MEASURE RIGIDITY AND EQUIDISTRIBUTION FOR FRACTAL CARPETS

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ABSTRACT. Let θ be a Bernoulli measure which is stationary for a random walk generated by finitely many contracting rational affine dilations of $\mathbb{R}^d$, and let $\mathcal{K} = \mathrm{supp}(\theta)$ be the corresponding attractor. An example in dimension $d = 1$ is the Hausdorff measure on Cantor's middle thirds set, and examples in higher dimensions include missing digits sets, Sierpiński carpets and Menger sponges. Let ν denote the image of θ under the map $\mathcal{K} \to \mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$ which sends $\mathbf{x}$ to the lattice $\Lambda_{\mathbf{x}} = \mathrm{span}_{\mathbb{Z}}(\mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{e}_{d+1} + (\mathbf{x}, 0))$. We prove equidistribution of the pushforward measures $a_{n*}\nu$ along any diverging sequence of diagonal matrices $(a_n) \subset \mathrm{SL}_{d+1}(\mathbb{R})$ that expand the first d coordinates under a natural non-escape of mass condition. The latter condition is known to hold whenever θ is absolutely friendly. We also show that weighted badly approximable vectors and Dirichlet-improvable vectors (for arbitrary norm) form a subset of $\mathcal{K}$ of θ-measure zero. The key ingredient is a measure classification theorem for the stationary measures of an associated random walk on an S-arithmetic space, introduced by two of the authors in [KL23]. A new feature of this setting is that this random walk admits stationary measures which are not invariant.

1. Introduction

This paper proves equidistribution results for pushforwards of certain fractal measures on certain homogeneous spaces, by analyzing certain random walks adapted to the fractal measures. These dynamical results, are then applied to problems about the Diophantine properties of typical points on certain fractals. We refer the reader to [KLW04, BQ11, SW19, PSS23, KL23, BHZ24, DJ24] for related work.

Let k and d be positive integers with $k \geq 2$ and let $\Phi = \{f_1, \ldots, f_k\}$ be a collection of affine maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$f_i(\mathbf{x}) = \varrho \mathbf{x} + \mathbf{y}_i, \quad \text{where } \mathbf{y}_i \in \mathbb{Q}^d,\ \varrho \in \mathbb{Q},\ 0 < |\varrho| < 1. \tag{1.1}$$

We will refer to such a collection Φ as a *carpet IFS*. This terminology is motivated by the example of the Sierpiński carpet. The *attractor* of Φ is the unique nonempty compact subset of $\mathbb{R}^d$ satisfying $\mathcal{K} = \bigcup f_i(\mathcal{K})$. We say Φ is *irreducible* if there is no finite collection of proper affine subspace of $\mathbb{R}^d$ which is left invariant by each of the maps f_i. For a probability vector $\mathbf{p} = (p_1, \ldots, p_k)$, let $\theta = \theta(\Phi, \mathbf{p})$ be the associated Bernoulli measure supported on its attractor $\mathcal{K}$. These terms are explained in §3.1, and the sets $\mathcal{K}$ and measures θ which arise in this way form a fairly large class of self-similar fractal sets and measures; for the purpose of this introduction it is enough to note that commonly studied self-similar sets, like Cantor's middle thirds set, missing digit sets in $d \geq 1$ dimensions, or the Sierpiński carpet, can arise as $\mathcal{K}$, and their Hausdorff measure can arise as the Bernoulli measure θ.

Let $\mathscr{X}_{d+1}$ denote the space of lattices of covolume one. This space is naturally identified with the quotient of Lie groups $\mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z})$, via the map $g\mathrm{SL}_{d+1}(\mathbb{Z}) \mapsto g\mathbb{Z}^{d+1}$, and this identification equips $\mathscr{X}_{d+1}$ with the topology of a non-compact manifold and with the measure $m_{\mathscr{X}_{d+1}}$, which is the unique $\mathrm{SL}_{d+1}(\mathbb{R})$-invariant measure. Let

$$u : \mathbb{R}^d \to \mathrm{SL}_{d+1}(\mathbb{R}), \quad u(\mathbf{x}) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathrm{Id} & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix}, \quad U \stackrel{\mathrm{def}}{=} \left\{ u(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^d \right\}. \tag{1.2}$$

For $g \in \mathrm{SL}_{d+1}(\mathbb{R})$, let

$$\Lambda_g \stackrel{\mathrm{def}}{=} g\mathbb{Z}^{d+1}, \quad \text{let } \Lambda_{\mathbf{x}} \stackrel{\mathrm{def}}{=} \Lambda_{u(\mathbf{x})},$$

and denote by ν_g the pushforward of θ under the map $\mathbf{x} \mapsto \Lambda_{u(\mathbf{x})g}$.

Given a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{SL}_{d+1}(\mathbb{R})$, we say that θ is *uniformly non-divergent along* (a_n) if for every $\varepsilon > 0$, there is a compact set $K \subset \mathscr{X}_{d+1}$ so that for all g, for all large enough n we have

$$a_{n*}\nu_g(K) \geq 1 - \varepsilon.$$

With this notation we are ready to state our main equidistribution result.

Theorem 1.1. *Let Φ be an irreducible carpet-IFS, let $\theta = \theta(\Phi, \mathbf{p})$ be a Bernoulli measure, for some probability vector $\mathbf{p}$, and let ν_0 be the pushforward of θ under $\mathbf{x} \mapsto \Lambda_{\mathbf{x}}$. Let $(a_n)_n$ be any sequence of diagonal matrices tending to ∞ in $\mathrm{SL}_{d+1}(\mathbb{R})$. If θ is uniformly non-divergent along (a_n), then*

$$\lim_{n\to\infty} a_{n*}\nu_0 = m_{\mathscr{X}_{d+1}}, \tag{1.3}$$

with respect to the weak-$*$ topology on the space of probability measures on $\mathscr{X}_{d+1}$.

As we will see in §3.2, conditions on θ and (a_n) which guarantee uniform non-divergence are well-understood. Clearly, non-divergence of the sequence $a_{n*}\nu_0$ is a necessary condition for (1.3), and all known methods for establishing this non-divergence estimate actually yield the stronger uniform version we use here. In particular, uniform non-divergence is known to hold when (a_n) is *drifting away from walls* and θ is *absolutely friendly*; cf. §3 for definitions. Here we mention that absolute friendliness is satisfied for large classes of examples including the Hausdorff measure on $\mathcal{K}$ under the open set condition, and for all Bernoulli measures under the strong separation condition.

Results about limits of measures as in (1.3) have a long history in homogeneous dynamics, and we briefly mention a few that are relevant to our discussion. In the case $a_n = \mathrm{diag}\left(e^{n/d}, \ldots, e^{n/d}, e^{-n}\right)$, the group U is the expanding horospherical group for the action of (a_n), and in this case, if θ is Lebesgue measure on $\mathbb{R}^d$ then (1.3) is an easy consequence of the Howe-Moore theorem on mixing on $\mathscr{X}_{d+1}$. Also in the case of Lebesgue measure, for more general sequences (a_n) which expand U by conjugations, (1.3) was established in [KW08] and [KM12].

The first equidistribution results for a measure which is singular with respect to Lebesgue measure were given by Shah in [Sha09], in the case that θ is the length measure on an analytic nondegenerate curve, and these results were later extended by various authors (see [SY21] and references therein). The first equidistribution result for fractal measures came in the paper [KL23], where an additional hypothesis was imposed, namely an inequality involving the contractions in the IFS Φ and the coefficients of the probability vector $\mathbf{p}$. Under these conditions it was shown that (1.3) holds in an effective form. Further effective results in case $d = 1$ were obtained by Datta and Jana in [DJ24], under an assumption involving Fourier decay, and in great generality in a breakthrough paper of Bénard, He and Zhang in [BHZ24]. The techniques of this paper are different from those used in [KL23, DJ24, BHZ24].

