MIXING, RESONANCES, AND SPECTRAL GAPS ON GEOMETRICALLY FINITE LOCALLY SYMMETRIC SPACES

OSAMA KHALIL

Abstract. We study the distribution of the Pollicott-Ruelle resonances for the geodesic flow on geometrically finite locally symmetric spaces of negative curvature with respect to the Bowen-Margulis-Sullivan measure of maximal entropy. Our main result shows that the Laplace transform of the correlation function of smooth observables extends meromorphically to the entire complex plane in the convex cocompact case and to a strip of explicit size beyond the imaginary axis in the case the manifold admits cusps. The method is dynamical in nature and is based on constructing anisotropic Banach spaces on which resolvents of the generator of the flow admit essential spectral gaps of size depending only on the critical exponent and the ranks of the cusps of the manifold (if any). A key ingredient is the construction of a Margulis function establishing quantitative recurrence of orbits to compact sets when the manifold has cusps. Our analysis also yields a large deviations estimate on the measure of the set of orbits which spend definite proportions of time outside large compact sets.

1. Introduction

1.1. **Pollicott-Ruelle resonances.** Let $\mathcal{X}$ be the unit tangent bundle of a quotient of a real, complex, quaternionic, or a Cayley hyperbolic space by a discrete, geometrically finite, non-elementary group of isometries Γ. Denote by g_t the geodesic flow on $\mathcal{X}$ and by $\mathrm{m}^{\mathrm{BMS}}$ the Bowen-Margulis-Sullivan probability measure of maximal entropy for g_t. Let δ_Γ be the critical exponent of Γ. We refer the reader to Section 2 for definitions.

Given two bounded functions f and g on $\mathcal{X}$, the associated correlation function is defined by

$$\rho_{f,g}(t) := \int_{\mathcal{X}} f \circ g_t \cdot g \, d\mathrm{m}^{\mathrm{BMS}}, \qquad t \in \mathbb{R}.$$

Its (one-sided) Laplace transform is defined for any $z \in \mathbb{C}$ with positive real part $\mathrm{Re}(z)$ as follows:

$$\hat{\rho}_{f,g}(z) := \int_0^\infty e^{-zt} \rho_{f,g}(t) \, dt.$$

Let δ_Γ denote the critical exponent of Γ and define

$$\sigma(\Gamma) := \begin{cases} \infty, & \text{if } \Gamma \text{ is convex cocompact,} \\ \min\{\delta_\Gamma, 2\delta_\Gamma - k_{\max}, k_{\min}\}, & \text{otherwise,} \end{cases} \tag{1.1}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively; cf. Section 3.1 for the definition of the rank of a cusp.

The following is the main result of the article.

Theorem 1.1. *Let $r \in \mathbb{N}$. For all $f, g \in C_c^{r+2}(\mathcal{X})$, $\hat{\rho}_{f,g}$ is analytic in the half plane $\mathrm{Re}(z) > 0$ and admits a meromorphic continuation to the half plane:*

$$\mathrm{Re}(z) > -\min\{r, \sigma(\Gamma)/2\},$$

with 0 being the only pole on the imaginary axis. In particular, when Γ is convex cocompact and $f, g \in C_c^\infty(\mathcal{X})$, $\hat{\rho}_{f,g}$ admits a meromorphic extension to the entire complex plane.

Theorem 1.1 is deduced from an analogous result on the meromorphic continuation of the family of resolvent operators $z \mapsto R(z)$,

$$R(z) := \int_0^\infty e^{-zt} \mathcal{L}_t \, dt : C_c(\mathcal{X}) \to C(\mathcal{X}), \tag{1.2}$$

defined initially for z with large enough $\mathrm{Re}(z)$, where $\mathcal{L}_t$ is the transfer operator given by $f \mapsto f \circ g_t$; cf. Theorem 6.4 for a precise statement. Analogous results regarding resolvents were obtained for Anosov flows in [GLP13] and Axiom A flows in [DG16, DG18] leading to a resolution of a conjecture of Smale on the meromorphic continuation of the Ruelle zeta function; cf. [Sma67]. We refer the reader to [GLP13] for a discussion of the history of the latter problem.

1.2. **Exponential recurrence from the cusp.** Our proof of Theorem 1.1 also yields the following exponential decay result on the measure of the set of orbits with long cusp excursions, which is of independent interest. Denote by N^+ the expanding horospherical group associated to g_t for $t > 0$, the orbits of which give rise to the strong unstable foliation. Let N_r^+ be the r-ball around identity in N^+ (cf. Section 2.5 for definition of the metric on N^+). Finally, let $\Omega \subseteq \mathcal{X}$ be the non-wandering set for the geodesic flow; i.e. the closure of the set of its periodic orbits.

Theorem 1.2. *Let $\sigma(\Gamma)$ be as in (1.1) and let $0 < \beta < \sigma(\Gamma)/2$ be given. For every $\varepsilon > 0$, there exists a compact set $K \subseteq \Omega$ and $T_0 > 0$ such that the following holds for all $T > T_0, 0 < \theta < 1$ and $x \in \Omega$. Let χ_K be the indicator function of K. Then,*

$$\mu_x^u \left(n \in N_1^+ : \int_0^T \chi_K(g_t n x) \, dt \leq (1-\theta) T \right) \ll_{\beta,x,\varepsilon} e^{-(\beta\theta - \varepsilon)T} \mu_x^u(N_1^+).$$

The implicit constant is uniform as x varies in any fixed compact set.

The reader is referred to Theorem 7.13 for a stronger and more precise statement. Theorem 1.2 implies that the Hausdorff dimension of the set of points in $N_1^+ x$ whose forward orbit asymptotically spends all of its time in the cusp is at most $\sigma(\Gamma)/2$. This bound is not sharp and can likely be improved using a refinement of our methods. We hope to return to this problem in future work.

1.3. **Outline of the argument.** The article has several parts that can be read independently of one another. For the convenience of the reader, we give a brief outline of those parts.

The first part consists of Sections 2-5. After recalling some basic facts in Section 2, we prove a key doubling result, Proposition 3.1, in Section 3 for the conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the strong unstable foliation.

In Section 4, we construct a Margulis function which shows, roughly speaking, that generic orbits with respect to $\mathrm{m}^{\mathrm{BMS}}$ are biased to return to the thick part of the manifold. In Section 5, we prove a statement on average expansion of vectors in linear representations which is essential for our construction of the Margulis function. The main difficulty in the latter result in comparison with the classical setting lies in controlling the *shape* of sublevel sets of certain polynomials in order to estimate their measure with respect to conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the unstable foliation.

The second part consists of Sections 6 and 7. In Section 6, we define anisotropic Banach spaces arising as completions of spaces of smooth functions with respect to a dynamically relevant norm and study the norm of the transfer operator as well as the resolvent in their actions on these spaces in Section 7. The proof of Theorem 1.1 is completed in Section 7. The approach of these two sections follows closely the ideas of [GL06, GL08, AG13], originating in [BKL02]. Theorem 1.2 is deduced from this analysis in Section 7.7.

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2. Preliminaries

We recall here some background and definitions on geometrically finite manifolds.

2.1. **Geometrically finite manifolds.** The standard reference for the material in this section is [Bow93]. Suppose G is the group of orientation preserving isometries of a real, complex, quaternionic or Cayley hyperbolic space, denoted $\mathbb{H}^d_{\mathbb{K}}$, of dimension $d \geq 2$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In the case $\mathbb{K} = \mathbb{O}$, then $d = 2$.

Fix a basepoint $o \in \mathbb{H}^d_{\mathbb{K}}$. Then, G acts transitively on $\mathbb{H}^d_{\mathbb{K}}$ and the stabilizer K of o is a maximal compact subgroup of G. We shall identify $\mathbb{H}^d_{\mathbb{K}}$ with $K\backslash G$. Denote by $A = \{g_t : t \in \mathbb{R}\}$ a 1-parameter subgroup of G inducing the geodesic flow on the unit tangent bundle of $\mathbb{H}^d_{\mathbb{K}}$. Let $M < K$ denote the centralizer of A inside K so that the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ may be identified with $M\backslash G$. In Hopf coordinates, we can identify $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ with $\mathbb{R} \times (\partial\mathbb{H}^d_{\mathbb{K}} \times \partial\mathbb{H}^d_{\mathbb{K}} \setminus \Delta)$, where $\partial\mathbb{H}^d_{\mathbb{K}}$ denotes the boundary at infinity and Δ denotes the diagonal.

Let $\Gamma < G$ be an infinite discrete subgroup of G. The limit set of Γ, denoted Λ_Γ, is the set of limit points of the orbit $\Gamma \cdot o$ on $\partial\mathbb{H}^d_{\mathbb{K}}$. Note that the discreteness of Γ implies that all such limit points belong to the boundary. Moreover, this definition is independent of the choice of o in view of the negative curvature of $\mathbb{H}^d_{\mathbb{K}}$. We often use Λ to denote Λ_Γ when Γ is understood from context. We say Γ is *non-elementary* if Λ_Γ is infinite.

The *hull* of Λ_Γ, denoted $\mathrm{Hull}(\Lambda_\Gamma)$, is the smallest convex subset of $\mathbb{H}^d_{\mathbb{K}}$ containing all the geodesics joining points in Λ_Γ. The convex core of the manifold $\mathbb{H}^d_{\mathbb{K}}/\Gamma$ is the smallest convex subset containing the image of $\mathrm{Hull}(\Lambda_\Gamma)$. We say $\mathbb{H}^d_{\mathbb{K}}/\Gamma$ is *geometrically finite* (resp. *convex cocompact*) if the closed 1-neighborhood of the convex core has finite volume (resp. is compact), cf. [Bow93]. The non-wandering set for the geodesic flow is the closure of the set of vectors in the unit tangent bundle whose orbit accumulates on itself. In Hopf coordinates, this set, denoted Ω, coincides with the projection of $\mathbb{R} \times (\Lambda_\Gamma \times \Lambda_\Gamma - \Delta)$ mod Γ.

A useful equivalent definition of geometric finiteness is that the limit set of Γ consists entirely of radial and bounded parabolic limit points; cf. [Bow93]. This characterization of geometric finiteness will be of importance to us and so we recall here the definitions of these objects.

A point $\xi \in \Lambda$ is said to be a *radial point* if any geodesic ray terminating at ξ returns infinitely often to a bounded subset of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$. The set of radial limit points is denoted by Λ_r.

Denote by N^+ the expanding horospherical subgroup of G associated to g_t, $t \geq 0$. A point $p \in \Lambda$ is said to be a *parabolic point* if the stabilizer of p in Γ, denoted by Γ_p, is conjugate in G to an unbounded subgroup of MN^+. A parabolic limit point p is said to be *bounded* if $(\Lambda - \{p\})/\Gamma_p$ is compact. An equivalent charachterization is that $p \in \Lambda$ is parabolic if and only if any geodesic ray terminating at p eventually leaves every compact subset of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$. The set of parabolic limit points will be denoted by Λ_p.

Given $g \in G$, we denote by g^+ the coset of P^-g in the quotient $P^-\backslash G$, where $P^- = N^-AM$ is the stable parabolic group associated to $\{g_t : t \geq 0\}$. Similarly, g^- denotes the coset P^+g in $P^+\backslash G$. Since M is contained in $P^\pm$, such a definition makes sense for vectors in the unit tangent bundle $M\backslash G$. Geometrically, for $v \in M\backslash G$, v^+ (resp. v^-) is the forward (resp. backward) endpoint of the geodesic determined by v on the boundary of $\mathbb{H}^d_{\mathbb{K}}$. Given $x \in G/\Gamma$, we say $x^\pm$ belongs to Λ if the same holds for any representative of x in G; this notion being well-defined since Λ is Γ invariant.

Notation. Throughout the remainder of the article, we fix a discrete non-elementary geometrically finite group Γ of isometries of some (irreducible) rank one symmetric space $\mathbb{H}^d_{\mathbb{K}}$ and denote by X the quotient G/Γ, where G is the isometry group of $\mathbb{H}^d_{\mathbb{K}}$.

2.2. **Standard horoballs.** Since parabolic points are fixed points of elements of Γ, Λ contains only countably many such points. Moreover, Γ contains at most finitely many conjugacy classes of parabolic subgroups. This translates to the fact that Λ_p consists of finitely many Γ orbits.

Let $\{p_1, \dots, p_s\} \subset \partial\mathbb{H}^d_{\mathbb{K}}$ be a maximal set of nonequivalent parabolic fixed points under the action of Γ. As a consequence of geometric finiteness of Γ, one can find a finite disjoint collection of *open* horoballs $H_1, \dots, H_s \subset \mathbb{H}^d_{\mathbb{K}}$ with the following properties (cf. [Bow93]):

(1) H_i is centered on p_i, for $i = 1, \dots, s$.
(2) $\overline{H_i}\Gamma \cap \overline{H_j}\Gamma = \emptyset$ for all $i \neq j$.
(3) For all $i \in \{1, \dots, s\}$ and $\gamma_1, \gamma_2 \in \Gamma$

$$\overline{H_i}\gamma_1 \cap \overline{H_i}\gamma_2 \neq \emptyset \Longrightarrow \overline{H_i}\gamma_1 = \overline{H_i}\gamma_2, \gamma_1^{-1}\gamma_2 \in \Gamma_{p_i}.$$

(4) $\mathrm{Hull}(\Lambda_\Gamma) \setminus (\bigcup_{i=1}^s H_i\Gamma)$ is compact mod Γ.

Remark 2.1. We shall assume throughout the remainder of the article that our fixed basepoint o lies outside these standard horoballs, i.e.

$$o \notin \bigcup_{i=1}^{s} \overline{H_i}\Gamma.$$

2.3. **Conformal Densities and the BMS Measure.** The *critical exponent*, denoted δ_Γ, is defined to be the infimum over all real number $s \geq 0$ such that the Poincaré series

$$P_\Gamma(s, o) := \sum_{\gamma \in \Gamma} e^{-s\mathrm{dist}(o, \gamma\cdot o)} \tag{2.1}$$

converges. We shall simply write δ for δ_Γ when Γ is understood from context. The Busemann function is defined as follows: given $x, y \in \mathbb{H}^d_{\mathbb{K}}$ and $\xi \in \partial\mathbb{H}^d_{\mathbb{K}}$, let $\gamma : [0, \infty) \to \mathbb{H}^d_{\mathbb{K}}$ denote a geodesic ray terminating at ξ and define

$$\beta_\xi(x, y) = \lim_{t\to\infty} \mathrm{dist}(x, \gamma(t)) - \mathrm{dist}(y, \gamma(t)).$$

A Γ-invariant conformal density of dimension s is a collection of Radon measures $\left\{\nu_x : x \in \mathbb{H}^d_{\mathbb{K}}\right\}$ on the boundary satisfying

$$\gamma_*\nu_x = \nu_{\gamma x}, \qquad \text{and} \qquad \frac{d\nu_y}{d\nu_x}(\xi) = e^{s\beta_\xi(x,y)}, \qquad \forall x, y \in \mathbb{H}^d_{\mathbb{K}}, \xi \in \partial\mathbb{H}^d_{\mathbb{K}}, \gamma \in \Gamma.$$

Given a pair of conformal densities $\{\mu_x\}$ and $\{\nu_x\}$ of dimensions s_1 and s_2 respectively, we can form a Γ invariant measure on $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$, denoted by $m^{\mu,\nu}$ as follows: for $x = (\xi_1, \xi_2, t) \in \mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$

$$dm^{\mu,\nu}(\xi_1, \xi_2, t) = e^{s_1\beta_{\xi_1}(o,x) + s_2\beta_{\xi_2}(o,x)}\, d\mu_o(\xi_1)\, d\nu_o(\xi_2)\, dt. \tag{2.2}$$

Moreover, the measure $m^{\mu,\nu}$ is invariant by the geodesic flow.

When Γ is geometrically finite and $\mathbb{K} = \mathbb{R}$, Patterson [Pat76] and Sullivan [Sul79] showed the existence of a unique (up to scaling) Γ-invariant conformal density of dimension δ_Γ, denoted $\left\{\mu^{\mathrm{PS}}_x : x \in \mathbb{H}^d_{\mathbb{R}}\right\}$. Geometric finiteness also implies that the measure $m^{\mu^{\mathrm{PS}},\mu^{\mathrm{PS}}}$ descends to a finite measure of full support on Ω and is the unique measure of maximal entropy for the geodesic flow. This measure is called the Bowen-Margulis-Sullivan (BMS for short) measure and is denoted $\mathrm{m}^{\mathrm{BMS}}$.

Since the fibers of the projection from G/Γ to $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}/\Gamma$ are compact and parametrized by the group M, we can lift such a measure to one G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, by taking locally the product with the Haar probability measure on M. Since M commutes with the geodesic flow, this lift is invariant under the group A. We refer the reader to [Rob03] and [PPS15] and references therein for details of the construction in much greater generality than that of $\mathbb{H}^d_{\mathbb{K}}$.

2.4. **Stable and unstable foliations and leafwise measures.** The fibers of the projection $G \to \mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ are given by the compact group M, which is the centralizer of A inside the maximal compact group K. In particular, we may lift $\mathrm{m}^{\mathrm{BMS}}$ to a measure on G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, and given locally by the product of $\mathrm{m}^{\mathrm{BMS}}$ with the Haar probability measure on M. The leafwise measures of $\mathrm{m}^{\mathrm{BMS}}$ on N^+ orbits are given as follows:

$$d\mu_x^u(n) = e^{\delta_\Gamma \beta_{(nx)^+}(o,nx)} d\mu_o^{\mathrm{PS}}((nx)^+). \tag{2.3}$$

They satisfy the following equivariance property under the geodesic flow:

$$\mu_{g_t x}^u = e^{\delta t} \mathrm{Ad}(g_t)_* \mu_x^u. \tag{2.4}$$

Moreover, it follows readily from the definitions that for all $n \in N^+$,

$$(n)_* \mu_{nx}^u = \mu_x^u, \tag{2.5}$$

where $(n)_*\mu_{nz}^u$ is the pushforward of μ_{nz}^u under the map $u \mapsto un$ from N^+ to itself. Finally, since M normalizes N^+ and leaves $\mathrm{m}^{\mathrm{BMS}}$ invariant, this implies that these conditionals are $\mathrm{Ad}(M)$-invariant:

$$\mu_{mx}^u = \mathrm{Ad}(m)_* \mu_x^u, \qquad m \in M. \tag{2.6}$$

2.5. **Cygan metrics.** We recall the definition of the Cygan metric on N^+, denoted d_{N^+}. These metrics are right invariant under translation by N^+, and satisfy the following convenient scaling property under conjugation by g_t. For all $r > 0$, if N_r^+ denotes the ball of radius r around identity in that metric and $t \in \mathbb{R}$, then

$$\mathrm{Ad}(g_t)(N_r^+) = N_{e^t r}^+. \tag{2.7}$$

To define the metric, we need some notation which we use throughout the article. For $x \in \mathbb{K}$, denote by $\bar{x}$ its $\mathbb{K}$-conjugate and by $|x| := \sqrt{\bar{x}x}$ its modulus. This modulus extends to a norm on $\mathbb{K}^n$ by setting

$$\|u\|^2 := \sum_i |u_i|^2, \qquad u = (u_1, \dots, u_n) \in \mathbb{K}^n.$$

We let $\mathrm{Im}\mathbb{K}$ denote those $x \in \mathbb{K}$ such that $\bar{x} = -x$. For example, $\mathrm{Im}\mathbb{K}$ is the pure imaginary numbers and the subspace spanned by the quaternions i, j and k in the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$ respectively. For $u \in \mathbb{K}$, we write $\mathrm{Re}(u) = (u + \bar{u})/2$ and $\mathrm{Im}(u) = (u - \bar{u})/2$.

The Lie algebra $\mathfrak{n}^+$ of N^+ splits under $\mathrm{Ad}(g_t)$ into eigenspaces as $\mathfrak{n}_\alpha^+ \oplus \mathfrak{n}_{2\alpha}^+$, where $\mathfrak{n}_{2\alpha}^+ = 0$ if and only if $\mathbb{K} = \mathbb{R}$. Moreover, we have the identification $\mathfrak{n}_\alpha^+ \cong \mathbb{K}^{d-1}$ and $\mathfrak{n}_{2\alpha}^+ \cong \mathrm{Im}(\mathbb{K})$ as real vector spaces; cf. [Mos73, Section 19]. We denote by $\|\cdot\|'$ the following quasi-norm on $\mathfrak{n}^+$:

$$\|(u,s)\|' := \left(\|u\|^4 + |s|^2\right)^{1/4}, \qquad (u,s) \in \mathfrak{n}_\alpha^+ \oplus \mathfrak{n}_{2\alpha}^+. \tag{2.8}$$

With this notation, the distance of $n := \exp(u,s)$ to identity is given by:

$$d_{N^+}(n, \mathrm{id}) := \|(u,s)\|'. \tag{2.9}$$

Given $n_1, n_2 \in N^+$, we set $d_{N^+}(n_1, n_2) = d_{N^+}(n_1 n_2^{-1}, \mathrm{id})$.

