L^2-FLATTENING OF SELF-SIMILAR MEASURES ON NON-DEGENERATE CURVES

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Abstract. Let μ be a non-atomic self-similar measure on $\mathbb{R}$, and let ν be its pushforward to a non-degenerate curve in $\mathbb{R}^d, d \geq 1$. We show that for every $\varepsilon > 0$ there is $p > 1$, so that $\|\hat{\nu}\|^p_{L^p(B(R))} = O_\varepsilon(R^\varepsilon)$ for all $R > 1$, where $B(R)$ is the R-ball about the origin. As a corollary, we show that convolution with ν quantitatively improves L^2-dimension.

1. Introduction

1.1. **Statement of Main results.** Let $U \subseteq \mathbb{R}$ be an open interval. We call a C^{d+1} curve $Q : U \to \mathbb{R}^d$ *non-degenerate* if

$$\det[Q^{(1)}(x)Q^{(2)}(x)\cdots Q^{(d)}(x)] \neq 0 \text{ for all } x \in U. \tag{1.1}$$

When Q is real analytic, we call it *non-trapped* if the determinant above has at most finitely many zeros in any compact interval; this is equivalent to the trace of Q not being contained in a proper affine hyperplane of $\mathbb{R}^d$ (Lemma 2.12). Such curves have become standard objects in harmonic analysis, see e.g. [FO14, GGW24] for some discussion and applications in modern projection theory.

The purpose of this paper is to study the L^p-norm of the Fourier transform of self-similar measures on $\mathbb{R}$, when they are pushed-forward by either a non-trapped or a non-degenerate curve.

Theorem 1.1. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a non-atomic self-similar measure, and let $d \geq 2$. Let U be an open interval containing* $\operatorname{supp}(\mu)$*, and let $Q : U \to \mathbb{R}^d$ be either a non-trapped analytic curve, or a C^{d+1} non-degenerate curve. Let $\nu = Q\mu$ be the pushforward of μ via Q. Then,*

$$\forall \varepsilon > 0 \quad \exists p > 1 \quad \forall R > 0 : \|\hat{\nu}\|^p_{L^p(B(R))} = O_\varepsilon(R^\varepsilon), \tag{1.2}$$

where $B(R)$ is the R-ball around 0 in $\mathbb{R}^d$.

Recall that a self-similar measure μ on the line is a Borel probability measure satisfying the stationarity condition, for some strictly positive probability vector $\mathbf{p}$,

$$\mu = \sum_{i=1}^n \mathbf{p}_i \cdot f_i\mu, \text{ where all } f_i \in \operatorname{Aff}(\mathbb{R}) \text{ are s.t. } |f_i'| \in (0,1), \text{ and } f_i\mu \text{ is the push forward of } \mu \text{ by } f_i.$$

See Section 2.1 for more discussion about them. Also, the notation O_ε means the implicit constant may depend on ε; it may also depend on μ, d, Q and the maps f_i. Next, we note that the non-degeneracy condition (1.1), and the non-trapped condition, cannot be relaxed from the Theorem. Indeed, (1.2) cannot hold for any measure ν living on a proper affine subspace, since then $|\hat{\nu}(\xi)| \sim 1$ for all frequencies ξ that are nearly orthogonal to the subspace supporting ν. In particular, such frequencies contribute non-trivially to the L^p-norm of $\hat{\nu}$.

Finally, let us explain the relation between the title of the paper and our main result. For $m \in \mathbb{N}$, let $\mathcal{D}_m$ be the dyadic partition of $\mathbb{R}^d$ given by translates of $2^{-m}[0,1)^d$ by $2^{-m}\mathbb{Z}^d$. Given $q \geq 1$, we define the moment sum corresponding to m, q, and a Borel probability measure ν, by

$$s_m(\nu, q) \stackrel{\text{def}}{=} \sum_{Q \in \mathcal{D}_m} \nu(Q)^q.$$

For $q = \infty$, we set

$$s_m(\nu, \infty) \stackrel{\text{def}}{=} \max\{\nu(Q) : Q \in \mathcal{D}_m\}.$$

A standard computation (Lemma 2.13) shows that if ν is compactly supported, then for every $R > 1$, letting $m = [\log_2 R] \in \mathbb{N}$ be the integer part of $\log_2 R$, we have

$$s_m(\nu, 2) \asymp 2^{-dm} \|\hat{\nu}\|^2_{L^2(B(R))}.$$

Thus, the conclusion of Theorem 1.1 is equivalent to the following flattening phenomenon of moment sums: for every $\varepsilon > 0$ there exists $p \in \mathbb{N}$ such that for all m large enough,

$$s_m(\nu^{*p}, 2) = O_\varepsilon\left(2^{m(\varepsilon - d)}\right). \tag{1.3}$$

Here $\mu * \nu$ means the usual Euclidean convolution of measures (pushforward of $\mu \times \nu$ by $(x, y) \mapsto x + y$), and ν^{*p} means the p-fold self-convolution of ν.

Theorem 1.1 has the following consequences. Recall that for a Borel probability ν on $\mathbb{R}^d$ and $1 < q < \infty$, one defines its L^q-dimension via

$$\dim_q(\nu) = \frac{\tau_q(\nu)}{q-1}, \qquad \text{where } \tau_q(\nu) = \liminf_{m \to \infty} \frac{-\log s_m(\nu, q)}{m}.$$

For $q = \infty$, we set $\dim_\infty(\nu) = \tau_\infty(\nu)$. The L^∞-dimension is also known as the *Frostman exponent* of ν. With this notation, we have the following corollary of Theorem 1.1.

Corollary 1.2. *Let ν be as in Theorem 1.1. Then,*

(1) For all $q \in [2, \infty]$ we have $\lim_{n \to \infty} \dim_q(\nu^{*n}) = d$.

(2) For every $\gamma > 0$, there is $\eta = \eta(\gamma) > 0$ and $m_0 = m_0(\gamma, \eta) > 1$ such that the following holds for all integers $m \geq m_0$: For every Borel probability measure θ,

$$s_m(\theta, 2) > 2^{m(\gamma - d)} \quad \Longrightarrow \quad s_m(\theta * \nu) \leq 2^{-\eta m} s_m(\theta, 2). \tag{1.4}$$

Indeed, both statements hold true for any measure ν satisfying (1.2).

Proof. Part (1) follows directly for $q = 2$ from Theorem 1.1, in its equivalent form (1.3). Hence, the case $q = \infty$ then follows by Young's inequality; cf. [MS18, Lemma 5.2], which then yields the claim for all other values of q. Part (2) follows from Theorem 1.1 similarly to the proof of [MS18, Theorem 4.1]. □

We proceed now to discuss some prior results in harmonic analysis, fractal geometry, and dynamical systems, and how Theorem 1.1 and its proof compare to them.

1.2. **Prior results.** We begin by comparing Theorem 1.1 to recent results in harmonic analysis. Recall that a probability measure ν on $\mathbb{R}^d$ is called an s-Frostman measure, where $s \in [0, d]$, if $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$. In [Orp23], Orponen proved that if ν is an s-Frostman measure on the truncated parabola $\mathbb{P} := \{(x, x^2) : [-1,1]\}$, then $\|\hat{\nu}\|^4_{L^4(B(R))} \ll R^{2-2s}$ for all $R \geq 1$. For $s \in (0,1)$ and $p > 4$ he was able to obtain an $\varepsilon = \varepsilon(p,s)$ improvement $\|\hat{\nu}\|^p_{L^p(B(R))} \ll R^{2-2s-\varepsilon}$. Orponen also related this problem to the sum-product phenomenon and the Borel subring problem [Orp23, Section 1.2]. He conjectured that for every $s \in [0,1]$ and $\varepsilon > 0$ there exists some $p = p(\varepsilon, s) \geq 1$ such that for every s-Frostman measure on $\mathbb{P}$, $\|\hat{\nu}\|^p_{L^p(B(R))} \ll R^{2-\min\{3s,1+s\}+p\varepsilon}$, see [Orp23, Conjecture 1.6]. Furthermore, exploiting the relation with iterated sum-sets, it was shown in [Orp23, Example 1.8] that the threshold $\min\{3s, 1+s\}$ cannot be further improved in this generality. Dasu and Demeter [DD24] later extended these results for s-Frostman measures ν that are supported on the graph of a $C^3([-1,1],\mathbb{R})$ function γ such that $\min_{x\in[-1,1]} |\gamma''(x)| > 0$; they established an Orponen-like bound of the form $\|\hat{\nu}\|^p_{L^6(B(R))} \ll R^{2-2s-\beta}$ where $\beta = \beta(s)$, $s \in (0,1)$. The conjectured bound from [Orp23, Conjecture 1.6] was then established by Orponen, Puliatti, and Pyörälä [OPP24], along with further applications and analogies with the dimensions of iterated sum-sets on $\mathbb{P}$; some problems in this direction, however, still remain open [OPP24, Questions 1 and 2]. Finally, for measures ν supported on graphs of functions as in the work of Dasu and Demeter [DD24], it was very recently shown by Demeter and Wang [DW25] that if ν is s-Frostman where $s \in (0, \frac{1}{2})$ then $\|\hat{\nu}\|^6_{L^6(B(R))} \ll R^{2-2s-\frac{s}{4}+\varepsilon}$ for all $\varepsilon > 0$. See also [Yi24] for further research in this direction.

To compare Theorem 1.1 to these results, we first note that all non-atomic self-similar measures are s-Frostman; this was first proved by Feng and Lau [FL09] (Proposition 2.6). The precise computation of the best possible s given the generating IFS is a subtle problem, that is still open in general; see Shmerkin's work [Shm19] for some recent progress on it. Nonetheless, Theorem 1.1 dramatically improves upon the results in [Orp23, DD24, OPP24, DW25] as it shows optimal flattening regardless of the precise value of s, for arbitrary non-trapped or non-degenerate curves in every dimension. In particular, we show that for self-similar measures the general threshold [Orp23, Example 1.8] can be substantially improved. However, unlike [Orp23, OPP24] where the key step involves a reduction to a problem about Furstenberg sets, or [DD24, DW25] that use various decoupling arguments, the stationary structure of self-similar measures allows for a wider array of tools. It is thus interesting to ask how much regularity the measure should enjoy so that it can break the general threshold given in [Orp23, Example 1.8]. In the same spirit, one may ask whether Theorem 1.1 holds true for non-atomic Ahlfors-David regular measures (note, though, that not all self-similar measures have this property).