1.1. **Applications to Diophantine approximations.** As with all of the previously mentioned dynamical results, Theorem 1.1 is motivated by questions in Diophantine approximations. Our Diophantine applications, which we now state, concern so-called weighted approximation. Given a *weight vector* $\mathbf{r} = (r_1, \ldots, r_d)$, with

$$\sum_{i=1}^{d} r_i = 1, \qquad \forall i,\ r_i > 0,$$

and $\mathbf{x} = (x_1, \ldots, x_d)$, we say that $\mathbf{x}$ is $\mathbf{r}$-*badly approximable* if there is $c > 0$ such that for all $Q \in \mathbb{N}$ and all $(P_1, \ldots, P_d) \in \mathbb{Z}^d$, we have

$$Q \max_i \left(|Qx_i - P_i|^{1/r_i}\right) \geq c. \tag{1.4}$$

Let

$$a_t^{(\mathbf{r})} \overset{\text{def}}{=} \operatorname{diag}\left(e^{r_1 t}, \ldots, e^{r_d t}, e^{-t}\right) \subset \mathrm{SL}_{d+1}(\mathbb{R}). \tag{1.5}$$

Given a norm $\|\cdot\|$ on $\mathbb{R}^{d+1}$, we set

$$\varepsilon_{\|\cdot\|} \overset{\text{def}}{=} \sup\{\varepsilon > 0 : \text{ there is } \Lambda \in \mathscr{X}_{d+1} \text{ such that } B_{\|\cdot\|}(0,\varepsilon) \cap \Lambda = \{0\}\}, \tag{1.6}$$

and say that $\mathbf{x} \in \mathbb{R}^d$ is $(\mathbf{r}, \|\cdot\|)$-*Dirichlet improvable* if there is $\varepsilon < \varepsilon_{\|\cdot\|}$ such that for all sufficiently large t, the lattice $a_t^{(\mathbf{r})}\Lambda_{\mathbf{x}}$ contains vectors $\mathbf{y}$ with $0 < \|y\| \leq \varepsilon$. The case in which $\|\cdot\| = \|\cdot\|_\infty$ is the supremum norm was studied by Davenport and Schmidt [DS70] and the case of general norms was studied by Kleinbock and Rao [KR21], see §3.3 for more details. With these notations we have:

Theorem 1.2. *Let Φ and θ be as in Theorem 1.1. Then for any weight vector $\mathbf{r}$, and for any norm $\|\cdot\|$ on $\mathbb{R}^{d+1}$, the set of $(\mathbf{r}, \|\cdot\|)$-Dirichlet improvable vectors has θ-measure zero. In particular, the set of $\mathbf{r}$-badly approximable points has θ-measure zero.*

The case $\mathbf{r} = (1/d, \cdots, 1/d)$ is called the case of *equal weights*. In case $d = 1$ the second assertion was proved by Einsiedler, Fishman and Shapira [EFS11], and the case of equal weights in all dimensions was settled in [SW19]. Note that in case $d = 1$ the only weights are the equal weights, thus our results are only new when $d > 1$. The first results about general weight vectors $\mathbf{r}$ were obtained by Prohaska, Sert and Shi [PSS23]; however, they could only treat measures defined by an IFS of affine maps, which depends on $\mathbf{r}$. Finally note that an effective version of Theorem 1.1 would have additional important Diophantine applications. Namely, in case $d = 1$ it was used in [BHZ24] to settle a question attributed to Mahler; the analogous conjecture in higher dimensions is still open.

1.2. **Random walks on S-arithmetic spaces.** The results of this paper rely on a description of stationary measures for certain random walks on an S-arithmetic space, which depends on the IFS Φ. Let $d \in \mathbb{N}$ and denote by $\mathbf{G}$ the automorphism group of the $\mathbb{Q}$-algebra Mat_{d+1}, i.e., $\mathbf{G} = \mathrm{PGL}_{d+1}$; cf. [KL23, §3.1]. For a finite collection of places $S = \{\infty, p_1, \ldots, p_\ell\}$, where $p_1, \ldots, p_\ell$ are primes, we define an S-arithmetic homogeneous space

$$\mathscr{X}_{d+1}^S \overset{\text{def}}{=} G^S/\Lambda^S,$$

where

$$G^S \overset{\text{def}}{=} \mathbf{G}(\mathbb{R}) \times \prod_{j=1}^{\ell} \mathbf{G}(\mathbb{Q}_{p_j}),$$

and Λ^S is the diagonal embedding in G^S of

$$\mathbf{G}\left(\mathbb{Z}\left[\frac{1}{p_1}, \cdots, \frac{1}{p_\ell}\right]\right).$$

The homogeneous space $\mathscr{X}_{d+1}^S$ is equipped with a unique G^S-invariant probability measure $m_{\mathscr{X}_{d+1}^S}$. Let $G \overset{\text{def}}{=} \mathbf{G}(\mathbb{R})$. Notice that the space of lattices $\mathscr{X}_{d+1}$ can also be identified with $G/\mathbf{G}(\mathbb{Z})$, and the projection $G^S \to G$ induces a well-defined surjective map $\mathscr{X}_{d+1}^S \to \mathscr{X}_{d+1}$. This map has a compact fiber (see e.g. [KL23, §3.1 & Appendix B]). There is a transitive action of G^S on $\mathscr{X}_{d+1}^S$ by left translations, and since we have an inclusion $G \to G^S$, we get an action of G on $\mathscr{X}_{d+1}^S$, for which the projection $\mathscr{X}_{d+1}^S \to \mathscr{X}_{d+1}$ is equivariant. For more information about S-arithmetic groups and S-arithmetic homogeneous spaces, we refer the reader to [PRR23, Rat98, Tom00].

Following [KL23], we now define a random walk on $\mathscr{X}_{d+1}^S$, depending on the carpet IFS Φ and a probability vector $\mathbf{p}$. We will define elements $h_1, \ldots, h_k \in G^S$ below. Then, fixing a probability vector $\mathbf{p}$, the Φ-*adapted random walk* is obtained by moving from $x \in \mathscr{X}_{d+1}^S$ to $h_i x$, independently, with probability p_i.

Let $S_{\mathrm{f}} \stackrel{\mathrm{def}}{=} \{p_1, \ldots, p_\ell\}$ be all the primes which appear in the denominators of all the coefficients of the maps of the IFS Φ, as well as the numerator of the contraction ratio ϱ, and let $S \stackrel{\mathrm{def}}{=} S_{\mathrm{f}} \cup \{\infty\}$. That is, S_{f} consists of the primes appearing in the decomposition of rq, where

$$\varrho = \frac{r}{q}, \quad \gcd(r, q) = 1,$$

and the denominators of all the coefficients of the translation vectors $\mathbf{y}_i$. This choice means that if we write

$$\mathbb{Z}\left[\frac{1}{S_{\mathrm{f}}}\right] \stackrel{\mathrm{def}}{=} \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_\ell}\right],$$

then S_{f} is the smallest set of primes such that ϱ is invertible in $\mathbb{Z}\left[\frac{1}{S_{\mathrm{f}}}\right]$ and each $\mathbf{y}_i$ belongs to $\left(\mathbb{Z}\left[\frac{1}{S_{\mathrm{f}}}\right]\right)^d$.

Here and in what follows, we will define elements of G^S by specifying $|S|$-tuples of matrices in $\mathrm{GL}_{d+1}(\mathbb{Q}_\sigma)$, where σ ranges over S; the reader should keep in mind that each of these matrices should be thought of as a coset representatives modulo the center of $\mathrm{GL}_{d+1}(\mathbb{Q}_\sigma)$. For

$$i = 1, \ldots, k \quad \text{and } f_i \in \Phi, \qquad f_i(\mathbf{x}) = \varrho\mathbf{x} + \mathbf{y}_i,$$

we define h_i in G^S by $h_i = \left(h_i^{(\sigma)}\right)_{\sigma \in S}$, where

$$h_i^{(\sigma)} \stackrel{\mathrm{def}}{=} \begin{cases} \begin{pmatrix} \varrho \, \mathrm{Id}_d & -\mathbf{y}_i \\ \mathbf{0} & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{f}} \\ \begin{pmatrix} \varrho \, \mathrm{Id}_d & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} & \text{if } \sigma = \infty. \end{cases} \tag{1.7}$$

For a $\mathbb{Q}$-algebraic group $\mathbf{J}$ and $S_0 \subset S$ we will write

$$\mathrm{J}_{S_0} \stackrel{\mathrm{def}}{=} \prod_{\sigma \in S_0} \mathbf{J}(\mathbb{Q}_\sigma), \tag{1.8}$$

where $\mathbb{Q}_\infty$ is another notation for $\mathbb{R}$. Let

$$S_{\mathrm{ue}} \stackrel{\mathrm{def}}{=} \{p \in S_{\mathrm{f}} : p | q\} \quad \text{and} \quad S_{\mathrm{ue}}^c \stackrel{\mathrm{def}}{=} S \smallsetminus S_{\mathrm{ue}}. \tag{1.9}$$

Let $\mathbf{U} \subset \mathbf{G}$ denote the algebraic group consisting of elements of the form appearing in (1.2), and let

$$W^{\mathrm{st}} \stackrel{\mathrm{def}}{=} \mathrm{U}_{S_{\mathrm{ue}}} \times \mathrm{G}_{S_{\mathrm{ue}}^c}. \tag{1.10}$$

The subscript 'ue' stands for 'uniform expansion', and the superscript 'st' stands for 'stable'.

Let μ be the measure on G^S given by

$$\mu = \sum_{i=1}^{k} p_i \delta_{h_i}. \tag{1.11}$$

By a *measure on* $\mathscr{X}_{d+1}^S$ we mean a finite regular Borel measure. Recall that a measure ν on $\mathscr{X}_{d+1}^S$ is called *μ-stationary* if $\mu * \nu = \nu$, where $\mu * \nu$ is the measure on $\mathscr{X}_{d+1}^S$ defined by

$$\forall f \in C_c\left(\mathscr{X}_{d+1}^S\right), \qquad \int_{\mathscr{X}_{d+1}^S} f \, \mathrm{d}(\mu * \nu) = \int_G \int_{\mathscr{X}_{d+1}^S} f \, \mathrm{d}g_*\nu \, \mathrm{d}\mu(g).$$

The collection of stationary probability measures is a closed convex set in the space of probability measures on $\mathscr{X}_{d+1}^S$, and ν is called *μ-ergodic* if it an extreme point of this convex set.