2.6. **Local stable holonomy.** We recall the definition of (stable) holonomy maps. We give a simplified discussion of this topic which is sufficient in our homogeneous setting. Let $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. Since the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism near identity, we can choose the norm on the Lie algebra so that the following holds. We can find maps $p^- : N_1^+ \to P^- = N^- AM$ and $u^+ : N_2^+ \to N^+$ so that

$$nu^- = p^-(n) u^+(n), \qquad \forall n \in N_2^+. \tag{2.10}$$

Then, it follows by (2.3) that for all $n \in N_2^+$, we have

$$d\mu_y^u(u^+(n)) = e^{-\delta \beta_{(nx)^+}(u^+(n)y, nx)} d\mu_x^u(n).$$

Moreover, by further scaling the metrics if necessary, we can ensure that these maps are diffeomorphisms onto their images. In particular, writing $\Phi(nx) = u^+(n)y$, we obtain the following change of variables formula: for all $f \in C(N_2^+)$,

$$\int f(n)\, d\mu_x^u(n) = \int f((u^+)^{-1}(n))e^{\delta\beta_{\Phi^{-1}(ny)^+}(ny,\Phi^{-1}(ny))}\, d\mu_y^u(n). \tag{2.11}$$

Remark 2.2. To avoid cluttering the notation with auxiliary constants, we shall assume that the N^- component of $p^-(n)$ belongs to N_2^- for all $n \in N_2^+$ whenever u^- belongs to N_1^-.

2.7. **Notational convention.** Throughout the article, given two quantities A and B, we use the Vinogradov notation $A \ll B$ to mean that there exists a constant $C \geq 1$, possibly depending on Γ and the dimension of G, such that $|A| \leq CB$. In particular, this dependence on Γ is suppressed in all of our implicit constants, except when we wish to emphasize it. The dependence on Γ may include for instance the diameter of the complement of our choice of cusp neighborhoods inside Ω and the volume of the unit neighborhood of Ω. We write $A \ll_{x,y} B$ to indicate that the implicit constant depends parameters x and y. We also write $A = O_x(B)$ to mean $A \ll_x B$.

3. Doubling Properties of Leafwise Measures

The goal of this section is to prove the following useful consequence of the global measure formula on the doubling properties of the leafwise measures. The result is an immediate consequence of Sullivan's shadow lemma in the case Γ is convex cocompact. In particular, the content of the following result is the uniformity, even in the case Ω is not compact. The argument is based on the topological transitivity of the geodesic flow when restricted to Ω.

Define the following exponents:

$$\begin{aligned} \Delta &:= \min\{\delta, 2\delta - k_{\max}, k_{\min}\}, \\ \Delta_+ &:= \max\{\delta, 2\delta - k_{\min}, k_{\max}\}. \end{aligned} \tag{3.1}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively. If Γ has no parabolic points, we set $k_{\max} = k_{\min} = \delta$, so that $\Delta = \Delta_+ = \delta$.

Proposition 3.1 (Global Doubling and Decay)**.** *For every $0 < \sigma \leq 5$, $x \in N_2^-\Omega$ and $0 < r \leq 1$, we have*

$$\mu_x^u(N_{\sigma r}^+) \ll \begin{cases} \sigma^{\Delta} \cdot \mu_x^u(N_r^+) & \forall 0 < \sigma \leq 1, 0 < r \leq 1, \\ \sigma^{\Delta_+} \cdot \mu_x^u(N_r^+) & \forall \sigma > 1, 0 < r \leq 5/\sigma. \end{cases}$$

Remark 3.2. The above proposition has very different flavor when applied with $\sigma < 1$, compared with $\sigma > 1$. In the former case, we obtain a global rate of decay of the measure of balls on the boundary, centered in the limit set. In the latter case, we obtain the so-called Federer property for our leafwise measures.

Remark 3.3. The restriction that $r \leq 5/\sigma$ in the case $\sigma > 1$ allows for a uniform implied constant. The proof shows that in fact, when $\sigma > 1$, the statement holds for any $0 < r \leq 1$, but with an implied constant depending on σ.

3.1. **Global Measure Formula.** Our basic tool in proving Proposition 3.1 is the extension of Sullivan's shadow lemma known as the global measure formula, which we recall in this section.

Given a parabolic fixed point $p \in \Lambda$, with stabilizer $\Gamma_p \subset \Gamma$, we define *the rank of* p to be twice the critical exponent of the Poincaré series $P_{\Gamma_p}(s, o)$ associated with Γ_p; cf. (2.1).

Given $\xi \in \partial\mathbb{H}_{\mathbb{K}}^d$, we let $[o\xi)$ denote the geodesic ray. For $t \in \mathbb{R}_+$, denote by $\xi(t)$ the point at distance t from o on $[o\xi)$. For $x \in \mathbb{H}_{\mathbb{K}}^d$, define the $\mathcal{O}(x)$ to be the *shadow* of unit ball $B(x,1)$ in $\mathbb{H}_{\mathbb{K}}^d$ on the boundary as viewed from o. More precisely,

$$\mathcal{O}(x) := \left\{\xi \in \partial\mathbb{H}_{\mathbb{K}}^d : [o\xi) \cap B(x,1) \neq \emptyset\right\}.$$

Shadows form a convenient, dynamically defined, collection of neighborhoods of points on the boundary.

The following generalization of Sullivan's shadow lemma gives precise estimates on the measures of shadows with respect to Patterson-Sullivan measures.

Theorem 3.4 (Theorem 3.2, [Sch04])**.** *There exists $C = C(\Gamma, o) \geq 1$ such that for every $\xi \in \Lambda$ and all $t > 0$,*

$$C^{-1} \leq \frac{\mu_o^{\mathrm{PS}}(\mathcal{O}(\xi(t)))}{e^{-\delta t} e^{d(t)(k(\xi(t))-\delta)}} \leq C,$$

where

$$d(t) = \mathrm{dist}(\xi(t), \Gamma \cdot o),$$

and $k(\xi(t))$ denotes the rank of a parabolic fixed point p if $\xi(t)$ is contained in a standard horoball centered at p and otherwise $k(\xi(t)) = \delta$.

A version of Theorem 3.4 was obtained earlier for real hyperbolic spaces in [SV95] and for complex and quaternionic hyperbolic spaces in [New03].

3.2. **Proof of Proposition 3.1.** Assume that $\sigma \leq 1$, the proof in the case $\sigma > 1$ is similar.

Fix a non-negative C^∞ bump function ψ supported inside N_1^+ and having value identically 1 on $N_{1/2}^+$. Given $\varepsilon > 0$, let $\psi_\varepsilon(n) = \psi(\mathrm{Ad}(g_{-\log\varepsilon})(n))$. Note that the condition that $\psi_\varepsilon(\mathrm{id}) = \psi(\mathrm{id}) = 1$ implies that for $x \in X$ with $x^+ \in \Lambda$,

$$\mu_x^u(\psi_\varepsilon) > 0, \qquad \forall \varepsilon > 0. \tag{3.2}$$

Note further that for any $r > 0$, we have that $\chi_{N_r^+} \leq \psi_r \leq \chi_{N_{2r}^+}$.

First, we establish a uniform bound over $x \in \Omega$. Consider the following function $f_\sigma : \Omega \to (0, \infty)$:

$$f_\sigma(x) = \sup_{0<r\leq 1} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)}.$$

We claim that it suffices to prove that

$$f_\sigma(x) \ll \sigma^\Delta, \tag{3.3}$$

uniformly over all $x \in \Omega$ and $0 < \sigma \leq 1$. Indeed, fix some $0 < r \leq 1$ and $0 < \sigma \leq 1$. By enlarging our implicit constant if necessary, we may assume that $\sigma \leq 1/4$. From the above properties of ψ, we see that

$$\mu_x^u(N_{\sigma r}^+) \leq \mu_x^u(\psi_{(4\sigma)(r/2)}) \ll \sigma^\Delta \mu_x^u(\psi_{r/2}) \leq \sigma^\Delta \mu_x^u(N_r^+).$$

Hence, it remains to prove (3.3). By [Rob03, Lemme 1.16], for each given $r > 0$, the map $x \mapsto \mu_x^u(\psi_{\sigma r})/\mu_x^u(\psi_r)$ is a continuous function on Ω. Indeed, the weak-$*$ continuity of the map $x \mapsto \mu_x^u$ is the reason we work with bump functions instead of indicator functions directly. Moreover, continuity of these functions implies that f_σ is lower semi-continuous.

The crucial observation regarding f_σ is as follows. In view of (2.4), we have for $t \geq 0$,

$$f_\sigma(g_t x) = \sup_{0<r\leq e^{-t}} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)} \leq f_\sigma(x).$$

Hence, for all $B \in \mathbb{R}$, the sub-level sets $\Omega_{<B} := \{f_\sigma < B\}$ are invariant by g_t for all $t \geq 0$. On the other hand, the restriction of the (forward) geodesic flow to Ω is topologically transitive. In particular, any invariant subset of Ω with non-empty interior must be dense in Ω. Hence, in view of the lower semi-continuity of f_σ, to prove (3.3), it suffices to show that f_σ satisfies (3.3) for all x in some open subset of Ω.

Recall we fixed a basepoint $o \in \mathbb{H}_{\mathbb{K}}^d$ belonging to the hull of the limit set. Let $x_o \in G$ denote a lift of o whose projection to G/Γ belongs to Ω. Let E denote the unit neighborhood of x_o. We show that $E \cap \Omega \subset \left\{ f_\sigma \ll \sigma^\Delta \right\}$. Without loss of generality, we may further assume that $\sigma < 1/2$, by enlarging the implicit constant if necessary.

First, note that the definition of the conditional measures μ_x^u immediately gives

$$\mu_x^u|_{N_4^+} \asymp \mu_o^{\mathrm{PS}}|_{(N_4^+ \cdot x)^+}, \qquad \forall x \in E.$$

It follows that

$$\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+) \ll \mu_x^u(\psi_r) \ll \mu_o^{\mathrm{PS}}((N_{2r}^+ \cdot x)^+),$$

for all $0 \leq r \leq 2$ and $x \in E$. Hence, it will suffice to show that for all $0 < \sigma < 1$,

$$\frac{\mu_o^{\mathrm{PS}}((N_{\sigma r}^+ \cdot x)^+)}{\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+)} \ll \sigma^\Delta.$$

To this end, there is a constant $C_1 \geq 1$ such that the following holds; cf. [Cor90, Theorem 2.2][1]. For all $x \in E$, if $\xi = x^+$, then, the shadow $S_r = \{(nx)^+ : n \in N_r^+\}$ satisfies

$$\mathcal{O}(\xi(|\log r| + C_1)) \subseteq S_r \subseteq \mathcal{O}(\xi(|\log r| - C_1)), \qquad \forall 0 < r \leq 2. \tag{3.4}$$

Here, and throughout the rest of the proof, if $s \leq 0$, we use the convention

$$\mathcal{O}(\xi(s)) = \mathcal{O}(\xi(0)) = \partial \mathbb{H}_{\mathbb{K}}^d.$$

Fix some arbitrary $x \in E$ and let $\xi = x^+$. To simplify notation, set for any $t, r > 0$,

$$t_\sigma := \max\{|\log \sigma r| - C_1, 0\}, \qquad t_r := |\log r| + C_1,$$
$$d(t) := \mathrm{dist}(\xi(t), \Gamma \cdot o), \qquad k(t) := k(\xi(t)),$$

where $k(\xi(t))$ is as in the notation of Theorem 3.4.

By further enlarging the implicit constant, we may assume for the rest of the argument that

$$-\log \sigma > 2C_1.$$

This insures that $t_\sigma \geq t_r$ and avoids some trivialities.

Let $0 < r \leq 1$ be arbitrary. We define constants $\sigma_0 := \sigma \leq \sigma_1 \leq \sigma_2 \leq \sigma_3 := 1$ as follows. If $\xi(t_\sigma)$ is in the complement of the cusp neighborhoods, we set $\sigma_1 = \sigma$. Otherwise, we define σ_1 by the property that $\xi(|\log \sigma_1 r|)$ is the first point along the geodesic segment joining $\xi(t_\sigma)$ and $\xi(t_r)$ (traveling from the former point to the latter) meets the boundary of the horoball containing $\xi(t_\sigma)$. Similarly, if $\xi(t_r)$ is outside the cusp neighborhoods, we set $\sigma_2 = 1$. Otherwise, we define σ_2 by the property that $\xi(|\log \sigma_2 r|)$ is the first point along the same segment, now traveling from $\xi(t_r)$ towards $\xi(t_\sigma)$, which intersects the boundary of the horoball containing $\xi(t_r)$. Define

$$t_{\sigma_0} := t_\sigma, \qquad t_{\sigma_3} := t_r, \qquad t_{\sigma_i} := |\log \sigma_i r| \quad \text{for } i = 1, 2.$$

In this notation, we first observe that $k(t_{\sigma_1}) = k(t_{\sigma_2}) = \delta$. In particular, Theorem 3.4 yields

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \ll \left(\frac{\sigma_1}{\sigma_2}\right)^\delta.$$

Note further that since geodesics in $\mathbb{H}_{\mathbb{K}}^d$ are unique distance minimizers, we have that the distance between $\xi(t_{\sigma_i})$ and $\xi(t_{\sigma_{i+1}})$ is equal to $|t_{\sigma_i} - t_{\sigma_{i+1}}|$, for $i = 0, 2$. Moreover, by our choice of basepoint o and standard horoballs (cf. Remark 2.1), we have that

$$\Gamma \cdot o \cap \bigcup_{j=1}^s H_j = \emptyset.$$

[1]The quoted result in [Cor90] is stated in terms of the so-called Carnot-Caratheodory metric d_{cc} on N^+, which enjoys the same scaling property in (2.7). In particular, this metric is Lipschitz equivalent to the Cygan metric in (2.9) by compactness of the unit sphere in the latter and continuity of the map $n \mapsto d_{cc}(n, \mathrm{id})$.

Let $H(\sigma_0)$ denote the element of the collection of standard horoballs $\Gamma \cdot H_j$, $j = 1, \ldots, s$, which contains the point $\xi(t_{\sigma_0})$ if the latter point is inside a cusp neighborhood, and otherwise set $H(\sigma_0)$ to be the unit ball around o. Then, there is a constant $C_2 \geq 1$, depending only on on the constant C_1 as well as the distance between the orbit $\Gamma \cdot o$ and the standard horoballs H_j, such that

$$\begin{aligned} d(t_{\sigma_0}) &\leq \operatorname{dist}(\xi(t_{\sigma_0}), \partial H(\sigma_0)) + \operatorname{dist}(\partial H(\sigma_0), \Gamma \cdot o) \\ &\leq \operatorname{dist}(\xi(t_{\sigma_0}), \xi(t_{\sigma_1})) + \operatorname{dist}(\partial H(\sigma_0), \Gamma \cdot o) \leq -\log(\sigma_0/\sigma_1) + C_2, \end{aligned}$$

where $\partial H(\sigma_0)$ denotes the boundary of $H(\sigma_0)$. Similarly, we also get that

$$d(t_{\sigma_3}) \leq \operatorname{dist}(\xi(t_{\sigma_2}), \xi(t_{\sigma_3})) + C_2 \leq -\log(\sigma_2/\sigma_3) + C_2.$$

Hence, it follows using Theorem 3.4 and the above discussion that

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \ll \left(\frac{\sigma_0}{\sigma_1}\right)^{\delta} e^{d(t_{\sigma_0})(k(t_{\sigma_0})-\delta)} \ll \begin{cases} \left(\frac{\sigma_0}{\sigma_1}\right)^{2\delta - k(t_{\sigma_0})} & \text{if } k(t_{\sigma_0}) \geq \delta, \\ \left(\frac{\sigma_0}{\sigma_1}\right)^{\delta} & \text{otherwise.} \end{cases}$$

Similarly, we obtain

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_3 r})} \ll \left(\frac{\sigma_2}{\sigma_3}\right)^{\delta} e^{-d(t_{\sigma_3})(k(t_{\sigma_3})-\delta)} \ll \begin{cases} \left(\frac{\sigma_2}{\sigma_3}\right)^{k(t_{\sigma_3})} & \text{if } k(t_{\sigma_3}) \leq \delta, \\ \left(\frac{\sigma_2}{\sigma_3}\right)^{\delta} & \text{otherwise.} \end{cases}$$

In all cases, we get for $i = 0, 1, 2$ that

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_i r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_{i+1} r})} \ll \left(\frac{\sigma_i}{\sigma_{i+1}}\right)^{\Delta},$$

where Δ is as in the statement of the proposition. Therefore, using the following trivial identity

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma r})}{\mu_o^{\mathrm{PS}}(S_r)} = \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_r)},$$

we see that $f(x) \ll \sigma^{\Delta}$. As $x \in E$ was arbitrary, we find that $E \subset \left\{f_\sigma \ll \sigma^{\Delta}\right\}$, thus concluding the proof in the case $\sigma \leq 1$. Note that in the case $\sigma > 1$, the constants σ_i satisfy $\sigma_i/\sigma_{i+1} \geq 1$, so that combining the 3 estimates requires taking the maximum over the exponents, yielding the bound with Δ_+ in place of Δ in this case.

Now, let $r \in (0, 1]$ and suppose $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. By [Cor90, Theorem 2.2], the analog of (3.4) holds, but with shadows from the viewpoint of x and y, in place of the fixed basepoint o. Recalling the map $n \mapsto u^+(n)$ in (2.10), one checks that this implies that this map is Lipschitz on N_1^+ with respect to the Cygan metric, with Lipschitz constant $\asymp C_1$. Moreover, the Jacobian of the change of variables associated to this map with respect to the measures μ_x^u and μ_y^u is bounded on N_1^+, independently of y and u^-; cf. (2.11) for a formula for this Jacobian. Hence, the estimates for $x \in N_2^- \Omega$ follow from their counterparts for points in Ω.

4. Margulis Functions In Infinite Volume

We construct Margulis functions on Ω which allow us to obtain quantitative recurrence estimates to compact sets. Our construction is similar to the one in [BQ11] in the case of lattices in rank 1 groups. We use geometric finiteness of Γ to establish the analogous properties more generally. The idea of Margulis functions originated in [EMM98].

Throughout this section, we assume Γ is a non-elementary, geometrically finite group containing parabolic elements. The following is the main result of this section. A similar result in the special case of quotients of $\mathrm{SL}_2(\mathbb{R})$ follows from combining Lemma 9.9 and Proposition 7.6 in [MO23].

Theorem 4.1. *Let $\Delta > 0$ denote the constant in (3.1). For every $0 < \beta < \Delta/2$, there exists a proper function $V_\beta : N_1^-\Omega \to \mathbb{R}_+$ such that the following holds. There is a constant $c \geq 1$ such that for all $x \in N_1^-\Omega$ and $t \geq 0$,*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_t n x)\, d\mu_x^u(n) \leq c e^{-\beta t} V_\beta(x) + c.$$

Our key tool in establishing Theorem 4.1 is Proposition 4.2, which is a statement regarding average expansion of vectors in linear represearntations of G. The fractal nature of the conditional measures μ_x^u poses serious difficulties in establishing this latter result.

4.1. **Construction of Margulis functions.** Let $p_1, \ldots, p_d \in \Lambda$ be a maximal set of inequivalent parabolic fixed points and for each i, let Γ_i denote the stabilizer of p_i in Γ. Let $P_i < G$ denote the parabolic subgroup of G fixing p_i. Denote by U_i the unipotent radical of P_i and by A_i a maximal $\mathbb{R}$-split torus inside P_i. Then, each U_i is a maximal connected unipotent subgroup of G admitting a closed (but not necessarily compact) orbit from identity in G/Γ. As all maximal unipotent subgroups of G are conjugate, we fix elements $h_i \in G$ so that $h_i U_i h_i^{-1} = N^+$. Note further that G admits an Iwasawa decomposition of the form $G = K A_i U_i$ for each i, where K is our fixed maximal compact subgroup.

Denote by W the adjoint representation of G on its Lie algebra. The specific choice of representation is not essential for the construction, but is convenient for making some parameters more explicit. We endow W with a norm that is invariant by K.