Let us now place Theorem 1.1 within recent literature on fractal geometry and dynamical systems. It can be considered a variant of the Fourier decay problem, which asks about optimal *pointwise* estimates for the decay rate of $\hat{\nu}(\xi)$ when ν is a stationary measure. For self-similar measures, it is of central importance to understand when have polynomial decay ($\hat{\nu}(\xi) = O(\|\xi\|^{-\alpha})$ for some $\alpha > 0$) due to e.g. the relation between this property and the absolute continuity of ν [Shm14]. Solomyak [Sol21], extending the classical Erdős-Kahane argument, had shown that polynomial decay is generic among self-similar measures, and a full characterization of the Rajchman property for them had been given by Li-Sahlsten [Li22], Brémont [Bré21] (see also Varjú-Yu [VY22]), and Rapaport [Rap22]. However, the only known *explicit* examples of self-similar measures with polynomial decay are due to Dai, Feng, and Wang [DFW07], a work that was recently extended by Streck [Str23]. We remark that there are many explicit examples of measures with logarithmic decay ($\hat{\nu}(\xi) = O((\log|\xi|)^{-\alpha})$ for some $\alpha > 0$) under various Diophantine conditions, see e.g. [Li22, VY22, AHW22, BKS24].

Another closely related direction concerns pointwise decay rates for non-linear push-forwards of self-similar measures. It was first observed by Kaufman [Kau84] that, among other things, if g is any C^2 diffeomorphism on $[0,1]$ such that $g'' > 0$ then the pushforward of the Cantor-Lebesgue measure on the middle thirds Cantor set does have polynomial Fourier decay (even though the original measure is not even Rajchman). This was extended to all uniformly contracting self-similar measures by Mosquera and Shmerkin [MS18], and then to all non-atomic self-similar measures by Algom, Chang, Meng Wu, and Yu-Liang Wu [ACWW25], and simultaneously and independently by Baker and Banaji [BB25]. We note that, combined with the earlier works of Algom, Rodriguez Hertz, and Wang [ARW23] and Baker and Sahlsten [BS23], this shows that all self-conformal measures with respect to non-affine real analytic IFSs have polynomial Fourier decay. These results were extended to higher dimensions in various directions by Algom, Rodriguez Hertz, and Wang [ARW24], Baker, Khalil, and Sahlsten [BKS24], and Banaji and Yu [BY25].

The proof of Theorem 1.1 is related to the approach of [ACWW25, BKS24], where the main tool exploited is the following large deviations estimate for the Fourier transform. Let μ be a non-atomic self-similar measure on $\mathbb{R}$. Then,

$$\forall\varepsilon > 0\,\exists\delta > 0 \text{ s.t. } \forall T \gg 1, \text{ we can cover } \{|\xi| \leq T : |\hat{\mu}(\xi)| \geq T^{-\delta}\} \text{ by } O_\varepsilon(T^\varepsilon) \text{ intervals of size } 1. \tag{1.5}$$

This was first observed by Kaufman [Kau84] for some Bernoulli convolutions. Tsujii [Tsu15] proved (1.5) for all non-atomic self-similar measures. Mosquera and Shmerkin [MS18] proved effective (quantitative) versions of (1.5); these in turn allow one to give explicit lower bounds on the Fourier dimension of non-linear images of self-similar measures; see also [BY25] for related results. Note that large deviation estimates of the form (1.5) are equivalent to the L^p-norm bounds in (1.2).

A general criterion implying that (1.5) holds for arbitrary Borel probability measures (not necessarily self-similar) was established in [Kha23]. Namely, it is shown that ν satisfies (1.2) if for all $\varepsilon > 0$, there is $\delta > 0$ such that for all proper affine subspaces W, we have

$$\nu(W(\delta r) \cap B(x,r)) \leq \varepsilon\, \nu\left(B(x, O(r))\right), \tag{1.6}$$

for all $x \in \operatorname{supp}(\nu)$ outside an exponentially small exceptional set, and all but a small proportion of scales $r > 0$; cf. [Kha23, Cor. 1.7 and 6.4] for precise statements. Here, $W(\delta r)$ and $B(x,r)$ denote the δr-neighborhood of W and the r-ball around x respectively.

The proof this result involves producing conditions on measures that are *not* L^2-improving in the sense of (1.4), generalizing Shmerkin's 1-dimensional inverse theorem [Shm19] to higher dimensions; cf. [Kha23, Prop. 11.10]. In particular, it is shown that large subsets in the supports of such measures must locally concentrate near proper subspaces at many scales. A similar result was recently obtained by Shmerkin in [Shm25], where the local structure of the non-improved measure θ in (1.4) was also described. In both instances, the proof relies on Hochman's inverse theorem for entropy [Hoc15] and the asymmetric Balog-Szemerédi-Gowers Lemma [TV06]. To obtain (1.2), this multi-scale concentration is ruled out using (1.6) through induction on scales [Kha23, Cor. 6.4].

A natural question is to find weaker non-concentration conditions than (1.6) under which (1.5) (and hence (1.2)) can be shown to hold. This question served as one of the major motivations of this work. In this vein, Theorem 1.1 provides a natural class of examples where the local non-concentration estimate (1.6) fails on large sets at every scale (due to local concentration of curves along their tangent lines), while the flattening estimate (1.2) holds. In particular, such local concentration is a serious obstacle to carrying out the approach of [Kha23], and indeed, our proof of Theorem 1.1 uses different techniques.

1.3. **On the proof of Theorem 1.1.** The proof of Theorem 1.1 consists of two main parts, corresponding to two ranges of frequencies which we now introduce. For $R \geq 1$ and $\varepsilon > 0$, let

$$C_{R,\varepsilon} \stackrel{\text{def}}{=} \left\{(\theta,\zeta) \in \mathbb{R}\times\mathbb{R}^{d-1} : |\theta|^{\varepsilon} \leq \|\zeta\| \leq R\right\},$$
$$E_{R,\varepsilon} \stackrel{\text{def}}{=} \left\{(\theta,\zeta) \in \mathbb{R}\times\mathbb{R}^{d-1} : \|\zeta\| \leq |\theta|^{\varepsilon} \leq R^{\varepsilon}\right\}.$$

In particular, we have the following decomposition of $B(R)$:

$$B(R) = C_{R,\varepsilon} \bigcup E_{R,\varepsilon}. \tag{1.7}$$

To describe the first ingredient, let $d \geq 1$ and define the moment curve

$$V_d(x) \stackrel{\text{def}}{=} (x, x^2, \dots, x^d) \text{ if } d \geq 2, \qquad \text{and} \qquad V_1(x) = x \text{ otherwise.}$$

Theorem 1.3. *Let μ be a non-atomic self-similar measure on $\mathbb{R}$. Let $d \geq 2$ and assume that*

$$\forall \varepsilon > 0 \quad \exists p > 1 \quad \forall R > 0 : \left\|\widehat{V_{d-1}\mu}\right\|^p_{L^p(B(R))} = O_\varepsilon(R^\varepsilon). \tag{1.8}$$

Let $g : U \to \mathbb{R}^{d-1}$ be a map defined on an open neighborhood U of $\operatorname{supp}(\mu)$, such that $Q(x) = (x, g(x))$ is either a non-trapped analytic curve, or a C^{d+1} non-degenerate curve. Let $\nu = Q\mu$ be the pushforward of μ to the graph of g. Then, for every $\varepsilon > 0$, there is $p_0 = p_0(\mu, g, \varepsilon) > 1$ such that for all $p \geq p_0$, we have

$$\int_{C_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi = O_\varepsilon(R^\varepsilon).$$

The following is the second main ingredient in our proof which handles the region $E_{R,\varepsilon}$.

Proposition 1.4. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a non-atomic self-similar measure and $d \geq 2$. Let $g : U \to \mathbb{R}^{d-1}$ be a C^1-map defined on an open neighborhood U of $\operatorname{supp}(\mu)$. For $Q(x) = (x, g(x))$, let $\nu = Q\mu$ be the pushforward of μ to the graph of g. Then, for every $\varepsilon > 0$, there exists $p = p(\varepsilon, \mu) > 1$ such that for all $R \geq 1$, we have*

$$\int_{E_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi = O_{\varepsilon,\mu}(R^\varepsilon).$$

First, let us show how Theorem 1.1 follows quickly from Theorem 1.3 and Proposition 1.4.

Proof of Theorem 1.1 assuming Theorem 1.3 and Prop. 1.4. We proceed by induction on the ambient dimension d. By Tsujii's Theorem (Corollary 2.9) for $d = 2$, and by induction for $d > 2$, we may assume that Hypothesis (1.8) holds for $\nu = V_{d-1}\mu$, where V_{d-1} is the moment curve in $\mathbb{R}^{d-1}$. By replacing μ with an affine image of smaller diameter, and replacing U with a smaller neighborhood if necessary, we may assume, after an affine change of coordinates, that $Q(x)$ takes the form $(x, g(x))$, for a map $g : U \to \mathbb{R}^{d-1}$ as in Theorem 1.3. Hence, in view of (1.7), Theorem 1.3 and Proposition 1.4 together imply Theorem 1.1. $\square$

Next, we describe the ideas behind the above two intermediate results, beginning with Proposition 1.4. Here, the main observation is that, via the coordinate relation in $E_{R,\varepsilon}$ and the Lipchitz continuity of $\hat{\nu}$, we can fully reduce it to an estimate about the Fourier transform of μ itself (Lemma 4.3). This estimate is then obtained in Lemma 4.4, where the end-game step follows from Tsujii's large deviations estimate (1.5) (Corollary 2.9). The details are given in Section 4.

The key to the proof of Theorem 1.3 is the following proposition providing *polynomial pointwise decay* in a large region of of frequencies, from which Theorem 1.3 follows immediately; cf. Corollary 3.7 for this deduction.

Proposition 1.5. *Let μ be a non-atomic self-similar measure on $\mathbb{R}$. Let $d \geq 2$ and assume that the flattening estimate (1.8) holds for $V_{d-1}\mu$. Let $g : U \to \mathbb{R}^{d-1}$ be as in Theorem 1.3. For $Q(x) = (x, g(x))$, let $\nu = Q\mu$ be the pushforward of μ to the graph of g. Then, there exists $\gamma = \gamma(\mu, g) > 0$ such that for every $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ with $\zeta \neq \mathbf{0}$, we have $|\hat{\nu}(\xi)| \ll \|\zeta\|^{-\gamma}$.*

Proposition 1.5 is closely related to the aforementioned pointwise Fourier decay results for non-linear pushforwards of self-similar measures obtained in [ACWW25, BB25], as well as to forthcoming work of Banaji and Yu for pushforwards of self-similar measures on $\mathbb{R}^k$ to graphs of analytic maps $g : \mathbb{R}^k \to \mathbb{R}^{k+d}$. We give a somewhat different proof of this result here, which we believe will be useful for future generalizations of Theorem 1.1.