Theorem 1.3. *Let Φ be an irreducible carpet IFS, let h_i be as in* (1.7)*, let $\mathbf{p}$ be a probability vector, and let μ be as in* (1.11)*. Then for any ergodic μ-stationary measure ν, one of the following holds:*

(1) $\nu\left(\left\{x \in \mathscr{X}_{d+1}^S : \mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}\right\}\right) = 1.$
(2) $\nu = m_{\mathscr{X}_{d+1}^S}$.

In §4 we will give examples of μ-stationary measures, showing that the two cases in Theorem 1.3 do in fact arise.

The following Corollary of Theorem 1.3 will be very useful. Let $b \stackrel{\mathrm{def}}{=} \left(b^{(\sigma)}\right)_{\sigma \in S}$, where

$$b^{(\sigma)} = \begin{cases} \begin{pmatrix} \varrho^{-1}\,\mathrm{Id} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} & \sigma \in S_{\mathrm{ue}}, \\ \mathrm{Id} & \sigma \in S_{\mathrm{ue}}^c. \end{cases} \tag{1.12}$$

For a sequence of measures $(\nu_k)_{k\in\mathbb{N}}$, we will write $\nu_k \to 0$ if the sequence (ν_k) converges in the weak-* topology to the zero measure. In other words, for every compact set K, $\nu_k(K) \to 0$. We will refer to this as *complete escape of mass.*

Corollary 1.4. *Let μ be as in (1.11), and suppose ν is an ergodic μ-stationary measure on $\mathscr{X}_{d+1}^S$. Then one of the following holds:*

(1) $b_*^k \nu \to 0$ *as* $k \to \infty$.
(2) $\nu = m_{\mathscr{X}_{d+1}^S}$.

1.3. **Organization of the paper.** Theorem 1.3, which provides a description of stationary measures for a certain random walk, is the main result of this paper. In §3, we introduce some some standard facts and introduce our notation. In §4 we illustrate some stationary measures which arise for the random walk we consider. As these examples show, the random walk we consider is not *stiff*, i.e., there are stationary measures which are not invariant under individual elements in $\mathrm{supp}(\mu)$. This sets the random walk considered here apart from many of the setups studied in prior works, and is a major complication in our setup. In §5 we deduce the other main results of the paper from Theorem 1.3. Note that our argument for equidistribution is quite different from the one used in prior work; the crucial point for these deductions is that the maps considered in (1.3) and in Corollary 1.4 commute with all elements in $\mathrm{supp}(\mu)$, and thus send stationary measures to stationary measures.

In §6 we begin the proof of Theorem 1.3. We introduce an auxiliary random walk $\bar{\mu}$. The trajectories for the $\bar{\mu}$-random walk stay within a bounded distance from those of the μ-random walk, and treating $\bar{\mu}$ makes it possible to avoid some technical complications. We state Theorem 6.1, which is the analogue of Theorem 1.3 for this random walk, and show, by considering a random walk on a common cover, that Theorem 6.1 implies Theorem 1.3. The proof of Theorem 6.1 occupies the rest of the paper. The steps and main ideas for its proof are described in §2 below.

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2. Outline of the Proof

For convenience of the reader, we provide an informal outline of the proof of Theorem 6.1. The proof follows the exponential drift strategy of Benoist and Quint [BQ11], but requires substantial adaptations to our non-stiff setup. In this section we recall the strategy of [BQ11], indicate the complications which arise in our setting, and explain how we deal with them. For the purpose of this section, there will be no harm in ignoring the distinction between μ and $\bar{\mu}$, that is, assume that the random walk is the one described in §1.2.

Let $B = (\operatorname{supp}\bar{\mu})^{\mathbb{N}}$ be the space of infinite words in the random walk and $\beta = \bar{\mu}^{\otimes\mathbb{N}}$ be the associated Bernoulli measure. Let ν be a $\bar{\mu}$-stationary measure. Recall that we have a disintegration $\nu = \int_B \nu_b \, \mathrm{d}\beta(b)$ in terms of Furstenberg limit measures $\nu_b = \lim_{n\to\infty}(b_1 \cdots b_n)_*\nu$; cf. §3.4 for details. Let W_b denote the subgroup tangent to the directions that are contracted to 0 under the adjoint action of $b_n^{-1} \cdots b_1^{-1}$ as $n \to \infty$. In our setting, this group is deterministic (i.e., independent of b) and is a proper subgroup of W^{st}.

We assume that item (1) of Theorem 1.3 does not hold, which by ergodicity, implies that ν-almost every point has trivial stabilizer in W^{st}. The goal of the exponential drift argument, implemented in §10, is to show that ν_b is invariant under a one-parameter (necessarily unipotent) subgroup of W_b for almost every b. The result will then follow by an application of Ratner's theorem along with the special structure of our random walk that is used to rule out other homogeneous measures besides $m_{\mathscr{X}^S_{d+1}}$. This is carried out in §11.

Following [BQ11], given a generic b, with the aid of Lusin's theorem, we find two ε-close points x, y in the support of ν_b, so that the leafwise measures of ν_b along the W_b-orbits of x, y are also close. We then apply the random walk to this configuration with the goal of finding two new points x_2, y_2, in the support of ν_{b_2}, for a suitable b_2, which satisfy the following:

(1) The distance between x_2, y_2 is $\asymp 1$.
(2) The displacement between x_2 and y_2 essentially points in the direction of W_{b_2}.
(3) The leafwise measures of ν_{b_2} along the W_{b_2}-orbits of x_2, y_2 remain close.

Taking ε to 0, and repeating this process, produces points with displacement belonging to $W_{b'}$ for suitable b', and whose leafwise measures agree, thus implying the desired invariance.

The random walk maneuvers that produce such x_2, y_2 consist of first deleting the length-n prefix of b, $n = n(\varepsilon)$, and then adjoining a suitably chosen length-m prefix, $m = m(n)$. The first leg (resp. second) of this itinerary is referred to as the backward (resp. forward) random walk. Item (2) follows from general properties of linear random walks, roughly that vectors tend to point towards the direction of a suitable top Lyapunov space under the random walk; cf. Lemmas 7.2 and 7.3.

The role of the no-rotations hypothesis. The reason the strategy involves both going backward and forward by the random walk is to achieve item (3), where n and m are chosen so that the distortion of the leafwise measures in the two legs nearly cancel each other out. Here, and in all prior works, the key property needed is the conformality of the action of the random walk on these leafwise measures. In our setting, this is the reason we assume the linear part of our IFS has no rotations. Indeed, otherwise, these rotations may generate a non-compact group over one of the primes in the definition of the induced random walk on $\mathscr{X}^S_{d+1}$. Such non-compactness would lead to a non-conformal action on the Lyapunov space corresponding to W_b.

The role of the group W^{st}. To achieve (1), we must ensure that x, y are not aligned along directions that may contract by going either backward or forward by the random walk. In [BQ11], and almost all prior works, this is done by ensuring that ν_b gives 0 mass to W_b-orbits, so that backward motion does not contract the displacement, while relying on growth properties of random walks to ensure that, for any given vector in the tangent space, most forward random walk trajectories will cause it to grow (with additional complications arising from neutral/central directions).

By contrast, our random walk admits a non-trivial deterministic *contracting* subspace under *every* forward random walk trajectory. The group W^{st} is thus defined to be the group generated by deterministic forward-contracting *and* backward-contracting groups, in addition to the centralizer of the random walk. A key step in carrying out the above strategy is thus to show that ν_b gives 0 mass to W^{st}-orbits. This is Theorem 9.1. Due to the mixed behavior of W^{st} (some of its directions expand in the future while others expand in the past), the proof of this result represents the major departure in our proof compared to prior works. A key ingredient is a projection argument introduced in Lemma 9.4 to separate these mixed behaviors allowing us to handle them individually. This is the critical step where the hypothesis that points have trivial stabilizers in W^{st} is used.

To get that most forward trajectories of our random walk expand a given transverse direction to W^{st}, we note that such directions point along $\mathrm{G}_{S_{\mathrm{ue}}}$ to which we apply the expansion results of Simmons and the third author [SW19]; cf. §7. Note that the results of [SW19] hold for a certain real random walk, which nonetheless has the same block structure as the restriction of the random walk considered in this article to the S_{ue}-adic places.

Non-atomicity in the presence of contracting spaces. Non-alignment along W^{st}-orbits is done by applying the forward and backward random walks to separately contract the respective pieces of the orbit, reducing the problem to showing almost sure non-atomicity of the measures ν_b. This is established in the strong form needed for the proof in Theorem 8.1. The presence of a deterministic forward contracting space poses significant difficulties in this step. For instance, the standard arguments involving a 'Margulis inequality' (see [BQ11, Prop. 3.9 & §6.2]) do not seem applicable in this setting. Note also that such non-atomicity fails to hold without the hypothesis on trivial stabilizers in W^{st}, as shown by the examples of stationary, non-invariant measures given in §4.

