}{\ddots + \cfrac{1}{t_{a_1}}}}}. \tag{2}$$

In fact, *we could take* (1) *and* (2) *as the definition of a basic tangle*.

These forms for basic tangles suggest associating to the basic tangles (1) and (2) the arithmetic continued fractions

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}} \quad \text{and} \quad \cfrac{1}{a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_1}}}}. \tag{3}$$

Each of these continued fractions equals a rational fraction which we call the *fraction of the tangle* and denote it by $F(b)$.

In particular, $F(t_a) = a$ and $F(t'_a) = F(1/t_a) = 1/a$.

PROPOSITION 5. *If $b$ is a basic tangle, then*

(1) $F(1/b) = 1/F(b)$, *independent of which inverse we use*.

(2) *If $t_a + b$ is also a basic tangle, then* $F(t_a + b) = F(t_a) + F(b) = a + F(b)$.

(3) $F(-t_a) = F(t_{-a}) = -a = -F(t_a)$.

$$F(-t'_a) = F(t'_{-a}) = F(1/t_{-a}) = -1/a = -F(1/t_a) = -F(t'_a).$$

(4) $F(-b) = -F(b)$.

*Proof.* Only (4) is not immediate from the definitions and Proposition 4. A formal proof of (4) requires induction, but we shall just illustrate the reasoning with an example. Let

$$b = t_3 + \cfrac{1}{t_4 + \cfrac{1}{t_{-5}}}.$$

Then, by Proposition 4,

$$-b = -t_3 - \cfrac{1}{t_4 + \cfrac{1}{t_{-5}}}$$

$$= -t_3 + \cfrac{1}{-t_4 - \cfrac{1}{t_{-5}}}$$

$$= -t_3 + \cfrac{1}{-t_4 + \cfrac{1}{-t_{-5}}}$$

$$= -t_3 + \cfrac{1}{-t_4 + \cfrac{1}{t_5}}.$$

Therefore

$$F(-b) = -3 + \cfrac{1}{-4 + \cfrac{1}{5}} = -\left[3 + \cfrac{1}{4 + \cfrac{1}{-5}}\right] = -F(b).$$

We now have the concepts and tools to state and prove John Conway's fundamental theorem.

CONWAY'S THEOREM. *Let $T_1$ and $T_2$ be basic tangles. If $F(T_1) = F(T_2)$, then $T_1$ is ambient isotopic to $T_2$.*

*Remarks.* The converse is in fact true, namely, *if $T_1$ is ambient isotopic to $T_2$, then $F(T_1) = F(T_2)$*. Later we will give a different interpretation of the fraction of a tangle, which allows us to prove this. Thus the fraction is a complete invariant for the equivalent of basic tangles. Moreover, since any rational tangle is ambient isotopic to a basic tangle (by an algorithm), we have an algorithmic procedure for deciding the equivalence of any two rational tangles.

*Proof.* Given a basic tangle $T$ in the continued fraction form of Eq. (1) with fraction $p/q$, we will show that it is ambient isotopic to a basic tangle $T$ with fraction $p/q$, whose continued fraction (3) satisfies $a_i > 0$ for $i = 1, \ldots, n-1$. Such a continued fraction is called regular. Since the regular continued fraction of $p/q$ is unique (up to the last term which can be written as $a_n - 1 + 1/1$—see [5]), this means that any tangle with fraction $p/q$ is ambient isotopic to $T'$ and we will be finished.

We begin arithmetically with the continued fraction (3) of the fraction $F(T)$ and show how to convert it into the regular continued fraction with the same sum. The key step is the following formula of Lagrange (See Lagrange's Appendix to Euler's Algebra [5]),

$$a - \frac{1}{b} = (a-1) + \cfrac{1}{1 + \cfrac{1}{(b-1)}}, \tag{4}$$

which is trivial to verify.

Applying Lagrange's formula to convert a continued fraction to a regular one is best illustrated with an example. Consider the continued fraction

$$\frac{43}{62} = 1 + \cfrac{1}{-3 + \cfrac{1}{-4 + \cfrac{1}{5}}} = 1 - \cfrac{1}{3 + \cfrac{1}{4 - \cfrac{1}{5}}} \tag{5}$$

and move all negative signs to the numerator as shown. We find the last appearance of a negative sign reading down the fraction (in this case $-1/5$) and apply Eq. (4) to $4 - 1/5$ ($a = 4$, $b = 5 - 1/5$). Then

$$4 - \frac{1}{5} = 3 + \cfrac{1}{1 + \cfrac{1}{4}},$$

which, substituted into the right-hand side of (5) yields

$$\frac{43}{62} = 1 - \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}}, \tag{6}$$

a continued fraction for $43/62$ with one less negative sign. Now find the last negative sign in (6), which is the only one, and apply Eq. (4) to (6) with $a = 1$ and

$$b = 3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}},$$

which gives

$$\frac{43}{62} = a - \frac{1}{b} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4}}}},$$

a continued fraction with all terms positive. This procedure, namely using Lagrange's formula to reduce the number of negative signs one step at a time, works in general yielding a continued fraction in which a negative sign can only appear in front of the first term.

Now we mirror this procedure topologically. Let the numbers $a$ and $b$ represent rational tangles with fractions $a$ and $b$ respectively. Then Fig. 2.10 shows the construction of a rational tangle with fraction $1/(1 + 1/(b - 1))$, which corresponds to part of the right-hand side of (4).