It remains to explain the proof of Proposition 1.5. First, via Taylor expansion and a change of coordinates, we will deduce Proposition 1.4 from the special case $Q(x) = V_d(x)$, where $V_d(x)$ is the moment curve. This deduction is carried out in Section 3.2.

The special case of the moment curve is carried out in Proposition 3.1. The key observation in the proof of the latter, which is based upon a construction of Feng and Käenmäki [FK18, Lemma 3.1], is that the pushed self-similar measure $\nu = V_d\mu$ is, in fact, self-affine: that is, there exist $A_i \in \mathrm{GL}(\mathbb{R}^d)$ with $\|A_i\| < 1$, and $b_i \in \mathbb{R}^d$, such that for $F_i(\mathbf{x}) = A_i\mathbf{x} + b_i$, we have

$$\nu = \sum_{i=1}^{n} \mathbf{p}_i \cdot F_i\mu.$$

See Lemma 2.10 below for the precise statement. Note that the probability vector $\mathbf{p}$ is the same one that is used to define μ.

This observation is then used to express the Fourier coefficient $\hat{\nu}(\xi)$ as an *average* over many Fourier coefficients of the original measure μ. Roughly speaking, since the derivative of V_d is essentially given by a copy of V_{d-1}, the set of frequencies we average over arise from a *projection* of $V_{d-1}\mu$ in a suitable direction determined by the original frequency ξ.

We then take advantage of our inductive flattening hypothesis (1.8) to ensure that *all* projections admit a uniform lower bound on their *upper Frostman exponent*. Indeed, the latter property in fact holds for any measure satisfying (1.8); cf. Theorem 2.4. It is likely that such Frostman bounds could be verified directly without appealing to (1.8); cf. [KLW04, Theorems 2.1(b) and 2.3] for special cases of this statement.

The Frostman bound on the projections of $V_{d-1}\mu$ in turn ensures that our average is sampled along a well-separated set of frequencies, thus enabling us to apply Tsujii's large deviations estimate (1.5) to conclude the proof.

1.4. **Notation.** Throughout the article, given two quantities A and B, we use the Vinogradov notation $A \ll B$ to mean that there exists a constant $C \geq 1$, possibly depending on the self-similar measure μ, ambient dimension d, and the analytic map g, such that $|A| \leq CB$. In particular, we suppress these dependencies except when we wish to emphasize them. We write $A \ll_{x,y} B$ to indicate that the implicit constant depends on parameters x and y. We also write $A = O_x(B)$ to mean $A \ll_x B$. Finally, we write $A \asymp_x B$ to mean $A \ll_x B$ and $B \ll_x A$. For $x \in \mathbb{R}$,

$$e(x) \stackrel{\text{def}}{=} e^{2\pi i x}.$$

Finally, for $v \in \mathbb{R}^n$, $\|v\|$ denotes the max norm.

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2. Preliminaries

2.1. **Self-similar measures.** Let

$$\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i \in I}, \ |I| = n > 1,$$

be a finite set of invertible strictly contracting similitudes preserving a compact interval $J \subseteq \mathbb{R}$ (often we will work with $J = [0,1]$). In particular, $0 < |\lambda_i| < 1$ for all $i \in I$. We call Φ a *self-similar IFS* (Iterated Function System). It is well known that there exists a unique compact non-empty set $K = K_\Phi \subseteq J$ such that

$$K = \bigcup_{i \in I} f_i(K).$$

The set K is called the *attractor* of Φ, and the *self-similar set* generated by it. We always make the assumption that there exist $i, j \in I$ such that the fixed point of f_i differs from that of f_j; this is known to imply that K is infinite (in fact, has positive Hausdorff dimension).

Next, let $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ be a (strictly positive) probability vector: For all i we have $\mathbf{p}_i > 0$, and $\sum_i \mathbf{p}_i = 1$. Then it is well known that there exists a unique Borel probability measure μ such that

$$\mu = \sum_i \mathbf{p}_i \cdot f_i \mu, \text{ where } f_i\mu \text{ is the push forward of } \mu \text{ by } f_i.$$

The measure μ is called a *self-similar measure*, and is supported on K. Occasionally we will use the notation *weighted IFS* and write the pair $(\Phi, \mathbf{p})$ to indicate a self-similar IFS paired with a probability vector. Under our assumptions (K is infinite and $\mathbf{p}$ is strictly positive) the measure μ is known to be non-atomic. Thus, all self-similar measures considered in this paper are non-atomic.

We next define cut-sets related with Φ and μ. For more details about this, see e.g. [BP17, Chapter 2]. For $\omega = \omega_1 \dots \omega_n \in I^n, n \in \mathbb{N}$, we define $f_\omega := f_{\omega_1} \circ \dots f_{\omega_n}$.

Definition 2.1. Let $\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i\in I}$ be a self-similar IFS on $\mathbb{R}$, and let $\tau \in (0,1)$. The cut-set corresponding to τ is defined by

$$P_\tau := \left\{\omega = (\omega_1, \ldots, \omega_k) \in I^* : |\lambda_{\omega_1} \cdot \ldots \lambda_{\omega_k}| < \tau \text{ yet } \left|\lambda_{\omega_1} \cdot \ldots \lambda_{\omega_{k-1}}\right| \geq \tau\right\}.$$

Given a probability vector $\mathbf{p}$, and a self-similar measure μ associated to $(\Phi, \mathbf{p})$, we set

$$\mu_\tau \stackrel{\text{def}}{=} \sum_{\omega \in P_\tau} \mathbf{p}_\omega \delta_{t_\omega},$$

where

$$t_\omega = f_\omega(0) \text{ for } \omega \in P_\tau.$$

We record a number of standard facts regarding cut-sets. See e.g. [ACWW25, Lemma 2.2] for a closely related discussion.

Lemma 2.2. *Let μ be a non-atomic self-similar measure on $\mathbb{R}$ with respect to the weighted IFS $(\Phi, \mathbf{p})$. If $\tau \in (0,1)$ is sufficiently small then:*

(1) We have

$$\mu = \sum_{\omega \in P_\tau} \mathbf{p}_\omega \cdot f_\omega \mu, \ \ \textit{where } \mathbf{p}_\omega = \prod_{i=1}^{|\omega|} \mathbf{p}_\mathbf{i}, \ \ \textit{and } f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_{|\omega|}}.$$

In particular,

$$\widehat{\mu} = \sum_{\omega \in P_\tau} \mathbf{p}_\omega \cdot \widehat{f_\omega \mu}.$$

(2) Let $\Lambda_\tau \subseteq (-1,1)$ denote the set

$$\Lambda_\tau := \{r_\omega : \omega \in P_\tau\}, \ \ \textit{where } r_\omega = r_{\omega_1} \cdot \ldots r_{\omega_{|\omega|}}.$$

Then, $\#\Lambda_\tau \ll (-\log \tau)^{n+1}$, where $n = |\Phi|$.

Proof. The proof of Part (1) follows from a standard stopping time type argument, see e.g. [BP17, Definition 2.2.3 and Lemma 2.2.4].

As for the numerical estimate in Part (2), let us first estimate, for $m \in \mathbb{N}$

$$|\{r_\eta : \eta \in \mathcal{I}^m\}|.$$

That is, we count how many different contraction ratios the maps in the IFS

$$\Phi^m := \{f_\eta : \eta \in \mathcal{I}^m\}$$

can admit. Note that for any map $f_\eta(x) = r_\eta \cdot x + t_\eta \in \Phi^m$, r_η only depends on the amount of times each r_i appears in η, for $i \in \mathcal{A}$. So, writing

$$n_i = n_i(\eta) = |\{1 \leq j \leq m : \eta_j = i\}|,$$

we have

$$\log |r_\eta| = \sum_{i=1}^n n_i \log |r_i|, \qquad \text{and } \sum_{i=1}^n n_i = m.$$

By a standard combinatorial argument, there are at most

$$\binom{n+m-1}{m}$$

different possible values for this sum. It follows that

$$|\{r_\eta : \eta \in \mathcal{A}^m\}| \leq O(m^n),$$

where the implicit constant is bounded independently of m.

Finally, since $\min_{i\in\mathcal{A}} |r_i| > 0$, there exists some $d \in \mathbb{N}$ such that: For all $\tau > 0$ and all $\omega \in \mathcal{A}^{\mathbb{N}}$,

$$\max\{|\omega| : \omega \in P_\tau\} \leq -d \cdot \log \tau.$$

Thus, by definition

$$\mathcal{P}_\tau \subseteq \bigcup_{i=1}^{[-\log\tau]+1} \left\{r_\eta : \eta \in \mathcal{I}^{d\cdot i}\right\},$$

where $[-\log\tau]$ is the integer part of $-\log\tau$. So, via our previous estimates,

$$|\{r_\eta : \eta \in \mathcal{P}_\tau\}| \leq \sum_{i=1}^{[-\log\tau]+1} O((d\cdot i)^n) \leq ([-\log\tau]+1)\cdot O((d\cdot([-\log\tau]+1))^n) = O\left((-\log\tau)^{n+1}\right).$$

This is the required bound. ☐

2.2. **Frostman exponents of projections and discretizations.** Let μ be a non-atomic self-similar measure on $\mathbb{R}$ for an IFS Φ and a strictly positive probability vector $\mathbf{p}$. We require the following uniform estimate on the Frostman exponent of *projections* of pushforwards of the discrete measures μ_τ to non-degenerate curves. For an affine subspace $W \subset \mathbb{R}^d$ and $\varepsilon > 0$, we denote by $W(\varepsilon)$ be the open ε-neighborhood of W.