Let $0 \neq v_0 \in W$ denote a vector that is fixed by N^+. In particular, v_0 is a highest weight vector for the diagonal group A (with respect to the ordering determined by declaring the roots in N^+ to be positive). Let $v_i = h_i v_0 / \|h_i v_0\|$. Note that each of the vectors v_i is fixed by U_i and is a weight vector for A_i. In particular, there is an additive character $\chi_i : A_i \to \mathbb{R}$ such that

$$a \cdot v_i = e^{\chi_i(a)} v_i, \qquad \forall a \in A_i. \tag{4.1}$$

We denote by A_i^+ the subsemigroup of A_i which expands U_i (i.e. the positive Weyl chamber determined by U_i). We let $\alpha_i : A_i \to \mathbb{R}$ denote the simple root of A_i in $\mathrm{Lie}(U_i)$. Then,

$$\chi_i = \chi_{\mathbb{K}} \alpha_i, \qquad \chi_{\mathbb{K}} = \begin{cases} 1, & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}. \end{cases} \tag{4.2}$$

Given $\beta > 0$, we define a function $V_\beta : G/\Gamma \to \mathbb{R}_+$ as follows:

$$V_\beta(g\Gamma) := \max_{w \in \bigcup_{i=1}^d g\Gamma \cdot v_i} \|w\|^{-\beta/\chi_{\mathbb{K}}}. \tag{4.3}$$

The fact that $V_\beta(g\Gamma)$ is indeed a maximum will follow from Lemma 4.6.

4.2. **Linear expansion.** The following result is our key tool in establishing the contraction estimate on V_β in Theorem 4.1.

Proposition 4.2. *For every $0 \leq \beta < \Delta/2$, there exists $C = C(\beta) \geq 1$ so that for all $t > 0$, $x \in N_1^-\Omega$, and all non-zero vectors v in the orbit $G \cdot v_0 \subset W$, we have*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathbb{K}}}\, d\mu_x^u(n) \leq C e^{-\beta t} \|v\|^{-\beta/\chi_{\mathbb{K}}}.$$

We postpone the proof of Proposition 4.2 to Section 5. Let $\pi_+ : W \to W^+$ denote the projection onto the highest weight space of g_t. The difficulty in the proof of Proposition 4.2 beyond the case $G = \mathrm{SL}_2(\mathbb{R})$ lies in controlling the *shape* of the subset of N^+ on which $\|\pi_+(n \cdot v)\|$ is small, so that we may apply the decay results from Proposition 3.1, that are valid only for balls of the form N_ε^+. We deal with this problem by using a convexity trick. A suitable analog of the above result holds for any non-trivial linear representation of G.

The following proposition establishes several geometric properties of the functions V_β which are useful in proving, and applying, Theorem 4.1. This result is proved in Section 4.4.

Proposition 4.3. *Suppose V_β is as in (4.3). Then,*

(1) For every x in the unit neighborhood of Ω, we have that

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathbb{K}}/\beta}(x),$$

where $\mathrm{inj}(x)$ denotes the injectivity radius at x. In particular, V_β is proper on Ω.

(2) For all $g \in G$ and all $x \in X$,

$$\|g\|^{-\beta} V_\beta(x) \leq V_\beta(gx) \leq \left\|g^{-1}\right\|^{\beta} V_\beta(x).$$

(3) There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_0$.

4.3. **Proof of Theorem 4.1.** In this section, we use Proposition 4.3 to translate the linear expansion estimates in Proposition 4.2 into a contraction estimate for the functions V_β.

Let $t_0 > 0$ be be given and define

$$\omega_0 := \sup_{n \in N_1^+} \max\left\{ \|g_{t_0} n\|^{1/\chi_{\mathbb{K}}}, \left\|(g_{t_0} n)^{-1}\right\|^{1/\chi_{\mathbb{K}}} \right\},$$

where $\|\cdot\|$ denotes the operator norm of the action of G on W. Then, for all $n \in N_1^+$ and all $x \in X$, we have

$$\omega_0^{-1} V_1(x) \leq V_1(g_{t_0} n x) \leq \omega_0 V_1(x), \tag{4.4}$$

where $V_1 = V_\beta$ for $\beta = 1$.

Let ε_0 be as in Proposition 4.3(3). Suppose $x \in X$ is such that $V_1(x) \leq \omega_0/\varepsilon_0$. Then, by (4.4), for any $\beta > 0$, we have that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x)\, d\mu_x^u(n) \leq B_0 := (\omega_0^2 \varepsilon_0^{-1})^\beta. \tag{4.5}$$

Now, suppose $x \in N_1^- \Omega$ is such that $V_1(x) \geq \omega_0/\varepsilon_0$ and write $x = g\Gamma$ for some $g \in G$. Then, by Proposition 4.3(3), there exists a unique vector $v_\star \in \bigcup_i g\Gamma \cdot v_i$ satisfying $V_1(x) = \|v_\star\|^{-1/\chi_{\mathbb{K}}}$. Moreover, by (4.4), we have that $V_1(g_{t_0} n x) \geq 1/\varepsilon_0$ for all $n \in N_1^+$. And, by definition of ω_0, for all $n \in N_1^+$, $\|g_{t_0} n v_\star\|^{1/\chi_{\mathbb{K}}} \leq \varepsilon_0$. Thus, applying Proposition 4.3(3) once more, we see that $g_{t_0} n v_\star$ is the unique vector in $\bigcup_i g_{t_0} n g\Gamma \cdot v_i$ satisfying

$$V_\beta(g_{t_0} n x) = \|g_{t_0} n v_\star\|^{-1/\chi_{\mathbb{K}}}, \qquad \forall n \in N_1^+.$$

Moreover, since the vectors v_i all belong to the G-orbit of v_0, it follows that $v_\star$ also belongs to $G \cdot v_0$. Thus, we may apply Proposition 4.2 as follows. Fix some $\beta > 0$ and let $C = C(\beta) \geq 1$ be the constant in the conclusion of the proposition. Then,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x) d\mu_x^u = \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_{t_0} n v_\star\|^{-\beta/\chi_{\mathbb{K}}} d\mu_x^u \leq C e^{-\beta t_0} \|v_\star\|^{-\beta/\chi_{\mathbb{K}}} = C e^{-\beta t_0} V_\beta(x).$$

Combining this estimate with (4.5), we obtain for any fixed t_0,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x)\, d\mu_x^u(n) \leq C e^{-\beta t_0} V_\beta(x) + B_0, \tag{4.6}$$

for all $x \in \Omega$. We claim that there is a constant $c_1 = c_1(\beta) > 0$ such that, if t_0 is large enough, depending on β, then

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{k t_0} n x)\, d\mu_x^u(n) \leq c_1^k e^{-\beta k t_0} V_\beta(x) + 3 c_1 B_0, \tag{4.7}$$

for all $k \in \mathbb{N}$. By Proposition 4.3, this claim completes the proof since $V_\beta(g_t y) \ll V_\beta(g_{\lfloor t/t_0 \rfloor t_0} y)$, for all $t \geq 0$ and $y \in X$, with an implied constant depending only on t_0 and β.

The proof of (4.7) is by now a standard argument, with the key ingredient in carrying it out being the doubling estimate Proposition 3.1. We proceed by induction. Let $k \in \mathbb{N}$ be arbitrary and assume that (4.7) holds for such k. Let $\left\{ n_i \in \mathrm{Ad}(g_{kt_0})(N_1^+) : i \in I \right\}$ denote a finite collection of points in the support of $\mu^u_{g_{kt_0}x}$ such that $N_1^+ n_i$ covers the part of the support inside $\mathrm{Ad}(g_{kt_0})(N_1^+)$. We can find such a cover with uniformly bounded multiplicity, depending only on N^+. That is

$$\sum_{i \in I} \chi_{N_1^+ n_i}(n) \ll \chi_{\cup_i N_1^+ n_i}(n), \qquad \forall n \in N^+.$$

Let $x_i = n_i g_{kt_0} x$. By (4.6), and a change of variable, cf. (2.4) and (2.5), we obtain

$$e^{\delta k t_0} \int_{N_1^+} V_\beta(g_{(k+1)t_0} n x) \, d\mu^u_x \leq \sum_{i \in I} \int_{N_1^+} V_\beta(g_{t_0} n x_i) \, d\mu^u_{x_i} \leq \sum_{i \in I} \mu^u_{x_i}(N_1^+) \left(C e^{-\beta t_0} V_\beta(x_i) + B_0 \right).$$

It follows using Proposition 4.3 that $\mu^u_y(N_1^+) V_\beta(y) \ll \int_{N_1^+} V_\beta(ny) \, d\mu^u_y(n)$ for all $y \in X$. Hence,

$$\int_{N_1^+} V_\beta(g_{(k+1)t_0} n x) \, d\mu^u_x(n) \ll e^{-\delta k t_0} \sum_{i \in I} \int_{N_1^+} \left(C e^{-\beta t_0} V_\beta(n x_i) + B_0 \right) \, d\mu^u_{x_i}(n).$$

Note that since g_t expands N^+ by at least e^t, we have

$$\mathcal{A}_k := \mathrm{Ad}(g_{-kt_0}) \left(\bigcup_i N_1^+ n_i \right) \subseteq N_2^+.$$

Using bounded multiplicity property of the cover, for any non-negative function φ, we have

$$\sum_{i \in I} \int_{N_1^+} \varphi(n x_i) \, d\mu^u_{x_i} = \int_{N^+} \varphi(n g_{kt_0} x) \sum_{i \in I} \chi_{N_1^+ n_i}(n) \, d\mu^u_{g_{kt_0}x} \ll \int_{\bigcup_i N_1^+ n_i} \varphi(n g_{kt_0} x) \, d\mu^u_{g_{kt_0}x}.$$

Changing variables back so the integrals take place against μ^u_x, we obtain

$$\begin{aligned} e^{-\delta k t_0} \sum_{i \in I} \int_{N_1^+} \left(C e^{-\beta t_0} V_\beta(n x_i) + B_0 \right) \, d\mu^u_{x_i} &\ll \int_{\mathcal{A}_k} \left(C e^{-\beta t_0} V_\beta(g_{kt_0} n x) + B_0 \right) \, d\mu^u_x \\ &\leq C e^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} n x) \, d\mu^u_x + B_0 \mu^u_x(N_2^+). \end{aligned}$$

To apply the induction hypothesis, we again pick a cover of N_2^+ by balls of the form $N_1^+ n$, for a collection of points $n \in N_2^+$ in the support of μ^u_x. We can arrange for such a collection to have a uniformly bounded cardinality and multiplicity. By essentially repeating the above argument, and using our induction hypothesis for k, in addition to the doubling property in Prop. 3.1, we obtain

$$C e^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} n x) \, d\mu^u_x + B_0 \mu^u_x(N_2^+) \ll (C c_1^k e^{-\beta(k+1)t_0} V_\beta(x) + 2 B_0 C e^{-\beta t_0} + B_0) \mu^u_x(N_1^+),$$

where we also used Prop. 4.3 to ensure that $V_\beta(nx) \ll V_\beta(x)$, for all $n \in N_3^+$. Taking c_1 to be larger than the product of C with all the uniform implied constants accumulated thus far in the argument, we obtain

$$\frac{1}{\mu^u_x(N_1^+)} \int_{N_1^+} V_\beta(g_{(k+1)t_0} n x) \, d\mu^u_x(n) \leq c_1^{k+1} e^{-\beta(k+1)t_0} V_\beta(x) + 2 c_1 e^{-\beta t_0} B_0 + c_1 B_0.$$

This completes the proof.

4.4. **Geometric properties of Margulis functions and proof of Proposition 4.3.** In this section, we give a geometric interpretation of the functions V_β which allows us to prove Proposition 4.3. Item (2) follows directly from the definitions, so we focus on the remaining properties.

The data in the definition of V_β allows us to give a linear description of cusp neighborhoods as follows. Given $g \in G$ and i, write $g = kau$ for some $k \in K$, $a \in A_i$ and $u \in U_i$. Geometrically, the size of the A component in the Iwasawa decomposition $G = KA_iU_i$ corresponds to the value of the Busemann cocycle $|\beta_{p_i}(Kg, o)|$, where Kg is the image of g in $K\backslash G$; cf. [BQ16, Remark 6.5] and the references therein for the precise statement. This has the following consequence. We can find $0 < \varepsilon_i < 1$ such that

$$\left\|\mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)}\right\| < \varepsilon_i \Longleftrightarrow Kg \in H_{p_i}, \tag{4.8}$$

where H_{p_i} is the standard horoball based at p_i in $\mathbb{H}^d_{\mathbb{K}} \cong K\backslash G$.

The functions $V_\beta(x)$ roughly measure how far into the cusp x is. More precisely, we have the following lemma.

Lemma 4.4. *The restriction of V_β to any bounded neighborhood of Ω is a proper map.*

Proof. In view of Property (2) of Proposition 4.3, it suffices to prove that V_β is proper on Ω. Now, suppose that for some sequence $g_n \in G$, we have $g_n\Gamma$ tends to infinity in Ω. Then, since Γ is geometrically finite, this implies that the injectivity radius at $g_n\Gamma$ tends to 0. Hence, after passing to a subsequence, we can find $\gamma_n \in \Gamma$ such that $g_n\gamma_n$ belongs to a single horoball among the horoballs constituting our fixed standard cusp neighborhood; cf. Section 2.2. By modifying γ_n on the right by a fixed element in Γ if necessary, we can assume that $Kg_n\gamma_n$ converges to one of the parabolic points p_i (say p_1) on the boundary of $\mathbb{H}^d_{\mathbb{K}} \cong K\backslash G$.

Moreover, geometric finiteness implies that $(\Lambda_\Gamma \setminus \{p_1\})/\Gamma_1$ is compact. Thus, by multiplying $g_n\gamma_n$ by an element of Γ_1 on the right if necessary, we may assume that $(g_n\gamma_n)^-$ belongs to a fixed compact subset of the boundary, which is disjoint from $\{p_1\}$.

Thus, for all large n, we can write $g_n\gamma_n = k_na_nu_n$, for $k_n \in K$, $a_n \in A_i$ and $u_n \in U_i$, such that the eigenvalues of $\mathrm{Ad}(a_n)$ are bounded above; cf. (4.8). Moreover, as $(g_n\gamma_n)^-$ belongs to a compact set that is disjoint from $\{p_1\}$ and $(g_n\gamma_n)^+ \to p_1$, the set $\{u_n\}$ is bounded. To show that $V_\beta(g_n\Gamma) \to \infty$, since U_i fixes v_i and K is a compact group, it remains to show that a_n contracts v_i to 0. Since $g_n\gamma_n$ is unbounded in G while k_n and u_n remain bounded, this shows that the sequence a_n is unbounded. Upper boundedness of the eigenvalues of $\mathrm{Ad}(a_n)$ thus implies the claim. ☐

Remark 4.5. The above lemma is false without restricting to Ω in the case Γ has infinite covolume since the injectivity radius is not bounded above on G/Γ. Note also that this lemma is false in the case Γ is not geometrically finite, since the complement of cusp neighborhoods inside Ω is compact if and only if Γ is geometrically finite.

The next crucial property of the functions V_β is the following linear manifestation of the existence of cusp neighborhoods consisting of disjoint horoballs. This lemma implies Proposition 4.3(3).

Lemma 4.6. *There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_0$.*

Remark 4.7. The constant ε_0 roughly depends on the distance from a fixed basepoint to the cusp neighborhoods.

Proof of Lemma 4.6. Let $g \in G$ and i be given. Write $g = kau$, for some $k \in K$, $a \in A_i$ and $u \in U_i$. Since U_i fixes v_i and the norm on W is K-invariant, we have $\|g \cdot v_i\| = \|a \cdot v_i\| = e^{\chi_i(a)}$; cf. (4.1). Moreover, since W is the adjoint representation, we have

$$\left\|\mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)}\right\| \asymp e^{\chi_i(a)},$$

and the implied constant, denoted C, depends only on the norm on the Lie algebra.

Let $0 < \varepsilon_i < 1$ be the constants in (4.8) and define $\varepsilon_0 := \min_i \varepsilon_i / C$. Let $x = g\Gamma \in G/\Gamma$. Suppose that there are elements $\gamma_1, \gamma_2 \in \Gamma$ and vectors v_{i_1}, v_{i_2} in our finite fixed collection of vectors v_i such that $\left\| g\gamma_j \cdot v_{i_j} \right\| < \varepsilon_0$ for $j = 1, 2$. Then, the above discussion, combined with the choice of ε_i in (4.8), imply that $Kg\gamma_j$ belongs to the standard horoball H_j in $\mathbb{H}^d_{\mathbb{K}}$ based at p_{i_j}. However, this implies that the two standard horoballs $H_1\gamma_1^{-1}$ and $H_2\gamma_2^{-1}$ intersect non-trivially. By choice of these standard horoballs, this implies that the two horoballs $H_j\gamma_j^{-1}$ are the same and that the two parabolic points p_{i_j} are equivalent under Γ. In particular, the two vectors v_{i_1}, v_{i_2} are in fact the same vector, call it v_{i_0}. It also follows that $\gamma_1^{-1}\gamma_2$ sends H to itself and fixes the parabolic point it is based at. Thus, $\gamma_1^{-1}\gamma_2$ fixes v_{i_0} by definition. But, then, we get that

$$g\gamma_2 \cdot v_{i_0} = g\gamma_1(\gamma_1^{-1}\gamma_2) \cdot v_{i_0} = g\gamma_1 \cdot v_{i_0}.$$

This proves uniqueness of the vector in $\bigcup_i g\Gamma \cdot v_i$ of norm $\leq \varepsilon_0$, if it exists, and concludes the proof. □

The following lemma verifies Proposition 4.3(1) relating the injectivity radius to V_β.

Lemma 4.8. *For all x in the unit neighborhood of Ω, we have*

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathbb{K}}/\beta}(x),$$

where $\chi_{\mathbb{K}}$ is given in (4.2).

Proof. Let $x \in \Omega$ and set $\tilde{x}_0 = Kx$. Let $x_0 \in K\backslash G \cong \mathbb{H}^d_{\mathbb{K}}$ denote a lift of $\tilde{x}_0$. Then, x_0 belongs to the hull of the limit set of Γ; cf. Section 2.

Since $\mathrm{inj}(\cdot)^{-1}$ and V_β are uniformly bounded above and below on the complement of the cusp neighborhoods inside Ω, it suffices to prove the lemma under the assumption that x_0 belongs to some standard horoball H based at a parabolic fixed point p. We may also assume that the lift x_0 is chosen so that p is one of our fixed finite set of inequivalent parabolic points $\{p_i\}$.

Geometric finiteness of Γ implies that there is a compact subset $\mathcal{K}_p$ of $\partial\mathbb{H}^d_{\mathbb{K}} \setminus \{p\}$, depending only on the stabilizer Γ_p in Γ, with the following property. Every point in the hull of the limit set is equivalent, under Γ_p, to a point on the set of geodesics joining p to points in $\mathcal{K}_p$. Thus, after adjusting x_0 by an element of Γ_p if necessary, we may assume that x_0 belongs to this set. In particular, we can find $g \in G$ so that $x_0 = Kg$ and g can be written as kau in the Iwasawa decomposition associated to p, for some $k \in K, a \in A_p$, and $u \in U_p$[2] with the property that $\mathrm{Ad}(a)$ is contracting on U_p and u is of uniformly bounded size.

Note that it suffices to prove the statement assuming the injectivity radius of x is sufficiently small, depending only on the metric on G, while the distance of x_0 to the boundary of the cusp horoball H_p is at least 1. Now, let $\gamma \in \Gamma$ be a non-trivial element such that $x_0\gamma$ is at distance at most $2\mathrm{inj}(x)$ from x_0. Then, this implies that both x_0 and $x_0\gamma$ belong to H_p. Let $v = \gamma - \mathrm{id}$. In view of the discreteness of Γ, we have that $\|v\| \gg 1$. Since the exponential map is close to an isometry near the origin, we see that

$$\mathrm{dist}(g\gamma g^{-1}, \mathrm{id}) \asymp \left\| g\gamma g^{-1} - \mathrm{id} \right\| = \left\| gvg^{-1} \right\| = \|\mathrm{Ad}(au)(v)\| \geq e^{\chi_{\mathbb{K}}\alpha(a)} \|\mathrm{Ad}(u)(v)\|,$$

where $\chi_{\mathbb{K}}$ is given in (4.2) and we used K-invariance of the norm. Here, α is the simple root of A_p in the Lie algebra of U_p and $e^{\chi_{\mathbb{K}}\alpha(a)}$ is the smallest eigenvalue of $\mathrm{Ad}(a)$ on the Lie algebra of the parabolic group stabilizing p. Note that since x_0 belongs to H_p, $\alpha(a)$ is strictly negative.

Recalling that u belongs to a uniformly bounded neighborhood of identity in G and that $\|v\| \gg 1$, it follows that $\mathrm{dist}(g\gamma g^{-1}, \mathrm{id}) \gg e^{\chi_{\mathbb{K}}\alpha(a)}$. Since γ was arbitrary, this shows that the injectivity radius at x satisfies the same lower bound.

[2]The groups A_p and U_p were defined at the beginning of the section.

Finally, let $v_p \in \{v_i\}$ denote the vector fixed by U_p. Using the above Iwasawa decomposition, we see that $V_\beta^{1/\beta}(x) \geq \|av_p\|^{-1/\chi_{\mathbb{K}}} = e^{-\chi_p(a)/\chi_{\mathbb{K}}}$, where χ_p is the character on A_p determined by v_p, cf. (4.1). This concludes the proof in view of (4.2) and the fact that $\chi_p = \chi_{\mathbb{K}}\alpha$. □

Finally, we record the following useful quantitative form of Lemma 4.4 which follows by similar arguments to those discussed in this section. We leave the details to the reader.

