Multi-fractal spectrum of planar harmonic measures

joint work in writing with I.Binder (University of Toronto)

Adi Glücksam

Northwestern University

Midwest dynamics, October 2023

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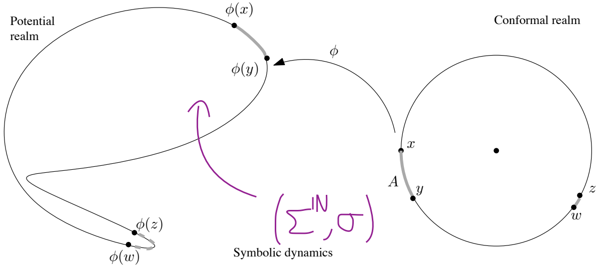

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- **Makarov dimension theorem:** The dimension of harmonic measure is equal to 1.

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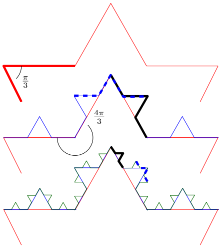

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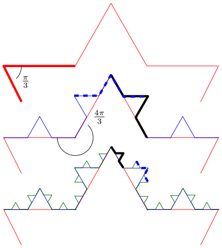

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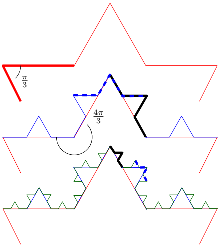

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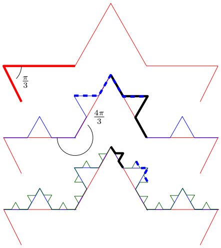

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→ Replacing the triangle by another polygon you get Carleson domain.

→ Allowing expanding conformal maps (instead of only linear ones) you get Jordan Repellers.

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The **Minkowski dimension spectrum**, *denoted* f, *is the dimension of the set in* $\partial\Omega$ *where the harmonic measure behaves approximately like* δ^{α}.

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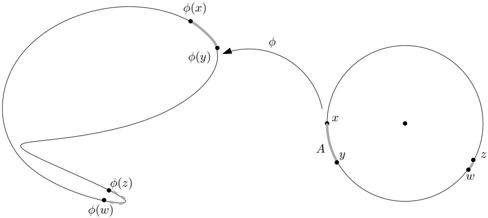

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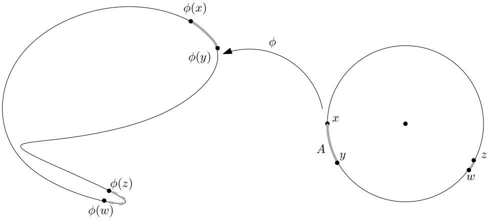

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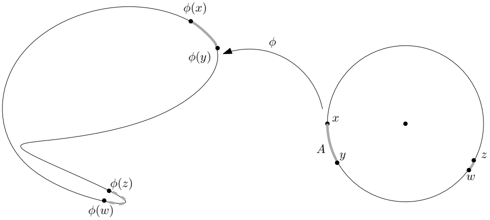

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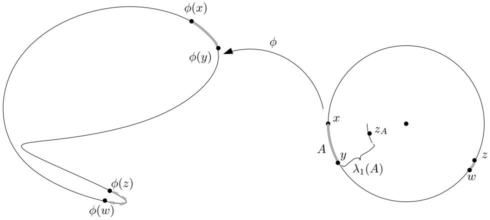

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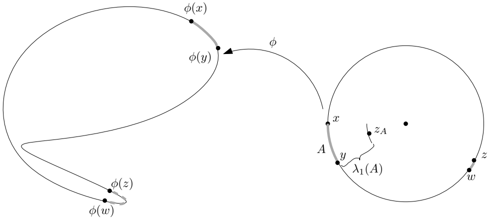

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By Koebe's distortion theorem,

$$|\phi'(z_A)| \asymp \frac{dist(\phi(z_A),\partial\Omega)}{1-|z_A|^2} \asymp \frac{\lambda_1(\phi(A))}{\omega(B(z,\delta))} \asymp \delta^{1-\alpha} = \left(\frac{1}{1-r}\right)^{1-\frac{1}{\alpha}}.$$