Proposition 2.3. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a non-atomic self-similar measure. Let $d \geq 1$ and let $g : U \to \mathbb{R}^d$ be a Lipschitz continuous map defined on an open neighborhood U of* $\operatorname{supp}(\mu)$. *Let $Q(x) = (x, g(x))$ and $\tau \in (0,1)$ and consider the measures ν, $\nu_\tau \in \mathcal{P}(\mathbb{R}^{d+1})$ defined by*

$$\nu := Q\mu, \quad \text{and} \quad \nu_\tau := Q\mu_\tau.$$

Suppose that ν satisfies (1.2). Then, there are $\beta, \varrho > 0$ and $C \geq 1$ such that for every proper affine subspace $W \subset \mathbb{R}^d$ we have

$$\nu_\tau(W(\varepsilon)) \leq C\varepsilon^\beta, \quad \text{for all } \varepsilon > \tau^\varrho.$$

Proposition 2.3 is a rather direct consequence of the following Theorem:

Theorem 2.4 ([Kha23, Theorem 6.23])**.** *Let ν be a compactly supported probability measure on $\mathbb{R}^d$ satisfying (1.2). Then, there are $\beta > 0$ and $C \geq 1$ such that $\nu(W(\varepsilon)) \leq C\varepsilon^\beta$ for all proper affine subspaces $W \subset \mathbb{R}^d$ and $\varepsilon > 0$.*

Remark 2.5. The statement of Theorem 2.4 is different from the reference [Kha23, Theorem 6.23], however the proof of the latter is written for measures satisfying (1.2).

Proof of Proposition 2.3. First, since the IFS is uniformly contracting, for some $\gamma > 0$ we have

$$|\mu_\tau(\varphi) - \mu(\varphi)| \ll \tau^\gamma \|\varphi\|_{\mathrm{Lip}},$$

for any Lipschitz function φ, where $\|\varphi\|_{\mathrm{Lip}}$ is the Lipschitz constant of φ. This follows for instance by straightforward adaptation of the argument of [Hut81, Theorem 4.4.1(ii)] to averages over general cut-sets. Hence, the same bound holds for ν and ν_τ in place of μ and μ_τ. To conclude the proof, let $B \subset \mathbb{R}^d$ be a large ball containing the supports of ν and ν_τ for all $\tau \in (0,1)$. Let φ be a Lipschitz function that is identically 1 on $W(\varepsilon) \cap B$ and vanishing outside $W(2\varepsilon)$. In particular, $\|\varphi\|_{\mathrm{Lip}} \ll \varepsilon^{-1}$. Combined with Theorem 2.4, we obtain

$$\nu_\tau(W(\varepsilon)) \leq \nu(\varphi) + O(\varepsilon^{-1}\tau^\gamma) \leq \nu(W(2\varepsilon)) + O(\varepsilon^{-1}\tau^\gamma) = O(\varepsilon^\beta + \varepsilon^{-1}\tau^\gamma),$$

for some $\alpha > 0$. The above bound is thus $O(\varepsilon^\beta)$ whenever $\varepsilon > \tau^{\gamma/(1+\beta)}$. □

We also recall the following fact, due to Feng and Lau [FL09], that non-atomic self-similar measures are always upper Frostman (Hölder) regular. See [GKM22] for a more recent and more general version.

Proposition 2.6 ([FL09, Proposition 2.2]). *Let μ be a non-atomic self-similar measure on $\mathbb{R}$. Then, there exists some $s_0 = s_0(\mu) > 0$ such that*

$$\sup_{x\in\mathbb{R}} \mu\left(B(x,r)\right) \ll_\mu r^{s_0}, \qquad \textit{for all } r > 0.$$

In particular, there is $\varrho > 0$ such that for every $\tau \in (0,1)$, we have that

$$\sup_{x\in\mathbb{R}} \mu_\tau\left(B(x,r)\right) \ll_\mu r^{s_0}, \qquad \textit{for all } r > \tau^\varrho,$$

where μ_τ is the discretization of μ defined in Def. 2.1.

Remark 2.7. The reference [FL09, Proposition 2.2] proves the first assertion of Prop. 2.6. The second assertion concerning the discrete measures μ_τ follows from the first by the same argument in the proof of Proposition 2.3, or from [GKM22].

2.3. **Tsujii's large deviations estimate and its consequences.** In this section, we recall a result of Tsujii that plays a key role in our analysis.

Theorem 2.8 ([Tsu15]). *Let μ be a non-atomic self-similar measure on $\mathbb{R}$. Then for every $\varepsilon > 0$ there exists some $c_0 = c_0(\varepsilon,\mu)$ such that for all $t \gg 1$*

$$\mathit{Leb}\left(\left\{\xi \in (-e^t, e^t) : |\hat{\mu}(\xi)| \geq |\xi|^{-c_0}\right\}\right) \leq e^{t\varepsilon}.$$

Theorem 2.8 immediately implies the following version of Theorem 1.1 for the self-similar measure μ itself.

Corollary 2.9. *Let μ be a non-atomic self-similar measure on $\mathbb{R}$ and let $B(R)$ be the R-ball around 0. Then,*

$$\forall\varepsilon > 0 \quad \exists p \in \mathbb{N} : \|\hat{\mu}\|^p_{L^p(B(R))} = O_\varepsilon(R^\varepsilon).$$

Moreover, for every $\varepsilon > 0$, there is $\delta > 0$ so that the set of $\xi \in B(R)$ with $|\hat{\mu}(\xi)| > R^{-\delta}$ can be covered with $O_{\varepsilon,\mu}(R^\varepsilon)$ intervals of length 1.

Proof. Let $\varepsilon > 0$ and let $p \in \mathbb{N}$ to be chosen later. Applying Theorem 2.8 with $t = \log R$ and letting c_0 be as in the Theorem, we see that

$$\int_{-R}^{R} |\hat{\mu}(\xi)|^p \, d\xi \leq \int_{\{\xi \in B(R) : \, |\hat{\mu}(\xi)| < |\xi|^{-c_0}\}} |\hat{\mu}(\xi)|^p \, d\xi + R^{\varepsilon} \leq \int_{-R}^{R} |\xi|^{-c_0 \cdot p} \, d\xi + R^{\varepsilon} = O(R^{-c_0 \cdot p + 1}) + R^{\varepsilon}.$$

Taking $p \gg 1$ in a manner that depends on c_0, we obtain the first conclusion. The second conclusion follows from Theorem 2.8 using the fact that $\hat{\mu}$ is Lipschitz and hence is slowly varying on small intervals; cf. [ACWW25, Corollary 2.5] for a detailed proof. □

2.4. **Self-similar sets and measures on the moment curve.** A key observation in our analysis is that if we are push a self-similar set (resp. measure) to the moment curve $(x, x^2, ..., x^\ell)$ then the resulting set is, in fact, a self-affine set (resp. measure). The proof is based upon a construction of Feng and Käenmäki [FK18, Lemma 3.1]:

Lemma 2.10. *Suppose K_Φ is a self-similar set generated by the IFS $\Phi = \{f_i(x) = \lambda_i \cdot x + t_i\}_{i=1}^n$ on $[0, 1]$. Let $P_\ell(x) = (x, x^2, ..., x^\ell)$. Then $P_\ell(K)$ is the attractor of the self affine IFS*

$$\Psi := \left\{ F_i(x) = A_i \cdot x - A_i \left(-\frac{t_i}{\lambda_i}, \left(-\frac{t_i}{\lambda_i}\right)^2, \ldots, \left(-\frac{t_i}{\lambda_i}\right)^\ell \right) \right\},$$

where

$$A_i = \begin{pmatrix} \lambda_i c_{i,1,1} & 0 & 0 & \ldots & 0 \\ \lambda_i^2 c_{i,2,1} & \lambda_i^2 c_{i,2,2} & 0 & \ldots & 0 \\ \lambda_i^3 c_{i,3,1} & \lambda_i^3 c_{i,3,2} & \lambda_i^3 c_{i,3,3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_i^\ell c_{i,\ell,1} & \lambda_i^\ell c_{i,\ell,2} & \lambda_i^n c_{i,\ell,3} & \ldots & \lambda_i^\ell c_{i,\ell,\ell} \end{pmatrix},$$

$$\text{and } c_{i,k,j} := \binom{k}{j} \cdot \left(\frac{t_i}{\lambda_i}\right)^{k-j} \text{ for } (i, k, j) \in \{1, ..., n\} \times \{1, ..., \ell\}^2.$$

We remark that a direct computation shows that if $\max_i |\lambda_i| < \frac{1}{2^{2\ell}\sqrt{\ell}}$ then $||T_i|| < 1$ for all i, whence the IFS Ψ will be uniformly contracting. Clearly we can induce the original IFS to achieve this (assuming as we may that ℓ is given). So, in application, we can assume without the loss of generality that Ψ is a uniformly contracting IFS.

Proof. Let $x \in \mathbb{R}$. Then for every $1 \leq k \leq \ell$ and $1 \leq i \leq n$ we have

$$\left(x - \left(-\frac{t_i}{\lambda_i} \right) \right)^k = \sum_{j=1}^{k} c_{i,k,j} \left(x^j - \left(-\frac{t_i}{\lambda_i} \right)^j \right).$$

It follows that for every $1 \leq i \leq n$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} F_i(x, x^2, \ldots, x^\ell) &= T_i \left(x - \left(-\frac{t_i}{\lambda_i}\right), x^2 - \left(-\frac{t_i}{\lambda_i}\right)^2, \ldots, x^\ell - \left(-\frac{t_i}{\lambda_i}\right)^\ell \right) \\ &= \left(\lambda \left(x - \left(-\frac{t_i}{\lambda_i}\right) \right), \lambda^2 \left(x^2 - \left(-\frac{t_i}{\lambda_i}\right) \right)^2, \ldots, \lambda^\ell \left(x - \left(-\frac{t_i}{\lambda_i}\right) \right)^\ell \right) \\ &= \left(\lambda_i \cdot x + t_i, (\lambda_i \cdot x + t_i)^2, \ldots, (\lambda_i \cdot x + t_i)^\ell \right). \end{aligned}$$

It follows that, writing $\tilde{K} = P_\ell(K)$, for every $i \in \{1, ..., n\}$, we have

$$T_i(\tilde{K}) = \left\{ \left(f_i(x),\ f_i^2(x), \ldots, f_i^\ell(x) \right) : x \in K \right\}$$

Therefore,

$$\bigcup_{i=1}^n F_i(\tilde{K}) = \bigcup_{i=1}^n \left\{ \left(f_i(x),\ f_i^2(x),\ \ldots, f_i^\ell(x) \right) :\ x \in K \right\} = \{P_\ell(x) : x \in K\} = P_\ell(K) = \tilde{K}.$$

The proof is complete. □

Corollary 2.11. *Let $\mu \in \mathcal{P}([0,1])$ be a self-similar measure with respect to the self-similar IFS $\Phi = \{f_i(x) = \lambda_i \cdot x + t_i\}_{i=1}^n$ and the probability vector $\mathbf{p}$. Then $\nu = P_\ell \mu$ is a self-affine measure with respect to the IFS Ψ defined in Lemma 2.10, and the same probability vector $\mathbf{p}$.*

Proof. This is a direct consequence of the previous Lemma and its proof. □

2.5. **Basic properties of analytic maps.** The following lemma is a consequence of finiteness of the number of zeros of non-identically vanishing analytic maps in a compact interval, combined with upper Frostman regularity of self-similar measures.

