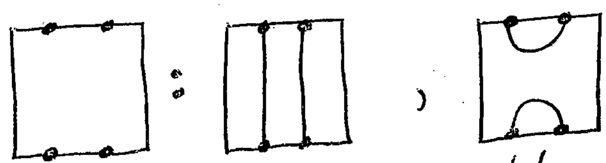
The Temperley-Lieb Category

by Louis H Kauffman

This short paper is about an algebraic system that is as fundamental as the mathematics of permutations. Take two rows of points (we'll start with the same number of points in each row). The first row of points is at the top of a rectangle; the second row is at the bottom of the rectangle. Connect points to points so that all points are used and <u>no connecting lines intersect one another</u>. For example,



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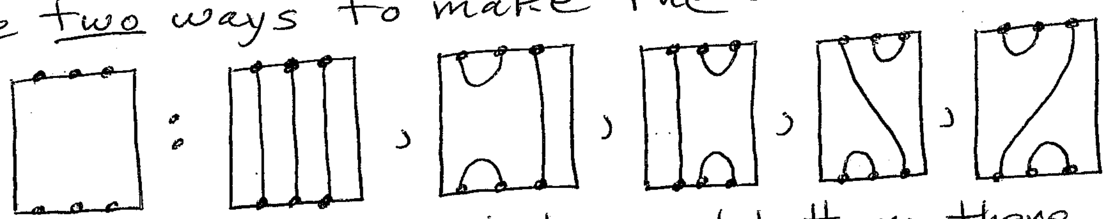
With two points at top and bottom, there are <u>two</u> ways to make the connections.

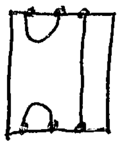
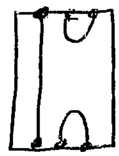
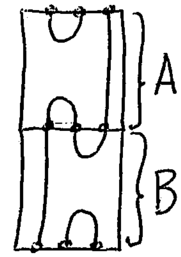


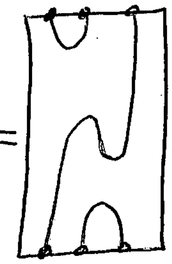
With three points at top and bottom, there are <u>five</u> ways to make the connections.

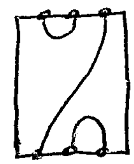
In general, with n points at top and bottom, there are $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)[n!]^2}$ ways to make the connections. C_n $(n = 1, 2, 3, 4, \ldots)$ are the <u>Catalan numbers</u>. For example, $C_3 = \frac{6!}{4(3!)^2} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2} = 5.$

Can you prove it?

With the same number of points at the top and bottom of the rectangle, we can <u>multiply</u> connection structures by matching tops and bottoms as shown below:

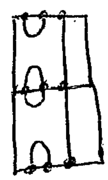
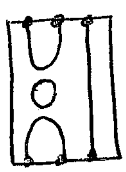
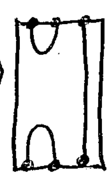
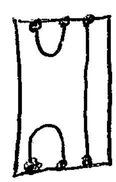
A =  , B =  , AB = 

= 

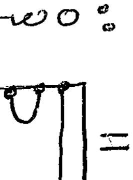
Thus we have AB =  = C.

When we multiply structures, we erase the common boundary between the two matched rectangles, making a single new rectangle with its resulting connection structure.

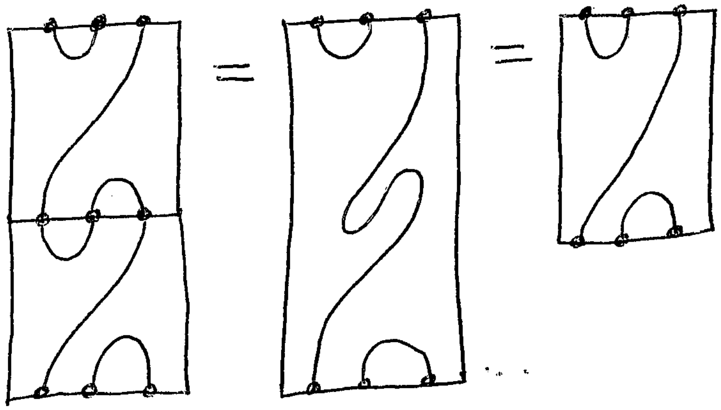
Here is another example or two:

A^2 =  =  = ○  = δ  = δA.