Lemma 4.9. *For all x in a bounded neighborhood of Ω, we have $e^{\mathrm{dist}(x,o)} \ll V_\beta(x)^{O_\beta(1)}$.*

5. Shadow Lemmas, Convexity, and Linear Expansion

The goal of this section is to prove Proposition 4.2 estimating the average rate of expansion of vectors with respect to leafwise measures. This completes the proof of Theorem 4.1.

5.1. **Proof of Proposition 4.2.** We may assume without loss of generality that $\|v\| = 1$. Let W^+ denote the highest weight subspace of W for $A_+ = \{g_t : t > 0\}$. Denote by π_+ the projection from W onto W^+. In our choice of representation W, the eigenvalue of A_+ in W^+ is $e^{\chi_{\mathbb{K}}t}$, , where $\chi_{\mathbb{K}}$ is given in (4.2). It follows that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u(n) \leq e^{-\beta t} \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|\pi_+(n \cdot v)\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u(n).$$

Hence, it suffices to show that, for a suitable choice of β, the integral on the right side is uniformly bounded, independently of v and x (but possibly depending on β).

For simplicity, set $\beta_{\mathbb{K}} = \beta/\chi_{\mathbb{K}}$. A simple application of Fubini's Theorem yields

$$\int_{N_1^+} \|\pi_+(n \cdot v)\|^{-\beta_{\mathbb{K}}} \, d\mu_x^u(n) = \int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n \cdot v)\|^{\beta_{\mathbb{K}}} \leq t^{-1}\Big) \, dt.$$

For $v \in W$, we define a polynomial map on N^+ by $n \mapsto p_v(n) := \|\pi_+(n \cdot v)\|^2$ and set

$$S(v, \varepsilon) := \left\{n \in N^+ : p_v(n) \leq \varepsilon\right\}.$$

To apply Proposition 3.1, we wish to efficiently estimate the radius of a ball in N^+ containing the sublevel sets $S\big(v, t^{-2/\beta_{\mathbb{K}}}\big) \cap N_1^+$. We have the following claim.

Claim 5.1. There exists a constant $C_0 > 0$, such that, for all $\varepsilon > 0$, the diameter of $S(v, \varepsilon) \cap N_1^+$ is at most $C_0 \varepsilon^{1/4\chi_{\mathbb{K}}}$.

We show how this claim concludes the proof. By estimating the integral over $[0, 1]$ trivially, we get

$$\int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n \cdot v)\|^{\beta_{\mathbb{K}}} \leq t^{-1}\Big) \, dt \leq \mu_x^u(N_1^+) + \int_1^\infty \mu_x^u\Big(S\big(v, t^{-2/\beta_{\mathbb{K}}}\big) \cap N_1^+\Big) \, dt. \tag{5.1}$$

Claim 5.1 implies that if $\mu_x^u\left(S(v, \varepsilon) \cap N_1^+\right) > 0$ for some $\varepsilon > 0$, then $S(v, \varepsilon) \cap N_1^+$ is contained in a ball of radius $2C_0 \varepsilon^{1/4\chi_{\mathbb{K}}}$, centered at a point in the support of the measure $\mu_x^u|_{N_1^+}$. Recalling that $\beta_{\mathbb{K}} = \beta/\chi_{\mathbb{K}}$, we thus obtain

$$\int_1^\infty \mu_x^u\Big(S\big(v, t^{-2/\beta_{\mathbb{K}}}\big) \cap N_1^+\Big) \, dt \leq \int_1^\infty \sup_{n \in \mathrm{supp}(\mu_x^u) \cap N_1^+} \mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big) \, dt, \tag{5.2}$$

where for $n \in N^+$ and $r > 0$, $B_{N^+}(n, r)$ denotes the ball of radius r centered at n.

To estimate the integral on the right side of (5.2), we use the doubling results in Proposition 3.1. Note that if $n \in \mathrm{supp}(\mu_x^u)$, then $(nx)^+$ belongs to the limit set Λ_Γ. Since $x \in N_1^- \Omega$ by assumption, this implies that nx belongs to $N_2^-\Omega$ for all $n \in N_1^+$ in the support of μ_x^u; cf. Remark 2.2. Hence, changing variables using (2.5) and applying Proposition 3.1, we obtain for all $n \in \mathrm{supp}(\mu_x^u) \cap N_1^+$,

$$\mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big) = \mu_{nx}^u\Big(B_{N^+}\big(\mathrm{id}, 2C_0 t^{-1/2\beta}\big)\Big) \ll t^{-\Delta/2\beta}\mu_{nx}^u(N_1^+).$$

Moreover, for $n \in N_1^+$, we have, again by Proposition 3.1, that

$$\mu_{nx}^u(N_1^+) \leq \mu_x^u(N_2^+) \ll \mu_x^u(N_1^+).$$

Put together, this gives

$$\int_1^\infty \sup_{n \in \mathrm{supp}(\mu_x^u)\cap N_1^+} \mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big)\, dt \ll \mu_x^u(N_1^+)\int_1^\infty t^{-\Delta/2\beta}\, dt.$$

The integral on the right side above converges whenever $\beta < \Delta/2$, which concludes the proof.

5.2. **Preliminary facts.** Towards the proof of Claim 5.1, we begin by recalling the Bruhat decomposition of G. Denote by P^- the subgroup MAN^- of G.

Proposition 5.2 (Theorem 5.15, [BT65]). *Let $w \in G$ denote a non-trivial Weyl "element" satisfying $wg_tw^{-1} = g_{-t}$. Then,*

$$G = P^-N^+ \bigsqcup P^-w. \tag{5.3}$$

We shall need the following result, which is yet another reflection in linear representations of G of the fact that G has real rank 1.

Proposition 5.3. *Let V be a normed finite dimensional representation of G, and $v_0 \in V$ be any highest weight vector for g_t $(t > 0)$ with weight $e^{\lambda t}$ for some $\lambda \geq 0$. Let v be any vector in the orbit $G \cdot v_0$ and define*

$$G(v, V^{<\lambda}(g_t)) = \left\{ g \in G : \lim_{t\to\infty} \frac{\log \|g_t g v\|}{t} < \lambda \right\}.$$

Then, there exists $g_v \in G$ such that

$$G(v, V^{<\lambda}(g_t)) \subseteq P^- g_v.$$

Proof. Let $h \in G$ be such that $v = hv_0$ and let $g \in G(v, V^{<\lambda}(g_t))$. By the Bruhat decomposition, either $gh = pn$ for some $p \in P^-$ and $n \in N^+$, or $gh = pw$ for some $p \in P^-$ and w being the long Weyl "element". Suppose we are in the first case, and note that N^+ fixes v_0 since it is a highest weight vector for g_t. Moreover, $\mathrm{Ad}(g_t)(p)$ converges to some element in G as t tends to ∞. Since $g_t g v = e^{\lambda t}\mathrm{Ad}(g_t)(p)v_0$, we see that $\log \|g_t g v\|/t \to \lambda$ as t tends to ∞, thus contradicting the assumption that g belongs to $G(v, V^{<\lambda}(g_t))$. Hence, gh must belong to P^-w. This implies the conclusion by taking $g_v := wh^{-1}$. □

The following immediate corollary is the form we use this result in our arguments.

Corollary 5.4. *Let the notation be as in Proposition 5.3. Then, $N^+ \cap G(v, V^{<\lambda}(g_t))$ contains at most one point.*

Proof. Recall the Bruhat decomposition of G in Proposition 5.2. Let $g_v \in G$ be as in Proposition 5.3 and suppose that $n_0 \in P^-g_v \cap N^+$. Let $p_0 \in P^-$ be such that $n_0 = p_0 g_v$.

First, assume $g_v = p_v n_v$ for some $p_v \in P^-$ and $n_v \in N^+$. Then, $n_0 = p_0 p_v n_v$ and, hence, $n_0 n_v^{-1} \in P^- \cap N^+ = \{\mathrm{id}\}$. In particular, $n_0 = n_v$, and the claim follows in this case.

Now assume that $g_v = p_v w$ for some $p_v \in P^-$, so that $n_0 = p_0 p_v w \in P^-w \cap N^+$. This is a contradiction, since the latter intersection is empty as follows from the Bruhat decomposition. □

5.3. **Convexity and proof of Claim 5.1.** Let $B_1 \subset \mathrm{Lie}(N^+)$ denote a compact convex set whose image under the exponential map contains N_1^+ and denote by B_2 a compact convex set containing B_1 in its interior.

Define $\mathfrak{n}_1^+$ to be the unit sphere in the Lie algebra $\mathfrak{n}^+$ of N^+ in the following sense:

$$\mathfrak{n}_1^+ := \left\{ u \in \mathfrak{n}^+ : d_{N^+}(\exp(u), \mathrm{id}) = 1 \right\},$$

where d_{N^+} is the Cygan metric on N^+; cf. Sec. 2.5. Given $u, b \in \mathfrak{n}^+$, define a line $\ell_{u,b} : \mathbb{R} \to \mathfrak{n}^+$ by

$$\ell_{u,b}(t) := tu + b,$$

and denote by $\mathcal{L}$ the space of all such lines $\ell_{u,b}$ such that $u \in \mathfrak{n}_1^+$. We endow $\mathcal{L}$ with the topology inherited from its natural identification with its $\mathfrak{n}_1^+ \times \mathfrak{n}^+$. Then, the subset $\mathcal{L}(B_1)$ of all such lines such that b belongs to the compact set B_1 is compact in $\mathcal{L}$.

Recall that a vector $v \in W$ is said to be unstable if the closure of the orbit $G \cdot v$ contains 0. Highest weight vectors are examples of unstable vectors. Let $\mathcal{N}$ denote the null cone of G in W, i.e., the closed cone consisting of all unstable vectors. Let $\mathcal{N}_1 \subset \mathcal{N}$ denote the compact set of unit norm unstable vectors. Note that, for any $v \in \mathcal{N}$, the restriction of p_v to any $\ell \in \mathcal{L}$ is a polynomial in t of degree at most that of p_v. We note further that the function

$$\rho(v, \ell) := \sup \left\{ p_v(\ell(t)) : \ell(t) \in B_2 \right\}$$

is continuous and non-negative on the compact space $\mathcal{N}_1 \times \mathcal{L}(B_1)$. We claim that

$$\rho_\star := \inf \left\{ \rho(v, \ell) : (v, \ell) \in \mathcal{N}_1 \times \mathcal{L}(B_1) \right\}$$

is strictly positive. Indeed, by continuity and compactness, it suffices to show that ρ is non-vanishing. Suppose not and let (v, ℓ) be such that $\rho(v, \ell) = 0$. Since B_1 is contained in the interior of B_2, the intersection

$$I(\ell) := \{ t \in \mathbb{R} : \ell(t) \in B_2 \}$$

is an interval (by convexity of B_2) with non-empty interior. Since $p_v(\ell(\cdot))$ is a polynomial vanishing on a set of non-empty interior, this implies it vanishes identically. On the other hand, Corollary 5.4 shows that p_v has at most 1 zero in all of $\mathfrak{n}^+$, a contradiction.

Positivity of $\rho_\star$ has the following consequence. Our choice of the representation W implies that the degree of the polynomial p_v is at most $4\chi_{\mathbb{K}}$, where $\chi_{\mathbb{K}}$ is given in (4.2). This can be shown by direct calculation in this case.[3] By the so-called (C, α)-good property (cf. [Kle10, Proposition 3.2]), we have for all $\varepsilon > 0$

$$|\left\{ t \in I(\ell) : p_v(\ell(t)) \leq \varepsilon \right\}| \leq C_d \left(\varepsilon / \rho_\star \right)^{1/4\chi_{\mathbb{K}}} |I(\ell)|,$$

where $C_d > 0$ is a constant depending only on the degree of p_v, and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

To use this estimate, we first note that the length of the intervals $I(\ell)$ is uniformly bounded over $\mathcal{L}(B_1)$. Indeed, suppose for some $u = (u_\alpha, u_{2\alpha}), b \in \mathfrak{n}^+$ and $\ell = \ell_{u,b} \in \mathcal{L}(B_1)$, $I(\ell)$ has endpoints $t_1 < t_2$ so that the points $\ell(t_i)$ belong to the boundary of B_2. Recall that the Lie algebra $\mathfrak{n}^+$ of N^+ decomposes into g_t eigenspaces as $\mathfrak{n}_\alpha^+ \oplus \mathfrak{n}_{2\alpha}^+$, where $\mathfrak{n}_{2\alpha}^+ = 0$ if and only if $\mathbb{K} = \mathbb{R}$. Set $x_1 = \ell(t_1)$ and $x_2 = \ell(t_2)$. Since N^+ is a nilpotent group of step at most 2, the Campbell-Baker-Hausdorff formula implies that $\exp(x_2)\exp(-x_1) = \exp(Z)$, where $Z \in \mathfrak{n}^+$ is given by

$$Z = x_2 - x_1 + \frac{1}{2}[x_2, -x_1] = (t_2 - t_1)u + \frac{1}{2}(t_2 - t_1)[b, u].$$

[3] In general, such a degree can be calculated from the largest eigenvalue of g_t in W; for instance by restricting the representation to suitable subalgebras of the Lie algebra of G that are isomoprhic to $\mathfrak{sl}_2(\mathbb{R})$ and using the explicit description of $\mathfrak{sl}_2(\mathbb{R})$ representations.

Note that since $\mathfrak{n}^+_{2\alpha}$ is the center of $\mathfrak{n}^+$, $[b,u]=[b,u_\alpha]$ belongs to $\mathfrak{n}^+_{2\alpha}$. Hence, we have by (2.9) that

$$d_{N^+}(\exp(x_1),\exp(x_2)) = \left((t_2-t_1)^4\left\|u_\alpha\right\|^4 + (t_2^2-t_1^2)^2\left\|u_{2\alpha}+\frac{1}{2}[b,u]\right\|^2\right)^{1/4}.$$

Since $\exp(u)$ is at distance 1 from identity, at least one of $\|u_\alpha\|$ and $\|u_{2\alpha}\|$ is bounded below by 10^{-1}. Moreover, we can find a constant $\theta\in(0,10^{-2})$ so that for all $b\in B_1$ and all $y_\alpha\in\mathfrak{n}^+_\alpha$ with $\|y_\alpha\|\leq\theta$ such that $\|[b,y_\alpha]\|\leq 10^{-2}$. Together this implies that

$$\min\left\{t_2-t_1,(t_2^2-t_1^2)^{1/2}\right\}\ll \operatorname{diam}(B_1),$$

where $\operatorname{diam}(B_1)$ denotes the diameter of B_1. This proves that $|I(\ell)|=t_2-t_1\ll 1$, where the implicit constant depends only on the choice of B_1. We have thus shown that

$$|\{t\in I(\ell): p_v(\ell(t))\leq\varepsilon\}|\ll \varepsilon^{1/4\chi_{\mathbb{K}}}. \tag{5.4}$$

We now use our assumption that v belongs to the G orbit of a highest weight vector v_0. Since v_0 is a highest weight vector, it is fixed by N^+. Hence, the Bruhat decomposition, cf. (5.3) with the roles of P^- and P^+ reversed, implies that the orbit $G\cdot v_0$ can be written as

$$G\cdot v_0 = P^+\cdot v_0\bigsqcup P^+w\cdot v_0,$$

where w is the long Weyl "element". Recall that $P^+=N^+MA$, where M is the centralizer of $A=\{g_t\}$ in the maximal compact group K. In particular, M preserves eigenspaces of A and normalizes N^+. Recall further that the norm on W is chosen to be K-invariant.

First, we consider the case $v\in P^+w\cdot v_0$ and has unit norm. For $v'\in W$, we write $[v']$ for its image in the projective space $\mathbb{P}(W)$. Then, since $w\cdot v_0$ is a joint weight vector of A, we see that the image of $P^+w\cdot v_0$ in $\mathbb{P}(W)$ has the form $N^+M\cdot[w\cdot v_0]$. Setting $v_1:=w\cdot v_0$, we see that

$$S(nm\cdot v_1,\varepsilon)=S(mv_1,\varepsilon)\cdot n^{-1}=\operatorname{Ad}(m^{-1})(S(v_1,\varepsilon))\cdot n^{-1}, \tag{5.5}$$

where we implicitly used the fact that M commutes with the projection π_+ and preserves the norm on W. Since the metric on N^+ is right invariant under translations by N^+ and is invariant under $\operatorname{Ad}(M)$, the above identity implies that it suffices to estimate the diameter of $S(v_1,\varepsilon)\cap N_1^+$ in the case $v\in P^+w\cdot v_0$. Similarly, in the case $v\in P^+\cdot v_0$, it suffices to estimate the diameter of $S(v_0,\varepsilon)\cap N_1^+$.

Let $\tilde{S}(v,\varepsilon)=\log S(v,\varepsilon)$ denote the pre-image of $S(v,\varepsilon)$ in the Lie algebra $\mathfrak{n}^+$ of N^+ under the exponential map. By Corollary 5.4, for any non-zero $v\in\mathcal{N}$, either $S(v,\varepsilon)$ is empty for all small enough ε, or there is a unique global minimizer of $p_v(\cdot)$ on N^+, at which p_v vanishes. In either case, for any given $v\in\mathcal{N}\setminus\{0\}$ in the null cone, the set $\tilde{S}(v,\varepsilon)$ is convex for all small enough $\varepsilon>0$, depending on v. Let $s_0>0$ be such that $\tilde{S}(v,\varepsilon)$ is convex for $v\in\{v_0,v_1\}$ and for all $0\leq\varepsilon\leq s_0$.

Fix some $v\in\{v_0,v_1\}$ and $\varepsilon\in[0,s_0]$. Suppose that $x_1\neq x_2\in\tilde{S}(v,\varepsilon)\cap B_1$. Let r denote the distance $d_{N^+}(x_1,x_2)$. Let $u'=x_2-x_1$, $u=u'/r$ and $b=x_1$. Set $\ell=\ell_{u,b}$ and note that $\ell_{u,b}(0)=x_1$ and $\ell_{u,b}(r)=x_2$. Since B_1 is convex, the set $\tilde{S}(v,\varepsilon)\cap B_1$ is also convex. Hence, the entire interval $(0,r)$ belongs to the set on the left side of (5.4) and, hence, that $r\ll\varepsilon^{1/4\chi_{\mathbb{K}}}$. Since x_1 and x_2 were arbitrary, this shows that the diameter of $\tilde{S}(v,\varepsilon)\cap B_1$ is $O(\varepsilon^{1/4\chi_{\mathbb{K}}})$ as desired.

6. Anisotropic Banach Spaces and Transfer Operators

In this section, we define the Banach spaces on which the transfer operator and resolvent associated to the geodesic flow have good spectral properties.

The transfer operator, denoted $\mathcal{L}_t$, acts on continuous functions as follows:

$$\mathcal{L}_t f := f\circ g_t, \qquad f\in C(X), t\in\mathbb{R}. \tag{6.1}$$

For $z \in \mathbb{C}$, the resolvent $R(z) : C_c(X) \to C(X)$ is defined formally as follows:

$$R(z)f := \int_0^\infty e^{-zt} \mathcal{L}_t f \; dt.$$

If Γ is not convex cocompact, we fix a choice of $\beta > 0$ so that Theorem 4.1 holds and set $V = V_\beta$. If Γ is convex cocompact, we take $V = V_\beta \equiv 1$ and we may take β as large as we like in this case. Note that the conclusion of Theorem 4.1 holds trivially with this choice of V. In particular, we shall use its conclusion throughout the argument regardless of whether Γ admits cusps.

Denote by $C_c^{k+1}(X)^M$ the subspace of $C_c^{k+1}(X)$ consisting of M-invariant functions, where M is the centralizer of the geodesic flow inside the maximal compact group K. In particular, $C_c^{k+1}(X)^M$ is naturally identified with the space of C_c^{k+1} functions on the unit tangent bundle of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$; cf. Section 2. The following is the main result of this section.

Theorem 6.1 (Essential Spectral Gap). *Let $k \in \mathbb{N}$ be given. Then, there exists a seminorm $\|\cdot\|_k$ on $C_c^{k+1}(X)^M$, non-vanishing on functions whose support meets Ω, and such that for every $z \in \mathbb{C}$, with $\mathrm{Re}(z) > 0$, the resolvent $R(z)$ extends to a bounded operator on the completion of $C_c^{k+1}(X)^M$ with respect to $\|\cdot\|_k$ and having spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ is bounded above by $1/(\mathrm{Re}(z) + \sigma_0)$, where*

$$\sigma_0 := \min\{k, \beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0 = k$.

By the completion of a topological vector space V with respect to a seminorm $\|\cdot\|$, we mean the Banach space obtained by completing the quotient topological vector space V/W with respect to the induced norm, where W is the kernel of $\|\cdot\|$.