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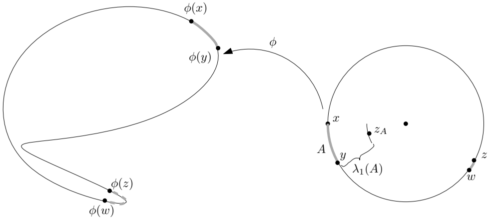

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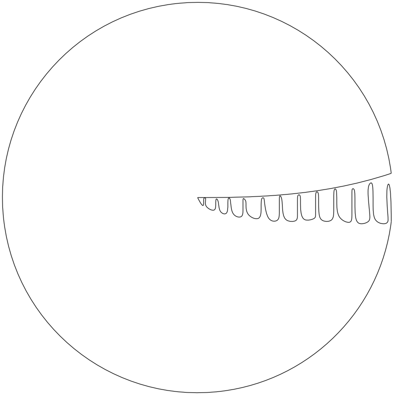

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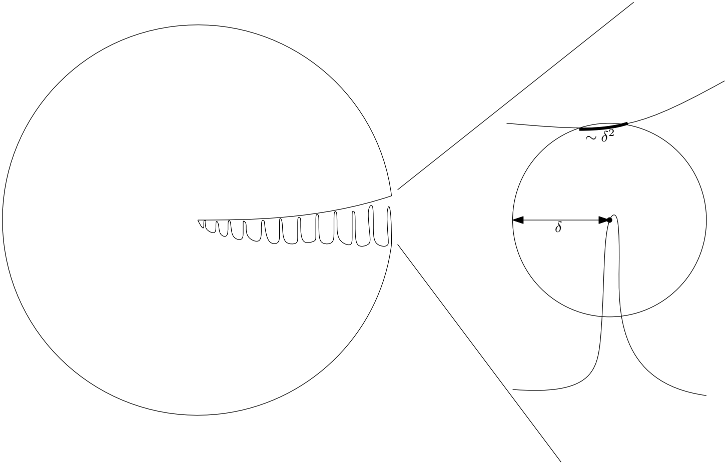

The Feisty Pac-Man: One curve is too short and carries most of the harmonic measure. The other curve is long, but carries very little harmonic measure.

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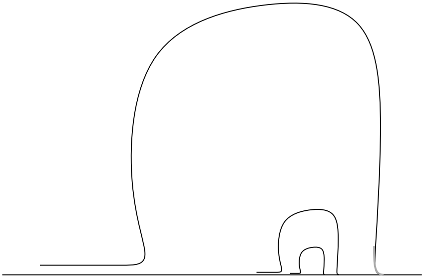

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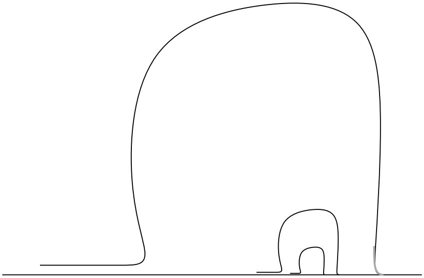

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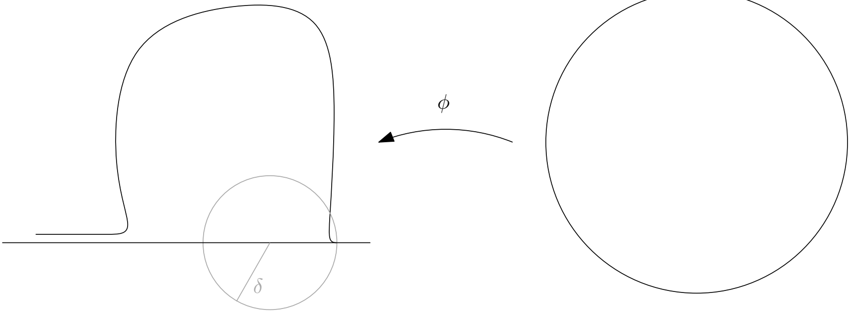