Instead, we argue directly by a delicate local analysis. We first show that averaging on the set of words which grow a given displacement vector to a macroscopic size satisfies a certain pointwise ergodic theorem in the underlying Bernoulli shift space; cf. Theorem 8.2. Our hypothesis that the IFS has a single contraction ratio is used in this step to ensure that this set of words has nearly full measure (indeed, otherwise these words will be given polynomially decaying measure in their length as can be checked by a direct computation). We believe it is possible to push our arguments to bypass this difficulty, and hope to return to this problem in future work.

Equipped with the above ergodic theorem, if the measures ν_b were atomic, we would obtain a contradiction by finding words that simultaneously push generic points towards one another (by continuity and the aforementioned prefix ergodic theorem), and away from one another (by the expansion of the action transverse to orbits of the forward-contracted group). The fact that the words have comparable norm in every irreducible component allows us to control the speed at which our points diverge from one another and capture our pair of generic points just as they are starting to separate from one another.

3. Preliminaries

In this section we collect some standard facts about our objects of study.

3.1. Attractors of IFS's.

A mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be a *contracting affine half-dilation* if it is of the form $f(\mathbf{x}) = \varrho\mathbf{x} + \mathbf{y}$, where $\varrho \in \mathbb{R}$, $0 < |\varrho| < 1$ is the *contraction ratio* and $\mathbf{y}$ is the *translation.* The nomenclature 'half-dilation' is due to the fact that we allow negative contraction ratios. Note that when $d > 1$, the semigroup of contracting affine dilations is strictly contained in the well-studied semigroup of contracting similarity maps, in which one is allowed to compose the maps f as above with orthogonal transformations. We say that f is *rational* if $c \in \mathbb{Q}$ and $\mathbf{y} \in \mathbb{Q}^d$. A collection $\Phi = \{f_1, \ldots, f_k\}$ of maps is called an *iterated function system (IFS).* With this terminology, the carpet-IFS's we consider in this paper are iterated function systems of rational half-dilation affine maps with constant contraction ratio.

Let $B = \{1, \ldots, k\}^{\mathbb{N}}$ and let $\mathrm{cod} : B \to \mathbb{R}^d$ be the map defined by

$$\mathrm{cod}(b) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(\mathbf{x}_0), \qquad \text{where } b = (i_1, i_2, \ldots) \quad \text{and } \mathbf{x}_0 \in \mathbb{R}^d. \tag{3.1}$$

It is well-known that the limit in (3.1) exists for all b, is independent of the choice of $\mathbf{x}_0$, and that the map cod is continuous. The image of cod is called the *attractor* of Φ, and we denote it by $\mathcal{K} = \mathcal{K}(\Phi)$. Basic results about the attractor $\mathcal{K}$ were obtained in classical work of Moran and

Hutchinson [Mor46, Hut81]. Among other things, they showed that $\mathcal{K}$ is the unique non-empty compact subset of $\mathbb{R}^d$ satisfying the stationarity property

$$\mathcal{K} = \bigcup_{i=1}^{k} f_i(\mathcal{K}). \tag{3.2}$$

When the elements in this union are disjoint we say that Φ satisfies *strong separation.* If there is a non-empty open subset $U \subset \mathbb{R}^d$ such that $f_i(U) \subset U$ for all i and $f_i(U) \cap f_j(U) = \varnothing$ for $i \neq j$, we say that Φ satisfies the *open set condition.* It is known that strong separation implies the open set condition. Let $\mathbf{p} = (p_1, \dots, p_k)$ be a *probability vector of full support*, that is a k-tuple of real numbers such that

$$\sum_{i=1}^{k} p_i = 1, \quad \forall i, \ p_i > 0.$$

The *Bernoulli measure* $\theta = \theta(\Phi, \mathbf{p})$ on $\mathcal{K}$ is the image of the measure $(\sum_{i=1}^k p_i \delta_i)^{\mathbb{N}}$ under the map cod. It is the unique measure on $\mathbb{R}^d$ which satisfies the stationarity property

$$\theta = \sum_{i=1}^{k} p_i f_{i*}\theta. \tag{3.3}$$

Let $s = \dim(\mathcal{K})$ denote the Hausdorff dimension of $\mathcal{K}$. It is well-known that if one assumes the open set condition, then up to scaling, the restriction of s-dimensional Hausdorff measure to $\mathcal{K}$ is a Bernoulli measure. If in addition, one assumes that the contraction ratios are equal to each other, this Bernoulli measure is given by the uniform probability vector $\mathbf{p} = (1/k, \dots, 1/k)$.

We will be interested in the effect of conjugation of elements of Φ by an affine similarity mapping $f : \mathbb{R}^d \to \mathbb{R}^d$. If we let

$$\Phi = \{f_1, \dots, f_k\} \quad \text{and } \Phi' = \{f'_1, \dots, f'_k\}, \text{ where } f'_i = f \circ f_i \circ f^{-1}, \tag{3.4}$$

then it can be easily checked that

$$\mathcal{K}(\Phi') = f(\mathcal{K}(\Phi)) \quad \text{and} \quad \theta(\Phi', \mathbf{p}) = f_*\theta(\Phi, \mathbf{p}). \tag{3.5}$$

3.1.1. *Examples.* A *missing digit set* is a set of the form

$$\mathcal{K} = \left\{ \sum_{i=1}^{\infty} a_i b^{-1} : a_i \in D \right\},$$

where $b \geq 3$ and $D \subset \{0, \dots, b-1\}$, with $2 \leq |D| \leq b-1$. The standard example is given by the Cantor middle thirds set, with $b = 3$ and $D = \{0, 2\}$. These sets are attractors of the IFS in dimension $d = 1$, given by $f_i(x) = \frac{1}{b}x + \frac{a_i}{b}$, where $D = \{a_1, \dots, a_k\}$. This IFS satisfies strong separation if D does not contain consecutive indices, and satisfies the open set condition for any D. Thus the class of missing digit sets is obviously contained in the class of carpet IFS's which we consider in this paper. Other well-known examples of irreducible carpet IFS's satisfying the open set condition are those whose attractors are the *Sierpiński carpet* and *Menger sponge*, which are examples in dimensions $d = 2$ and $d = 3$ respectively.

An example of a well-known fractal not covered by our results is the Koch snowflake. This two-dimensional fractal is the attractor of an irreducible IFS on $\mathbb{R}^2$ satisfying the open set condition, where the maps in the IFS are similarities, but these similarities do not have rational coefficients and cannot be represented as dilations (rotations by multiples of $\pi/3$ are required). Another example not covered by our work is the translation of a Sierpiński carpet by an irrational vector. It would be interesting to know whether Bernoulli measures on these fractals satisfy the conclusion of Theorem 1.1.

3.2. **Friendliness of some fractal measures.** We now introduce some properties of a measure θ on $\mathbb{R}^d$, following [KLW04]. Let $B(\mathbf{x}, r)$ be the ball of radius r centered at $\mathbf{x}$, with respect to the Euclidean metric on $\mathbb{R}^d$. For a constant $D \geq 1$, we say that θ is *D-Federer* if for any $\mathbf{x} \in \mathrm{supp}(\theta)$ and any $r > 0$ we have

$$\theta(B(\mathbf{x}, 3r)) \leq D\, \theta(B(\mathbf{x}, r)). \tag{3.6}$$

For any $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, we write $A^{(\varepsilon)} = \bigcup_{a \in A} B(a, \varepsilon)$ for the ε-neighborhood of A. Given positive C, α, we say that θ is *(C, α)-absolutely decaying* if for any ball B centered in $\mathrm{supp}(\theta)$, any affine hyperplane $\mathcal{L}$, any $B = B(\mathbf{x}, r)$ with $\mathbf{x} \in \mathrm{supp}(\theta)$, $r \in (0, 1)$ and any $\varepsilon > 0$, we have

$$\theta\left(B \cap \mathcal{L}^{(\varepsilon)}\right) \leq C \left(\frac{\varepsilon}{r}\right)^{\alpha} \theta(B). \tag{3.7}$$

A measure θ for which there are D, C, α such that θ is D-Federer and (C, α)-absolutely decaying is called *absolutely friendly.*

We will need the following:

Proposition 3.1. *Let Φ be an irreducible carpet-IFS. Assume that θ is a measure on the attractor $\mathcal{K}$ of Φ, satisfying the conditions of Theorem 1.1, namely at least one of the following:*

(1) Φ satisfies the open set condition and θ is the Hausdorff measure on $\mathcal{K}$;
(2) Φ satisfies strong separation and $\theta = \theta(\Phi, \mathbf{p})$ is a Bernoulli measure, for some probability vector $\mathbf{p}$.

Then θ is absolutely friendly. Moreover, there are D, C, α such that for any conjugate Φ' of Φ by an affine dilation map, the Hausdorff measure on the attractor $\mathcal{K}'$ of Φ' is D-Federer and (C, α)-absolutely decaying.