The proof of Theorem 6.1 occupies Sections 6 and 7.

6.1. **Anisotropic Banach Spaces.** We construct a Banach space of functions on X containing C^∞ functions satisfying Theorem 6.1.

Given $r \in \mathbb{N}$, let $\mathcal{V}_r^-$ denote the space of all C^r vector fields on N^+ pointing in the direction of the Lie algebra $\mathfrak{n}^-$ of N^- and having norm at most 1. More precisely, $\mathcal{V}_r^-$ consists of all C^r maps $v : N^+ \to \mathfrak{n}^-$, with C^r norm at most 1. Similarly, we denote by $\mathcal{V}_r^0$ the set of C^r vector fields $v : N^+ \to \mathfrak{a} := \mathrm{Lie}(A)$, with C^r norm at most 1. Note that if $\omega \in \mathfrak{a}$ is the vector generating the flow g_t, i.e. $g_t = \exp(t\omega)$, then each $v \in \mathcal{V}_r^0$ is of the form $v(n) = \phi(n)\omega$, for some $\phi \in C^r(N^+)$ such that $\|\phi\|_{C^r(N^+)} \leq 1$. Define

$$\mathcal{V}_r = \mathcal{V}_r^- \cup \mathcal{V}_r^0.$$

For $v \in \mathrm{Lie}(G)$, denote by L_v the differential operator on $C^1(X)$ given by differentiation with respect to the vector field generated by v. Hence, for $\varphi \in C^1(G/\Gamma)$,

$$L_v\varphi(x) = \lim_{s \to 0} \frac{\varphi(\exp(sv)x) - \varphi(x)}{s}.$$

For each $k \in \mathbb{N}$, we define a norm on $C^k(N^+)$ functions as follows. Letting $\mathcal{V}^+$ be the unit ball in the Lie algebra of N^+, $0 \leq \ell \leq k$, and $\phi \in C^k(N^+)$, we define $c_\ell(\phi)$ to be the supremum of $|L_{v_1} \cdots L_{v_\ell}(\phi)|$ over N^+ and all tuples $(v_1, \ldots, v_\ell) \in (\mathcal{V}^+)^\ell$. We define $\|\phi\|_{C^k}$ to be $\sum_{\ell=0}^k c_\ell(\phi)/(2^\ell \ell!)$. One then checks that for all $\phi_1, \phi_2 \in C^k(N^+)$, we have

$$\|\phi_1\phi_2\|_{C^k} \leq \|\phi_1\|_{C^k} \|\phi_2\|_{C^k}. \tag{6.2}$$

Following [GL06, GL08], we define a norm on $C_c^{k+1}(X)$ as follows. Given $f \in C_c^{k+1}(X)$, k, ℓ non-negative integers, $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathcal{V}_{k+\ell}^\ell$ (i.e. ℓ tuple of $C^{k+\ell}$ vector fields) and $x \in X$, define

$$e_{k,\ell,\gamma}(f; x) := \frac{1}{V(x)} \sup \frac{1}{\mu_x^u(N_1^+)} \left| \int_{N_1^+} \phi(n) L_{\gamma_1(n)} \cdots L_{\gamma_\ell(n)}(f)(g_s n x) \; d\mu_x^u(n) \right|, \tag{6.3}$$

where the supremum is taken over all $s \in [0,1]$ and all functions $\phi \in C^{k+\ell}(N_1^+)$ which are compactly supported in the interior of N_1^+ and having $\|\phi\|_{C^{k+\ell}(N_1^+)} \leq 1$.

For $\gamma \in \mathcal{V}_{k+\ell+1}^{\ell}$, we define $e'_{k,\ell,\gamma}(f;x)$ analogously to $e_{k,\ell,\gamma}(f;x)$, but where we take $s = 0$ and take the supremum over $\phi \in C^{k+\ell+1}(N_{1/10}^+)$ instead[4] of $C^{k+\ell}(N_1^+)$. Given $r > 0$, set

$$\Omega_r^- := N_r^- \Omega. \tag{6.4}$$

We define

$$e_{k,\ell,\gamma}(f) := \sup_{x \in \Omega_1^-} e_{k,\ell,\gamma}(f;x), \qquad e_{k,\ell}(f) = \sup_{\gamma \in \mathcal{V}_{k+\ell}^{\ell}} e_{k,\ell,\gamma}(f). \tag{6.5}$$

Finally, we define $\|f\|_k$ and $\|f\|'_k$ by

$$\|f\|_k := \max_{0 \leq \ell \leq k} e_{k,\ell}(f), \qquad \|f\|'_k := \max_{0 \leq \ell \leq k-1} \sup_{\gamma \in \mathcal{V}_{k+\ell+1}^{\ell}, x \in \Omega_{1/2}^-} e'_{k,\ell,\gamma}(f;x). \tag{6.6}$$

Note that the (semi-)norm $\|f\|'_k$ is weaker than $\|f\|_k$ since we are using more regular test functions and vector fields, and we are testing fewer derivatives of f.

Remark 6.2. Since the suprema in the definition of $\|\cdot\|_k$ are restricted to points on Ω_1^-, $\|\cdot\|_k$ defines a seminorm on $C_c^{k+1}(X)^M$. Moreover, since Ω_1^- is invariant by g_t for all $t \geq 0$, the kernel of this seminorm, denoted W_k, is invariant by $\mathcal{L}_t$. The seminorm $\|\cdot\|_k$ induces a norm on the quotient $C_c^{k+1}(X)^M/W_k$, which we continue to denote $\|\cdot\|_k$.

Definition 6.3. We denote by $\mathcal{B}_k$ the Banach space given by the completion of the quotient $C_c^{k+1}(X)^M/W_k$ with respect to the norm $\|\cdot\|_k$, where $C_c^{k+1}(X)^M$ denotes the subspace consisting of M-invariant functions.

Note that since $\|\cdot\|'_k$ is dominated by $\|\cdot\|_k$, $\|\cdot\|'_k$ descends to a (semi-)norm on $C_c^{k+1}(X)^M/W_k$ and extends to a (semi-)norm on $\mathcal{B}_k$, again denoted $\|\cdot\|'_k$.

The following is a reformulation of Theorem 6.1 in the above setup.

Theorem 6.4. *For all $z \in \mathbb{C}$, with $\mathrm{Re}(z) > 0$, and for all $k \in \mathbb{N}$, the operator $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ acting on $\mathcal{B}_k$ is bounded above by $1/(\mathrm{Re}(z) + \sigma_0)$, where*

$$\sigma_0 := \min\{k, \beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0 = k$.

6.2. **Hennion's Theorem and Compact Embedding.** Our key tool in estimating the essential spectral radius is the following refinement of Hennion's Theorem, based on Nussbaum's formula.

Theorem 6.5 (cf. [Hen93] and Lemma 2.2 in [BGK07])**.** *Suppose that $\mathcal{B}$ is a Banach space with norm $\|\cdot\|$ and that $\|\cdot\|'$ is a seminorm on $\mathcal{B}$ so that the unit ball in $(\mathcal{B}, \|\cdot\|)$ is relatively compact in $\|\cdot\|'$. Suppose R is a bounded operator on $\mathcal{B}$ such that for some $n \in \mathbb{N}$, there exist constants $r > 0$ and $C > 0$ satisfying*

$$\|R^n v\| \leq r^n \|v\| + C\|v\|', \tag{6.7}$$

for all $v \in \mathcal{B}$. Then, the essential spectral radius of R is at most r.

The following proposition, roughly speaking, verifies the compactness assumption of Theorem 6.5 for $\|\cdot\|_k$ and $\|\cdot\|'_k$.

Proposition 6.6. *Let $K \subseteq X$ be such that*

$$\sup\{V(x) : x \in K\} < \infty.$$

Then, every sequence $f_n \in C_c^{k+1}(X)^M$, such that f_n is supported in K and has $\|f_n\|_k \leq 1$ for all n, admits a Cauchy subsequence in $\|\cdot\|'_k$.

[4]The restriction on the supports allows us to handle non-smooth conditional measures; cf. proof of Prop. 6.6.

6.3. **Proof of Proposition 6.6.** We adapt the arguments in [GL06,GL08] with the main difference being that we bypass the step involving integration by parts over N^+ since our conditionals μ_x^u need not be smooth in general. The idea is to show that since all directions in the tangent space of X are accounted for in the definition of $\|\cdot\|_k$ (differentiation along the weak stable directions and integration in the unstable directions), one can estimate $\|\cdot\|_k'$ using finitely many coefficients $e_k(f;x_i)$. More precisely, we first show that there exists $C \geq 1$ so that for all sufficiently small $\varepsilon > 0$, there exists a finite set $\Xi \subset \Omega$ so that for all $f \in C_c^{k+1}(X)^M$, which is supported in K,

$$\|f\|_k' \leq C\varepsilon \|f\|_k + C \sup \int_{N_1^+} \phi L_{v_1} \cdots L_{v_\ell} f \; d\mu_{x_i}^u, \tag{6.8}$$

where the supremum is over all $0 \leq \ell \leq k-1$, all $(v_1, \ldots, v_\ell) \in \mathcal{V}_{k+\ell+1}^\ell$, all functions $\phi \in C^{k+\ell+1}(N_2^+)$ with $\|\phi\|_{C^{k+\ell+1}} \leq 1$ and all $x_i \in \Xi$.

First, we show how (6.8) completes the proof. Let $f_n \in C_c^{k+1}(K)$ be as in the statement. Let $\varepsilon > 0$ be small enough so that (6.8) holds. Since $C^{k+\ell+1}(N_2^+)$ is compactly included inside $C^{k+\ell}(N_2^+)$, we can find a finite collection $\{\phi_j : j\} \subset C^{k+\ell}(N_2^+)$ which is ε dense in the unit ball of $C^{k+\ell+1}(N_2^+)$. Similarly, we can find a finite collection of vector fields $\{(v_1^m, \ldots, v_\ell^m) : m\} \subset \mathcal{V}_{k+\ell}^\ell$ which is ε dense in $\mathcal{V}_{k+\ell+1}^\ell$ in the $C^{k+\ell+1}$ topology. Then, we can find a subsequence, also denoted f_n, so that the finitely many quantities

$$\left\{ \int_{N_1^+} \phi_j L_{v_1^m} \cdots L_{v_\ell^m} f_n \; d\mu_{x_i}^u : i, j, m \right\}$$

converge. Together with (6.8), this implies that

$$\|f_{n_1} - f_{n_2}\|_k' \ll \varepsilon,$$

for all large enough n_1, n_2, where we used the fact that $\|f_n\|_k \leq 1$ for all n. As ε was arbitrary, one can extract a Cauchy subequence by a standard diagonal argument. Thus, it remains to prove (6.8).

Fix some $f \in C_c^{k+1}(X)^M$ which is supported inside K. Let an arbitrary tuple $\gamma = (v_1, \ldots, v_\ell) \in \mathcal{V}_{k+\ell+1}^\ell$ be given and set

$$\psi = L_{v_1} \cdots L_{v_\ell} f.$$

Let $\phi \in C^{k+\ell+1}(N_{1/10}^+)$ and write $Q = N_{1/10}^+$. To estimate $e'_{k,\ell,\gamma}(f;z)$ using the right side of (6.8), we need to estimate integrals of the form

$$\frac{1}{V(z)} \frac{1}{\mu_z^u\left(N_1^+\right)} \int_{N_1^+} \phi(n)\psi(nz) \; d\mu_z^u(n), \tag{6.9}$$

for all $z \in \Omega_{1/2}^-$.

Denote by $\rho : X \to [0,1]$ a smooth function which is identically one on the 1-neighborhood Ω^1 of Ω and vanishes outside its 2-neighborhood. Note that if f is supported outside of Ω^1, then the integral in (6.9) vanishes for all z and the estimate follows. The same reasoning implies that

$$\|\rho f\|_k = \|f\|_k, \qquad \|\rho f\|_k' = \|f\|_k'.$$

Hence, we may assume that f is supported inside the intersection of K with Ω^1. In particular, for the remainder of the argument, we may replace K with (the closure of) its intersection with Ω^1.

This discussion has the important consequence that we may assume that K is a compact set in light of Proposition 4.3. Let K_1 denote the 1-neighborhood of K and fix some $z \in K_1 \cap \Omega_{1/2}^-$. By shrinking ε, we may assume it is smaller than the injectivity radius of K_1. Hence, we can find a finite cover $B_1, \ldots, B_M$ of $K_1 \cap \Omega_{1/2}^-$ with flow boxes of radius ε and with centers $\Xi := \{x_i\} \subset \Omega_{1/2}^-$.

Step 1: We first handle the case where z belongs to the same unstable manifold as one of the x_i's. Note that we may assume that Q intersects the support of μ_z^u non-trivially, since otherwise the integral in question is 0. Let $u \in Q$ be one point in this intersection and let $x = uz$. Thus, by (2.5), we get

$$\int_{N_1^+} \phi(n)\psi(nz)\, d\mu_z^u(n) = \int_Q \phi(n)\psi(nz)\, d\mu_z^u(n) = \int_{Qu^{-1}} \phi(nu)\psi(nx)\, d\mu_x^u(n).$$

Let $\phi_u(n) := \phi(nu)$. Then, ϕ_u is supported inside Qu^{-1}. Moreover, since $u \in Q$, $Q_u := Qu^{-1}$ is a ball of radius $1/10$ containing the identity element. Hence, $Qu^{-1} \subset N_1^+$ and, thus,

$$\int_{Q_u} \phi(nu)\psi(nx)\, d\mu_x^u(n) = \int_{N_1^+} \phi_u(n)\psi(nx)\, d\mu_x^u(n).$$

Fix some $\varepsilon > 0$. We may assume that $\varepsilon < 1/10$. Note that x belongs to the 1-neighborhood of K. Then, $x = u_2^{-1}x_i$ for some i and some $u_2 \in N_\varepsilon^+$, by our assumption in this step that z belongs to the unstable manifold of one of the x_i's. By repeating the above argument with z, u, x, Q and ϕ replaced with x, u_2, x_i, Q_u and ϕ_u respectively, we obtain

$$\int_{N_1^+} \phi_u(n)\psi(nx)\, d\mu_x^u(n) = \int_{Q_u u_2^{-1}} \phi_u(nu_2)\psi(nx_i)\, d\mu_{x_i}^u(n).$$

Note that Q_u is contained in the ball of radius $1/5$ centered around identity. Since $u_2 \in N_\varepsilon^+$ and $\varepsilon < 1/10$, we see that $Q_u u_2^{-1} \subset N_1^+$. It follows that

$$\int_{N_1^+} \phi_u(n)\psi(nx_i)\, d\mu_{x_i}^u(n) = \int_{N_1^+} \phi_{u_2u}(n)\psi(nx_i)\, d\mu_{x_i}^u(n),$$

where $\phi_{u_2u}(n) = \phi_u(nu_2) = \phi(nu_2u)$. The function ϕ_{u_2u} satisfies $\|\phi_{u_2u}\|_{C^{k+\ell+1}} = \|\phi\|_{C^{k+\ell+1}} \leq 1$. Finally, let $\varphi_1, \varphi_2 : N^+ \to [0,1]$ be non-negative bump C^0 functions where $\varphi_1 \equiv 1$ on N_1^+ and while φ_2 is equal to 1 at identity and its support is contained inside N_1^+. Since $y \mapsto \mu_y^u(\varphi_i)$ is continuous for $i = 1, 2$, by [Rob03, Lemme 1.16], and is non-zero on Ω_1^-, we can find, by compactness of K_1, a constant $C \geq 1$, depending only on K (and the choice of φ_1, φ_2), such that

$$1/C \leq \mu_y^u\left(N_1^+\right) \leq C, \qquad \forall y \in K_1 \cap \Omega_1^-. \tag{6.10}$$

Hence, recalling that $\psi = L_{v_1} \cdots L_{v_\ell} f$ and that $V(z) \gg 1$, we conclude that the integral in (6.9) is bounded by the second term in (6.8).

Step 2: We reduce to the case where z is contained in the unstable manifolds of the x_i's. Let i be such that $z \in B_i$. Set $z_1 = z$ and let $z_0 \in (N_\varepsilon^+ \cdot x_i)$ be the unique point in the intersection of $N_\varepsilon^+ \cdot x_i$ with the local weak stable leaf of z_1 inside B_i. Let $p_1^- \in P^- := MAN^-$ be an element of the ε neighborhood of identity P_ε^- in P^- such that $z_1 = p_1^- z_0$.

We will estimate the integral in (6.9) using integrals at z_0. The idea is to perform weak stable holonomy between the local strong unstable leaves of z_0 and z_1. To this end, we need some notation. Let $Y \in \mathfrak{p}^-$ be such that $p_1^- = \exp(Y)$ and set

$$p_t^- = \exp(tY), \qquad z_t = p_t^- z_0,$$

for $t \in [0,1]$. Let us also consider the following maps $u_t^+ : N_1^+ \to N^+$ and $\tilde{p}_t^- : N_1^+ \to P^-$ defined by the following commutation relations

$$np_t^- = \tilde{p}_t^-(n)u_t^+(n), \qquad \forall n \in N_1^+.$$

Recall we are given a test function $\phi \in C^{k+\ell+1}(N_{1/10}^+)$. We can rewrite the integral we wish to estimate as follows:

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u(n) = \int_{N_1^+} \phi(n)\psi(np_1^- z_0)\, d\mu_{z_1}^u(n) = \int \phi(n)\psi(\tilde{p}_1^-(n)u_1^+(n)z_0)\, d\mu_{z_1}^u(n).$$

Let $U_t^+ \subset N^+$ denote the image of u_t^+. Note that if ε is small enough, $U_t^+ \subseteq N_2^+$ for all $t \in [0,1]$. We may further assume that ε is small enough so that the map u_t^+ is invertible on U_t^+ for all $t \in [0,1]$ and write $\phi_t := \phi \circ (u_t^+)^{-1}$. For simplicity, set

$$p_t^-(n) := \tilde{p}_t^-((u_t^+)^{-1}(n)).$$

Write $m_t(n) \in M$ and $b_t^-(n) \in AN^-$ for the components of $p_t^-(n)$ along M and AN^- respectively so that

$$p_t^-(n) = m_t(n)b_t^-(n).$$

We denote by J_t the Radon-Nikodym derivative of the pushforward of $\mu_{z_1}^u$ by u_t^+ with respect to $\mu_{z_t}^u$; cf. (2.11) for an explicit formula. Thus, changing variables using $n \mapsto u_1^+(n)$, and using the M-invariance of f, we obtain

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u = \int \phi_1(n)\psi(p_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u = \int \phi_1(n)\tilde{\psi}_1(b_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u,$$

where $\tilde{\psi}_t$ is given by

$$\tilde{\psi}_t := L_{\tilde{v}_1^t} \cdots L_{\tilde{v}_\ell^t} f, \qquad \tilde{v}_i^t(n) := \mathrm{Ad}(m_t((u_t^+)^{-1}(n)))(v_i((u_t^+)^{-1}(n))).$$

Here, we recall that $\mathrm{Ad}(M)$ commutes with A and normalizes N^- so that $\tilde{v}_i^t$ is a vector field with the same target as v_i.

Let $\mathfrak{b}^-$ denote the Lie algebra of AN^- and denote by $\tilde{w}_t' : U_t^+ \times [0,1] \to \mathfrak{b}^-$ the vector field tangent to the paths defined by b_t^-. More explicitly, $\tilde{w}_t'$ is given by the projection of tY to $\mathfrak{b}^-$. Denote $\tilde{w}_t(n) := \mathrm{Ad}(m_t(n))(\tilde{w}_t'(n))$. Then, using the M-invariance of f as above once more, we can write

$$\psi(b_1^-(n)nz_0) - \psi(nz_0)) = \int_0^1 \frac{\partial}{\partial t}\tilde{\psi}_t(b_t^-(n)nz_0)\, dt = \int_0^1 L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)\, dt.$$

To simplify notation, let us set $w_t = \tilde{w}_t \circ u_t^+$, and

$$F_t := L_{\tilde{v}_1^t \circ u_t^+} \cdots L_{\tilde{v}_\ell^t \circ u_t^+} f.$$

Using a reverse change of variables, we obtain for every $t \in [0,1]$ that

$$\begin{aligned}\int \phi_1(n)L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u &= \int (\phi_1 J_1) \circ u_t^+(n) L_{w_t}(F_t)(\tilde{p}_t^-(n)u_t^+(n)z_0)J_t^{-1}(n)\, d\mu_{z_t}^u \\ &= \int (\phi_1 J_1) \circ u_t^+(n) \cdot L_{w_t}(F_t)(nz_t) \cdot J_t^{-1}(n)\, d\mu_{z_t}^u(n),\end{aligned}$$

where we used the identities $\tilde{p}_t^-(n)u_t^+(n) = np_t^-$ and $z_t = p_t^- z_0$. Let us write

$$\Phi_t(n) := (\phi_1 J_1) \circ u_t^+(n) \cdot J_t^{-1}(n),$$

which we view as a test function[5]. Hence, the last integral above amounts to integrating $\ell + 1$ weak stable derivatives of f against a $C^{k+\ell}$ function. Moreover, since ϕ is supported in $N_{1/10}^+$, we may assume that ε is small enough so that Φ_t is supported in N_1^+ for all $t \in [0,1]$, and meets the requirements on the test functions in the definition of $\|f\|_k$. Since $z = z_1$ belongs to $\Omega_{1/2}^-$ by assumption, we may further shrink ε if necessary so that the points z_t all[6] belong to Ω_1^-. Thus, decomposing w_t into its A and N^- components, and noting that $\|w_t\| \ll \varepsilon$, we obtain the estimate

$$\int \Phi_t(n) \cdot L_{w_t}(F_t)(nz_t)\, d\mu_{z_t}^u(n) \ll \varepsilon \|f\|_k V(z_t)\mu_{z_t}^u(N_1^+). \tag{6.11}$$

[5]The Jacobians are smooth maps as they are given in terms of Busemann functions; cf. (2.11).