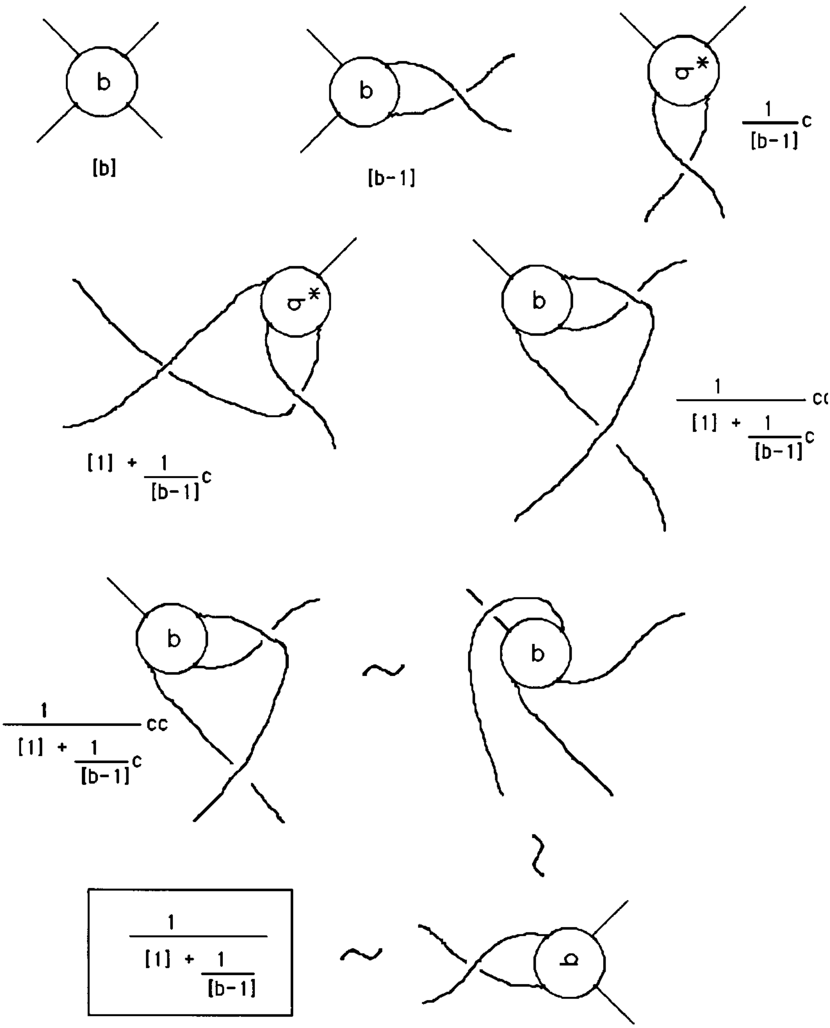



FIGURE 2.10

In Fig. 2.11, we use the rational tangle from Fig. 2.10 to form a rational tangle corresponding to $(a - 1) + 1/(1 + 1/(b - 1))$ and then topologically deform this tangle until it is the rational tangle for the fraction $a + 1/(-b)$. Thus we have the topological version of Lagrange's formula.

Now we are essentially done. We start with a basic tangle $T$ in the continued fraction form of Eq. (1) with fraction $p/q$ given by the continued fraction (3). We have shown how to convert the continued fraction into a regular one, which is essentially unique. Then we showed how to convert the original tangle to an ambient isotopic tangle corresponding to the regular continued fraction for $p/q$. Hence given any two tangles $T_1$ and $T_2$ with the same fraction $p/q$, they are ambient isotopic to the same tangle $T'$ which corresponds to the unique regular continued fraction for $p/q$. Therefore $T_1$ and $T_2$ are ambient isotopic. Moreover, our proof actually gives an algorithm for converting $T_1$ to $T_2$.

Note that the basic tangles associated to regular continued fractions are *alternating*, i.e., as we walk along a strand, the crossings alternate between under and over crossings. So we have proved that any basic tangle is ambient isotopic to an alternating tangle.

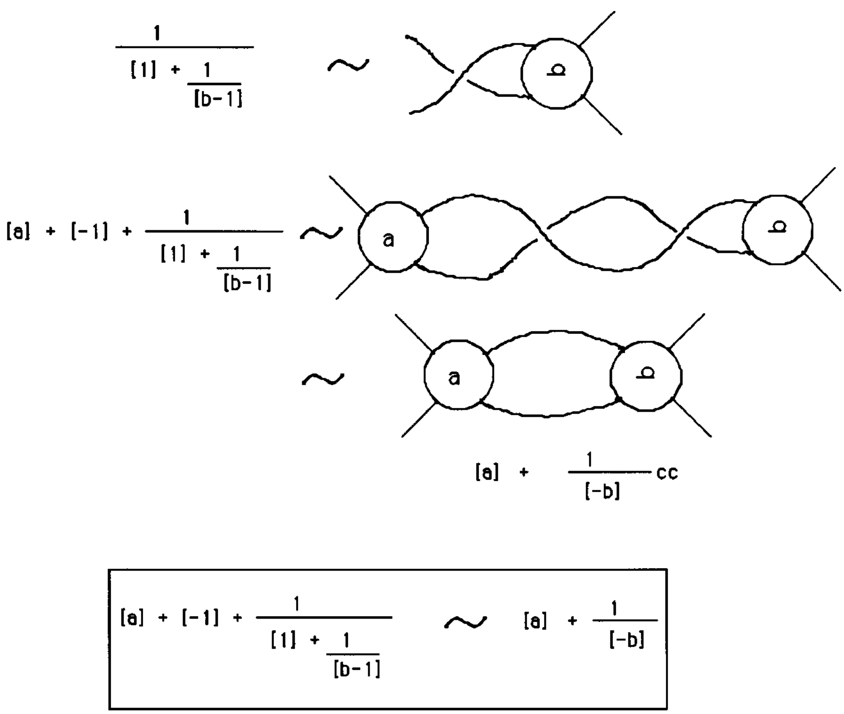



FIGURE 2.11

## 3. THE CONDUCTANCE INVARIANT AND THE BRACKET

We have seen how to define the fraction of a rational tangle and that tangles with the same fraction are ambient isotopic. In this section we shall prove the converse—that if two tangles are ambient isotopic, then they have the same fraction. This will complete our goal of showing that the fraction of a rational tangle completely classifies its topological type.

