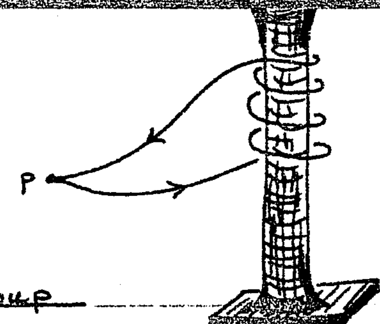


V. Fundamental Group

Topological space X, $p \in X$ a basepoint.

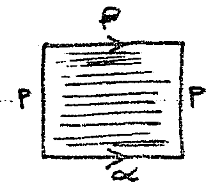
$\mathcal{L}(X, p) = \{\alpha : I \longrightarrow X \mid \alpha(0) = \alpha(1) = p\}$, α contin., $I = [0,1]$.

$\alpha, \beta \in \mathcal{L}$, define $\alpha * \beta \in \mathcal{L}$:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & , 0 \le t \le 1/2 \\ \beta(2t-1) & , \frac{1}{2} \le t \le 1 \end{cases} .$$

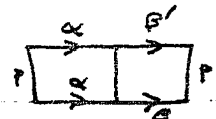
Def. $\alpha \sim \beta$ (α homotopic to β) if $\exists F : I \times I \longrightarrow X$ s.t.

$$\left.\begin{aligned} F(0,t) &= \alpha(t) \\ F(1,t) &= \beta(t) \\ F(t,0) &= F(t,1) = p \end{aligned}\right\} \forall t .$$

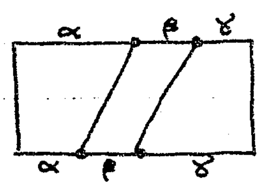


Def. $\pi_1(X, p) = \mathcal{L}(X, p)/\sim$. $[\alpha]$ denotes the equivalence class of α. Define $[\alpha][\beta] = [\alpha * \beta]$. $\pi = \pi_1(X, p)$ is called the fundamental group of X (based at p).

1° Multiplication is well-defined on π_1:

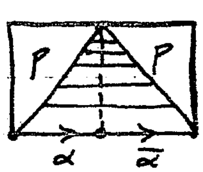
Pf:  extends to .

Hence $\beta \sim \beta' \Rightarrow \alpha * \beta \sim \alpha * \beta'$. //

2° Associative  $\Rightarrow \alpha * (\beta * \gamma) \sim (\alpha * \beta) * \gamma$.

3° Inverses Let $\bar{\alpha}(t) = \alpha(1-t)$.

Then $\alpha * \bar{\alpha} \sim *$, $(*(t) = p \ \ \forall t)$



Hence $\pi_1(X, p)$ is a group.

Let $f : (X, p) \longrightarrow (Y, q)$ be a map of $(f(p) = q)$ topological spaces (a continuous map). Define $f_* : \pi_1(X, p) \longrightarrow \pi_1(Y, q)$ by $f_*[\alpha] = [f \circ \alpha]$.
(<u>exercise</u>: check well-defined)

<u>Lemma</u>. $f_* : \pi_1(X, p) \longrightarrow \pi_1(Y, q)$ is a homomorphism of groups.

<u>Pf</u>:
$$\begin{aligned} f_*([\alpha][\beta]) &= f_*([\alpha * \beta]) \\ &= [f \circ (\alpha * \beta)] \\ &= [(f \circ \alpha) * (f \circ \beta)] \quad \text{(from defn of } *) \\ &= [f \circ \alpha][f \circ \beta] \\ &= f_*[\alpha] f_*[\beta] . \end{aligned}$$

Thus if f is a <u>homeomorphism</u>, then f_* is an <u>isomorphism</u> (why?). So the group <u>$\pi_1(X, p)$ is a topological invariant</u> of the space X.

<u>Exercise</u>. If $\exists$ a path $\beta : I \longrightarrow X$ such that $\beta(0) = p$, $\beta(1) = q$ then $\pi_1(X, p) \cong \pi_1(X, q)$.

<u>Proposition</u>: Let S^1 denote the circle. Then $\pi_1(S^1, p) \cong \mathbb{Z}$.

<u>Proof</u>. $S^1 = \{ e^{2\pi i t} \mid t \in \mathbb{R} \}$
$$p : \mathbb{R} \longrightarrow S^1 , \quad p(t) = e^{2\pi i t} .$$
This mapping has the following properties:

a) $p^{-1}(1) = \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\} \subset \mathbb{R}$.

b) For each $x \in S^1$ $\exists$ a sufficiently small nbhd U_x s.t. $p^{-1}(U_x) =$ a countable disjoint union of nbhds each mapping homeomorphically to U via p.