Lemma 2.12. *Let $g : U \to \mathbb{R}^{d-1}$ be an analytic map defined on an open neighborhood U of $[0,1]$, and so that its graph is not contained in a proper affine subspace of $\mathbb{R}^d$. Let $G(x)$ denote the $((d-1)\times(d-1))$-matrix $[g^{(2)}(x)g^{(3)}(x)\cdots g^{(d)}(x)]$. Then, for all sufficiently small $\delta > 0$, depending on g, there exists a set $E = E(\delta) \subseteq [0,1]$ such that:*

(1) E is a union of $O_g(1)$-many intervals.

(2) There exist $c_1 \geq 1$, depending only on g, such that

$$\min \{|\det(G(x))| : x \in [0,1] \setminus E\} \gg_g \delta^{c_1},$$

where det *denotes the matrix determinant.*

Proof. Since $x \mapsto D(x) \stackrel{\text{def}}{=} \det(G(x))$ is analytic, it either vanishes identically, or has finitely many zeros in $[0,1]$. Moreover, it is known (cf. [BD10]) that identical vanishing of $D(x)$ is equivalent to the graph of g being contained in a proper affine subspace, and thus, D can only vanish at finitely many points. Let $Z = \{x \in [0,1] : D(x) = 0\}$ be this finite set. Let $Z^{(\delta)}$ be the δ-neighborhood of Z. Then, $Z^{(\delta)}$ is a union of $O_g(1)$ intervals for all small enough δ. Here, the implicit constant depends only on (the number and order of vanishing of zeros of) D. Finally, it is a standard consequence of the properties of zeros of real analytic functions that there exists some $c_1 = c_1(g) \geq 1$ such that

$$\min \left\{ |D(y)| : y \in [0,1] \setminus Z^{(\delta)} \right\} \gg_g \delta^{c_1}.$$

This completes the proof by taking $E = Z^{(\delta)}$. □

2.6. **Moment sums vs Fourier transforms.** Recall the definition of the dyadic partitions $\mathcal{D}_m$ and moment sums given in the introduction. For $x \in \mathbb{R}^d$, we denote by $\mathcal{D}_m(x)$ the unique element of $\mathcal{D}_m$ containing x.

The following lemma is a consequence of Plancherel's theorem, and relates Fourier analytic properties of measures to moment sums of their discretizations.

Lemma 2.13. *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. Then, for every $R > 1$, letting $m = [\log_2 R] \in \mathbb{N}$ be the integer part of $\log_2 R$, we have*

$$s_m(\mu, 2) \asymp 2^{-dm} \|\hat{\mu}\|^2_{L^2(B(R))},$$

where $B(R)$ denotes the R-ball around the origin.

Proof. The proof follows similar lines to [FNW02, Proof of Claim 2.8], where the inequality

$$2^{-dm} \|\hat{\mu}\|^2_{L^2(B(R))} \ll s_m(\mu, 2)$$

was essentially proved for $d = 1$; cf. [Kha23, Eq. (6.28) and (6.29)]. It can also be deduced by a very similar argument to the one we give below for the reverse inequality.

Let φ be a Schwartz function on $\mathbb{R}^d$ satisfying $\varphi \geq 1$ on the unit ball $B(1)$, $\hat{\varphi} \geq 0$, and $\mathrm{supp}(\hat{\varphi}) \subset B(1)$; cf. [Mat15, Example 3.2] for a construction of such function. For $R > 1$, let $\varphi_R(x) = R^d \varphi(Rx)$. Then, $\widehat{\varphi_R}(\xi) = \hat{\varphi}(\xi/R)$.

Note that $\widehat{\varphi_R} \ll_\varphi 1$ on $\mathrm{supp}(\widehat{\varphi_R}) \subset B(R)$. Hence, by Plancherel's formula, we have

$$\int_{\mathbb{R}^d} |\varphi_R * \mu|^2 \, dx \simeq \int_{\mathbb{R}^d} |\widehat{\varphi_R * \mu}|^2 \, d\xi = \int_{\mathbb{R}^d} |\widehat{\varphi_R}|^2 |\hat{\mu}|^2 \, d\xi \ll \int_{B(R)} |\hat{\mu}|^2 \, d\xi.$$

On the other hand, we have

$$\varphi_R * \mu(x) = R^d \int \varphi(R(x-y)) \, d\mu(y) \geq R^d \mu(B(x, 1/R)).$$

And, hence, we get

$$R^{2d} \int \mu(B(x, 1/R))^2 \, dx \ll \|\hat{\mu}\|^2_{L^2(B(R)}. \tag{2.1}$$

It remains to bound the left-hand side of (2.1) from below using a suitable moment sum. To this end, let $m = [\log_2 R] \in \mathbb{N}$, and let $m' = m + O_d(1) \in \mathbb{N}$ be such that for every cube $P \in \mathcal{D}_{m'}$ and every $x \in P$, we have

$$B(x, 1/R) \supseteq P.$$

This yields the lower bound

$$\int \mu(B(x, 1/R))^2 \, dx = \sum_{P \in \mathcal{D}_{m'}} \int_P \mu(B(x, 1/R))^2 \, dx \gg 2^{-dm'} \sum_{P \in \mathcal{D}_{m'}} \mu(P)^2,$$

where we used the fact that each P has Lebesgue measure $\asymp 2^{-dm'}$ in the last inequality. Moreover, by the Cauchy-Schwartz inequality, and the fact that each $Q \in \mathcal{D}_m$ contains $O_d(1)$ boxes $P \in \mathcal{D}_{m'}$, we get

$$\sum_{P \in \mathcal{D}_{m'}} \mu(P)^2 = \sum_{Q \in \mathcal{D}_m} \sum_{P \in \mathcal{D}_{m'}, P \subseteq Q} \mu(P)^2 \gg_d \sum_{Q \in \mathcal{D}_m} \left(\sum_{P \in \mathcal{D}_{m'}, P \subseteq Q} \mu(P) \right)^2 = \sum_{Q \in \mathcal{D}_m} \mu(Q)^2.$$

Combining the above estimates, we obtain

$$R^{2d} 2^{-dm'} s_m(\mu, 2) \ll \|\hat{\mu}\|^2_{L^2(B(R)}.$$

This implies the desired inequality $s_m(\mu, 2) \ll 2^{-dm} \|\hat{\mu}\|^2_{L^2(B(R))}$ since $m' = m + O_d(1)$. ∎

3. Uniform pointwise Fourier decay away from the first coordinate

The goal of this Section is to prove Theorem 1.3. The key step, Proposition 1.5, is to show that for a self-similar measure pushed to a non-degenerate curve ν as in Theorem 1.1, there is an $\alpha > 0$ such that: $|\hat{\nu}(\theta, \zeta)| \ll \|\zeta\|^{-\alpha}$ for every $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$, with $\zeta \neq \mathbf{0}$. In the general context of Theorem 1.1, this will take care of the case when $\|\zeta\| \gg |\theta|^{\varepsilon}$. We begin with the moment curve.

3.1. The case of moment curves.

Let $d \geq 2$ and recall the moment curves V_d:

$$V_d(x) = (x, x^2, \ldots, x^d).$$

Proposition 3.1. *Let μ be a non-atomic self-similar measure for a uniformly contracting weighted IFS $(\Phi, \mathbf{p})$ on $\mathbb{R}$. Let $d \geq 2$ and assume that $V_{d-1}\mu$ satisfies the flattening estimate (1.8). Let $\nu = V_d\mu$. Then, there exists $\alpha > 0$ such that for every $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ with $\zeta \neq \mathbf{0}$, we have $|\hat{\nu}(\xi)| \ll \|\zeta\|^{-\alpha}$.*

Proof of Proposition 3.1. By replacing μ with an affine image of itself, we shall assume its support is contained in $[0,1]$. Let $\Phi = \{f_i : f_i(x) = \lambda_i x + t_i\}_{i \in I}$. Apply Corollary 2.11 to find the IFS $\Psi = \{F_i : i \in I\}$ on $\mathbb{R}^d$ so that each F_i is of the form $F_i(x) = A_i x + v_i$ for some $v_i \in \mathbb{R}^d$, and for some contracting lower triangular matrices A_i as in the corollary.

Let $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ be a frequency with $\|\zeta\| > 1$. Fix a parameter $\varrho \in (0, 1/2)$ to be chosen using Lemma 3.3 below. By abuse of notation, we use P_ζ to denote the cut-set P_τ defined in Definition 2.1 for $\tau = \|\zeta\|^{-(1-\varrho)}$. That is,

$$P_\zeta \overset{\text{def}}{=} P_{\|\zeta\|^{-(1-\varrho)}}$$

Let $\Lambda_\zeta = \{\lambda_\omega : \omega \in P_\zeta\}$ denote the set of all contraction ratios of the maps in the one-dimensional IFS Φ corresponding to words in P_ζ. For $\lambda \in \Lambda_\zeta$, define a measure γ_λ on $\mathbb{R}^{d-1}$ by

$$\gamma_\lambda = \sum_{\omega \in P_\zeta, \lambda_\omega = \lambda} \mathbf{p}_\omega \delta_{(2t_\omega, 3t_\omega^2, \ldots, dt_\omega^{d-1})}.$$

Note that γ_λ is not a probability measure in general, as in may have total mass less than 1. Denote by $\check{\gamma}_\lambda$ the image of γ_λ under the map $x \mapsto -x$.

Lemma 3.2. *We have*

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda \in \Lambda_\zeta} \int |\hat{\mu}(\lambda\zeta \cdot y)| \, d(\gamma_\lambda * \check{\gamma}_\lambda)(y) + O(\#\Lambda_\zeta^2 \times \|\zeta\|^{2\varrho - 1}),$$

where $\gamma_\lambda * \check{\gamma}_\lambda$ is the additive convolution of the two measures.