In order to show that ambient isotopic tangles have the same fraction, we shall build an *invariant*, $C(T)$, for arbitrary tangles $T$ and then show

1. If $T$ is ambient isotopic to $S$, then $C(T) = C(S)$.
2. If $T$ is rational, then $C(T) = F(T)$, the fraction of $T$.

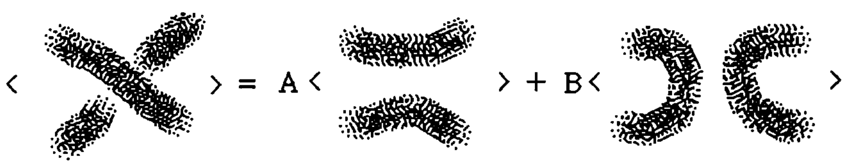
This is a case where it is actually easier to solve a more general problem (of finding an invariant for arbitrary tangles rather than just for rational tangles.)

We shall call the invariant $C(T)$ the *conductance* of $T$, because it is the (generalized) conductance of a (generalized) electrical network associated with $T$. The full theory of $C(T)$ is explained in our paper [7], but here we shall give a different and elementary development of $C(T)$ that is based on the bracket polynomial. In order to do this we will first give a quick introduction to the bracket. Readers interested in more information on the bracket polynomial should consult [8, 9, or 11]. The relationship between the bracket (Jones polynomial) and the conductance invariant is explained in [7].

### *The Bracket*

The bracket invariant $\langle K \rangle$ is defined on all knots and links $K$ (not yet on tangles) by the formulas:

(1)




(2) $\langle OK \rangle = \delta \langle K \rangle,$
$\langle O \rangle = 1,$

where $\delta = -A^2 - A^{-2}$ and $B = A^{-1}$. The assignment of $A$ and $B = A^{-1}$ in (1) follows the labeling convention—*the regions swept out as the over crossing strand is rotated counterclockwise to the under crossing strand are labeled with $A$; the other regions are labeled with $A^{-1}$*. The symbol $A$ is thus associated with splicing the crossing from the $A$ to the $A$ regions and $B$ with splicing the crossing from the $A^{-1}$ to the $A^{-1}$ regions. See [8] for the motivation behind this definition. In the formula (1) the small diagrams stand for otherwise identical parts of larger diagrams. In formula (2) it is stated that the appearance of an extra loop $(O)$ multiplies the bracket by $\delta = -A^2 - A^{-2}$, and that the solitary loop receives the value 1. (In other words, $\langle\ \rangle = 1/\delta$. The unmarked plane has bracket value $1/\delta$.)

These rules give a well-defined polynomial $\langle K \rangle$ in $A$ and $A^{-1}$ associated to a given link diagram $K$ (see [8, 9, 11] and Exercise 0 below).

*Exercises in Bracketology*

The following exercises provide a self-contained introduction to various basic notions and to the bracket polynomial. Some of these results will be used later.

0. Information on the Reidemeister Moves. In [10] Reidemeister proved that the set of two-dimensional moves on diagrams shown in Fig. 3.1 are sufficient to capture the concept of ambient isotopy of knots and links in three-dimensional space. That is, two knots and links are ambient isotopic if and only if diagrammatic "snapshots" of each (projections on a plane where at most two strands meet at a crossing) are equivalent by a sequence of Reidemeister moves combined with homeomorphisms of the diagrams in the plane that do not change the crossing structure.

For an exercise, unknot the knot shown in Fig. 3.2 using only Reidemesiter moves. For a second exercise, turn the figure-eight knot in Fig. 3.2 into its mirror image via the Reidemeister moves. For the latter, it may help to make a model of the figure eight from rope and see by direct topological manipulation that the figure-eight knot is indeed equivalent to its mirror image.

1. Let $\langle K \rangle$ denote the bracket polynomial in independent commuting variables $A$, $B$, and $\delta$. (We do not yet assume that $B = A^{-1}$ or that $\delta = -A^2 - A^{-2}$.) Verify that the recursion formulas (1) and (2) in the definition of the bracket polynomial yield a *well-defined* polynomial function of link diagrams in the three variables $A$, $B$, and $\delta$.

2. Continue with the assumptions of problem 1 and show that the three-variable bracket polynomial has the following behavior under Reide-