This map has particularly nice lifting properties.

<u>Lemma</u>: Let $f: I$ or $I \times I \longrightarrow S^1$ be a continuous map. Let $\mathcal{D}$ denote I or $I \times I$ and $q \in \mathcal{D}$ a chosen point. Let $q' \in p^{-1}(f(q))$ be a given point in $\mathbb{R}$. <u>Then</u> there exists a <u>unique</u> <u>lifting</u> $\hat{f}: \mathcal{D} \longrightarrow \mathbb{R}$ such that $p \circ \hat{f} = f$ and $\hat{f}(q) = q'$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathcal{D} \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

<u>Pf</u>: (omitted from notes)(Use a partitioning argument.).

Now define $\lambda : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ by $\lambda[\alpha] = \hat{\alpha}(1)$ where $\hat{\alpha}$ = unique lift of α such that $\hat{\alpha}(0) = 0$. It follows from the lemma that $\alpha \sim \beta \Rightarrow \hat{\alpha}(1) = \hat{\beta}(1)$ (by continuity + discreteness of $\mathbb{Z}$). Now we claim that λ is injective, surjective and a homomorphism. Details omitted from the notes. ∎

For path-connected spaces we will often omit mention of the base point. The next result shows how to piece together $\pi_1(X)$ from its subspaces.

<u>Given</u>: X path conn. space.
$X \supset A, B$ open, path conn. sets.
$C = A \cap B$ path conn.
<u>and</u> $X = A \cup B$.

<u>Thus</u>

$$\begin{tikzcd} & \pi(A) \arrow[dr, "k"] & \\ \pi(C) \arrow[ur, "i"] \arrow[dr, "j"'] & & \pi(X) \\ & \pi(B) \arrow[ur, "l"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \pi(A) \arrow[dr, "k"] & \\ \pi(C) \arrow[ur, "i"] \arrow[dr, "j"'] & & \pi(X) \\ & \pi(B) \arrow[ur] & \end{tikzcd}$$

Theorem (Van-Kampen). $\pi(X) \cong \dfrac{\pi(A) * \pi(B)}{\langle i(h) * j(h^{-1}) \mid h \in \pi(C) \rangle}$

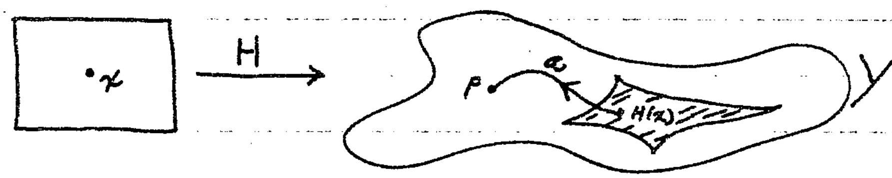
Here $G * H$ denotes the free product of groups and $\langle \ \mid \ \rangle$ denotes normal subgroup generated by $\cdots$. Thus $\pi(X)$ is a free product with amalgamation.

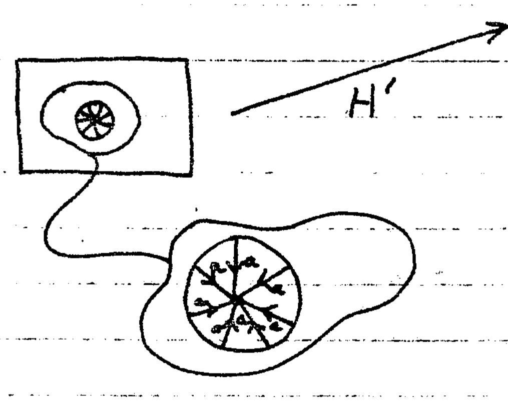
The following lemma will be used in the proof:

Lemma. $H : I \times I \longrightarrow Y$, Y path connected. Let $p \in Y$ and $x_1, \ldots, x_r$ be a finite set of points in $I \times I$. Then H is homotopic to $H' : I \times I \longrightarrow Y$ such that $H(x_i) = p$ for each i and H' agrees with H outside a tiny disk about each x_i (half-disks for those x_i on bndry $I \times I$).