Proof. The first assertion is proved in [KLW04, §8] in case (1), and in [DFSU21, Thm. 1.7] in case (2) (the latter proof is based on [Urb05]). For the second assertion we use (3.5), and note that an affine similarity map sends Euclidean balls to Euclidean balls, and hence does affect the validity of (3.6) and (3.7). □

We will also need the following 'self-similar Lebesgue density theorem':

Proposition 3.2. *Let Φ and θ be as in Proposition 3.1, let $\mathrm{cod} : \mathbb{R}^d \to \mathcal{K}$ be the coding map, and let $B \subset \mathbb{R}^d$ be a Borel set. Then for θ-a.e. $\mathbf{x} \in B$ we have that*

$$\lim_{n \to \infty} \frac{\theta\left(B \cap f_n(\mathcal{K})\right)}{\theta\left(f_n(\mathcal{K})\right)} = 1, \tag{3.8}$$

where $\mathbf{x} = \mathrm{cod}(i_1, i_2, \ldots)$ and $f_n = f_{i_1} \circ \cdots \circ f_{i_n}$.

Note that in the Lebesgue density theorem, valid for all finite Borel measures on $\mathbb{R}^d$ (see [Mat95, Cor. 2.14]), the appearance of $f_n(\mathcal{K})$ is replaced with a ball centered at $\mathbf{x}$, shrinking as $n \to \infty$. The self-similar structure allows us to replace balls with shrinking copies of the attractor.

Proof. For each n, let $\mathcal{B}_n$ be the finite σ-algebra of subsets of $\mathcal{K}$ generated by the sets $f(\mathcal{K})$, where f ranges over all the compositions $f_{i_1} \circ \cdots \circ f_{i_n}$. We claim that under the assumptions on θ we have that for any $i \neq j$, $\theta(f_i(\mathcal{K}) \cap f_j(\mathcal{K})) = 0$. Indeed, under the strong separation condition $\theta(f_i(\mathcal{K}) \cap f_j(\mathcal{K})) = \varnothing$, and for the Hausdorff measure on $\mathcal{K}$ we use (3.3) and the scaling of Hausdorff measure under similarity maps.

This implies that in the covering (3.2), the sets are disjoint up to θ-nullsets. Thus the atoms of $\mathcal{B}_n$ are (up to θ-nullsets) the sets $f(\mathcal{K})$ themselves, and the quotient on the left-hand side of (3.8) is the conditional expectation of the indicator function $\mathbf{1}_B$ w.r.t. the σ-algebra $\mathcal{B}_n$, evaluated at $\mathbf{x}$. Since the diameter of the sets $f_n(\mathcal{K})$ goes to zero, the σ-algebra generated by $\bigcup_n \mathcal{B}_n$ is the Borel σ-algebra on $\mathcal{K}$. The increasing Martingale theorem (see [EW11, Chap. 5.2]) now implies that for θ-a.e. $\mathbf{x}$, the left hand side of (3.8) converges to $\mathbf{1}_B(\mathbf{x})$. This completes the proof. □

3.3. **The Dani-Kleinbock correspondence and homogeneous dynamics.** The so-called *Dani correspondence* was developed by Dani in [Dan85]. He showed that for equal weights approximation, a vector $\mathbf{x} \in \mathbb{R}^d$ is badly approximable if and only if the trajectory $\{a_t \Lambda_{\mathbf{x}} : t > 0\}$ is bounded in $\mathscr{X}_{d+1}$, where $a_t = a_t^{\mathbf{r}_0}$ is the one-parameter group as in (1.5), corresponding to equal weights $\mathbf{r}_0 = (1/d, \cdots, 1/d)$. The correspondence was later developed and extended to the notions of Dirichlet improvability and weighted approximation in the papers [Kle98, KW08, KR21]. In particular the following hold:

Proposition 3.3. *Let $\mathbf{x} \in \mathbb{R}^d$, $\Lambda_{\mathbf{x}}$ and $\mathbf{r}$ be as above. Then the following hold:*

(1) $\mathbf{x}$ is $\mathbf{r}$-badly approximable if and only if the trajectory $\left\{a_t^{(\mathbf{r})} \Lambda_{\mathbf{x}} : t > 0\right\}$ is bounded in $\mathscr{X}_{d+1}$.
(2) Denoting the sup-norm by $\|\cdot\|_\infty$, we have that $\mathbf{x}$ is $(\mathbf{r}, \|\cdot\|_\infty)$-Dirichlet improvable if and only if for every $\varepsilon \in (0,1)$, for all sufficiently large T, there are $Q \in \mathbb{N}$ and $(P_1, \ldots, P_d) \in \mathbb{Z}^d$ such that $Q \le T$ and $\max_i \left(|Qx_i - P_i|^{1/r_i}\right) \le \frac{\varepsilon}{T}$.

We can view the expression appearing on the left-hand side of (1.4) as the sup-norm of the vector $Q(|Qx_1 - P_1|^{1/r_1}, \ldots, |Qx_d - P_d|^{1/r_d})$. It follows from item (1) of Proposition 3.3, and can also be easily inferred directly from the definition, that the property of being $\mathbf{r}$-badly approximable does not depend on the choice of a norm on $\mathbb{R}^d$. On the other hand, Dirichlet improvability depends rather delicately on the chosen norm. Note that following [KR21], we have defined the notion of Dirichlet improvability for a general norm in terms of the dynamical behavior of the trajectory $\left\{a_t^{(\mathbf{r})} \Lambda_{\mathbf{x}} : t > 0\right\}$. Item (2) of Proposition 3.3 shows that for the sup-norm, there is a simple interpretation of this property in terms of Diophantine inequalities, and indeed, this was the original definition introduced by Davenport and Schmidt [DS70]. Similar Diophantine interpretations can be given for norms on $\mathbb{R}^{d+1}$ arising from a norm $\|\cdot\|'$ on $\mathbb{R}^d$ via the formula $\|(y_1, \ldots, y_{d+1})\| = \max(\|(y_1, \ldots, y_d)\|, |y_{d+1}|)$, but there is no such interpretation for general norms on $\mathbb{R}^{d+1}$. It was shown by Davenport and Schmidt (see [DS70]) that $\mathbf{r}$-badly approximable are necessarily $(\mathbf{r}, \|\cdot\|_\infty)$-Dirichlet improvable. For general norms this implication does not necessarily hold, see [KR21].

3.3.1. *Quantitative nondivergence for friendly measures.* Given a probability measure ν on $\mathscr{X}_{d+1}$ and a sequence of elements $(a_j) \subset \mathrm{SL}_{d+1}(\mathbb{R})$, we say that ν has *no escape of mass under* (a_j) if any weak-* accumulation point of $(a_{j*}\nu)$ is a probability measure; equivalently, for any $\varepsilon > 0$ there is a compact subset $K \subset \mathscr{X}_{d+1}$ such that for all large enough j, $a_{j*}\nu(K) \ge 1 - \varepsilon$.

Let $A \subset G$ be the group of diagonal matrices of determinant one. Each $a \in A$ can be represented as

$$\exp(\mathbf{X}) = \mathrm{diag}\left(e^{X_1}, \ldots, e^{X_{d+1}}\right)$$

where

$$\mathbf{X} = (X_1, \ldots, X_{d+1}), \quad X_i \in \mathbb{R}, \quad X_1 + \cdots + X_{d+1} = 0.$$

Following [KW08] we say that a sequence $a_n = \exp\left(\mathbf{X}^{(n)}\right)$ *drifts away from walls* if

$$\left\lfloor \mathbf{X}^{(n)} \right\rfloor \to \infty, \quad \text{where } \lfloor \mathbf{X} \rfloor \stackrel{\text{def}}{=} \min\left\{X_i : i = 1, \ldots, d\right\}.$$

The following result, proved in [KLW04], is useful for exhibiting no escape of mass.

Proposition 3.4. *Let θ be an absolutely friendly measure on $\mathbb{R}^d$, let ν be the pushforward of θ under the map $\mathbf{x} \mapsto \Lambda_{\mathbf{x}}$, and let (a_j) be a sequence in A that drifts away from walls. Then ν has no escape of mass under (a_j).*

We will need the following strengthening of Proposition 3.4, which establishes the uniform nondivergence property of Theorem 1.1:

Proposition 3.5. *For any positive ε, C, α, and any $D \geq 1$ there is a compact $K \subset \mathscr{X}_{d+1}$ such that for any (C,α)-decaying and D-Federer compactly supported measure θ on $\mathbb{R}^d$, for any $g \in \mathrm{SL}_{d+1}(\mathbb{R})$, the measure $\nu = \nu_g$ defined below (1.2) satisfies that for any sequence $\{a_j\} \subset A$ which drifts away from walls, for all sufficiently large j we have*

$$a_{j*}\nu(K) \geq 1 - \varepsilon.$$

The proof is essentially given in [KW08], based on [KLW04]. We sketch the argument for the reader's convenience.

Sketch of proof. We first claim that in proving this statement we may assume that $\mathrm{supp}(\theta) \subset \mathbf{B}$, where $\mathbf{B}$ is the unit ball around the origin in $\mathbb{R}^d$. Indeed, if the support of θ is not contained in $\mathbf{B}$, we can replace θ by $f_*\theta$, for a linear contraction $f : \mathbb{R}^d \to \mathbb{R}^d$ which maps $\mathrm{supp}(\theta)$ into $\mathbf{B}$. As in the proof of Proposition 3.1, such a map does not affect the constants D, C, α, and can be realized by the conjugation action of a_t on U for some $t_0 \in \mathbb{R}$. This amounts to replacing x_0 by $a_{t_0}x_0$, replacing ν by $a_{t_0 *}\nu$, and replacing (a_j) by $(a_j a_{t_0})$, which is also a sequence drifting away from walls.