[6]This type of estimate is the reason we use stable thickenings Ω_r^- of Ω in the definition of the norm instead of Ω.

To complete the argument, note that the integral we wish to estimate satisfies

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu^u_{z_1} = \int (\phi_1 J_1)(n)\psi(nz_0)\, d\mu^u_{z_0} + \int_0^1 \int \Phi_t(n) \cdot L_{w_t}(F_t)(nz_t)\, d\mu^u_{z_t}(n)\, dt. \quad (6.12)$$

Moreover, recall that z_0 belongs to the same unstable manifold as some $x_i \in \Xi$. Additionally, since ϕ is supported in $N^+_{1/10}$, by taking ε small enough, we may assume that ϕ_1 is supported inside $N^+_{1/5}$. Hence, arguing similarly to Step 1, viewing $\phi_1 J_1$ as a test function, we can estimate the first term on the right side above using the right side of (6.8).

The second term in (6.12) is also bounded by the right side of (6.8), in view of (6.11). Here we are using that $y \mapsto \mu^u_y(N_1^+)$ and $y \mapsto V(y)$ are uniformly bounded as y varies in the compact set K_1; cf. (6.10). This completes the proof of (6.8) in all cases, since ϕ and z were arbitrary.

7. The Essential Spectral Radius of Resolvents

In this section, we study the operator norm of the transfer operators $\mathcal{L}_t$ and the resolvents $R(z)$ on the Banach spaces constructed in the previous section. These estimates constitute the proof of Theorem 6.1. With these results in hand, we deduce Theorem 1.1 at the end of the section.

7.1. **Strong continuity of transfer operators.** Recall that a collection of measurable subsets $\{B_i\}$ of a space Y is said to have *intersection multiplicity* bounded by a constant $C \geq 1$ if for all i, the number of sets B_j in the collection that intersect B_i non-trivially is at most C. In this case, one has

$$\sum_i \chi_{B_i}(y) \leq C\chi_{\cup_i B_i}(y), \qquad \forall y \in Y.$$

The following lemma implies that the operators $\mathcal{L}_t$ are uniformly bounded on $\mathcal{B}_k$ for $t \geq 0$.

Lemma 7.1. *For every $k, \ell \in \mathbb{N} \cup \{0\}$, $\gamma \in \mathcal{V}^\ell_{k+\ell}$, $t \geq 0$, and $x \in \Omega_1^-$,*

$$e_{k,\ell,\gamma}(\mathcal{L}_t f; x) \ll_\beta e^{-\varepsilon(\gamma)t} e_{k,\ell,\gamma}(f)(e^{-\beta t} + 1/V(x)),$$

where $\varepsilon(\gamma) \geq 0$ is the number of stable derivatives determined by γ. In particular, $\varepsilon(\gamma) = 0$ if and only if $\ell = 0$ or all components of γ point in the flow direction.

Proof. Fix some $x \in \Omega$ and $\gamma = (v_1, \dots, v_\ell) \in \mathcal{V}^\ell_{k+\ell}$. Since the Lie algebra of N^- has the orthogonal decomposition $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, where α is the simple positive root in $\mathfrak{g}$ with respect to g_t, we have that g_t contracts the norm of each stable vector $v \in \mathcal{V}^-_{k+\ell}$ by at least e^{-t}. It follows that for all $v \in \mathcal{V}^-_{k+\ell}$ and $w \in \mathcal{V}^0_{k+\ell}$,

$$L_v(\mathcal{L}_t f)(x) = e^{-t} L_{\bar{v}^t}(f)(g_t x), \qquad L_w(\mathcal{L}_t f)(x) = L_w(f)(g_t x), \quad (7.1)$$

for all $f \in C^{k+1}(X)^M$, where $v^t = \mathrm{Ad}(g_t)(v)$ and $\bar{v}^t = e^t v^t$ if $v \in \mathcal{V}^-_{k+\ell}$ and $\bar{v}^t = v^t$ if $v \in \mathcal{V}^0_{k+\ell}$. Moreover, we have

$$\left\| v^t \right\| \leq e^{-t} \left\| v \right\| \leq e^{-t}.$$

Let ϕ be a test function, $f \in C^{k+1}(X)^M$, and set $\psi = L_{\bar{v}_1^t} \cdots L_{\bar{v}_\ell^t} f$. Then, we get

$$\left| \int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\mathcal{L}_t f)(nx)\, d\mu^u_x(n) \right| = e^{-\varepsilon(\gamma)t} \left| \int_{N_1^+} \phi(n)\psi(g_t n x)\, d\mu^u_x(n) \right|.$$

Let $\{\rho_i : i \in I\}$ be a partition of unity of $\mathrm{Ad}(g_t)(N_1^+)$ so that each ρ_i is non-negative, C^∞, and supported inside some ball of radius 1 centered inside $\mathrm{Ad}(g_t)(N_1^+)$. Such a partition of unity can be chosen so that the supports of ρ_i have a uniformly bounded multiplicity[7], depending only on

[7]Note that the analog of the classical Besicovitch covering theorem fails to hold for N^+ with the Cygan metric when N^+ is not abelian; cf. [KR95, pg. 17]. Instead, such a partition of unity can be constructed using the Vitali covering lemma with the aid of the right invariance of the Haar measure. To obtain a uniform bound on the multiplicity here and throughout, it is important that such an argument is applied to balls with uniformly comparable radii.

N^+. Denote by $I(\Lambda)$ the subset of indices $i \in I$ such that there is $n_i \in N^+$ in the support of the measure $\mu^u_{g_t x}$ with the property that the support of ρ_i is contained in $N_1^+ \cdot n_i$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu^u_{g_t x}$ is the 0 measure. Then, using (2.4) to change variables, we obtain

$$\int_{\mathrm{Ad}(g_t)(N_1^+)} \phi(g_{-t} n g_t)\psi(n g_t x)\, d\mu^u_{g_t x}(n) = \sum_{i \in I(\Lambda)} \int_{N_1^+ \cdot n_i} \rho_i(n)\phi(g_{-t} n g_t)\psi(n g_t x)\, d\mu^u_{g_t x}(n).$$

Setting $x_i = n_i g_t x$ and changing variables using (2.5), we obtain

$$\int_{N_1^+} \phi(n)\psi(g_t n x)\, d\mu^u_x(n) = e^{-\delta t} \sum_{i \in I(\Lambda)} \int_{N_1^+} \rho_i(n n_i)\phi(g_{-t} n n_i g_t)\psi(n x_i)\, d\mu^u_{x_i}(n). \tag{7.2}$$

The bounded multiplicity of the partition of unity implies that the balls $N_1^+ \cdot n_i$ have intersection multiplicity bounded by a constant C_0, depending only on N^+. Enlarging C_0 if necessary, we may also choose ρ_i so that $\|\rho_i\|_{C^{k+\ell}} \leq C_0$. In particular, C_0 is independent of t and x.

For each i, let $\bar{\phi}_i(n) = \rho_i(n n_i)\phi(g_{-t} n n_i g_t)$. Since ρ_i is chosen to be supported inside $N_1^+ n_i$, then $\bar{\phi}_i$ is supported inside N_1^+. Moreover, since ρ_i is C^∞, $\bar{\phi}_i$ is of the same differentiability class as ϕ. Since conjugation by g_{-t} contracts N^+, we see that $\|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} \leq \|\phi\|_{C^{k+\ell}} \leq 1$ (note that the supremum norm of $\phi \circ \mathrm{Ad}(g_{-t})$ does not decrease, and hence we do not gain from this contraction). Hence, since $\|\rho_i\|_{C^{k+\ell}} \leq C_0$, (6.2) implies that $\|\bar{\phi}_i\|_{C^{k+\ell}} \leq C_0$.

First, let us suppose that $t \geq 1$. Then, using Remark 2.2, since $x \in N_1^- \Omega$, one checks that x_i belongs to $N_1^- \Omega$ as well for all i. Hence, we obtain

$$\begin{aligned} \left| \int_{N_1^+} \phi(n)\psi(g_t n x)\, d\mu^u_x \right| &\leq e^{-\delta t} \sum_{i \in I(\Lambda)} \left| \int_{N_1^+} \bar{\phi}_i(n)\psi(n x_i)\, d\mu^u_{x_i} \right| \\ &\leq C_0 e_{k,\ell,\gamma}(f)\, \|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}}\, e^{-\delta t} \sum_{i \in I(\Lambda)} \mu^u_{x_i}(N_1^+) V(x_i). \end{aligned} \tag{7.3}$$

By the log Lipschitz property of V provided by Proposition 4.3, and by enlarging C_0 if necessary, we have $V(x_i) \leq C_0 V(n x_i)$ for all $n \in N_1^+$. It follows that

$$\sum_{i \in I(\Lambda)} \mu^u_{x_i}(N_1^+) V(x_i) \leq C_0 \sum_{i \in I(\Lambda)} \int_{N_1^+} V(n x_i)\, d\mu^u_{x_i}(n).$$

Recall that the balls $N_1^+ \cdot n_i$ have intersection multiplicity at most C_0. Moreover, since the support of ρ_i is contained inside $\mathrm{Ad}(g_t)(N_1^+)$, the balls $N_1^+ n_i$ are all contained in $N_2^+ \mathrm{Ad}(g_t)(N_1^+)$. Hence, applying the equivariance properties (2.4) and (2.5) once more yields

$$\sum_{i \in I(\Lambda)} \int_{N_1^+} V(n x_i)\, d\mu^u_{x_i}(n) \leq C_0 \int_{N_2^+ \mathrm{Ad}(g_t)(N_1^+)} V(n g_t x)\, d\mu^u_{g_t x}(n) \leq C_0 e^{\delta t} \int_{N_3^+} V(g_t n x)\, d\mu^u_x(n).$$

Here, we used the positivity of V and that $\mathrm{Ad}(g_{-t})(N_2^+) N_1^+ \subseteq N_3^+$. Combined with (7.2) and the contraction estimate on V, Theorem 4.1, it follows that

$$\int_{N_1^+} \phi(n)\psi(g_t n x)\, d\mu^u_x \leq C_0^3 (c e^{-\beta t} V(x) + c) \mu^u_x(N_3^+) e_{k,\ell,\gamma}(f),$$

for a constant $c \geq 1$ depending on β. By Proposition 3.1, we have $\mu^u_x(N_3^+) \leq C_1 \mu^u_x(N_1^+)$, for a uniform constant $C_1 \geq 1$, which is independent of x. This estimate concludes the proof in view of (7.1).

Now, let $s \in [0,1]$ and $t \geq 0$. If $t+s \geq 1$, then the above argument applied with $t+s$ in place of t implies that

$$\left| \int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\mathcal{L}_t f)(g_s n x)\, d\mu_x^u \right| \ll_\beta e^{-\varepsilon(\gamma)t} e_{k,\ell,\gamma}(f)(e^{-\beta t} V(x) + 1)\mu_x^u(N_1^+),$$

as desired. Otherwise, if $t+s<1$, then by definition of $e_{k,\ell,\gamma}$, we have that

$$\left| \int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\mathcal{L}_t f)(g_s n x)\, d\mu_x^u \right| \leq e_{k,\ell,\gamma}(f) V(x) \mu_x^u(N_1^+).$$

Since t is at most 1 in this case, the conclusion of the lemma follows in this case as well.

□

As a corollary, we deduce the following strong continuity statement which implies that the infinitesimal generator of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. When restricted to $C_c^{k+1}(X)^M$, this generator is nothing but the differentiation operator in the flow direction.

Corollary 7.2. *The semigroup $\{\mathcal{L}_t : t \geq 0\}$ is strongly continuous; i.e. for all $f \in \mathcal{B}_k$,*

$$\lim_{t \downarrow 0} \|\mathcal{L}_t f - f\|_k = 0.$$

Proof. For all $f \in C_c^{k+1}(X)^M$, one easily checks that, since $V(\cdot) \gg 1$ on any bounded neighborhood of Ω, then

$$\|\mathcal{L}_t f - f\|_k \ll \sup_{0 \leq s \leq 1} \|\mathcal{L}_{t+s} f - \mathcal{L}_s f\|_{C^k(X)}.$$

Moreover, since f belongs to C^{k+1}, the right side above inequality tends to 0 as $t \to 0$ by the mean value theorem.

Now, let f be a general element of $\mathcal{B}_k$ and suppose that $\|\mathcal{L}_t f - f\|_k \nrightarrow 0$. Then, there is $t_n \to 0$ such that $\|\mathcal{L}_{t_n} f - f\|_k \to c \neq 0$. For every $j \in \mathbb{N}$, let $f_j \in C_c^{k+1}(X)^M$ be such that $\|f - f_j\|_k < 1/j$. For each j, let n_j be large enough such that $\left\|\mathcal{L}_{t_{n_j}} f_j - f_j\right\|_k < 1/j$. Then,

$$\left\|\mathcal{L}_{t_{n_j}} f - f\right\|_k \leq \left\|\mathcal{L}_{t_{n_j}} f - \mathcal{L}_{t_{n_j}} f_j\right\|_k + \left\|\mathcal{L}_{t_{n_j}} f_j - f_j\right\|_k + \|f_j - f\|_k.$$

The last two terms on the right side are each bounded by $1/j$ by construction. By Lemma 7.1, we also have that $\left\|\mathcal{L}_{t_{n_j}} f - \mathcal{L}_{t_{n_j}} f_j\right\|_k$ is $O(\|f - f_j\|_k)$. It follows that $\left\|\mathcal{L}_{t_{n_j}} f - f\right\|_k \ll 1/j \to 0$, which contradicts the hypothesis that $\|\mathcal{L}_{t_n} f - f\|_k \to c \neq 0$. □

7.2. **Towards a Lasota-Yorke inequality for the resolvent.** Recall that for all $n \in \mathbb{N}$,

$$R(z)^n = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t\, dt, \tag{7.4}$$

as follows by induction on n. The following corollary is immediate from Lemma 7.1 and the fact that

$$\left| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt}\, dt \right| \leq \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-\mathrm{Re}(z)t}\, dt = 1/\mathrm{Re}(z)^n, \tag{7.5}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Corollary 7.3. *For all $n, k, \ell \in \mathbb{N} \cup \{0\}$, $f \in C_c^{k+1}(X)^M$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we have*

$$e_{k,\ell}(R(z)^n f; x) \ll_\beta e_{k,\ell}(f) \left(\frac{1}{(\mathrm{Re}(z) + \beta)^n} + \frac{V(x)^{-1}}{\mathrm{Re}(z)^n} \right) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n.$$

In particular, $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$.

Note that Lemma 7.1 does not provide contraction in the part of the norm that accounts for the flow direction. In particular, the estimate in this lemma is not sufficient to control the essential spectral radius of the resolvent. The following lemma provides the first step towards a Lasota-Yorke inequality for resolvents for the coefficients $e_{k,\ell}$ when $\ell < k$. The idea, based on regularization of test functions, is due to [GL06]. The doubling estimates on conditional measures in Proposition 3.1 are crucial for carrying out the argument.

Lemma 7.4. *For all $t \geq 2$ and $0 \leq \ell < k$, we have*

$$e_{k,\ell}(\mathcal{L}_t f) \ll_{k,\beta} e^{-kt} e_{k,\ell}(f) + e'_{k,\ell}(f).$$

Proof. Fix some $0 \leq \ell < k$. Let $x \in \Omega_1^-$ and $\phi \in C^{k+\ell}(N_1^+)$. Let $(v_i)_i \in \mathcal{V}_{k+\ell}^\ell$ and set $F = L_{v_1} \cdots L_{v_\ell} f$. We wish to estimate the following:

$$\sup_{0 \leq s \leq 1} \int_{N_1^+} \phi(n) F(g_{t+s} n x) \, d\mu_x^u.$$

To simplify notation, we prove the desired estimate for $s = 0$, the general case being essentially identical.

Let $\varepsilon > 0$ to be determined and choose ψ_ε to be a C^∞ bump function supported inside N_ε^+ and satisfying $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$. Define the following regularization of ϕ

$$\mathcal{M}_\varepsilon(\phi)(n) = \frac{\int_{N^+} \phi(un) \psi_\varepsilon(u) \, du}{\int_{N^+} \psi_\varepsilon(u) \, du},$$

where du denotes the right-invariant Haar measure on N^+. Recall the definition of the coefficients c_r above (6.2). Let $0 \leq m < k + \ell$ and $(w_j) \in (\mathcal{V}^+)^m$. Then,

$$\begin{aligned} |L_{w_1} \cdots L_{w_m}(\phi - \mathcal{M}_\varepsilon(\phi))(n)| &\leq \frac{\int |L_{w_1} \cdots L_{w_m}(\phi)(n) - L_{w_1} \cdots L_{w_m}(\phi)(un)| \psi_\varepsilon(u) \, du}{\int \psi_\varepsilon(u) \, du} \\ &\ll c_{m+1}(\phi) \frac{\int \operatorname{dist}(n, un) \psi_\varepsilon(u) \, du}{\int \psi_\varepsilon(u) \, du}. \end{aligned}$$

Now, note that if $\psi_\varepsilon(u) \neq 0$, then $\operatorname{dist}(u, \mathrm{id}) \leq \varepsilon$. Hence, right invariance of the metric on N^+ implies that $c_m(\phi - \mathcal{M}_\varepsilon(\phi)) \ll \varepsilon c_{m+1}(\phi)$.

Moreover, we have that $c_m(\mathcal{M}_\varepsilon(\phi)) \leq c_m(\phi)$ for all $0 \leq m \leq k + \ell$. It follows that $c_{k+\ell}(\phi - \mathcal{M}_\varepsilon(\phi)) \leq 2c_{k+\ell}(\phi)$. Finally, given $(w_i) \in (\mathcal{V}^+)^{k+\ell+1}$, integration by parts gives

$$L_{w_1} \cdots L_{w_{k+\ell+1}}(\mathcal{M}_\varepsilon(\phi))(n) = \frac{-\int_{N^+} L_{w_2} \cdots L_{w_{k+\ell+1}}(\phi)(un) \cdot L_{w_1}(\psi_\varepsilon)(u) \, du}{\int_{N^+} \psi_\varepsilon(u) \, du}.$$

In particular, since $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$, we get $c_{k+\ell+1}(\mathcal{M}_\varepsilon(\phi)) \ll \varepsilon^{-1} c_{k+\ell}(\phi)$. Since g_t expands N^+ by at least e^t, this discussion shows that for any $t \geq 0$, if $\|\phi\|_{C^{k+\ell}} \leq 1$, then

$$\begin{aligned} \|(\phi - \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} &\ll \varepsilon \sum_{m=0}^{k+\ell-1} \frac{e^{-mt}}{2^m} + \frac{e^{-(k+\ell)t}}{2^{k+\ell}}, \\ \|\mathcal{M}_\varepsilon(\phi) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} &\ll \sum_{m=0}^{k+\ell} \frac{e^{-mt}}{2^m} + \frac{\varepsilon^{-1} e^{-(k+\ell+1)t}}{2^{k+\ell+1}}. \end{aligned} \tag{7.6}$$

Then, taking $\varepsilon = e^{-kt}$, we obtain

$$\begin{aligned} \int_{N_1^+} \phi(n) F(g_t n x) \, d\mu_x^u &= \int \phi(n) F(g_t n x) \, d\mu_x^u \\ &= \int (\phi - \mathcal{M}_\varepsilon(\phi))(n) F(g_t n x) \, d\mu_x^u + \int \mathcal{M}_\varepsilon(\phi)(n) F(g_t n x) \, d\mu_x^u. \end{aligned} \tag{7.7}$$

To estimate the second term, we recall that the test functions for the weak norm were required to be supported inside $N^+_{1/10}$. On the other hand, the support of $\mathcal{M}_\varepsilon(\phi)$ may be larger, but still inside $N^+_{1+\varepsilon}$. To remedy this issue, we pick a partition of unity $\{\rho_i : i \in I\}$ of N^+_2, so that each ρ_i is smooth, non-negative, and supported inside some ball of radius $1/20$. We also require that $\|\rho_i\|_{C^{k+\ell+1}} \ll_k 1$. We can find such a partition of unity with bounded cardinality and multiplicity, depending only on N^+ (through its dimension and metric).