[Def: $f, g : X \longrightarrow Y$ are homotopic if $\exists\ F : I \times X \longrightarrow Y$ s.t. $F(0,x) = f(x)$, $F(1,x) = g(x)$ (no other restrs in general)]

Pf (of lemma). Clearly, it suffices to prove this for $r = 1$. Say $x = x_1$.





H' is defined as follows: $I \times I - x$ is homeomorphic to $I \times I - D_\varepsilon(x)$. Let $g : (I \times I - D_\varepsilon(x)) \longrightarrow (I \times I - x)$ be such a homeomorphism. Let $a : I \longrightarrow Y$ be any path from $H(x)$ to p. This defines a map $A : D^2_\epsilon \longrightarrow Y$ such that $A|$(any ray from center to bndry) $\equiv a$. Thus $A|\partial D^2_\epsilon : \partial D^2_\epsilon \longrightarrow H(x)$. Now define H' by

$$H'(q) = \begin{cases} H \circ g(q) & , q \in I \times I - D_\epsilon(x) \\ A(q) & , q \in D_\epsilon(x). \end{cases}$$

H' is clearly continuous & it is easy to check that it is homotopic to H. ∎

<u>Proof (sketch) of Van Kampen Theorem:</u>

Have map $f : \pi(A) * \pi(B) \longrightarrow \pi(X)$ defined by

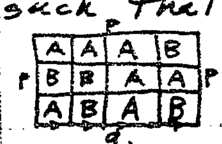
$$f(\alpha) = k(\alpha) \quad , \alpha \in \pi(A)$$
$$f(\beta) = \ell(\beta) \quad , \beta \in \pi(B).$$

<u>Exercise</u>. <u>f is surjective</u> (partition the interval). Thus, let $K = \mathrm{Kernel}(f)$. We want to show that $K = \langle i(\gamma) * j(\gamma^{-1}) \mid \gamma \in \pi(C) \rangle$. The subgroup on the right is certainly contained in the kernel.

Let $g \in K$. Then one has a homotopy

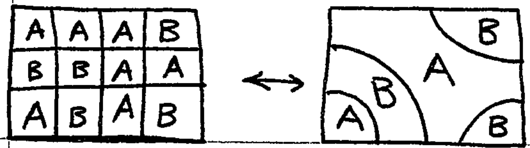
$\longrightarrow X$.

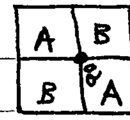
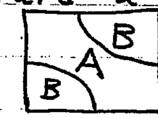
Since $\{A, B\}$ is an open cover for X there is a subdivision of the square into rectangles such that each rectangle goes entirely into A or B.

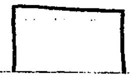
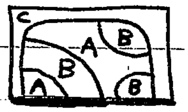


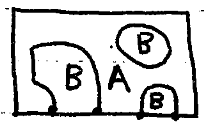
Note that along the bottom of the square, we may assume that the subdivision corresponds to the way g is written in $\pi(A) * \pi(B)$ as a product of elements living on A and B. Thus all vertices on the bottom edge go to base point p. Since the other sides of the square go to base-pt., their vertices do also.

Now we want to view the square as divided into A-regions and B-regions.



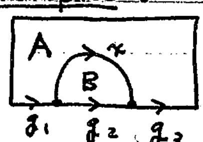
The point is that whenever  occurs then q goes to $A \cap B = C$ and $\therefore$ some nbhd of $q \longmapsto C$. Hence we may regard a nbhd of q as going into A and so write .

Furthermore  goes to $p \in C$ and $\therefore$ some nbhd of it goes to C. Therefore we may write  and then choose to regard this "collar" as going to either A or B, say A here:



Thus the square becomes divided by a collection of simple closed curves on its interior and arcs touching the bottom line. The decomposition can, of course, be somewhat complicated. Note that each circle or arc is a loop supported on C.