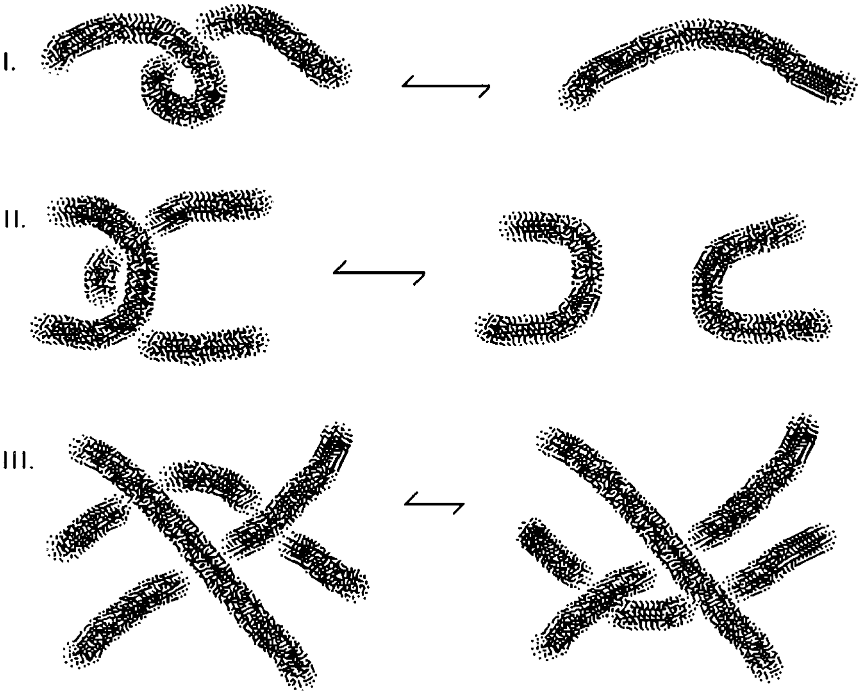



FIG. 3.1. Reidemeister moves.

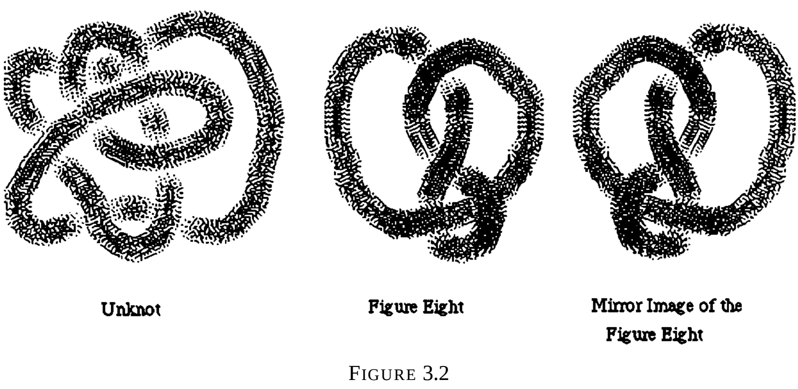
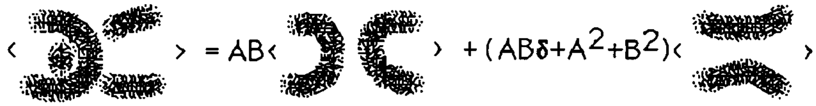
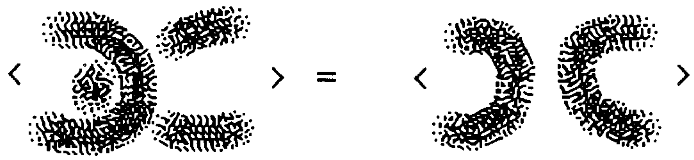



FIGURE 3.2

meister move II:



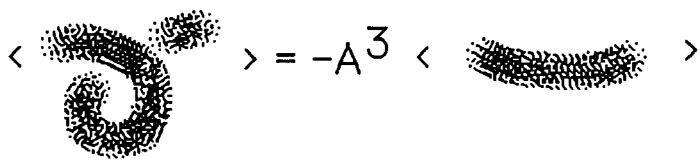

It follows from this formula that if we take $B = A^{-1}$ and $\delta = -A^2 - A^{-2}$, then

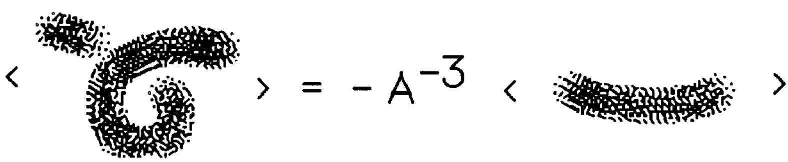



With this specialization of the variables, the bracket is invariant under the second Reidemeister move.

3. Show that with $B = A^{-1}$ and $\delta = -A^2 - A^{-2}$, the invariance of the bracket under the second Reidemeister move implies its invariance under the third Reidemeister move.

4. Show that $\langle K \rangle$ changes under the first Reidemeister move via the formulas



Note that in working with the bracket calculations we are not allowed to perform type I moves on the diagrams without applying these compensating formulas.

5. Show that if $K^*$ is the mirror image of $K$, obtained by switching all the crossings in the diagram for $K$, then $\langle K^*\rangle(A) = \langle K\rangle(A^{-1})$ where we now use the bracket with $B = A^{-1}$ and $\delta = -A^2 - A^2$. Compute the bracket polynomial in $A$ and $A^{-1}$ for the trefoil diagram in Fig. 3.3.

6. Let $K$ be an oriented link diagram and define the *writhe* of $K$, $w(K)$, by the formula $w(K)$ = the sum of $+1$ or $-1$ for each crossing in $K$ where the $+1$ or $-1$ is the sign of the crossing.

The sign of a crossing is obtained as shown in Fig. 3.4.

Now define the normalized bracket, $f_K(A)$, by the formula $f_K(A) = (-A^3)^{-w(K)}\langle K\rangle$ where we take $K$ to be oriented but forget the orientation when we compute the bracket. Show that $f_K(A)$ is an invariant of all three Reidemeister moves. Show that $f_{K^*}(A) = f_K(A^{-1})$. Use these facts and your calculation of the trefoil diagram (from Problem 4) to prove that the trefoil knot is chiral (i.e., inequivalent to its mirror image).

*This ends the exercises in bracketology.*

We now introduce the bracket for tangles.