Proof. By Lemma 2.2, we obtain

$$\hat{\nu}(\xi) = \sum_{\omega \in P_\zeta} \mathbf{p}_\omega \int e(\langle \xi, F_\omega(x) \rangle) \, d\nu(x) = \sum_{\lambda \in \Lambda_\zeta} \int \left(\sum_{\omega \in P_\zeta, \lambda_\omega = \lambda} \mathbf{p}_\omega e(\langle \xi, F_\omega(x) \rangle) \right) d\nu(x).$$

Hence, by Cauchy-Schwarz and Jensen inequalities,

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda \in \Lambda_\zeta} \int \left| \sum_{\omega \in P_\zeta, \lambda_\omega = \lambda} \mathbf{p}_\omega e(\langle \xi, F_\omega(x) \rangle) \right|^2 d\nu(x). \tag{3.1}$$

Expanding the square, we obtain

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda\in\Lambda_\zeta} \sum_{\substack{\omega_1,\omega_2\in P_\zeta,\\ \lambda_{\omega_1}=\lambda=\lambda_{\omega_2}}} \prod_{j=1}^{2} \mathbf{p}_{\omega_j} \left| \int e(\langle \xi(A_{\omega_1}-A_{\omega_2}), x\rangle)\, d\nu(x) \right|.$$

Let $\mathcal{C}^1_{\omega_1,\omega_2}$ denote the first column of the matrix $A_{\omega_1} - A_{\omega_2}$. By Lemma 2.10, we have for all $x \in \mathrm{supp}(\nu)$, and all $\lambda \in \Lambda_\zeta$, and $\omega_1, \omega_2 \in P_\zeta$ with $\lambda_{\omega_1} = \lambda = \lambda_{\omega_2}$, that

$$\langle \xi(A_{\omega_1}-A_{\omega_2}), x\rangle = (\zeta \cdot \mathcal{C}^1_{\omega_1,\omega_2})x_1 + O(\lambda^2 \|\zeta\|)).$$

It follows that, as the projection of ν on the first coordinate is our original measure μ,

$$\left| \int e(\langle \xi(A_{\omega_1}-A_{\omega_2}), x\rangle)\, d\nu(x) \right| = |\hat{\mu}(\zeta \cdot \mathcal{C}^1_{\omega_1,\omega_2}) + O(\lambda^2 \|\zeta\|)).$$

To simplify notation, for $\vec{\omega} = (\omega_1,\omega_2) \in P_\zeta^2$, we write $\mathbf{p}_{\vec{\omega}} = \prod_{j=1}^2 \mathbf{p}_{\omega_j}$. Recall that for all $\lambda \in \Lambda_\zeta$, we have $\lambda \asymp \|\zeta\|^{-(1-\varrho)}$. Hence, by combining the last estimate with the fact that the Fourier transform is Lipschitz continuous, we obtain

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda\in\Lambda_\zeta} \sum_{\substack{(\omega_1,\omega_2)\in P_\zeta^2,\\ \lambda_{\omega_1}=\lambda=\lambda_{\omega_2}}} \mathbf{p}_{\vec{\omega}} |\hat{\mu}\left(\zeta\cdot \mathcal{C}^1_{\omega_1,\omega_2}\right)| + O(\#\Lambda_\zeta^2 \times \|\zeta\|^{2\varrho-1}).$$

Next, for $\zeta = (\zeta_1,\ldots,\zeta_{d-1})$, we have by Lemma 2.10

$$\zeta \cdot \mathcal{C}^1_{\omega_1,\omega_2} = \sum_{k=2}^{d} \lambda k (t_{\omega_1}^{k-1} - t_{\omega_2}^{k-1})\zeta_{k-1}.$$

For $\lambda \in \Lambda_\zeta$ recall the definition of the measures γ_λ and $\check{\gamma}_\lambda$ given before the Lemma. Then, the above bound can be rewritten as follows

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda\in\Lambda_\zeta} \int |\hat{\mu}(\lambda\zeta\cdot y)|\, d(\gamma_\lambda * \check{\gamma}_\lambda)(y) + O(\#\Lambda_\zeta^2 \times \|\zeta\|^{2\varrho-1}),$$

This was our claim. □

We will need a uniform Frostman estimate on the projections of the various measures γ_λ. This is how we select the value of $\varrho > 0$ that appears in the definition of P_ζ, and it is the only point in the proof where the assumption on $V_{d-1}\mu$ is used.

Lemma 3.3. *There are constants $C \geq 1$, $\beta > 0$ and $0 < \varrho < 1/2$, depending only on μ and d, such that $\gamma_\lambda(W(\varepsilon)) \leq C\varepsilon^\beta$ for all proper affine subspaces W and all $\varepsilon > \|\zeta\|^{-\varrho}$.*

Proof. By our assumption,

$$\forall \varepsilon > 0 \quad \exists p > 1 \quad \forall R > 0 : \left\| \widehat{V_{d-1}\mu} \right\|^p_{L^p(B(R))} = O_\varepsilon(R^\varepsilon).$$

Since the curve

$$\widetilde{V}_{d-1}(x) = (2x, 3x^2, \ldots, dx^{d-1})$$

is a linear image of V_{d-1}, it follows that the measure $\tilde{\nu} \overset{\text{def}}{=} \widetilde{V}_{d-1}\mu$ also satisfies this property. Hence, by Proposition 2.3, there are $\beta > 0$ and $\varrho > 0$ such that for every $\tau > 0$ the discrete measure

$$\tilde{\nu}_\tau = \sum_{\omega \in P_\tau} \mathbf{p}_\omega \delta_{(2t_\omega, 3t_\omega^2, \dots, dt_\omega^{d-1})}$$

give mass $O(\varepsilon^\beta)$ to ε-neighborhoods of proper affine subspaces whenever $\varepsilon > \tau^\varrho$. Moreover, without loss of generality, the parameter ϱ may be taken $< 1/2$. Note that $\gamma_\lambda(A) \leq \tilde{\nu}_\tau(A)$ for all Borel sets A and all $\lambda \in \Lambda_\zeta$. Putting $\tau = \|\zeta\|^{-(1-\varrho)}$, the claim follows upon noting that $\|\zeta\|^{-\varrho} \geq \tau^\varrho = \|\zeta\|^{-\varrho(1-\varrho)}$. $\square$

We are now in position to complete the proof of Proposition 3.1.

Let $C \geq 1$ be such that the supports of all the measures $\gamma_\lambda * \check{\gamma}_\lambda$ are contained a ball of radius C around the origin. Let $\lambda \in \Lambda_\zeta$. By Corollary 2.9, for every $\varepsilon > 0$ there is $\delta > 0$ so that for $|z| \leq C\,\|\lambda\zeta\|$,

$$|\hat{\mu}(z)| \ll \|\lambda\zeta\|^{-\delta}$$

except for a set $B \subseteq \mathbb{R}$ of frequencies z that is a union of $O_\varepsilon(\|\lambda\zeta\|^\varepsilon)$ intervals of length 1. Since every $\lambda, \lambda' \in \Lambda_\zeta$ are $\lambda \asymp \lambda'$, we may assume this property (and B in particular) is independent of the choice of λ.

Fix $\varepsilon = \beta/2$, for β as in Lemma 3.3. Hence, recalling that $\lambda \asymp \|\zeta\|^{-(1-\varrho)}$ for all $\lambda \in \Lambda_\zeta$, via Lemma 3.2 and the bounds above, we obtain

$$|\hat{\nu}(\xi)|^2 \leq \#\Lambda_\zeta^2 \times \sum_{\lambda \in \Lambda_\zeta} \gamma_\lambda * \check{\gamma}_\lambda\left(\{y : \lambda\zeta \cdot y \in B\}\right) + O(\#\Lambda_\zeta^3 \times \|\zeta\|^{-\varrho\delta} + \#\Lambda_\zeta^2 \times \|\zeta\|^{2\varrho-1}).$$

Fix a unit-length interval $I = (a - 1/2, a + 1/2) \subseteq B$ for some $a \in \mathbb{R}$, and consider the affine hyperplane $W_a = \{x : \lambda\zeta \cdot x = a\}$. An elementary computation then shows that if y is such that $\lambda\zeta \cdot y \in I$, then $y \in W_a(\|\lambda\zeta\|^{-1})$. Thus, noting that

$$\gamma_\lambda * \check{\gamma}_\lambda(A) = \int \gamma_\lambda(A + z)\, d\gamma_\lambda(z) \text{ for any Borel set } A,$$

we obtain

$$\gamma_\lambda * \check{\gamma}_\lambda(\{y : \lambda\zeta \cdot y \in I\}) \leq \sup_{W = W_a + z} \gamma_\lambda(W(\|\lambda\zeta\|^{-1})) \leq C\,\|\lambda\zeta\|^{-\beta},$$

where the supremum runs over all translations of W_a, and we applied Lemma 3.3 in the last inequality.

Putting together the above estimates, we arrive at the bound

$$|\hat{\nu}(\xi)|^2 \ll \#\Lambda_\zeta^3 \times \|\zeta\|^{(\varepsilon-\beta)\varrho} + \#\Lambda_\zeta^3 \times \|\zeta\|^{-\varrho\delta} + \#\Lambda_\zeta^2 \times \|\zeta\|^{2\varrho-1}.$$

Finally, by Lemma 2.2, we have $\#\Lambda_\zeta \ll (\log\|\zeta\|)^A$, where $A = \#\Phi + 1$. Recalling that $\varepsilon = \beta/2$ and $\varrho < 1/2$, this bound completes the proof. $\square$

3.2. **General non-affine curves, and proof of Proposition 1.5.** The goal of this section is to deduce Proposition 1.5 from Proposition 3.1. The deduction is via Taylor expansion and a change of coordinates. Recall the notation V_d set before Proposition 3.1.

Proof of Proposition 1.5. We give the proof in the case $Q(x)$ is a non-trapped analytic curve. The proof in the case Q is a C^{d+1} non-degenerate curve is very similar, and in fact simpler, due to the non-vanishing of the determinant in (1.1) over an entire neighborhood of $\mathrm{supp}(\mu)$.

As usual, by replacing μ with an affine image of itself, we shall assume its support is contained in $[0,1]$. Let $\Phi = \{f_i : f_i(x) = \lambda_i x + t_i\}_{i\in I}$. Let $\alpha > 0$ be the exponent provided by Proposition 3.1. Fix a frequency $\xi = (\theta, \zeta) \in \mathbb{R}\times\mathbb{R}^{d-1}$, with $\|\zeta\| > 1$, and define

$$\tau = \|\zeta\|^{-(1+\alpha)/(1+d(1+\alpha))}.$$

Let P_τ be the cut-set defined in Def. 2.1. Without loss of generality, we shall assume over the course of the proof that $\|\zeta\|$ is sufficiently large, depending only on g and μ.