Examples:

1) 

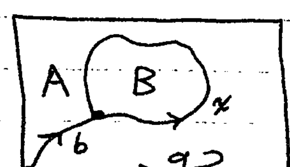
$x \in \pi(C)$

$i(x) \in \pi(A)$, $j(x) \in \pi(B)$

$$\left.\begin{aligned} g &= g_1 g_2 g_3 \\ j(x) &= g_2 \\ g_1 i(x) g_3 &= e \end{aligned}\right\} \text{ identities in } \pi(A) * \pi(B).$$

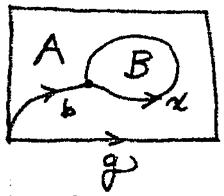
$$g = g_1 g_2 g_3 = (g_1 g_2 i(x^{-1}) g_1^{-1})(g_1 i(x) g_3)$$

$$\therefore \boxed{g = g_1 (j(x) i(x^{-1})) g_1^{-1}} \quad \text{as desired.}$$

2) 

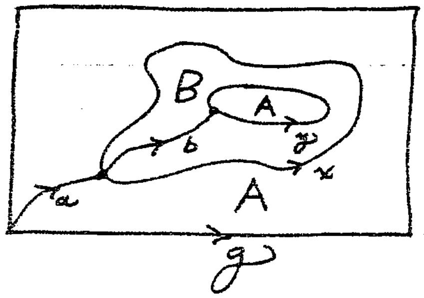
$x \in \pi(C)$. Here we use the lemma to make sure that x really is in $\pi(C)$ (by sending appropriate pt. to base-pt.).

Choose b s.t. $b(0) = p = b(1)$ is a loop coming from the A-region. Then



$g = b\, i(x)\, b^{-1}, \quad j(x) = e$

$\therefore \quad g = b\, i(x)\, j(x^{-1})\, b^{-1}$

3) 

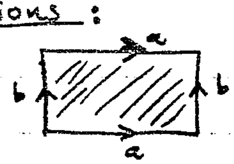
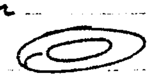
$x, y \in \pi(C)$

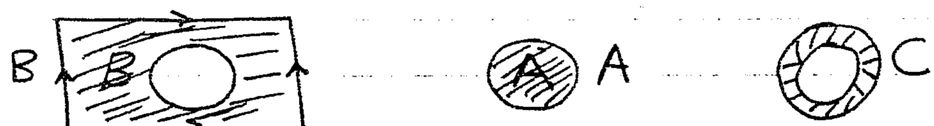
$$\begin{cases} g = a\, i(x)\, a^{-1} \\ j(x) = b\, j(y)\, b^{-1} \\ i(y) = e \end{cases}$$

$$\begin{aligned} g &= (a\, i(x)\, j(x^{-1})\, a^{-1})(a\, j(x)\, a^{-1}) \\ &= (a\, i(x)\, j(x^{-1})\, a^{-1})((ab)\, j(y)\, (ab)^{-1}) \\ &= (a\, i(x)\, j(x^{-1})\, a^{-1})((ab)\, i(y^{-1})\, j(y)\, (ab)^{-1}). \end{aligned}$$

The general proof proceeds along the same lines by induction on the number of arcs and circles in the decomposition. ∎

Applications:

1) T  identif diagram for torus 



$\pi(C) = \mathbb{Z}, \ \pi(A) = \{e\}, \ \pi(B) = F(a,b)$

$(F(a,b) = $ free group on a & b).

$$\left.\begin{aligned} i &: \pi(C) \longrightarrow \pi(A) \\ j &: \pi(C) \longrightarrow \pi(B) \end{aligned}\right\} \quad j(g) = aba^{-1}b^{-1}$$

$(g = $ gen of $\pi(C))$.

$$\therefore \ \pi(T) = \pi(A) * \pi(B) / \langle i(g) * j(g^{-1}) \rangle$$

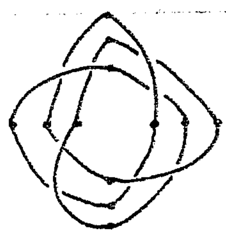
$$= F(a,b) / \langle aba^{-1}b^{-1} \rangle$$

$$\therefore \ \pi(T) = \mathbb{Z} \times \mathbb{Z}.$$

2) Same argument shows that if X_g = closed compact surface of genus g, then $\pi(X_g) = (a_1, b_1, \ldots, a_n, b_n \mid e = \prod_{i=1}^{g} [a_i, b_i])$ where $[x,y] = xyx^{-1}y^{-1}$.

3) 

$X = P^2$, the projective plane.
$\pi(P^2) = (a \mid a^2 = e) = \mathbb{Z}/2\mathbb{Z}$.