We now verify the assumptions of [KLW04, Thm. 4.3], with U a big enough ball containing $\mathbf{B}$ and with $h(u) = a_j \tau(u) g_0$, where τ is as in [KW08, Sec. 2.1], g_0 satisfies $x_0 = \pi(g_0)$, and with $\rho = 1$, and we verify the conditions for all j large enough. Assumption (1) is immediate from [KLW04, Lemma 4.1] and [KW08, Lemma 3.3] (where in [KW08, Lemma 3.3] we take $\mathbf{w}$ to be $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_j$, where $\mathbf{v}_i = g_0 \mathbf{u}_i$ and $\mathbf{u}_1, \ldots, \mathbf{u}_j$ generate $\mathbb{Z}^k \cap V$ as a $\mathbb{Z}$-module). Assumption (2) is verified in [KLW04, Pf. of Thm. 3.3] for the standard one parameter flow and for $g_0 = \mathrm{Id}$. For the general case we need here, we argue as in [KW08, Section 3.3], only using the drifting away from walls assumption. Namely, in the proof in [KW08, Proof of Thm. 3.5], the constant δ is allowed to depend on g_0. This does not cause any issues and the same proof goes through.

□

3.4. **Generalities on stationary measures.** We will use the following basic facts about stationary measures. Most of the results are valid in great generality but we only state the results which we will need here. See [BQ11, BQ24] for details and proofs.

Proposition 3.6. *Let G be a locally compact second countable group, acting continuously on a locally compact second countable space X, and let μ be a measure as in (1.11). Let*

$$B \stackrel{\mathrm{def}}{=} \{1, \ldots, k\}^{\mathbb{N}} \cong \{h_1, \ldots, h_k\}^{\mathbb{N}}, \qquad \beta \stackrel{\mathrm{def}}{=} \mu^{\mathbb{N}}. \tag{3.9}$$

and $T : B \to B$ the shift map. Then:

- *The collection of μ-stationary measures on X is a compact convex set in the space of probability measures on X. Ergodic stationary measures are extremal in this cone.*
- *If a homeomorphism $\varphi : X \to X$ commutes with each element in the support of μ, then it maps any (ergodic) μ-stationary measure to an (ergodic) μ-stationary measure.*
- *An ergodic stationary measure which assigns full measure to a countable subset of X is supported on a finite set and gives equal mass to all elements in this set.*
- *For any μ-stationary probability measure ν, for β-almost every sequence $b = (h_n)_n$, the limit*

$$\nu_b \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} (h_1 \circ \cdots \circ h_n)_* \nu \tag{3.10}$$

exists and is a probability measure on X. The collection (ν_b) of limit measures *satisfies*

$$\text{for every } i \text{ and } \beta\text{-a.e. } b \in B,\ \nu_{h_i b} = h_{i*}\nu_b, \tag{3.11}$$

and

$$\nu = \int \nu_b \, \mathrm{d}\beta(b). \tag{3.12}$$

- *Conversely, for any assignment* $b \mapsto \nu_b$*, where* ν_b *is a measure on* X *and the assignment satisfies* (3.11)*, the measure defined by* (3.12) *is stationary and the collection* $(\nu_b)_{b\in B}$ *is its system of leafwise measures.*
- *Also define*

$$B^X \stackrel{\mathrm{def}}{=} B \times \mathscr{X}^S_{d+1}, \quad T^X(b,x) = \left(Tb, h_{b_1}^{-1}x\right), \quad \beta^X \stackrel{\mathrm{def}}{=} \int_{B^X} \delta_b \otimes \nu_b \, \mathrm{d}\beta(b). \tag{3.13}$$

Then β^X *is* T^X*-invariant, and is ergodic if and only if* ν *is ergodic.*

We remark that what we denote here by T^X is the *backward random walk transformation* which is denoted in [BQ24] by $T^{\vee X}$. It should not be confused with the forward random walk transformation $(b,x) \mapsto (Tb, h_{b_1}x)$.

3.5. **Irreducible S-adic lattices.** In this subsection we record some simple properties of the groups introduced in §1.2 which we will use repeatedly.

Proposition 3.7. *For any* $x \in \mathscr{X}^S_{d+1}$ *and any nontrivial* $g = (g^{(\sigma)})_{\sigma\in S} \in G^S$ *in the stabilizer of* x*, we have*

$$\forall \sigma \in S, \qquad g^{(\sigma)} \neq \mathrm{Id}. \tag{3.14}$$

Proof. Since property (3.14) is invariant under conjugations, and the stabilizer of $x = \pi(g_0)$ is a conjugate of Λ^S, it suffices to consider the case that $x = \pi(\Lambda^S)$ and that $g \in \Lambda^S$. In this case (3.14) follows immediately from the definition of Λ^S as a diagonal embedding, □

Proposition 3.8. *For any* $\sigma \in S_{\mathrm{ue}}$*, there are no nontrivial elements of* $\mathbf{G}(\mathbb{Q}_\sigma)$ *which commute with all of the matrices* $h_1, \ldots, h_k$*.*

Proof. Suppose $\sigma \in S_{\mathrm{ue}}$, let

$$H \stackrel{\mathrm{def}}{=} \overline{\left\langle h_i^{(\sigma)} : i = 1, \ldots, k \right\rangle},$$

and let C be the centralizer of H. Since $\mathbf{G}(\mathbb{Q}_\sigma)$ is center-free, it suffices to prove that C is contained in the center of $\mathbf{G}(\mathbb{Q}_\sigma)$. Since $\mathbf{G}(\mathbb{Q}_\sigma)$ is a matrix group, it is enough to show that H is epimorphic in $\mathbf{G}(\mathbb{Q}_\sigma)$; that is, in any finite-dimensional representation of algebraic groups, any vector fixed by H is fixed by $\mathbf{G}(\mathbb{Q}_\sigma)$. In order to verify this statement, we can replace H by its Zariski closure, and by using [Sha96, Lemma 5.2], we see that it suffices to show that the Zariski closure of H contains the group

$$U^{(\sigma)} \stackrel{\mathrm{def}}{=} u^{(\sigma)}(\mathbb{Q}_\sigma^d), \quad \text{where } u^{(\sigma)} \colon \mathbb{Q}_\sigma^d \to \mathbf{G}(\mathbb{Q}_\sigma), \qquad u^{(\sigma)}(\mathbf{x}) \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathrm{Id} & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix}. \tag{3.15}$$

To this end, let $U_0 \stackrel{\mathrm{def}}{=} \bar{H}^Z \cap U^{(\sigma)}$, where $\bar{H}^Z$ denotes the Zariski closure of H. Since $U^{(\sigma)}$ is normalized by H, so is U_0. Also, for each i, j, U_0 contains the elements

$$h_i^{(\sigma)} \circ \left(h_j^{(\sigma)}\right)^{-1} = u^{(\sigma)}(\mathbf{y}_j - \mathbf{y}_i);$$

that is, U_0 contains the image under $u^{(\sigma)}$ of all the differences of the translations vectors appearing in the IFS Φ. Let V_0 be the subspace of $\mathbb{Q}_\sigma^d$ spanned by these differences. An easy calculation shows that the maps in Φ all preserve the affine subspace $(1-\varrho)^{-1}\mathbf{y}_1 + V_0$. Since Φ is irreducible, this means that $V_0 = \mathbb{Q}_\sigma^d$, and since U_0 is Zariski closed, this means that $U_0 = u^{(\sigma)}(V_0) = U^{(\sigma)}$, as required. □

4. Examples of Stationary Measures

In analogy with Theorem 1.1, we define a measure $\hat{\nu}_0$ on $\mathscr{X}_{d+1}^S$ as the pushforward of θ under the map $\mathbf{x} \mapsto u(\mathbf{x})\Lambda^S$. We denote the orbit of the identity coset Λ^S under the group U_S by $\mathscr{U}^S$. This is a compact subset of $\mathscr{X}_{d+1}^S$, which is isomorphic to the compact S-arithmetic quotient

$$\mathbf{J}(\mathbb{Q}_S)/\mathbf{J}\left(\mathbb{Z}\left[\frac{1}{S}\right]\right), \qquad \text{where } \mathbf{J} = (\mathbf{G}_\mathrm{a})^d$$

and $\mathbf{G}_\mathrm{a}$ is the additive group. We can also view $\mathscr{U}^S$ as a solenoid, that is, the natural extension of $\mathbb{R}^d/\mathbb{Z}^d$ for which the multiplication maps $\times_p$, $p \in S_\mathrm{f}$ are invertible; cf. [HR63, Chap. II.10] and [EL18, Ex. 10.3]. We denote the unique U_S-invariant measure on $\mathscr{U}^S$ by $m_{\mathscr{U}}$. We denote by $A \subset \mathrm{SL}_{d+1}(\mathbb{R})$ the group of diagonal matrices of determinant one.