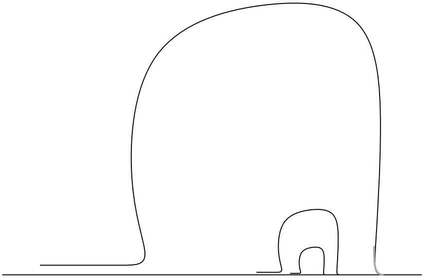

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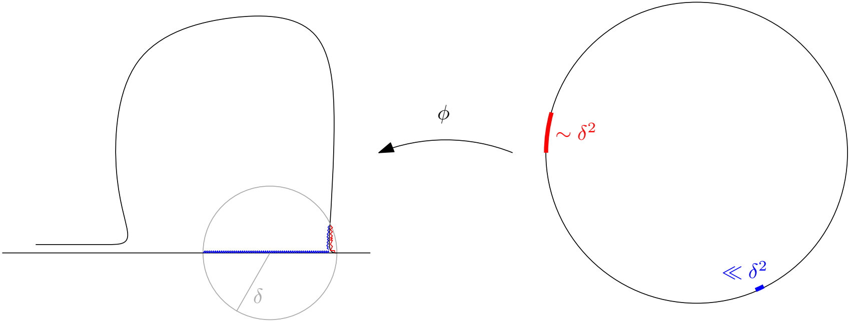

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Theorem (The Universal Counterparts: Carleson-Jones 1992, Makarov 1998, Binder-G., 2023?)

$$F(\alpha) := \sup_{\substack{\Omega \\ s.c}} f_\Omega(\alpha) = F^+(\alpha) = \sup_{F\ IFS} f^+_{\Omega_F}(\alpha),\ \textit{for all } \alpha > 0.$$

$$D(a) := \sup_{\substack{\Omega \\ s.c}} d_\Omega(a) = \sup_{F\ IFS} d_{\Omega_F}(a),\ \textit{for all } a > 0.$$

In particular,

$$D\left(1 - \frac{1}{\alpha}\right) = \frac{1}{\alpha} F(\alpha).$$

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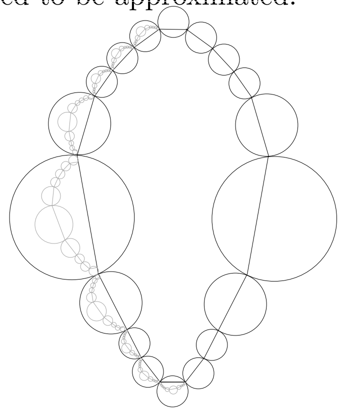

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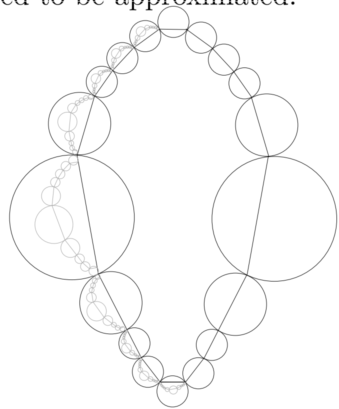

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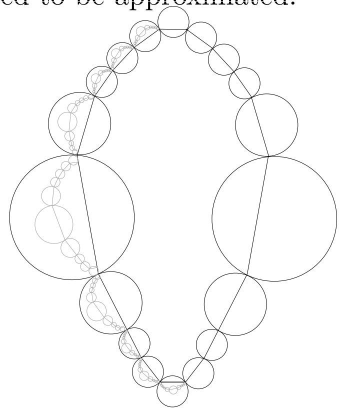

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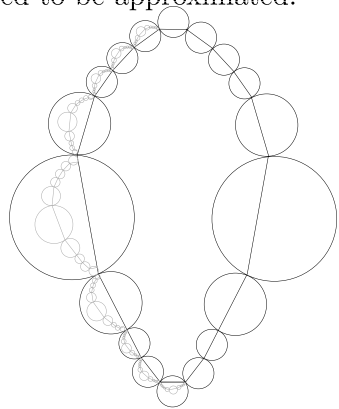

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5) Use multiplicativity of harmonic measure of Repellers (refined Carleson's estimate) to get a lower bound on the spectrum.


Thank you!!!