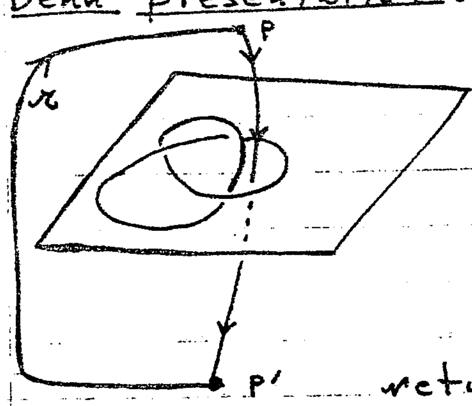
4) 
$K_{4,3}$. $K_{p,q}$ = torus knot of type p, q. $(\gcd(p,q) = 1)$

Then $\pi(S^3 - K_{pq}) = (a, b \mid a^p = b^q)$.

To see this, decompose $S^3 - K_{pq}$ as : innertorus, outer-torus (and intersection is torus minus the knot).

5) <u>The knot group</u>

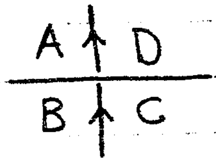
Van Kampen Theorem can be used to obtain presentation of the group of a knot. We omit details (see Rolfson) and discuss here the <u>Dehn presentation</u>. $\boxed{G = \pi_1(S^3 - K, p)}$



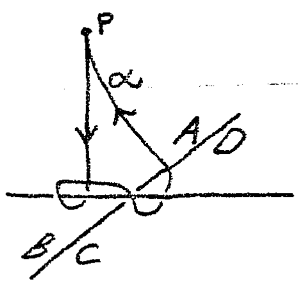
View the knot projection as lying on a plane. Let basepoint p be above the plane. Choose a mirror point p' below the plane and a standard path r that returns from p' to p intersecting the plane in the outermost (unbounded) region of the knot diagram. For each region R of the knot diagram we associate an element "R" $\in \pi_1(S^3 - K, p) = G$ by choosing a path from p to p' going thru R and returning

$C^{-1}DA^{-1}=1$
$A^{-1}DB^{-1}=1$
$DC^{-1}B^{-1}=1$

Each crossing gives rise to a relation:



$AB^{-1}CD^{-1}=1$.



To see this, convince yourself that $AB^{-1}CD^{-1}$ is homotopic to the curve α pictured here.

Theorem. Let K be a knot or link diagram and $G = \pi_1(S^3-K)$. Let $A_1, A_2, \ldots, A_n$ be the bounded regions of the diagram, E the unbounded region. Let $R_1, \ldots, R_m$ denote the crossing relations described above. (The region E is identified with $1 \in G$ if it occurs in a relation) Then G has presentation
$$G \cong (A_1, A_2, \ldots, A_n \mid R_1, R_2, \ldots, R_m).$$

** Exercise. *Read Rolfson's exposition (pp. 58–60) of the Wirtinger presentation. Adopt his argument and use Van Kampen to prove this theorem.

* Exercise. Let $G(K) = (A_1, \ldots, A_n \mid R_1, \ldots, R_m)$ be associated to each knot diagram by the prescription above. Show that if $K \sim K'$ (by elementary deformations), then $G(K) \cong G(K')$.
(This gives a completely elementary approach to the knot group.)

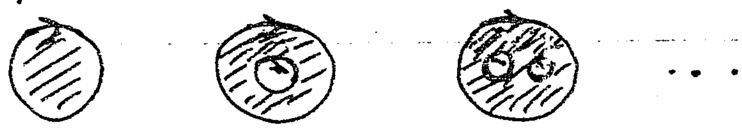
6) The First Homology Group $H_1(X)$

Associated to each space X there is an abelian group $H_1(X)$ that is a topological invariant of X. We shall give a geometric definition of $H_1(X)$ and then prove that for X path connected, $H_1(X) \cong G/G'$ where $G = \pi_1(X)$ and G' = the commutator subgroup of G.

Def. $C_1(X) = \{\alpha : \Lambda_K \longrightarrow X\}$ where α is continuous and $\Lambda_K = \circlearrowleft \circlearrowleft \circlearrowleft \cdots \circlearrowleft$, a disjoint union of K standardly oriented circles $(K = 0, 1, 2, \ldots)$. Let $0 \in C_1(X)$ denote the empty map. If $\alpha, \beta \in C_1(X)$, define $\alpha + \beta$ by taking disjoint union of maps.

Let $T : \Lambda_K \to \Lambda_K$ be the map that reverses orientations on all the circles. Define $-\alpha = \alpha \circ T$.