DEFINITION. Let $T$ be a tangle. Then

$$\langle T\rangle = \alpha(T)\langle \| \rangle + \beta(T)\langle = \rangle,$$

where the coefficients $\alpha(T)$ and $\beta(T)$ are obtained by starting with $T$ and using formulas (1) and (2) repeatedly until only the infinity and zero tangles are left. We let [0] denote the 0-tangle and [∞] denote the infinity tangle so that $\langle T\rangle = \alpha(T)\langle[\infty]\rangle + \beta(T)\langle[0]\rangle$. *Neither* ⟨[0]⟩ *nor* ⟨[∞]⟩ *are numbers*. They are place holders for the results of the computation of the bracket restricted to the tangle.

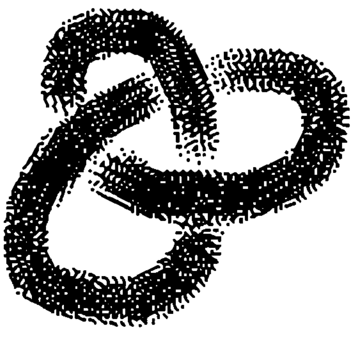


FIGURE 3.3

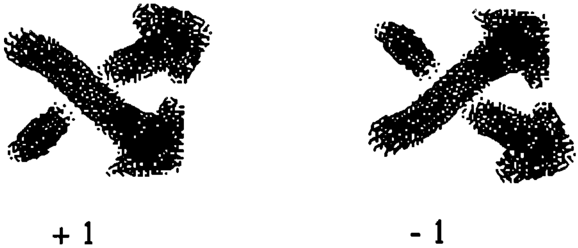



FIGURE 3.4

Figure 3.5 shows an example of a bracket calculation.

THEOREM 3.1. *For any choice of $A$, $R_T(A) = \alpha(T)/\beta(T)$ is an ambient isotopy invariant of tangles.*

*Proof.* The main point is to examine how $\alpha(T)$ and $\beta(T)$ behave under the Reidemeister moves. As outlined in the exercises above, one can expand the tangles in the same way that one expanded a knot or a link to calculate the bracket. In so doing, the verification of invariance under the second and third Reidemeister moves goes through in the same way as for links—with the caveat that one keep the ends of the tangle fixed and that

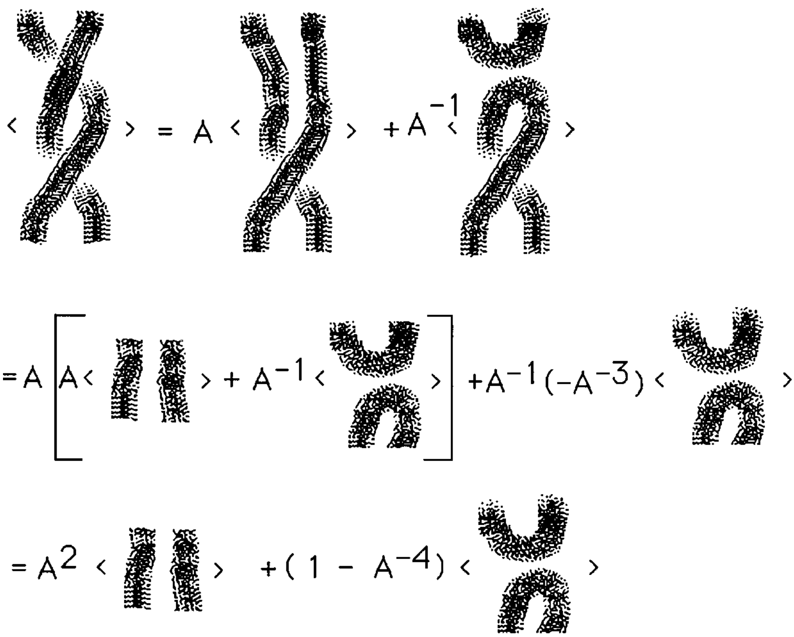



FIGURE 3.5

only isotopies within the tangle box are allowed. By the properties of the bracket expansion, $\alpha(T)$ and $\beta(T)$ are each invariant under Reidemeister II and Reidemeister III moves of the tangle. We know that the bracket behaves under a type-I move by multiplying by either $-A^3$ or by $-A^{-3}$. It is easy to see that both $\alpha(T)$ and $\beta(T)$ are multiplied by the same factor under a type-one move applied to the tangle. Hence the ratio $R_T(A) = \alpha(T)/\beta(T)$ is unchanged under the first Reidemeister move. Therefore the ratio $R_T(A)$ is an invariant of the tangle $T$. Note that $R(T)$ may take the value "infinity" under the circumstance that $\beta(T)$ is zero when $\alpha(T)$ is nonzero. This completes the proof of the theorem.

*Specializing the A*

Now let $A = \sqrt{i}$ $(i^2 = -1)$; hence $B = 1/\sqrt{i}$ and $\delta = -A^2 - A^{-2} = -i - 1/i = -i + i = 0$. Thus the loop value is 0 for this bracket. *From now on $\langle T \rangle$ denotes this specialization of the bracket*. Let

$$C(T) = -iR_T(\sqrt{i}).$$

We are now going to see that $C(T)$ has just the properties that we need to show that $C(T) = F(T)$ for rational tangles.

It is easy to see that $C(1/T) = (1/C(T))^*$, where * denotes complex conjugation, for

$$\langle T \rangle = \alpha(T)\langle [\infty] \rangle + \beta(T)\langle [0] \rangle$$
$$\Rightarrow \langle 1/T \rangle = \alpha(T)^*\langle [0] \rangle + \beta(T)^*\langle [\infty] \rangle,$$

since

$$1/[0] = [\infty] \text{ as tangles and } (1/\sqrt{i})^* = \sqrt{i}.$$

Thus $C(1/T) = \beta(T)^*/i\alpha(T)^* = i^*\beta(T)^*/\alpha(T)^* = (1/C(T))^*$.