We repeat here a simple computation from [KL23, §4] that is central to our approach. By identifying $\mathrm{SL}_{d+1}(\mathbb{R})$ with a subgroup of $\mathbf{G}(\mathbb{R})$, we consider the map u in (1.2) as a map $\mathbb{R}^d \to G^S$. From (1.7) one sees that for any $i \in \{1, \ldots, k\}$ and any $\mathbf{x} \in \mathbb{R}^d$ we have that $h_i u(\mathbf{x}) h_i^{-1} = u(\varrho\,\mathbf{x})$. Let λ_i be the diagonal embedding of $\begin{pmatrix} \varrho\,\mathrm{Id}_d & -\mathbf{y}_i \\ \mathbf{0} & 1 \end{pmatrix} \in \mathbf{GL}_{d+1}\left(\mathbb{Z}\left[\frac{1}{S}\right]\right)$, so that

$$\lambda_i \in \Lambda^S \quad \text{and} \quad h_i \lambda_i^{-1} = u(\mathbf{y}_i).$$

As a consequence

$$h_i u(\mathbf{x})\Lambda^S = u(\varrho\,\mathbf{x}) h_i \Lambda^S = u(\varrho\,\mathbf{x})\, u(\mathbf{y}_i)\Lambda^S = u(f_i(\mathbf{x}))\Lambda^S. \tag{4.1}$$

Proposition 4.1. *The measures $m_{\mathscr{X}_{d+1}^S}$, $\hat{\nu}_0$ and $m_{\mathscr{U}}$ are all μ-stationary. For any $a \in A$, the same holds for the pushforwards of these measures by a.*

Proof. Since each h_i preserves $m_{\mathscr{X}_{d+1}^S}$, it is clear that this measure is μ-stationary. For $\hat{\nu}_0$, we have from (4.1) that the map $\mathbf{x} \mapsto u(\mathbf{x})\Lambda^S$ is a conjugacy between the actions of the f_i on $\mathbb{R}^d$, and of the h_i on the orbit $U\Lambda^S \subset \mathscr{X}_{d+1}^S$. Stationarity for μ now follows from (3.3). For $m_{\mathscr{U}}$ we observe that each h_i belongs to the normalizer of U_S, and $h_i\lambda_i^{-1} \in \mathrm{U}_S$, and so by repeating the computation in (4.1) and using that U_S is abelian, we have that

$$\forall w \in \mathrm{U}_S, \quad h_i w \Lambda^S = u(\mathbf{y}_i)(h_i w h_i^{-1})\Lambda^S. \tag{4.2}$$

That is, applying h_i to $\mathscr{U}^S$ induces a solenoid affine endomorphism; i.e., a map which is the composition of a group endomorphism and a group translation. Since its inverse is given by applying h_i^{-1} this is actually an automorphism, and thus preserves Haar measure on $\mathscr{U}^S$. This implies that every h_i preserves $m_{\mathscr{U}}$, and in particular $m_{\mathscr{U}}$ is μ-stationary. The last assertion follows from Proposition 3.6. □

Note that $\hat{\nu}_0$ is supported on the solenoid $\mathscr{U}^S$ and is not invariant under the individual elements h_i.

Remark 4.2. The measures appearing in Proposition 4.1 are all ergodic. For $\hat{\nu}_0$, this follows from the uniqueness in (3.3). For $m_{\mathscr{X}_{d+1}^S}$ and for $m_{\mathscr{U}}$ one can give proofs along the lines of [CKS21, Lemma 7.1]; we will not be using this statement and so we omit the details.

We now construct some additional ergodic μ-stationary measures by 'finite-index perturbations' of the measures appearing in Proposition 4.1. These examples are not surprising but are useful for understanding some of the conditions appearing in our results.

Example 4.3. The orbit $U\Lambda^S$ is dense in the solenoid $\mathscr{U}^S$, and the measure $\hat{\nu}_0$ is supported on a compact subset of this orbit. Here we construct another μ-stationary measure, supported on a different U-orbit in the solenoid. As we saw in (4.2), this is equivalent to producing a stationary measure with this property on $\mathscr{U}^S$, invariant for the random walk by affine automorphisms.

Write $\mathbf{x} = \left(x^{(\sigma)}\right)_{\sigma \in S}$ for an element of $\mathbf{G}_{\mathrm{a}}^d$, and let $[\mathbf{x}]$ denote the corresponding coset in $\mathscr{U}^S$. In these coordinates, the U-orbits in $\mathscr{U}^S$ are obtained by adding real numbers to the $\sigma = \infty$ coordinate of $\mathbf{x}$, and $\hat{\nu}_0$ is supported on the orbit of the zero coset $[0]$. Let φ_i denote the affine automorphism induced by h_i. Then

$$\varphi_i([\mathbf{x}]) = \left[\left(\bar{x}^{(\sigma)}\right)_{\sigma \in S}\right], \qquad \text{where } \bar{x}^{(\sigma)} = \begin{cases} \varrho x^{(\sigma)} & \sigma \in S_{\mathrm{f}} \\ \varrho x^{(\sigma)} + \mathbf{y}_i & \sigma = \infty. \end{cases}$$

For a first example, suppose that $\varrho = \frac{1}{q}$ for some $q \in \mathbb{N}$, and that S_{f} consists of the prime divisors of q (in other words, the divisors of the translations $\mathbf{y}_i$ do not involve primes not dividing q). Let a be an integer coprime to q and let

$$\mathbf{z} = \left(z^{(\sigma)}\right), \quad \text{where } z^{(\infty)} = 0 \quad \text{and} \quad z^{(p)} \stackrel{\text{def}}{=} \sum_{i \geq 0} a q^i \in \mathbb{Q}_p \quad \text{for } p \in S_{\mathrm{f}}.$$

This choice implies that for $p \in S_{\mathrm{f}}$, $\varrho z^{(p)} = \frac{a}{q} + z^{(p)}$. Since we can replace a coset representative by subtracting $\frac{a}{q}$ from each coordinate, we see that the random walk fixes the U-orbit through $[\mathbf{z}]$, acting on this U-orbit by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{y}_i - \frac{a}{q}$. Thus if we define

$$\Phi' \stackrel{\text{def}}{=} \{f_1', \ldots, f_k'\}, \quad \text{where } f_i'(\mathbf{x}) = f(\mathbf{x}) - \frac{a}{q},$$

let θ' be the unique stationary measure for Φ' on $\mathbb{R}^d$ (in the sense of (3.3)), and let ν' be the pushforward of θ' under the map $\mathbf{x} \mapsto u(\mathbf{z} + \mathbf{x})\Lambda^S$, then ν' is a μ-stationary measure on $\mathscr{U}^S$. It differs from $\hat{\nu}_0$ since it is supported on the orbit of $[\mathbf{z}]$.

In this first example, the main property of the U-orbit of $u(\mathbf{z})\Lambda^S$ is that it is mapped to itself by the affine maps $\{\varphi_1, \ldots, \varphi_k)$. More complicated examples can be constructed by considering finite collections of U-orbits which are permuted by this collection of maps. In the preceding example, this will happen if in the definition of $\mathbf{z}$, we replace the series $\sum a q^i$ with a series with a periodic sequence of coefficients. We leave the details to the dedicated reader.

5. Deducing Theorems 1.1 and 1.2 and Proof of Corollary 1.4

5.1. **Proof of Theorems 1.1 and 1.2.** Let θ and ν_0 be as in Theorem 1.1. As in §4, let $\hat{\nu}_0$ denote the pushforward of θ under $\mathbf{x} \mapsto u(\mathbf{x})\Lambda^S$. Let $P : \mathscr{X}_{d+1}^S \to \mathscr{X}_{d+1}$ be the projection, so that $\nu_0 = P_* \hat{\nu}_0$. In this subsection we will assume Corollary 1.4 and show:

Theorem 5.1. *If $(a_n) \subset A$ is a sequence so that θ is uniformly non-divergent along (a_n), then*

$$\lim_{n \to \infty} a_{n*} \hat{\nu}_0 = m_{\mathscr{X}_{d+1}^S}. \tag{5.1}$$

Since the map P is equivariant for the action of $\mathrm{SL}_{d+1}(\mathbb{R})$, Theorem 1.1 follows immediately from Theorem 5.1. For the proof, the key property of the diagonal element b appearing in Corollary 1.4 is that it commutes with $\mathrm{PGL}_{d+1}(\mathbb{R})$. This follows from (1.12) and the fact that $\infty \notin S_{\mathrm{ue}}$.

Proof of Theorem 5.1 assuming Corollary 1.4. Let $\mathcal{T} = (a_n)$. The space of measures on $\mathscr{X}_{d+1}^S$ of total mass at most one, is compact with respect to the weak-* topology. Thus it suffices to show that (5.1) holds along any subsequence of $\mathcal{T}$ for which the left-hand side converges. We will take a subsequence of $\mathcal{T}$, which we still denote by (a_j) to simplify notation, and for which $\lim_{j \to \infty} a_{j*} \hat{\nu}_0 = \bar{\nu}_\infty$, where $\bar{\nu}_\infty$ is a measure on $\mathscr{X}_{d+1}^S$ of total mass at most one. We need to show $\bar{\nu}_\infty = m_{\mathscr{X}_{d+1}^S}$. Along the proof we will freely pass to subsequences, which we will continue to denote by (a_j).