When a loop ○ appears, we let it float out and call it δ. Note that in multiplying AB we allowed a wiggle  to contract to 

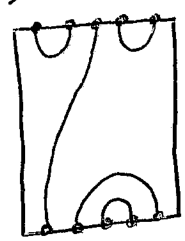
Now, before you turn the page, work out C^2 for C = .

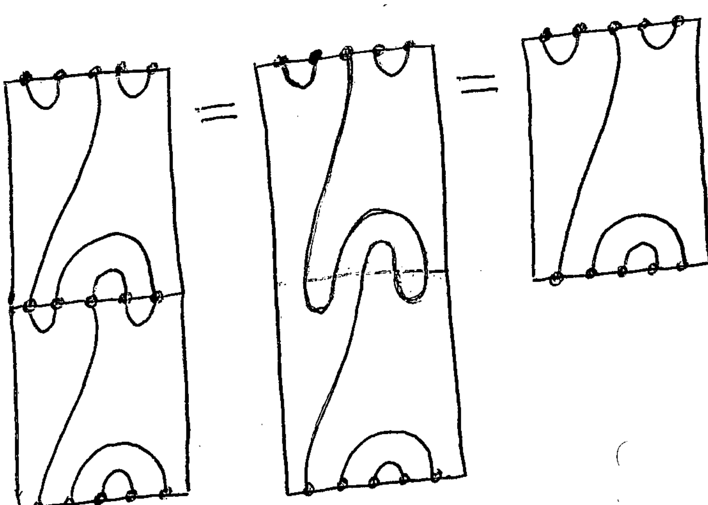
Here goes C^2:

$C^2 =$  $= C.$

Thus $C^2 = C$.

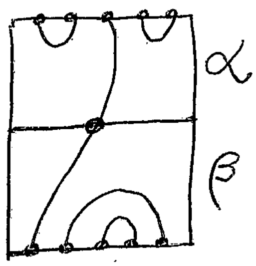
(By the way, this way of combining connection structures with n top points and n bottom points is called ~~the~~ Temperley-Lieb Algebra, and denoted by the symbol TL_n.)

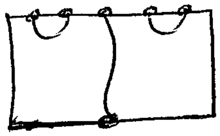
Look at this: $P =$ 

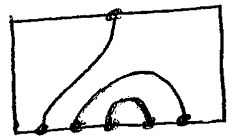
$P^2 =$  $= P.$

Can you find a way to construct all elements Q of TL_n such that $Q^2 = Q$?

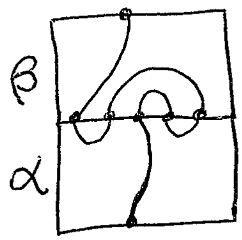
Here is a hint. Look at P, thinking of it as factored into two rectangles α and β:

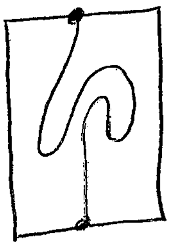
$P =$ 

$\alpha =$ 

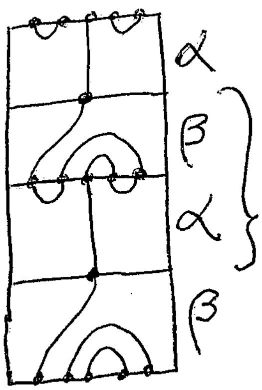
$\beta =$ 

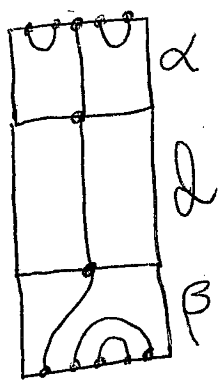
Individually, α and β have different numbers of points top and bottom, but $\alpha\beta$ makes sense becaus α has one bottom point, and β has one top point. $\beta\alpha$ also makes sense and in fact

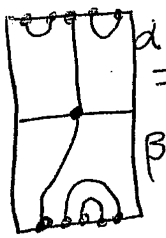
$\beta\alpha =$ 

$=$  $=$  $= \mathcal{I}$.