Note that when $C(T)$ is real, then $C(1/T) = 1/C(T)$ since a real number is conjugate to itself. We shall see shortly that the values of $C(T)$ on rational tangles are rational numbers and hence real.

Thus *for rational tangles, $C(1/T) = 1/C(T)$.*

EXAMPLE.

$\langle [1] \rangle = \langle$  $\rangle$

$$= \sqrt{i}\langle \| \rangle + (1/\sqrt{i})\langle = \rangle = \sqrt{i}\langle [\infty] \rangle + (1/\sqrt{i})\langle [0] \rangle.$$

Hence $C([1]) = (1/i)(\sqrt{i}\,/(1/\sqrt{i}\,)) = (1/i)(i/1) = 1$. In exactly the same way, we find that $C([-1]) = -1$. Note that $C([\infty]) = 0$ and $c([0]) = \infty$, with the convention that formally $1/0 = \infty$.

We now see that $C(T + S) = C(T) + C(S)$.

PROPOSITION 3.2. *If* $\langle T \rangle = a\langle[\infty]\rangle + b\langle[0]\rangle$ *and* $\langle S \rangle = c\langle[\infty]\rangle + d\langle[0]\rangle$ *then* $\langle T + S \rangle = (ad + bc)\langle[\infty]\rangle + bd\langle[0]\rangle$. *Hence* $C(T + S) = C(T) + C(S)$ *follows from the fact that* $(ad + bc)/bd = a/b + c/d$.

*Proof.* In this proof we shall proceed by a picture-writing technique (Fig. 3.6): the boxes stand for the tangles in question. We first expand on the tangle $T$, replacing it by the sum of two tangles with coefficients that corresponds to its bracket expansion. We then expand each of these pictures on the tangle $S$ to get the full sum that is evaluated. Thus $\langle T + S \rangle = (ad + bc)\langle[\infty]\rangle + bd\langle[0]\rangle$, completing the proof of the theorem.

This completes the proof of all the properties of $C(T)$ that we need. Since the generating tangles [1] and $[-1]$ take rational values under $C(T)$ it follows that tangles generated from them will also take (real) rational values so that $C(1/T) = 1/C(T)$ for any rational tangle. Then repeated application of this reciprocal formula, coupled with the addition theorem $C(T + S) = C(T) + C(S)$, implies that $C(T) = F(T)$ for rational tangles.

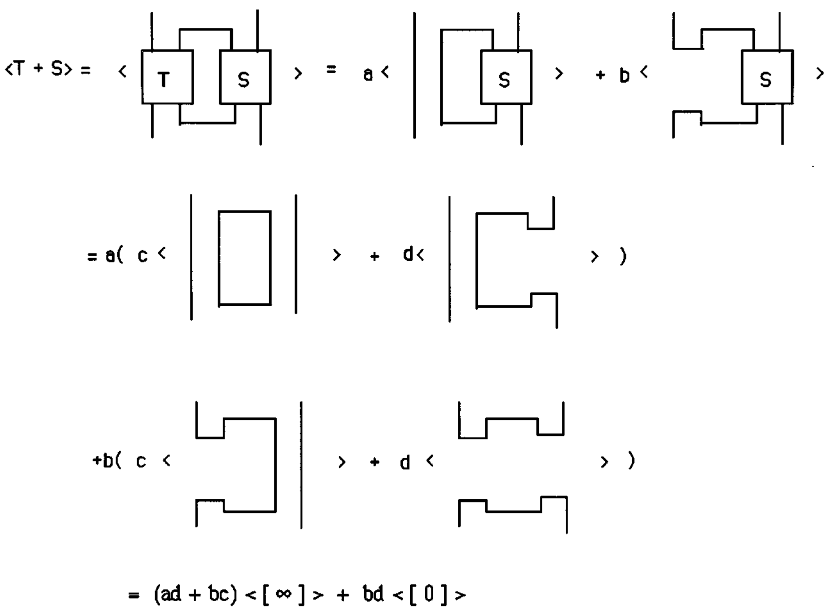



FIGURE 3.6

Our mission is complete. We have proved that the rational tangles are classified by their fractions.

*Remark on Numerators and Denominators.* To any tangle $T$ there are associated two links, the numerator of $T$, $N(T)$, and the denominator of $T$, $D(T)$. The numerator is obtained by tying the input strands to each other and the output strands to each other (Fig. 3.7). The denominator is obtained by tying the input strands to the output strands as indicated in Fig. 3.7. It is easy to see that the conductance, $C(T)$, of the tangle is $i$ times the ratio of the bracket evaluations of the numerator and the denominator of the tangle. That is,

THEOREM 3.3. $C(T) = -i\langle N(T)\rangle/\langle D(T)\rangle$.

*Proof.* When $\langle T\rangle = a\langle[\infty]\rangle + b\langle[0]\rangle$, it follows that $\langle N(T)\rangle = a$, while $\langle D(T)\rangle = b$. Thus $C(T) = -ia/b = -i\langle N(T)\rangle/\langle D(T)\rangle$.

In the next section we shall have occasion to use numerators and denominators of tangles associated with DNA recombination.