Similarly to Lemma 7.1, we denote by $I(\Lambda) \subseteq I$, the subset of those indices i such that there is some $n_i \in N^+$ in the support of of μ^u_x so that the support of ρ_i is contained inside $N^+_{1/10} \cdot n_i$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu^u_x$ is the 0 measure.

Now, observe that the functions $n \mapsto \rho_i(nn_i)\mathcal{M}_\varepsilon(\phi)(nn_i)$ are supported inside $N^+_{1/10}$. Thus, writing $x_i = n_i g_1 x$, using a change of variable, and arguing as in the proof of Lemma 7.1, cf. (7.3), we obtain

$$\begin{aligned}\int \mathcal{M}_\varepsilon(\phi)(n)F(g_t n x)\, d\mu^u_x &= e^{-\delta} \sum_{i\in I(\Lambda)} \int (\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-1})(n) F(g_{t-1} n g_1 x)\, d\mu^u_{g_1 x} \\ &\ll e'_{k,\ell}(f) \cdot \sum_{i\in I(\Lambda)} \|(\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} \cdot V(x_i) \mu^u_{x_i}(N^+_1).\end{aligned}$$

The point of replacing x with $g_1 x$ is that since x belongs to $N^-_1\Omega$, $g_1 x$ belongs to $N^-_{1/2}\Omega$, which satisfies the requirement on the basepoints in the definition of the weak norm.

Note that the bounded multiplicity property of the partition of unity, together with the doubling property in Proposition 3.1, imply that

$$\sum_{i\in I} \mu^u_{x_i}(N^+_1) \ll \mu^u_x(N^+_3) \ll \mu^u_x(N^+_1).$$

Moreover, combining the Leibniz estimate (6.2) with (7.6), we see that the $C^{k+\ell+1}$ norm of $(\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})$ is $O_k(1)$. Hence, by properties of the height function V in Proposition 4.3, it follows that

$$\int \mathcal{M}_\varepsilon(\phi)(n)F(g_t n x)\, d\mu^u_x \ll_k e'_{k,\ell}(f) V(x) \mu^u_x(N^+_1).$$

Using a completely analogous argument to handle the issues of the support of the test function, we can estimate the first term in (7.7) as follows:

$$\frac{1}{V(x)\mu^u_x(N^+_1)} \int_{N^+_1} (\phi - \mathcal{M}_\varepsilon(\phi))(n) F(g_t n x)\, d\mu^u_x \ll_k e^{-kt} e_{k,\ell}(f).$$

Since $(v_i) \in \mathcal{V}^\ell_{k+\ell}$, $x \in \Omega^-_1$ and $\phi \in C^{k+\ell}(N^+_1)$ were all arbitrary, this completes the proof. □

It remains to estimate the coefficients $e_{k,k}$. First, the following estimate in the case all the derivatives point in the stable direction follows immediately from Lemma 7.1.

Lemma 7.5. *For all $\gamma = (v_i) \in (\mathcal{V}^-_{2k})^k$, we have*

$$e_{k,k,\gamma}(R(z)^n f) \ll_\beta \frac{1}{(\mathrm{Re}(z) + k)^n} e_{k,k}(f).$$

Proof. Indeed, Lemma 7.1 shows that

$$e_{k,k,\gamma}(\mathcal{L}_t f) \ll e^{-kt} e_{k,k}(f).$$

Moreover, induction and integration by parts give $|\int_0^\infty t^{n-1} e^{-(z+k)t}/(n-1)! dt| \leq 1/(\mathrm{Re}(z)+k)^n$. This completes the proof. □

To give improved estimates on the the coefficient $e_{k,k,\gamma}$ in the case some of the components of γ point in the flow direction, the idea (cf. [AG13, Lem. 8.4] and [GLP13, Lem 4.5]) is to take advantage of the fact that the resolvent is defined by integration in the flow direction, which provides additional smoothing. This is leveraged through integration by parts to estimate the coefficient $e_{k,k}$ by $e_{k,k-1}$.

To see how such estimate can be turned into a gain on the norm of the resolvents, following [AG13], we define the following equivalent norms to $\|\cdot\|_k$. First, let us define the following coefficients:

$$e_{k,\ell,s} := \begin{cases} e_{k,\ell} & 0 \leq \ell < k, \\ \sup_{\gamma \in (\mathcal{V}_{2k}^-)^k} e_{k,k,\gamma} & \ell = k, \end{cases}, \qquad e_{k,k,\omega} := \sup_{\gamma \in \mathcal{V}_{2k}^k \setminus (\mathcal{V}_{2k}^-)^k} e_{k,k,\gamma}.$$

Given $B \geq 1$, define

$$\|f\|_{k,B,s} := \sum_{\ell=0}^{k} \frac{e_{k,\ell,s}(f)}{B^\ell}, \qquad \|f\|_{k,B,\omega} := \frac{e_{k,k,\omega}(f)}{B^k}.$$

Finally, we set

$$\|f\|_{k,B} := \|f\|_{k,B,s} + \|f\|_{k,B,\omega}. \tag{7.8}$$

Lemma 7.6. *Let $n, k \in \mathbb{N}$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ be given. Then, if B is large enough, depending on n, k, β and z, we obtain for all $f \in C_c^{k+1}(X)^M$ that*

$$\|R(z)^n f\|_{k,B,\omega} \leq \frac{1}{(\mathrm{Re}(z) + k + 1)^n} \|f\|_{k,B}.$$

Proof. Fix an integer $n \geq 0$. We wish to estimate integrals of the form

$$\int_{N_1^+} \phi(u) L_{v_1} \cdots L_{v_k} \left(\int_0^\infty \frac{t^n e^{-zt}}{n!} \mathcal{L}_{t+s} f \, dt \right)(ux) \, d\mu_x^u(u)$$
$$= \int_{N_1^+} \phi(u) \int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} \cdots L_{v_k} (\mathcal{L}_{t+s} f)(ux) \, dt \, d\mu_x^u(u),$$

with $0 \leq s \leq 1$ and at least one of the v_i pointing in the flow direction.

First, let us consider the case v_k points in the flow direction. Then, $v_k(u) = \psi_k(u)\omega$, where ω is the vector field generating the geodesic flow, for some function ψ_k in the unit ball of $C^{2k}(N^+)$. Hence, for a fixed $u \in N_1^+$, integration by parts in t, along with the fact that f is bounded, yields

$$\int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} L_{v_2} \cdots L_{v_k} (\mathcal{L}_{t+s} f)(ux) \, dt$$
$$= \psi_k(u) z \int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_{t+s} f)(ux) \, dt - \psi_k(u) \int_0^\infty \frac{t^{n-1} e^{-zt}}{(n-1)!} L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_{t+s} f)(ux) \, dt$$
$$= \psi_k(u) z L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_s R(z)^{n+1} f)(ux) - \psi_k(u) L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_s R^n(z) f)(ux).$$

Recall by Lemma 7.1 that $e_{k,\ell}(R(z)^n f) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n$ for all $n \in \mathbb{N}$; cf. Corollary 7.3. It follows that

$$e_{k,k,\gamma}(R(z)^{n+1} f) \leq e_{k,k-1}(R(z)^n f) + |z| e_{k,k-1}(R(z)^{n+1} f) \ll_\beta \left(\frac{\mathrm{Re}(z) + |z|}{\mathrm{Re}(z)^{n+1}} \right) e_{k,k-1}(f).$$

In the case v_k points in the stable direction instead, we note that $L_v L_w = L_w L_v + L_{[v,w]}$ for any two vector fields v and w, where $[v, w]$ is their Lie bracket. In particular, we can write $L_{v_1} \cdots L_{v_k}$ as a sum of at most k terms involving $k-1$ derivatives in addition to one term of the form $L_{w_1} \cdots L_{w_k}$, where w_k points in the flow direction. Each of the terms with one fewer derivative can be bounded by $e_{k,k-1}(R(z)^{n+1} f) \ll_\beta e_{k,k-1}(f)/\mathrm{Re}(z)^{n+1}$, while the term with k derivatives is controlled as in the previous case. Hence, taking the supremum over $\gamma \in \mathcal{V}_{2k}^k \setminus (\mathcal{V}_{2k}^-)^k$ and choosing B to be large enough, we obtain the conclusion. ☐

7.3. **Decomposition of the transfer operator according to recurrence of orbits.** In order to make use of the compact embedding result in Proposition 6.6, we need to localize our functions to a fixed compact set. This is done with the help of the Margulis function V. In this section, we introduce some notation and prove certain preliminary estimates for that purpose.

Recall the notation in Theorem 4.1. Let $T_0 \geq 1$ be a constant large enough so that $e^{\beta T_0} > 2$. We will enlarge T_0 over the course of the argument to absorb various auxiliary uniform constants. Define V_0 by

$$V_0 = e^{3\beta T_0}. \tag{7.9}$$

Let $\rho_{V_0} \in C_c^\infty(X)$ be a non-negative M-invariant function satisfying $\rho_{V_0} \equiv 1$ on the unit neighborhood of $\{x \in X : V(x) \leq V_0\}$ and $\rho_{V_0} \equiv 0$ on $\{V > 2V_0\}$. Moreover, we require that $\rho_{V_0} \leq 1$. Note that since T_0 is at least 1, we can choose ρ_{V_0} so that its C^{2k} norm is independent of T_0.

Let $\psi_1 = \rho_{V_0}$ and $\psi_2 = 1 - \psi_1$. Then, we can write

$$\mathcal{L}_{T_0} f = \tilde{\mathcal{L}}_1 f + \tilde{\mathcal{L}}_2 f,$$

where $\tilde{\mathcal{L}}_i f = \mathcal{L}_{T_0}(\psi_i f)$, for $i \in \{1,2\}$. It follows that for all $j \in \mathbb{N}$, we have

$$\mathcal{L}_{jT_0} f = \sum_{\varpi \in \{1,2\}^j} \tilde{\mathcal{L}}_{\varpi_1} \cdots \tilde{\mathcal{L}}_{\varpi_j} f = \sum_{\varpi \in \{1,2\}^j} \mathcal{L}_{jT_0}(\psi_\varpi f), \qquad \psi_\varpi = \prod_{i=1}^j \psi_{\varpi_i} \circ g_{-(j-i)T_0}. \tag{7.10}$$

Note that if $\varpi_i = 1$ for some $1 \leq i \leq j$, then, by Proposition 4.3, we have

$$\sup_{x \in \mathrm{supp}(\psi_\varpi)} V(x) \leq e^{\beta I_\varpi T_0} V_0, \qquad I_\varpi = j - \max\{1 \leq i \leq j : \varpi_i = 1\}. \tag{7.11}$$

The following lemma estimates the effect of multiplying by a fixed smooth function such as ψ_ϖ. To formulate the lemma, we need the following definition.

Definition 7.7. Given $\psi \in C^r(X)$, we use the notation $\|\psi\|_{C^r}^u$ to denote the C^r-norm of ψ along the unstable foliation. More precisely, we set

$$\|\psi\|_{C^r}^u = \sum_{i=0}^r \frac{c_i^u(\psi)}{2^i i!}, \tag{7.12}$$

where $c_i^u(\psi)$ denotes the maximum of the sup norm of all order-i derivatives of ψ along directions tangent to N^+.

Lemma 7.8. *Let $\psi \in C^{2k}(X)$ be given. Then, if $B \geq 1$ is large enough, depending on k and $\|\psi\|_{C^{2k}}$, we have*

$$\|\psi f\|_{k,B,s} \leq 2 \|\psi\|_{C^{2k}(X)}^u \|f\|_{k,B,s},$$

where $\|\psi\|_{C^{2k}(X)}^u$ is defined in (7.12).

Proof. Given $0 \leq \ell \leq k$ and $0 \leq s \leq 1$, we wish to estimate integrals of the form

$$\int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\psi f)(g_s n x)\, d\mu_x^u(n).$$

The term $L_{v_1} \cdots L_{v_\ell}(\psi f)$ can be written as a sum of 2^ℓ terms, each consisting of a product of an order-i derivative of ψ by an order-$(\ell - i)$ derivative of f, for $0 \leq i \leq \ell$. Viewing the product of ϕ

by an order-i derivative of ψ as a $C^{k+\ell-i}$ test function, and using (6.2) to bound the $C^{k+\ell-i}$ norm of such a product, we obtain a bound of the form

$$\begin{aligned} B^{-\ell} e_{k,\ell,s}(\psi f) &\leq B^{-1} C_{k,\psi} \sum_{i=0}^{\ell-1} \binom{\ell}{i} B^{-i} e_{k,i,s}(f) + B^{-\ell} \|\psi\|^u_{C^{2k}} e_{k,\ell,s}(f) \\ &\leq B^{-1} C_{k,\psi} 2^k \sum_{i=0}^{\ell-1} B^{-i} e_{k,i,s}(f) + B^{-\ell} \|\psi\|^u_{C^{2k}} e_{k,\ell,s}(f), \end{aligned}$$

for a suitably large constant depending on k and the C^{2k}-norm of ψ. Here, we note that the terms that contribute to the $e_{k,\ell,s}(f)$ term in the above sum all have the form $\int_{N_1^+} \phi\psi L_{v_1} \cdots L_{v_\ell}(f)\, d\mu^u_x$.

Summing over ℓ, we obtain

$$\begin{aligned} \|\psi f\|_{k,B,s} = \sum_{\ell=0}^{k} \frac{1}{B^\ell} e_{k,\ell,s}(\psi f) &\leq B^{-1} C_{k,\psi} 2^k \sum_{\ell=0}^{k} \sum_{i=0}^{\ell-1} B^{-i} e_{k,i,s}(f) + \|\psi\|^u_{C^{2k}} \|f\|_{k,B,s} \\ &\leq (B^{-1} C_{k,\psi} 2^k k + \|\psi\|^u_{C^{2k}}) \|f\|_{k,B,s}. \end{aligned}$$

Taking B large enough completes the proof of the lemma. □

The above lemma allows us to estimate the norms of the operators $\tilde{\mathcal{L}}_i$, for $i = 1, 2$ as follows.

Lemma 7.9. *There exists a constant $C_{k,\beta} \geq 1$, depending only on β and $\|\rho_{V_0}\|_{C^{2k}}$, such that for all large enough $B \geq 1$, we have*

$$\left\|\tilde{\mathcal{L}}_1 f\right\|_{k,B,s} \leq C_{k,\beta} \|f\|_{k,B,s}, \qquad \left\|\tilde{\mathcal{L}}_2 f\right\|_{k,B,s} \leq C_{k,\beta} e^{-\beta T_0} \|f\|_{k,B,s}.$$

Proof. The first inequality follows by Lemmas 7.1 and 7.8. The second inequality follows similarly since

$$\psi_2(g_{T_0} n x) \neq 0 \Longrightarrow V(g_{T_0} n x) \geq V_0.$$

By Proposition 4.3, this in turn implies that, whenever $\psi_2(g_{T_0} n x) \neq 0$ for some $n \in N_1^+$, then $V(x) \gg e^{\beta T_0}$, by choice of V_0. □

7.4. **Proof of Theorems 6.1 and 6.4.** Theorem 6.1 follows at once from 6.4. Theorem 6.4 will follow upon verifying the hypotheses of Theorem 6.5. The boundedness assertion follows by Corollary 7.3. It remains to estimate the essential spectral radius of the resolvent $R(z)$.

Write $z = a + ib \in \mathbb{C}$. Fix some parameter $0 < \theta < 1$ and define

$$\sigma := \min\{k, \beta\theta\}.$$

Let $0 < \epsilon < \sigma/5$ be given. We show that for a suitable choice of r and B, the following Lasota-Yorke inequality holds:

$$\left\|R(z)^{r+1} f\right\|_{k,B} \leq \frac{\|f\|_{k,B}}{(a + \sigma - 2\epsilon)^{r+1}} + C'_{k,\beta,B,r,T_0} \|\Psi_{r,\theta} f\|'_k, \tag{7.13}$$

where $C'_{k,\beta,B,r,T_0} \geq 1$ is a constant depending on the parameters in its subscript, while $\Psi_{r,\theta} : X \to [0,1]$ is a smooth function vanishing outside a sublevel set of the Margulis function V, and whose support depends on r and θ.

First, we show how (7.13) implies the result. Hennion's Theorem, Theorem 6.5, applied with the norm $\|\cdot\| = \|\cdot\|_{k,B}$ and the semi-norm $\|\cdot\|' = \|\Psi_{r,\theta} \bullet\|'_k$, implies that the essential spectral radius of $R(z)$, with respect to the norm $\|\cdot\|_{k,B}$, is at most $1/(a + \sigma - 2\epsilon)$. Equivalence of the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,B}$ implies that the same estimate also holds for the essential spectral radius $\rho_{ess}(R(z))$ with respect to $\|\cdot\|_k$. Note that the compact embedding requirement follows by Proposition 6.6

again by equivalence of the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,B}$. Since $\epsilon > 0$ was arbitrary, this shows that $\rho_{ess}(R(z)) \leq 1/(a+\sigma)$. Finally, as $0 < \theta < 1$ was arbitrary, we obtain that

$$\rho_{ess}(R(z)) \leq \frac{1}{\mathrm{Re}(z)+\sigma_0},$$

completing the proof.

To show (7.13), let an integer $r \geq 0$ be given and $J_r \in \mathbb{N}$ to be determined. Using (7.10) and a change of variable, we obtain

$$\begin{aligned} R(z)^{r+1} f &= \int_0^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt \\ &= \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt + \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt + \sum_{j=1}^{J_r} \int_{jT_0}^{(j+1)T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt. \end{aligned}$$

First, by Lemma 7.6, if B is large enough, depending on r, k and z, we obtain

$$\left\| R(z)^{r+1}(z) f \right\|_{k,B,\omega} \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

It remains to estimate $\left\| R(z)^{r+1} f \right\|_{k,B,s}$. Note that $\int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq T_0^{r+1}/r!$. Hence, taking r large enough, depending on k, a, β and T_0, and using Lemma 7.1, we obtain for any $B \geq 1$,

$$\left\| \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt \right\|_{k,B,s} \ll_\beta \|f\|_{k,B} \int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

Similarly, taking J_r to be large enough, depending on k, a, β, and r, we obtain for any $B \geq 1$,

$$\left\| \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt \right\|_{k,B,s} \ll_\beta \|f\|_{k,B} \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-at}}{r!} \, dt \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

To estimate the remaining term in $R(z)^{r+1} f$, let $1 \leq j \leq J_r$ and $\varpi = (\varpi_i)_i \in \{1,2\}^j$ be given. Let θ_ϖ denote the number of indices i such that $\varpi_i = 2$. Then, it follows from Lemma 7.1 and induction on Lemma 7.9 that

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \ll_\beta \|\mathcal{L}_{jT_0}(\psi_\varpi f)\|_{k,B,s} = \left\| \tilde{\mathcal{L}}_{\varpi_1} \circ \cdots \circ \tilde{\mathcal{L}}_{\varpi_j} f \right\|_{k,B,s} \leq C_{k,\beta}^j e^{-\beta \theta_\varpi T_0} \|f\|_{k,B,s}, \tag{7.14}$$

where $C_{k,\beta}$ is the constant provided by Lemma 7.9. We shall assume that $C_{k,\beta}$ is taken than the implicit constant in the first inequality.

Suppose $\theta_\varpi \geq \theta j$. Then, by taking T_0 to be large enough so that $C_{k,\beta}^{j+1} \leq e^{\epsilon j T_0}$, we obtain

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \leq e^{-(\beta\theta - \epsilon) j T_0} \|f\|_{k,B,s}.$$

The case $\theta_\varpi < \theta j$ is addressed in the following lemma. Its proof is given in Section 7.4.1 below and is an application of Lemmas 7.1, 7.4, and 7.8.

Lemma 7.10. *Assume $B \geq 1$ is chosen large enough, depending on k and r, and that $T_0 \geq 1$ is chosen large enough depending k, β and ϵ. Then, there exists a sublevel set $K_{r,\theta}$ of the Margulis function V and a smooth function $\Psi_{r,\theta} : X \to [0,1]$ vanishing outside the unit neighborhood of $K_{r,\theta}$ so that the following hold. For all $1 \leq j \leq J_r$, and all $\varpi \in \{1,2\}^j$ with $\theta_\varpi < \theta j$, we have*

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \leq e^{-(k-\epsilon)(t+jT_0)} \|f\|_{k,B,s} + C_{k,\beta,B,r,T_0} \|\Psi_{r,\theta} f\|'_k,$$

for a suitably large constant $C_{k,\beta,B,r,T_0} \geq 1$.