Suppose S is an oriented surface with boundary. Represent S via standard identif diagrams:

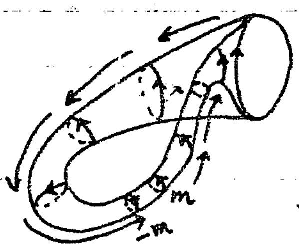


Given $F : S \longrightarrow X$ let $\partial F = F|\partial S$ where ∂S = oriented boundary of S. Thus $\partial F \in C_1(X)$.

Def. If $\alpha, \beta \in C_1(X)$ we say α is homologous to β $(\alpha \sim \beta)$ if $\exists\ F : S \longrightarrow X$, S a surface with boundary, such that $\partial F = \alpha - \beta$.

Define $H_1(X) = \mathcal{C}_1(X)/\sim$ and verify the following facts :

1) Letting $\langle\alpha\rangle$ denote the equiv class of $\alpha \in \mathcal{C}_1(X)$, define $\langle\alpha\rangle + \langle\beta\rangle = \langle\alpha+\beta\rangle$. Show that $H_1(X)$ is an abelian group under $+$. The zero element is the class of the empty map.
2) If $f: X \longrightarrow Y$ is a continuous map, then $f_*: H_1(X) \longrightarrow H_1(Y)$ defined by $f_*\langle\alpha\rangle = \langle f\circ\alpha\rangle$ is a well-defined homomorphism of abelian groups.
[Hence $H_1(X)$ is a topological invariant of X.]
3) If X is path-connected, then every element of $H_1(X)$ is represented by $\alpha: S^1 \longrightarrow X$.
4) Define $h: \pi_1(X,p) \longrightarrow H_1(X)$ by
$$h[\alpha] = \langle\alpha\rangle .$$
Then h is a homomorphism of groups, and h is surjective when X is path-connected.

ex: 

In a Klein bottle, the meridian m is homologous to $-m$.
$m \sim -m \Rightarrow m + m \sim 0$.
Thus the twist in the bottle gives rise to torsion in $H_1(\text{bottle})$.

Theorem. Let X be a pathwise connected topological space. Then if $\pi = \pi_1(X,p)$ and $\pi' =$ the commutator subgroup of π, $h: \pi \longrightarrow H_1(X)$ the (Hurewicz) homomorphism (4 above) then $\operatorname{Ker}(h) = \pi'$.
Hence $\mathcal{Ab}(\pi) = \pi/\pi' \cong H_1(X)$.

Proof. Let $K = \operatorname{Ker}(h)$. Then $\pi' \subset K$ since H_1 is abelian.

Thus suffices to show $K \subset \pi'$. Let $[\alpha] \in K$.
$h[\alpha] = 0 \Rightarrow \exists\ F : \mathcal{S} \longrightarrow X$ such that $\partial F = \alpha$. Let $i : \partial\mathcal{S} \hookrightarrow \mathcal{S}$ be the inclusion map. $\partial \mathcal{S} = S^1$.

$$\begin{tikzcd} \mathcal{S} \arrow[dr, "F"] & \\ S^1 \arrow[u, "i"] \arrow[r, "\alpha"'] & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \pi_1(\mathcal{S}) \arrow[dr, "F_*"] & \\ \pi_1(S^1) \arrow[u, "i_*"] \arrow[r, "\alpha_*"'] & \pi_1(X) \end{tikzcd}$$

Let g = generator of $\pi_1(S^1)$. Then $[\alpha] = \alpha_*(g) = F_*(i_* g)$. But $i_* g \in \pi_1(\mathcal{S})$ is a product of commutators (since it represents the boundary). Hence $F_*(i_* g) \in \pi_1(X)'$. Thus $[\alpha] \in \pi'$. This completes the proof. ∎

We can compute H_1 by abelianizing the Van Kampen Theorem. Let A, B, C ; $A \cap B = C$, $A \cup B = X$ be spaces satisfying the hypotheses of the Van Kampen Theorem. Then

$$H_1(C) \xrightarrow{p} H_1(A) \oplus H_1(B) \xrightarrow{q} H_1(X) \longrightarrow 0$$

$$p(x) = (i_* x, -j_* x)$$
$$q(a,b) = k_* a + l_* b$$

$$\begin{tikzcd} & A \arrow[dr, "k"] & \\ C \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\ & B \arrow[ur, "l"'] & \end{tikzcd}$$

The sequence above is exact.

$$H_1(X) \cong \frac{H_1(A) \oplus H_1(B)}{\langle (i_* x, -j_* x) \mid x \in H_1(C) \rangle}$$