*Remark on Graphs and Duals.* Before leaving this topic there is an observation about the usefulness of the bracket that is worth making. In order to make this remark we will assume the background of our paper [7] without further explanation. In that paper we associate to the tangle $T$ a signed plane graph called $G(T, v, v')$ with special vertices $v$ and $v'$ corresponding to the inputs and outputs of the tangle. We regard this signed plane graph as a generalized electrical network. We prove that $C(T)$ is equal to the (generalized) conductance of the network between the vertices $v$ and $v'$ where each edge has conductance $\pm 1$ according to its sign and negative conductances are handled algebraically in the same

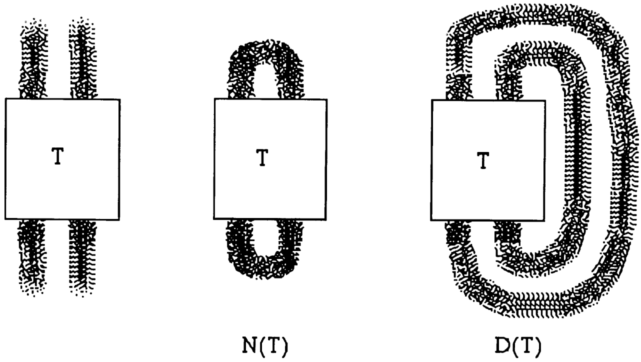



FIG. 3.7. Numerator and denominator.

manner as classical positive conductances in a linear circuit. Let $T^*$ denote any choice of inverse tangle for $T$. It follows that the graph $G(1/T, w, w')$ is the planar dual to the graph $G(T, v, v')$ and, by using the bracket, we conclude that the conductance of the graph is the inverse of the conductance of its planar dual. This result comes directly from our easy proof that $C(1/T) = 1/C(T)$. It is not at all obvious how to produce such a direct argument using pure graph combinatorics. This shows how the bracket model for conductance is not just a trick, but in fact it is a way to see more deeply the properties of both the topology and the electricity associated with plane graphs.

## IV. MOLECULAR BIOLOGY

In this section we sketch an application of fractions of rational tangles to molecular biology. This method of using tangles has been pioneered by DeWitt Sumners [14] and used in the work of Cozzarelli and Spengler [4]. See Kauffman [11, Part 2] and Adams [1] for introductions to the subject.

Recombination of DNA is the process of cutting two neighboring strands with an enzyme and then reconnecting them in a different way. The idea of applying tangle theory is to use the addition of tangles to write the equations for possible recombinations of DNA molecules. Then one uses topological information (such as the fraction of tangle) to obtain limitations on the possibilities for the products of the recombination. Recombination occurs in successive *rounds* for which the nature of the products can be known through a combination of electrophoresis and electron microscopy. In particular, electron microscopy provides the biologist with an enhanced image of the DNA molecule from which it is possible to see direct evidence of knotting and supercoiling. In the case of TN3 resolvase, a species of closed circular DNA is seen to produce very specific knots and links in successive rounds of recombination. By knowing these actual products of the rounds of recombination it is possible to use topology to deduce the mechanism for the recombination.

In order to apply the fraction of a tangle to molecular biology, we shall make the blanket assumption that *all products of recombination, starting from a given unknotted and unlinked form of double-stranded DNA, are closures (numerators) of rational tangles*. This is a reasonable assumption. It assumes that the knots or links that are built in the recombination process are obtained by a combination of simple twisting (of the sort that builds new rational tangle from old) and the addition of single crossings at a smoothing site. The latter operation is what is usually called *site specific recombination* by biologists (see Fig. 4.1). A crossing is created in place of the smoothing that is the local configuration of the ‘‘lined-up’’ sites. There

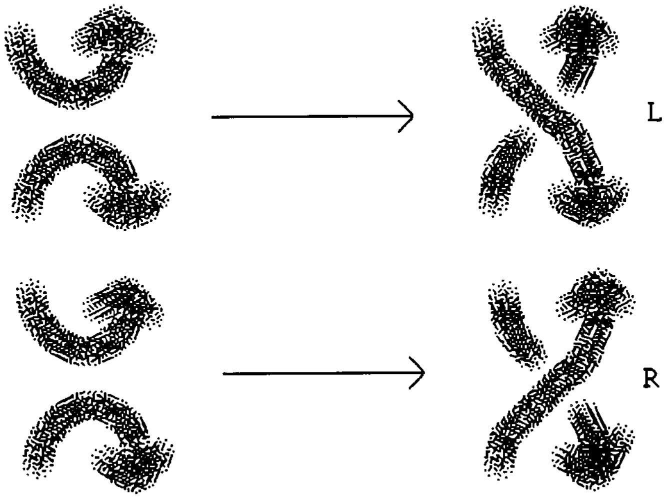



FIG. 4.1. Site specific recombination.

are two possibilities for such a crossing. We have called these possibilities **R** and **L**. In Fig. 4.2 we have illustrated the concept of a site specific recombination by drawing the DNA schematically in a single closed loop with local arrows at these sites. In order for the recombination to occur, the DNA must twist about to bring these two sites into proximity with the orientations lined up.

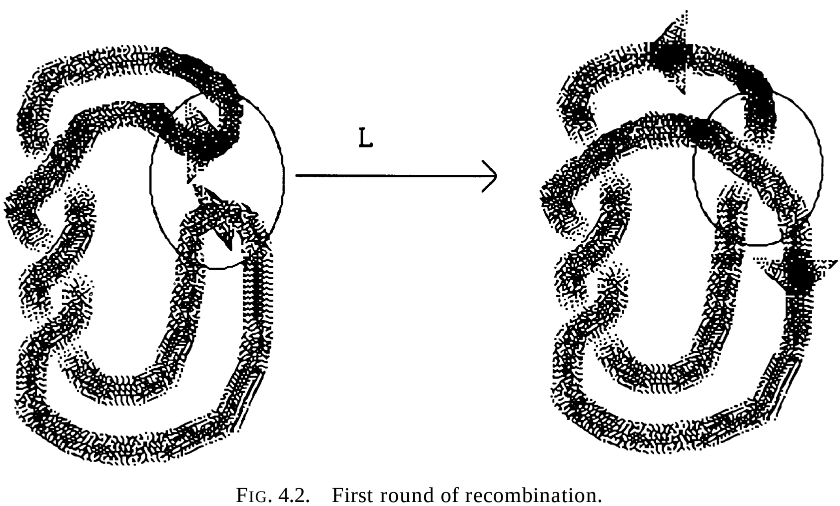



FIG. 4.2. First round of recombination.