Putting the above estimates together, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{J_r}\int_{jT_0}^{(j+1)T_0}\frac{t^r e^{-zt}}{r!}\mathcal{L}_t f\;dt\right\|_{k,B,s} &\le \sum_{j=1}^{J_r} e^{-ajT_0}\sum_{\varpi\in\{1,2\}^j}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}\left\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\right\|_{k,B,s}\;dt\\ &\le \|f\|_{k,B,s}\sum_{j=1}^{J_r} e^{-(a+\sigma-\epsilon)jT_0}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}dt\\ &+ C_{k,\beta,B,r,T_0}\left\|\Psi_r f\right\|_k'\sum_{j=1}^{J_r}2^j e^{-ajT_0}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}\;dt\\ &\le e^{(\sigma-\epsilon)T_0}\|f\|_{k,B,s}\int_1^{J_r}\frac{t^r e^{-(a+\sigma-\epsilon)t}}{r!}\;dt + C'_{k,\beta,B,r,T_0}\left\|\Psi_r f\right\|_k',\end{aligned}$$

where we take $C'_{k,\beta,B,r,T_0}\ge 1$ to be a constant large enough so that the last inequality holds.

Next, we note that

$$\int_1^{J_r}\frac{t^r e^{-(a+\sigma-\epsilon)t}}{r!}\;dt \le \int_0^{\infty}\frac{t^r e^{-(a+\sigma-\epsilon)t}}{r!}\;dt = \frac{1}{(a+\sigma-\epsilon)^{r+1}}.$$

Thus, taking r to be large enough depending on a and T_0, and combining the estimates on $\left\|R(z)^{r+1}f\right\|_{k,B,\omega}$ and $\left\|R(z)^{r+1}f\right\|_{k,B,s}$, we obtain (7.13) as desired.

7.4.1. *Proof of Lemma 7.10.* Let $\varpi\in\{1,2\}^j$ be such that $\theta_\varpi<\theta j$. By Lemma 7.4, for all $0\le\ell<k$, we have

$$e_{k,\ell}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_{k,\beta} e^{-k(t+jT_0)}e_{k,\ell}(\psi_\varpi f)+e'_{k,\ell}(\psi_\varpi f),$$

where we may assume that T_0 is at least 2 so that the hypothesis of Lemma 7.4. For the coefficient $e_{k,k}$, Lemma 7.1 shows that for any $\gamma\in(\mathcal{V}_{2k}^-)^k$, we have

$$e_{k,k,\gamma}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_\beta e^{-k(t+jT_0)}e_{k,k,s}(\psi_\varpi f).$$

Hence, summing over ℓ, we obtain

$$\left\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\right\|_{k,B,s}\le C_{k,\beta}e^{-k(t+jT_0)}\left\|\psi_\varpi f\right\|_{k,B,s}+C_{k,\beta,B}\left\|\psi_\varpi f\right\|_k',$$

for suitable constants $C_{k,\beta}\ge 1$ and $C_{k,\beta,B}\ge 1$ depending on the parameters in their respective subscripts.

Our next task is to remove the dependence over ϖ in the right side of the above estimate. By taking B large enough, depending on the maximum over $1\le j\le J_r$ and $\varpi\in\{1,2\}^j$ of the C^{2k}-norm of the functions ψ_ϖ, we may apply Lemma 7.8 to get

$$\left\|\psi_\varpi f\right\|_{k,B,s}\le 2\left\|\psi_\varpi\right\|_{C^{2k}}^u\|f\|_{k,B,s},$$

where the unstable norm $\|\cdot\|_{C^{2k}}^u$ is defined in (7.12).

By the formula (7.10) for ψ_ϖ, the functions ψ_ϖ are given by a product of j functions of the form ρ_{V_0} and $1-\rho_{V_0}$ composed by g_{-t} for suitable $t>0$. Since composition by g_{-t}, $t>0$, is non-expanding on the unstable norm $\|\cdot\|_{C^{2k}}^u$, we get

$$\left\|\psi_\varpi\right\|_{C^{2k}}^u\le\left\|\rho_{V_0}\right\|_{C^{2k}}^j.$$

By enlarging the constant $C_{k,\beta}$ if necessary, we may assume it is larger than $2\left\|\rho_{V_0}\right\|_{C^{2k}}$. Thus, we obtain the bound:

$$\left\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\right\|_{k,B,s}\le C_{k,\beta}^{j+1}e^{-k(t+jT_0)}\|f\|_{k,B,s}+C_{k,\beta,B}\left\|\psi_\varpi f\right\|_k'.$$

To put the term $\left\|\psi_\varpi f\right\|_k'$ in a form where we can apply Hennion's Theorem 6.5, we take advantage of the bound $\theta_\varpi<\theta j$. To this end, note that the bound $\theta_\varpi<\theta j$ and the formula (7.11) for the support of ψ_ϖ imply that there is a sublevel set $K_{r,\theta}$ of the Margulis function V, depending only on θ and J_r, such that the following holds. For every $1 \leq j \leq J_r$ and all $\varpi \in \{1,2\}^j$ with $\theta_\varpi < \theta j$, the function ψ_ϖ is supported inside $K_{r,\theta}$. Let $\Psi_{r,\theta} : X \to [0,1]$ denote a smooth bump function which is identically 1 on $K_{r,\theta}$ and vanishes outside the unit neighborhood of $K_{r,\theta}$. Then, for every ϖ with $\theta_\varpi < \theta j$, we have that $\psi_\varpi = \psi_\varpi \Psi_{r,\theta}$. Hence, arguing as in the proof of Lemma 7.8 with $\|\cdot\|_k'$ in place of $\|\cdot\|_{k,B,s}$, we obtain

$$\|\psi_\varpi f\|_k' = \|\psi_\varpi \Psi_{r,\theta} f\|_k' \ll_{k,T_0,J_r} \|\Psi_{r,\theta} f\|_k' .$$

Here, the dependence of the implicit constant arises from the norm $\|\psi_\varpi\|_{C^{2k}}$. Hence, taking T_0 large enough so that $C_{k,\beta}^{j+1} \leq e^{\epsilon k(t+jT_0)}$, and combining the above estimates, we obtain

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \leq e^{-(k-\epsilon)(t+jT_0)} \|f\|_{k,B,s} + C_{k,\beta,B,r,T_0} \|\Psi_{r,\theta} f\|_k' ,$$

for a suitably large constant $C_{k,\beta,B,r,T_0} \geq 1$.

7.5. **Proof of Theorem 1.1.** Recall the notation in the statement of the theorem. We note that switching the order of integration in the definition of the Laplace transform shows that

$$\hat{\rho}_{f,g}(z) = \int R(z)(f) g \; d\mathrm{m}^{\mathrm{BMS}}, \qquad \mathrm{Re}(z) > 0.$$

In particular, the poles of $\hat{\rho}_{f,g}$ form a subset of the set of poles the resolvent $R(z)$.

On the other hand, Corollary 7.2 implies that the infinitesimal generator $\mathfrak{X}$ of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. Moreover, $R(z)$ coincides with the resolvent operator $(\mathfrak{X} - z\mathrm{id})^{-1}$ associated to $\mathfrak{X}$, whenever z belongs to the resolvent set (complement of the spectrum) of $\mathfrak{X}$.

We further note that the spectra of $\mathfrak{X}$ and $R(z)$ are related by the formula $\sigma(\mathfrak{X}) = z - 1/\sigma(R(z))$. In particular, by Theorem 6.4, in the half plane $\mathrm{Re}(z) > -\sigma_0$, the poles of $R(z)$ coincide with the eigenvalues of $\mathfrak{X}$. In view of this relationship between the spectra, the fact that the imaginary axis does not contain any poles for the resolvent, apart from 0, follows from the mixing property of the geodesic flow with respect to $\mathrm{m}^{\mathrm{BMS}}$ as shown in Lemma 7.11 below.

Finally, we note that in the case Γ has cusps, β was an arbitrary constant in $(0, \Delta/2)$, so that we may take σ_0 in the conclusion of Theorem 6.4 to be the minimum of k and $\Delta/2$ in this case. This completes the proof of Theorem 1.1.

7.6. **Resonances on the imaginary axis.** In this section, we study the intersection of the spectrum of $\mathfrak{X}$ with the imaginary axis.

Lemma 7.11. *The intersection of the spectrum of $\mathfrak{X}$ with the imaginary axis consists only of the eigenvalue 0 which has algebraic multiplicity one.*

First, we need the following lemma relating our norms to correlation functions.

Lemma 7.12. *For all $f, \varphi \in C_c^2(X)^M$, we have that $\int f \cdot \varphi \; d\mathrm{m}^{\mathrm{BMS}} \ll_\varphi \|f\|_1'$, where the implied constant depends on $\|\varphi\|_{C^2}$ and the injectivity radius of its support.*

Proof. Using a partition of unity, we may assume φ is supported inside a flow box. The implied constant then depends on the number of elements of the partition of unity needed to cover the support of φ. Inside each such flow box, the measure $\mathrm{m}^{\mathrm{BMS}}$ admits a disintegration in terms of the conditional measures μ_x^u averaged against a suitable measure on the transversal to the strong unstable foliation. Thus, the lemma follows by definition of the norm by viewing the restriction of φ to each local unstable leaf as a test function. ∎

Proof of Lemma 7.11. In what follows, we endow elements φ of $C_c^2(X)$ with the norm $\|\varphi\|_{C^2}'$ given by multiplying the C^2-norm of φ with a suitable power of the reciprocal of the injectivity radius of its support so that $\|\varphi\|_{C^2}'$ dominates the implicit constant depending on φ in Lemma 7.12. Such power exists by the proof of the lemma. The dual space $C_c^2(X)^*$ is endowed with the corresponding dual norm.

First, we note that, since $\mathcal{B}_k \subseteq \mathcal{B}_1$ for all $k \geq 1$, it suffices to prove the lemma for the action of $\mathfrak{X}$ on $\mathcal{B}_1$. Let $\Phi : \mathcal{B}_1 \to C_c^2(X)^*$ denote the linear map which extends the mapping $f \mapsto (\varphi \mapsto \int f\varphi \, d\mathrm{m}^{\mathrm{BMS}})$ from $C_c^2(X)^M$ to the dual space $C_c^2(X)^*$. The fact that this mapping extends continuously to $\mathcal{B}_1$ follows by Lemma 7.12. We claim that Φ is injective. This claim is routine in the absence of cusps, and we briefly outline why it also holds in general.

To prove this claim, note first that the coefficients $e_{1,0}(\cdot;x)$ and $e_{1,1}(\cdot;x)$ extend from C_c^2 to define seminorms on $\mathcal{B}_1$. In particular, given any $f \in \mathcal{B}_1$ and $f_n \in C_c^2(X)^M$ tending to f in $\mathcal{B}_1$, we have $e_{1,\ell}(f;x) = \lim_{n\to\infty} e_{1,\ell}(f_n;x)$ for $\ell = 0,1$ and for every $x \in N_1^-\Omega$. Since the coefficient $e_{1,\ell}(f)$ is defined by taking a supremum over x, it follows that we can find a sequence $x_m \in N^-\Omega$ such that $e_{1,\ell}(f;x_m)$ converges to $e_{1,\ell}(f)$. In particular, we obtain

$$e_{1,\ell}(f) = \lim_{m\to\infty} \lim_{n\to\infty} e_{1,\ell}(f_n;x_m). \tag{7.15}$$

Now, suppose $f \in \mathcal{B}_1$ is in the kernel of Φ and let $f_n \in C_c^2(X)^M$ be a sequence of functions converging to f. By continuity, $\Phi(f_n)$ tends to 0 in $C_c^2(X)^*$. One then checks that this implies that for every fixed $x \in N_1^-\Omega$, we have that[8] $e_{1,0}(f_n;x) \to 0$ as $n \to \infty$. Hence, by (7.15), we get that $e_{1,0}(f) = 0$. Since $\|f\|_1' \leq e_{1,0}(f)$, this shows that $\|f\|_1' = 0$, and hence Φ is injective as claimed.

We now show that this injectivity implies the lemma. Via the relationship between the spectra of $\mathfrak{X}$ and the resolvents (cf. Section 7.5), Theorem 6.4 implies that the intersection of the spectrum $\sigma(\mathfrak{X})$ with the imaginary axis consists of a discrete set of eigenvalues. Similarly, finiteness of the multiplicities of each of these eigenvalues is a consequence of quasi-compactness of the resolvent.

Let $b \in \mathbb{R}$ be such that ib is one such eigenvalue with eigenvector $0 \neq f \in \mathcal{B}_1$ and note that this implies that $\mathcal{L}_t f = e^{ibt} f$. We show that $\Phi(f)$ is a multiple of the measure $\mathrm{m}^{\mathrm{BMS}}$. This implies that $b = 0$ by injectivity since $\mathrm{m}^{\mathrm{BMS}}$ is the image of the constant function 1 under Φ. To do so, we use the fact that the geodesic flow is mixing[9] with respect to $\mathrm{m}^{\mathrm{BMS}}$ by work of Rudolph [Rud82] and Babillot [Bab02]. Let $\varphi \in C_c^2(X)$ be arbitrary and let $\theta_n = \int f_n \, d\mathrm{m}^{\mathrm{BMS}}$ and $\xi = \int \varphi \, d\mathrm{m}^{\mathrm{BMS}}$. Then, for every $t \geq 0$ and $n \in \mathbb{N}$, we have

$$|\Phi(f)(\varphi) - \theta_n\xi| \leq \left|\Phi(f)(\varphi) - \int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right| + \left|\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}} - \theta_n\xi\right|. \tag{7.16}$$

By mixing, for every fixed n, the second term can be made arbitrarily small by taking t large enough. Moreover, since $\Phi(f) = e^{-ibt}\Phi(\mathcal{L}_t f)$, the first term is bounded by

$$\left|e^{-ibt}\Phi(\mathcal{L}_t f)(\varphi) - e^{-ibt}\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right| + |e^{-ibt} - 1|\left|\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right|. \tag{7.17}$$

The first term in (7.17) is equal to $|\Phi(\mathcal{L}_t(f-f_n)(\varphi)|$, which is $O_\varphi(\|f - f_n\|_1)$ in view of Lemmas 7.12 and 7.1. Similarly, since f_n converges to f in $\mathcal{B}_1$, the second term is $O_\varphi(|e^{-ibt} - 1| \, \|f\|_1)$. To bound this term, note that one can find arbitrarily large t so that e^{ibt} is arbitrarily close to 1.

Therefore, using a diagonal argument, this implies that we can find a sequence $t(n)$ tending to infinity so that the upper bound in (7.16) tends to 0 with n. If $\xi \neq 0$, the above argument implies that θ_n is $O_\varphi(\Phi(f)(\varphi))$ and hence converges (along a subsequence) to some $\theta \in \mathbb{R}$. In particular, the values of $\Phi(f)$ and $\theta\mathrm{m}^{\mathrm{BMS}}$ agree on φ in this case. If $\xi = 0$, then the above argument shows that $\Phi(f)(\varphi) = 0$ so that the same conclusion also holds.

[8]This is similar to the argument in the proof of (6.8). One proceeds by thickening test functions on $N_1^+ \cdot x$ to functions supported in a small box around x and controlling the difference between the integrals using $e_{1,0}(f_n;x)$ and the integral against the thickened functions using $e_{1,1}(f_n)$. The seminorms $e_{1,1}(f_n)$ remain bounded since $f_n \to f$, while the integrals against thickened functions tend to 0 since $\Phi(f_n) \to 0$.

[9]We refer the reader to [BDL18, Corollary 5.4] for this deduction using only ergodicity of the flow.

The assertion on the algebraic multiplicity, which in particular involves ruling out the presence of Jordan blocks, is standard and can be deduced from quasi-compactness of the resolvent and the bound on its norm given in Corollary 7.3 following very similar lines to [BDL18, Corollary 5.4] to which we refer the interested reader for details. ☐

7.7. **Exponential recurrence from the cusp and Proof of Theorem 1.2.** As a corollary of our analysis, we obtain the following stronger form of Theorem 1.2 regarding the exponential decay of the measure of orbits spending a large proportion of their time in the cusp. This result is crucial to our arguments in later sections. The deduction of Theorem 1.2 in its continuous time formulation from the following result follows using Proposition 4.3 and is left to the reader.

Theorem 7.13. *For every $\varepsilon > 0$, there exists $r_0 \asymp_\beta 1/\varepsilon$ such that the following holds for all $m \in \mathbb{N}, r \geq r_0, 0 < \theta < 1$ and $x \in N_1^- \Omega$. Let $H = e^{3\beta r_0}$, and let χ_H be the indicator function of the set $\{x : V(x) > H\}$. Then,*

$$\mu_x^u \left(n \in N_1^+ : \sum_{1 \leq \ell \leq m} \chi_H(g_{r\ell} nx) > \theta m \right) \leq e^{-(\beta\theta - \varepsilon)m} V(x) \mu_x^u(N_1^+).$$

Proof. The argument is very similar to the proof of the estimate (7.14), with small modifications allowing for the height H to be independent of the step size r. This subtle difference from (7.14) will be important in the application of this result to the proof of exponential mixing in the sequel.

Let $r_0 \geq 1$ to be chosen later in the argument depending on ε and β and set $V_0 = e^{2\beta r_0}$. As before, let $\rho_{V_0} : X \to [0,1]$ denote a smooth compactly supported function which is identically 1 on $\{V \leq V_0\}$ and vanishing outside $\{V > 2V_0\}$. Let $\psi = 1 - \rho_{V_0}$. Let $r \geq r_0$ and define the following operators:

$$\tilde{\mathcal{L}}_1(f) := \mathcal{L}_r f, \qquad \tilde{\mathcal{L}}_2(f) = \mathcal{L}_r(\psi f).$$

Note that, unlike our previous arguments, the operators $\tilde{\mathcal{L}}_i$ do not provide a decomposition of $\mathcal{L}_r$, i.e., $\mathcal{L}_r \neq \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2$. Given $m \in \mathbb{N}$ and $\varpi \in \{1,2\}^m$, let $\mathcal{L}_\varpi = \tilde{\mathcal{L}}_{\varpi_1} \circ \cdots \circ \tilde{\mathcal{L}}_{\varpi_m}$. We also have that

$$\mathcal{L}_\varpi(f) = \mathcal{L}_{mr}(\psi_\varpi f), \qquad \text{where} \qquad \psi_\varpi = \prod_{\ell : \varpi_\ell = 2} \psi \circ g_{(\ell - m)r}.$$

Similarly to Lemma 7.9, Lemma 7.1 implies the bounds

$$e_{1,0}(\tilde{\mathcal{L}}_1 f) \ll_\beta e_{1,0}(f), \qquad e_{1,0}(\tilde{\mathcal{L}}_2 f) \ll_\beta e^{-\beta r_0} e_{1,0}(\psi f) \ll e^{-\beta r_0} e_{1,0}(f). \tag{7.18}$$

Let $H = e^{3\beta r_0}$. We shall assume that r_0 is large enough so that $H > 2V_0$. Define

$$E_\varpi = \left\{ n \in N_1^+ : \varpi_\ell = 2 \Rightarrow V(g_{\ell r} nx) > H, \text{ for all } \ell = 1, \ldots, m \right\}.$$

Then, for all $n \in N_1^+$,

$$\psi_\varpi(g_{mr} nx) \geq \mathbb{1}_{E_\varpi}(n). \tag{7.19}$$

Indeed, if $\mathbb{1}_{E_\varpi}(n) = 1$, and ℓ is such that $\varpi_\ell = 2$, then $V(g_{\ell r} nx) > H > 2V_0$ and, hence, $\psi(g_{\ell r} nx) = 1$. It follows that

$$\psi_\varpi(g_{mr} nx) = \prod_{\ell : \varpi_\ell = 2} \psi(g_{\ell r} nx) = 1.$$

This verifies (7.19). Denote by θ_ϖ the number of indices ℓ for which $\varpi_\ell = 2$. Then, we see that

$$\left\{ n \in N_1^+ : \sum_{1 \leq \ell \leq m} \chi_H(g_{r\ell} nx) > \theta m \right\} \subseteq \bigcup_{\varpi : \theta_\varpi > \theta m} E_\varpi.$$

We wish to apply (7.18) with f the constant function on X. One checks that this f belongs to the space $\mathcal{B}_1$ and $e_{1,0}(f) \ll 1$. Let $C_1 \geq 1$ denote a constant larger than $e_{1,0}(f)$ and the two implicit constants in (7.18). Then, applying (7.18) iteratively m times, and using (7.19), we obtain

$$\mu_x^u(E_\varpi) \leq e_{1,0}(\mathcal{L}_\varpi(f)) \leq C_1^m e^{-\beta\theta_\varpi r_0} V(x)\mu_x^u(N_1^+)e_{1,0}(f) \leq C_1^{m+1} e^{-\beta\theta m r_0} V(x)\mu_x^u(N_1^+).$$

Since there are at most 2^m choices of ϖ, the result follows by taking r_0 large enough so that $2^m C_1^{m+1} \leq e^{\varepsilon m r_0}$. □

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Department of Mathematics, Statistics, and Computer Science, University of Illinois Chicago
Email address: okhalil@uic.edu