Let $\varrho > 0$ be the parameter provided by Proposition 2.6. Let $\varepsilon = \varepsilon(\alpha, g, d) \in (0, \varrho)$ to be chosen at the end of the proof to be sufficiently small depending only on g, the ambient dimension d, and the exponent α, and let

$$\delta = \tau^{\varepsilon}.$$

Let $E = E(\delta) \subset [0,1]$ be the set provided by Lemma 2.12 and let

$$P'_\tau = \{\omega \in P_\tau : \ f_\omega(0) \in E\}.$$

For a word $\omega \in I^*$, we write ν_ω for the pushforward of $f_\omega\mu$ under $x \mapsto (x, g(x))$.

Lemma 3.4. *For the Frostman exponent s_0 of μ we have*

$$|\widehat{\nu}(\xi)| \le \sum_{\omega\in P_\tau\setminus P'_\tau} \mathbf{p}_\omega |\widehat{\nu_\omega}(\xi)| + O_{g,\mu}\left(\delta^{s_0}\right).$$

Proof. By stationarity of μ (Lemma 2.2) and the triangle inequality, we have

$$|\widehat{\nu}(\xi)| \le \sum_{\omega\in P_\tau\setminus P'_\tau} \mathbf{p}_\omega |\widehat{\nu_\omega}(\xi)| + \sum_{\omega\in P'_\tau} \mathbf{p}_\omega.$$

By Lemma 2.12, E is a union of $O_g(1)$ δ-intervals. Therefore, by Proposition 2.6, since $\delta > \tau^{\varrho}$, for the Frostman exponent s_0 of μ we have

$$|\widehat{\nu}(\xi)| \le \sum_{\omega\in P_\tau\setminus P'_\tau} \mathbf{p}_\omega |\widehat{\nu_\omega}(\xi)| + \sum_{\omega\in P'_\tau} \mathbf{p}_\omega = \sum_{\omega\in P_\tau\setminus P'_\tau} \mathbf{p}_\omega |\widehat{\nu_\omega}(\xi)| + O_{g,\mu}\left(\delta^{s_0}\right).$$

The proof is complete. ∎

Thus, it remains to estimate the sum over $P_\tau \setminus P'_\tau$.

Lemma 3.5. *If τ is sufficiently small then for every $\omega \in P_\tau$ we have, writing $f_\omega(x) = \lambda_\omega x + t_\omega$,*

$$|\widehat{\nu_\omega}(\xi)| \le \left| \int e(\langle \xi_\omega, V_d(\lambda_\omega x)\rangle)\, d\mu(x) \right| + O(\|\zeta\|\, \tau^{d+1}).$$

Here, given a frequency $\xi = (\theta, \zeta) \in \mathbb{R}\times\mathbb{R}^{d-1}$, we define $\xi_\omega := (\theta_\omega, \zeta_\omega)$, where

$$\theta_\omega := \theta + \langle \zeta, g'(t_\omega)\rangle, \qquad \zeta_\omega := \left(\langle \zeta, g''(t_\omega)\rangle/2, \ldots, \langle \zeta, g^{(d)}(t_\omega)\rangle/d!\right).$$

Proof. Note that

$$\mathrm{diam}(\mathrm{supp}(f_\omega\mu)) \asymp \tau \text{ for every } \omega \in P_\tau.$$

We assume τ is small enough so that g can be written as a power series in a neighborhood of t_ω that contains $\text{supp}(f_\omega \mu)$. By considering the Taylor expansion of g about t_ω, we see that for every $x \in \text{supp}(f_\omega \mu)$,

$$\langle \xi, (x, g(x)) \rangle = \zeta \cdot g(t_\omega) + \xi_\omega \cdot V_d(x - t_\omega) + O(\|\zeta\| \, \tau^{d+1}),$$

where we recall that $V_d(y) = (y, y^2, \dots, y^d)$ is the moment curve. Hence, by the Lipchitz continuity of the Fourier transform, we obtain the bound

$$|\widehat{\nu}_\omega(\xi)| \leq \left| \int e(\langle \xi_\omega, V_d(x - t_\omega) \rangle) \, df_\omega \mu(x) \right| + O(\|\zeta\| \, \tau^{d+1}).$$

Recalling that $f_\omega(x) = \lambda_\omega x + t_\omega$ thus yields

$$|\widehat{\nu}_\omega(\xi)| \leq \left| \int e(\langle \xi_\omega, V_d(\lambda_\omega x) \rangle) \, d\mu(x) \right| + O(\|\zeta\| \, \tau^{d+1}).$$

This proves the Lemma □

Let $x \in [0,1]$ and $\omega \in P_\tau$. We define two $(d-1) \times (d-1)$ matrices $G(x), D_\omega$ via

$$G(x) \text{ is the } ((d-1) \times (d-1)) \text{ matrix with } k^{th} \text{ column given by the vector } g^{(k+1)}(x)$$

and

$$D_\omega \text{ is the } (d-1) \times (d-1) \text{ diagonal matrix with } k^{th} \text{ diagonal entry } \lambda_\omega^{k+1}/(k+1)!.$$

Let $F_\omega(x) = D_\omega \cdot G(x)$. When ω is fixed we simply write $F(x)$. Note that

$$\langle \xi_\omega, V_d(r_\omega x) \rangle = \langle (\lambda_\omega \theta_\omega, \zeta \cdot F(t_\omega)), V_d(x) \rangle.$$

Hence, by Proposition 3.1 and Lemma 3.5, we obtain for some $\alpha > 0$

$$|\widehat{\nu}_\omega(\xi)| \ll \|\zeta \cdot F(t_\omega)\|^{-\alpha} + \|\zeta\| \, \tau^{d+1}. \tag{3.2}$$

Recall the definition of $P_{\tau'}$ from before Lemma 3.4.

Lemma 3.6. *There is $C \geq 1$, depending only on g, such that for every $\omega \in P_\tau \setminus P'_\tau$*

$$\|\zeta \cdot F(x)\| \geq \|\zeta \cdot D_\omega\| \, |\det(G(x)|/C,$$

uniformly over $x \in [0,1]$ and $\zeta \in \mathbb{R}^{d-1}$, where $\det(-)$ *is the matrix determinant.*

Proof. Let $\sigma_{\min}(x)$ denote the singular value of smallest magnitude of $G(x)$. Then

$$\|\zeta \cdot F(x)\| \geq \sigma_{\min}(x) \, \|\zeta \cdot D_\omega\| \, .$$

Note that $\sigma_{\min}(x)$ is non-zero in view of Lemma 2.12 and the choice of $\omega \in P_{\tau'}$. Moreover,

$$\sigma_{\min}(x) \geq |\det(G(x)|/C(x), \text{ where } C(x) = \text{ the product of all the other singular values of } G(x).$$

Hence, since the singular values of $G(x)$ are non-vanishing and vary continuously with x, $C(x)$ is uniformly bounded above over $x \in [0,1]$. This implies the Lemma. □

We can now complete the proof. Let $\omega \in P_\tau \setminus P_{\tau'}$. Note that $\|\zeta \cdot D_\omega\| \gg \|\zeta\| \lambda_\omega^d$. Hence, by (3.2), Lemma 3.6 and Lemma 2.12, we obtain

$$|\widehat{\nu}_\omega(\xi)| \ll \delta^{-\alpha c_1} \|\zeta \cdot D_\omega\|^{-\alpha} + \|\zeta\| \tau^{d+1} \ll \delta^{-\alpha c_1} \lambda_\omega^{-d\alpha} \|\zeta\|^{-\alpha} + \|\zeta\| \tau^{d+1},$$

where $c_1 = c_1(g) \geq 1$ is the exponent provided by Lemma 2.12. Recalling that $\lambda_\omega \asymp \tau$, we obtain

$$|\widehat{\nu}_\omega(\xi)| \ll \delta^{-\alpha c_1} \tau^{-d\alpha} \|\zeta\|^{-\alpha} + \|\zeta\| \tau^{d+1} \leq \delta^{-\alpha c_1}(\tau^{-d\alpha} \|\zeta\|^{-\alpha} + \|\zeta\| \tau^{d+1}).$$

Setting $\beta = \alpha/(d(1+\alpha)+1)$, by our choice of τ the above bound becomes

$$|\widehat{\nu}_\omega(\xi)| \ll \delta^{-\alpha c_1} \|\zeta\|^{-\beta}.$$

Recall that $\delta = \tau^\varepsilon$. Thus, taking ε sufficiently small, depending on α and c_1, we can ensure that $\delta^{-\alpha c_1} \|\zeta\|^{-\beta}$ is at most $\|\zeta\|^{-\beta/2}$. Hence, the result follows by combining the above bound with Lemma 3.4. □

3.3. Average decay away from the first coordinate, and proof of Theorem 1.3.

Proposition 1.5 yields the following Corollary, which is a more precise form of Theorem 1.3.

Corollary 3.7. *Let μ, g, and ν be as in Theorem 1.3. For $R \geq 1$ and $\varepsilon > 0$, let*

$$C_{R,\varepsilon} = \left\{(\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1} : |\theta|^\varepsilon \leq \|\zeta\| \leq R\right\}.$$

Let $\gamma > 0$ be the parameter provided by Proposition 1.5. Then, for $p > d^3/\gamma\varepsilon^2$,

$$\int_{C_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi = O(R^\varepsilon).$$

Proof. Let $\delta = \varepsilon/d$. Let $D_{R,\delta} = C_{R,\delta} \setminus \{\xi : \|\xi\| \leq R^\delta\}$. Then, applying Proposition 1.5 on $D_{R,\varepsilon}$, using its notations and the trivial bound $|\widehat{\nu}(-)| \leq 1$ on its complement, we get

$$\int_{C_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi = \int_{D_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi + O(R^\varepsilon) \ll \int_{\{\xi = (\theta,\zeta) \in D_{R,\delta}\}} \|\zeta\|^{-\gamma p} \, d\xi + R^\varepsilon.$$

Finally, note that, on $D_{R,\delta}$, $\|\zeta\| \gg \|\xi\|^\delta > R^{\delta^2}$. The corollary follows since $D_{R,\delta}$ has measure $O(R^d)$. □

4. Average Decay Near the First Coordinate: Proof of Proposition 1.4

The purpose of this Section is to prove the following estimate on the average decay of curved self-similar measures for frequencies with dominant first coordinate. This immediately yields Proposition 1.4, which is the second main ingredient in the proof of Theorem 1.1. We recall the statement of that proposition for the reader's convenience.