The straightened structure $\mathcal{I}$ does not change anything it combines with, and now we see

$P^2 =$ 

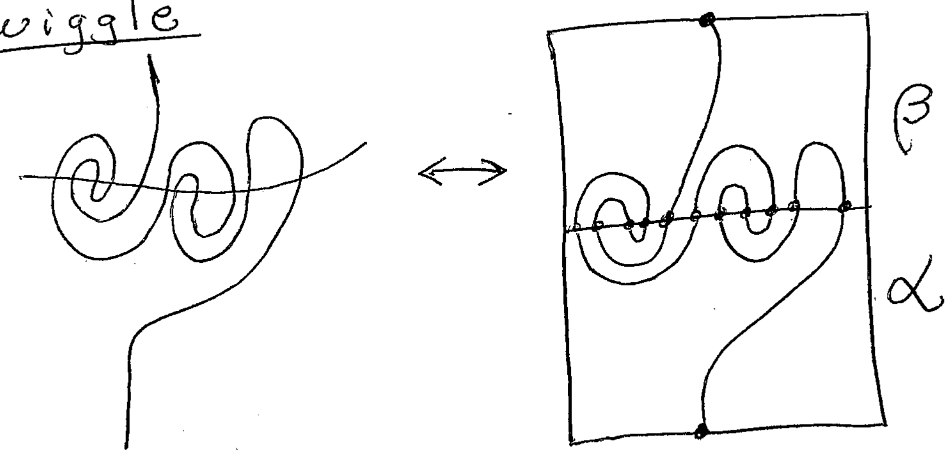
$=$ 

$=$ 

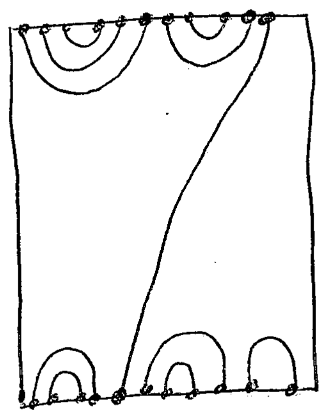
$= P$

$$P^2 = \alpha\beta\alpha\beta = \alpha\mathcal{I}\beta = \alpha\beta = P.$$

So you see, <u>you can factorize a wiggle</u>



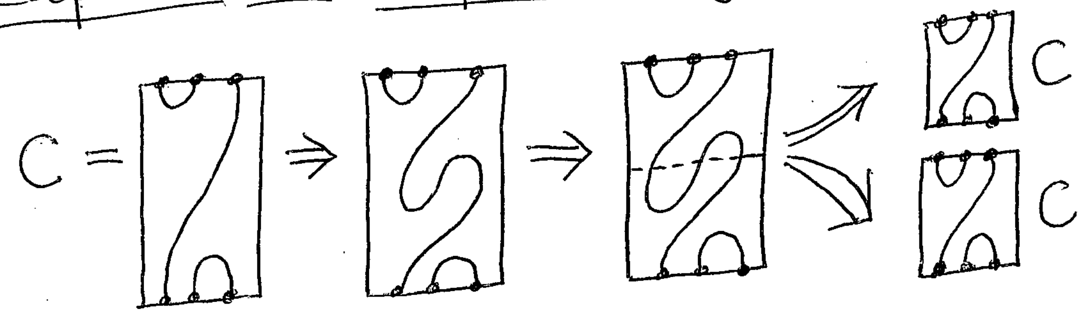
Then make $P = \alpha\beta$ with $\beta\alpha = \mathbb{1}$ and $P^2 = P$. Here

$P =$ 

<u>Factorized wiggles</u> are often called <u>meanders</u>.

The Temperley-Lieb algebras
have infinitely many elements P
with $P^2 = P$, and their structure
can be understood in terms of
generalized meanders and the
context of the <u>category</u> of
connection structures with
different numbers of top and bottom
points.

You can think of these idempotents P as little genetic creatures that are capable of reproducing themselves.



The Temperley–Lieb category lets us peek into a complex almost-biological mathematical world, a genuinely planar algebraic structure, and an entry point into topology and combinatorics.

References

1. L. Kauffman, "Knots and Physics", World Scientific 1991, 1994, 2001.
2. L. Kauffman and S. Lins, "Temperley–Lieb Recoupling Theory and Invariants of Three-Manifolds" Princeton Univ. Press 1994.
3. L. Kauffman, Biologic, AMS Contemp. Series, Vol. 304 (2002), pp. 313–340.