We regard the crossing **R** as a small tangle and differentiate it from its reverse version **L**. Now view Fig. 4.2. Here we see the result of a single round of recombination. First it is assumed that there is a total twist of the DNA and that the two sites are brought into proximity. In this form, the DNA can be described as the numerator, Num($[1/n] + U$), of the tangle sum $[1/n] + U$ where $U$ denotes the tangle that we have previously called [0]. The recombination then produces Num($[1/n] + S$), where $S$ equals **R** or **L**. In Fig. 4.2, we have taken $n = +3$ and $S = L$.

Successive rounds of recombination produce

$$\mathrm{Num}([1/n] + S + S),$$
$$\mathrm{Num}([1/n] + S + S + S),$$
$$\cdots.$$

For example, let $n = 3$ and $S = L$. Then the successive rounds of recombination are shown in Figs. 4.3 and 4.4. The first two rounds give a simple link and the figure-eight knot. The third round gives a link of two components with linking number zero. In fact, TN3 resolvase produces just these knots and links in its successive rounds. Here we have indicated a possible mechanism for TN3 resolvase. Is it the only possibility?

In order to answer this question in the context of our model, we consider the most general case of a sequence of rational tangles in the form $T, T + S, T + S + S, \ldots$ such that FRAC($T + S$) $= 1/3 - 1 = -2/3$ and FRAC($T + S + S$) $= 1/3 - 1 - 1 = -5/3$. Let $x =$ FRAC($T$) and $y =$ FRAC($S$). Then (using the fact that the fraction of a

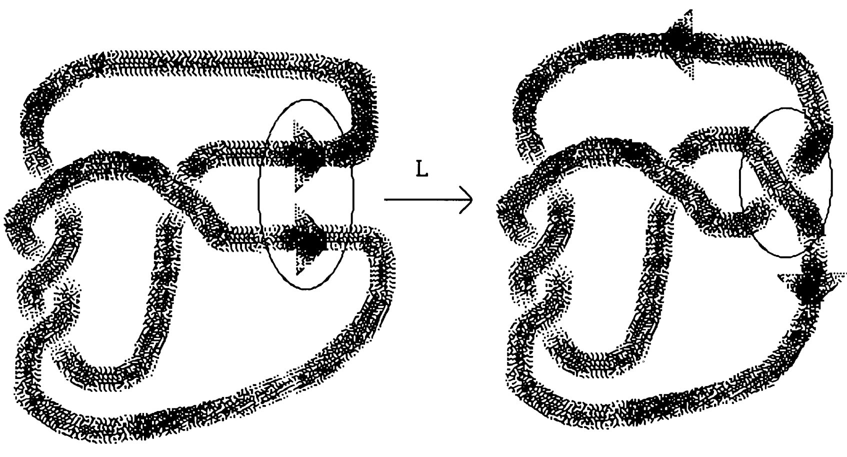



FIG. 4.3. Second round of recombination.

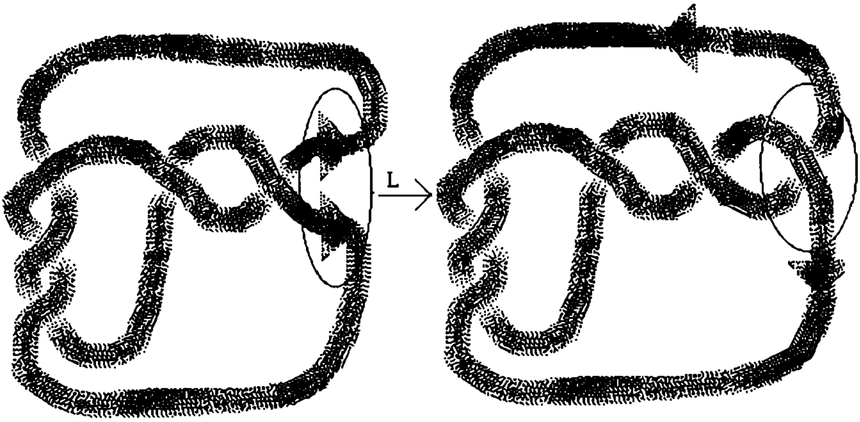



FIG. 4.4. Third round of recombination.

sum of tangles is the sum of the fractions of the summands) we have the equations

$$x + y = -2/3$$
$$x + 2y = -5/3.$$

The only solution to these equations is $x = 1/3$ and $y = -1$. By the Conway Fraction Theorem, the tangles $T$ and $S$ are identified as $T = [1/3]$ and $S = [-1]$. This shows how the topology can be used to pinpoint a biological mechanism.

## ACKNOWLEDGMENTS

It gives Lou Kauffman great pleasure to thank John Conway for numerous conversations about tangles and square dancing and David Krebes for insightful remarks on tangles and the bracket. We both thank DeWitt Sumners for his enthusiasm for the topology of DNA and Steve Bleiler for helpful conversations.

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