Proposition 4.1. *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a non-atomic self-similar measure and $d \geq 2$. Let $g : U \to \mathbb{R}^{d-1}$ be a C^1-map defined on an open neighborhood U of $\operatorname{supp}(\mu)$. For $Q(x) = (x, g(x))$, let $\nu = Q\mu$ be the pushforward of μ to the graph of g. For $R \geq 1$ and $\varepsilon > 0$, let*

$$E_{R,\varepsilon} = \left\{(\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1} : \|\zeta\| \leq |\theta|^\varepsilon \leq R^\varepsilon\right\}.$$

Then, for every $\varepsilon > 0$, there exists $p = p(\varepsilon, \mu) > 1$ such that for all $R \geq 1$, we have

$$\int_{\xi \in E_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi = O_{\varepsilon,\mu}(R^\varepsilon).$$

The rest of this section is dedicated to the proof of Proposition 4.1. By replacing μ with an affine image of itself, we shall assume its support is contained in $[0,1]$. Let $\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i \in I}$. As before, for a word $\omega \in I^*$, we write ν_ω for the pushforward of $f_\omega \mu$ under $x \mapsto (x, g(x))$. Recall the definition of the cut-set P_τ from Definition 2.1, and the notations set in Section 2.1.

Fix $\varepsilon > 0$ and let $\delta = \varepsilon/d$. We also fix $R > 1$, which we shall assume to be sufficiently large depending on ε and μ, and let

$$F_{R,\varepsilon} = E_{R,\varepsilon} \setminus \left\{ (\theta, \zeta) \in E_{R,\varepsilon} : |\theta| \leq R^\delta \right\}.$$

First, note that since $E_{R,\varepsilon} \setminus F_{R,\varepsilon}$ has measure $O(R^\varepsilon)$ and $|\hat{\nu}(-)| \leq 1$, we get

$$\int_{E_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi \leq \int_{F_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi + O(R^\varepsilon). \tag{4.1}$$

Hence, we focus on the region $F_{R,\varepsilon}$. By Lemma 2.2, for every $\tau > 0$ and every $\xi \in \mathbb{R}^d$ we have

$$\hat{\nu}(\xi) = \sum_{\omega \in P_\tau} \mathbf{p}_\omega \widehat{\nu_\omega}(\xi).$$

We have the following initial pointwise bound on terms of the above sum using the Fourier transform of the original self-similar measure.

Lemma 4.2. *For every $\tau \in (0,1)$, $\omega \in P_\tau$, and $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$, we have*

$$|\widehat{\nu_\omega}(\xi)| \leq |\hat{\mu}(\lambda_\omega \theta)| + O(\tau \, \|\zeta\|).$$

Proof. Note that, for every $\omega \in P_\tau$ and $x \in \text{supp}(\mu)$, $\lambda_\omega \leq \tau$, since g is C^1, we have $g(f_\omega(x)) = g(f_\omega(0)) + O(\tau)$. Thus, since $f_\omega(x) = \lambda_\omega \cdot x + t_\omega$, and $\hat{\mu}(-)$ is Lipschitz continuous, we get for every frequency $\xi = (\theta, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1}$ that

$$|\widehat{\nu_\omega}(\xi)| = \left| \int e((\theta f_\omega(x) + \zeta \cdot g(t_\omega)) \, d\mu(x) \right| + O(\|\zeta\| \, \tau) = |\hat{\mu}(\lambda_\omega \theta)| + O(\tau \, \|\zeta\|),$$

which is the claimed bound. $\square$

The rest of the proof of Proposition 4.1 is broken down into the following two Lemmas. In what follows, for $n \in \mathbb{N}$, we let for

$$q \overset{\text{def}}{=} 2^{1/2\varepsilon}, \qquad \tau_n \overset{\text{def}}{=} 2^{-n}, \qquad F^n_{R,\varepsilon} \overset{\text{def}}{=} \left\{ (\theta, \zeta) \in F_{R,\varepsilon} : q^n \leq |\theta| < q^{n+1} \right\}. \tag{4.2}$$

Lemma 4.3. *For all integers $p \geq 2d^2/\varepsilon$, we have*

$$\int_{F_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi \leq \sum_{n \in \mathbb{N}} \int_{F^n_{R,\varepsilon}} \left(\sum_{\omega \in P_{\tau_n}} \mathbf{p}_\omega \, |\hat{\mu}(\lambda_\omega \theta(\xi))|^p \right) d\xi + O_\varepsilon(1),$$

where $\theta(\xi)$ denotes the first coordinate of ξ.

Proof. First, since $F_{R,\varepsilon} = \cup_{n\in\mathbb{N}} F_{R,\varepsilon}^n$, we get

$$\int_{F_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi \le \sum_{n\in\mathbb{N}} \int_{F_{R,\varepsilon}^n} |\widehat{\nu}(\xi)|^p \, d\xi = \sum_{n\in\mathbb{N}} \int_{F_{R,\varepsilon}^n} \left| \sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \widehat{\nu_\omega}(\xi) \right|^p \, d\xi.$$

Note that the outer sum runs over n with $R^\delta \le q^{n+1} \le qR$. Applying Lemma 4.2 with $\tau = \tau_n$ for each n, we obtain

$$\int_{F_{R,\varepsilon}} |\widehat{\nu}(\xi)|^p \, d\xi \le \sum_{n\in\mathbb{N}} \int_{\xi=(\theta,\zeta)\in F_{R,\varepsilon}^n} \left| \sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \left[|\hat{\mu}(\lambda_\omega \cdot \theta)| + O(2^{-n} \|\zeta\|) \right] \right|^p \, d\xi.$$

By Jensen's inequality, since $p > 1$ and $\sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega = 1$, we obtain

$$\left| \sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \left[|\hat{\mu}(\lambda_\omega \cdot \theta)| + O(2^{-n} \|\zeta\|) \right] \right|^p \ll \|\zeta\|^p \cdot 2^{-np} + \sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \, |\hat{\mu}(\lambda_\omega \cdot \theta)|^p \, .$$

Integrating the second term over $F_{R,\varepsilon}^n$ and summing over $n \in \mathbb{N}$ gives the first term of the claimed bound. For the first term, note that for each $\xi = (\theta, \zeta) \in F_{R,\varepsilon}^n$, we have that $\|\zeta\| \le |\theta|^\varepsilon \le q^{\varepsilon(n+1)}$. The choice of q in (4.2) also gives $q^{\varepsilon(n+1)} 2^{-n} \le q^\varepsilon 2^{-n/2}$.

Thus, using that $F_{R,\varepsilon}$ has measure $O(R^d)$ and that the sum over n only involves terms satisfying $R^\delta \le q^{n+1} \le Rq$, we get

$$\sum_{n\in\mathbb{N}} \int_{\xi=(\theta,\zeta)\in F_{R,\varepsilon}^n} \|\zeta\|^p \cdot 2^{-np} \, d\xi \ll \sum_{n\in\mathbb{N}: q^{n+1}\ge R^\delta} 2^{-np/2} R^d \le R^{d-\delta p/2} q.$$

Taking $p \ge 2d/\delta = 2d^2/\varepsilon$, the above bound is $O_\varepsilon(1)$, thus completing the proof. □

The next lemma estimates the main term in the bound provided by Lemma 4.3. The key ingredient is Corollary 2.9 on L^2-flattening of self-similar measures.

Lemma 4.4. *For all $p \gg_{\varepsilon,\mu} 1$, we have*

$$\sum_{n\in\mathbb{N}} \int_{F_{R,\varepsilon}^n} \left(\sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \, |\hat{\mu}(\lambda_\omega \theta(\xi))|^p \right) d\xi = O_{\varepsilon,\mu}(R^{\varepsilon(d+3)}),$$

where $\theta(\xi)$ denotes the first coordinate of ξ.

Proof. Recalling the definition of $F_{R,\varepsilon}^n$ in (4.2), since the integrand depends only on the first coordinate of ξ, we get

$$\sum_{n\in\mathbb{N}} \int_{F_{R,\varepsilon}^n} \left(\sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \, |\hat{\mu}(\lambda_\omega \theta(\xi))|^p \right) d\xi \ll \sum_{n\in\mathbb{N}: R^\delta \le q^{n+1} \le Rq} q^{\varepsilon(n+1)(d-1)} \sum_{\omega\in P_{\tau_n}} \mathbf{p}_\omega \int_{q^n}^{q^{n+1}} |\hat{\mu}(\lambda_\omega \theta)|^p \, d\theta, \tag{4.3}$$

where, as in the proof of the previous lemma, we also used the fact that the outer sum on the left-hand side runs over $n \in \mathbb{N}$ satisfying $R^\delta \le q^{n+1} \le Rq$.

Using the change of variable $\lambda_\omega \theta \mapsto \theta$, and recalling that $\lambda_\omega \asymp \tau_n = 2^{-n}$ for all $\omega \in P_{\tau_n}$, we get

$$\begin{aligned}(4.3) &\ll \sum_{n\in\mathbb{N}: R^\delta \le q^{n+1} \le Rq} q^{\varepsilon(n+1)(d-1)} \sum_{\omega \in P_{\tau_n}} \mathbf{p}_\omega \int_{|\theta| \asymp (q/2)^n} |\hat{\mu}(\theta)|^p \, 2^n \, d\theta \\ &\ll \sum_{n\in\mathbb{N}: R^\delta \le q^{n+1} \le Rq} q^{\varepsilon(n+1)(d-1)} \times 2^n \times \int_{|\theta| \asymp (q/2)^n} |\hat{\mu}(\theta)|^p \, d\theta.\end{aligned}$$

Now, by Corollary 2.9, if p is large enough, depending only on μ and ε, for each n, the inner integral is $O_{\mu,\varepsilon}((q/2)^{\varepsilon n})$. Thus, recalling that $2 = q^{2\varepsilon}$, we arrive at the bound

$$(4.3) \ll_{\varepsilon,\mu} \sum_{n\in\mathbb{N}: q^n \le R} q^{\varepsilon n d} \times 2^n \times q^{\varepsilon n} \ll R^{\varepsilon(d+3)},$$

which completes the proof of the lemma. □

Since ε was arbitrary, Proposition 4.1 now follows from the combination of (4.1) with Lemmas 4.3 and 4.4.

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