We first claim that $\bar{\nu}_\infty$ is a probability measure. Indeed, by the assumption of uniform non-divergence along (a_n), there is no escape of mass in the real factor $\mathscr{X}_{d+1}$; that is, that any subsequential limit $\lim_{j \to \infty} a_{j*} \nu_0$ is a probability measure on $\mathscr{X}_{d+1}$. Since the projection P is a proper map, this also implies that there is no escape of mass in the S-adic extension $\mathscr{X}_{d+1}^S$.

Note that the projection of each of the elements h_i to $\mathrm{PGL}_{d+1}(\mathbb{R})$ is contained in the diagonal group, and thus commutes with the a_j. Also note from Proposition 4.1 that $\hat{\nu}_0$ is μ-stationary. It follows by Proposition 3.6 that $\bar{\nu}_\infty$ is a μ-stationary measure. By Corollary 1.4, there is $\eta \in [0,1]$ such that $\bar{\nu}_\infty = (1-\eta) m_{\mathscr{X}^S_{d+1}} + \eta \nu'$, where ν' is a μ-stationary measure such that for b as in (1.12), we have that $b^k_* \nu' \to_{k\to\infty} 0$. We want to prove $\eta = 0$.

Assume for the sake of contradiction that $\eta > 0$. By the assumption of uniform non-divergence along (a_n), there is a compact $K \subset \mathscr{X}_{d+1}$ such that for any $x_0 \in \mathscr{X}_{d+1}$, for all large enough j we have

$$a_{j*}\nu_{x_0}(K) \geq 1 - \frac{\eta}{3}, \tag{5.2}$$

where ν_{x_0} is the pushforward of θ under the map $\mathbf{x} \mapsto u(\mathbf{x})x_0$. Let $\bar{K} \stackrel{\text{def}}{=} P^{-1}(K) \subset \mathscr{X}^S_{d+1}$, where $P : \mathscr{X}^S_{d+1} \to \mathscr{X}_{d+1}$ is the projection. Since P is proper, $\bar{K}$ is compact.

By the definition of η, for some k_0 large enough we have

$$b^{k_0}_* \hat{\nu}_\infty(\bar{K}) < 1 - \frac{\eta}{2}. \tag{5.3}$$

Now, using (5.3), let f be a compactly supported continuous function on $\mathscr{X}^S_{d+1}$ satisfying the pointwise bound $\mathbf{1}_{\bar{K}} \leq f \leq 1$, and such that

$$\begin{aligned}
1 - \frac{\eta}{2} &> \int f \,\mathrm{d}(b^{k_0}_* \hat{\nu}_\infty) = \int f(b^{k_0}x)\mathrm{d}\hat{\nu}_\infty(x) = \lim_{j\to\infty} \int f(b^{k_0}a_j x)\mathrm{d}\hat{\nu}_0(x) \\
&\geq \lim_{j\to\infty} \int \mathbf{1}_{\bar{K}}(b^{k_0}a_j u(\mathbf{x})\Lambda^S)\mathrm{d}\theta(\mathbf{x}) = \lim_{j\to\infty} \int \mathbf{1}_{\bar{K}}(a_j u(\mathbf{x}) b^{k_0}\Lambda^S)\mathrm{d}\theta(\mathbf{x}) \\
&= \lim_{j\to\infty} \int \mathbf{1}_K(a_j x)\mathrm{d}\nu_{x_0}(x) = \lim_{j\to\infty} (a_{j*}\nu_{x_0})(K),
\end{aligned}$$

where $x_0 = P(b^{k_0}\Lambda^S) \in \mathscr{X}_{d+1}$. Note that in the second line we used the commutation relation between b and the elements of the random walk. We have obtained a contradiction to (5.2). ☐

Proof of Theorem 1.2. Given a norm $\|\cdot\|$ on $\mathbb{R}^{d+1}$, let $\varepsilon_{\|\cdot\|}$ be as in (1.6) and let $\varepsilon < \varepsilon_{\|\cdot\|}$. Let $\mathbf{r}$ be a weight vector. We need to show that for θ-a.e. $\mathbf{x} \in \mathbb{R}^d$, the trajectory $\left\{a^{(\mathbf{r})}_t \Lambda_{\mathbf{x}} : t > 0\right\}$ visits $\mathcal{U}$ along an unbounded set, where

$$\mathcal{U} = \left\{\Lambda \in \mathscr{X}_{d+1} : \Lambda \cap \overline{B}(0,\varepsilon) = \{0\}\right\}.$$

Here $\overline{B}(0,\varepsilon)$ is the closed ball of radius ε around the origin in $\mathbb{R}^d$, with respect to the given norm. Note that $\mathcal{U}$ is open, is nonempty by definition of $\varepsilon_{\|\cdot\|}$, and hence

$$\delta \stackrel{\text{def}}{=} m_{\mathscr{X}_{d+1}}(\mathcal{U}) \quad \text{satisfies } \delta > 0.$$

Assume for the sake of contradiction that there is $t_0 > 0$ such that

$$\theta(B) > 0, \qquad \text{where } B \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^d : \forall t \geq t_0,\ a^{(\mathbf{r})}_t \Lambda_{\mathbf{x}} \notin \mathcal{U}\right\}.$$

Using Proposition 3.2, there is $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form $f = f_{i_1} \circ \cdots \circ f_{i_n}$ such that

$$\frac{\theta(B \cap f(\mathcal{K}))}{\theta(f(\mathcal{K}))} \geq 1 - \frac{\delta}{2}.$$

Let $\theta_1 = f_*\theta$. Because of the self-similar structure, θ_1 is the normalized restriction of θ to $f(\mathcal{K})$, and we have $\theta_1(B) \geq 1 - \frac{\delta}{2}$. Let $\bar{\nu}_1$ be the measure obtained by pushing forward θ_1 under the map

$\mathbf{x} \mapsto u(\mathbf{x})\Lambda^S$. By Propositions 3.1 and 3.4, the measure θ_1 satisfies uniform non-divergence along any unbounded subsequence $(a_n) \subset \left\{a_t^{(\mathbf{r})}\right\}$. Also, the definition of B implies that for all $t \geq t_0$,

$$a_{t*}^{(\mathbf{r})}\bar{\nu}_1(\mathcal{U}) \leq \frac{\delta}{2}.$$

On the other hand, θ_1 is the self-similar measure for the conjugated IFS Φ' as in (3.4), and thus we can apply Theorem 5.1 to obtain that

$$\liminf_{t\to\infty} a_{t*}^{(\mathbf{r})}\bar{\nu}_1(\mathcal{U}) \geq m_{\mathscr{X}_{d+1}}(\mathcal{U}) = \delta.$$

This gives the desired contradiction. □

Remark 5.2. Following [KW08], one can define Dirichlet improvability along any unbounded sequence (a_n) of the diagonal group. Our arguments then show that for any measure θ as in Theorem 1.1, and any sequence (a_n) drifting away from walls, the set of $\mathbf{x}$ which are Dirichlet improvable along (a_n) are a nullset with respect to θ.

5.2. **Proof of Corollary 1.4.** The goal of this section is to establish complete escape of mass, for certain stationary measures which differ from $m_{\mathscr{X}_{d+1}^S}$, under the action of certain diagonal subgroups of G^S.

We first use the structure of the lattice Λ^S, to obtain the following statement:

Proposition 5.3. *Let W^{st} and b be as in (1.10) and (1.12), and let $x_0 \in \mathscr{X}_{d+1}^S$ such that $\mathrm{Stab}(x_0) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}$. Then $b^n x_0 \to \infty$ as $n \to \infty$.*

Proof. Recall from Proposition 3.7 that (3.14) holds for any nontrivial element in the stabilizer of any $x \in \mathscr{X}_{d+1}^S$. Let $w \in W^{\mathrm{st}} \smallsetminus \{\mathrm{Id}\}$ with $wx_0 = x_0$. Suppose by contradiction that for some subsequence $n_j \to \infty$ we have $b^{n_j}x_0 \to x_\infty$, for some $x_\infty \in \mathscr{X}_{d+1}^S$. The stabilizer of $b^{n_j}x_0$ contains $b^{n_j}wb^{-n_j}$. Thus any accumulation point w_∞ of the sequence $(b^{n_j}wb^{-n_j})_j$ belongs to the stabilizer of x_∞. By definition of S_{ue}, we have that $|\varrho^{-1}|_\sigma < 1$ for all $\sigma \in S_{\mathrm{ue}}$. In particular, the conjugation action of b on $\mathrm{U}_{\mathrm{ue}} \overset{\mathrm{def}}{=} \mathrm{U}_{S_{\mathrm{ue}}}$ is contracting. Hence, writing $w = \left(w^{(\sigma)}\right)_{\sigma\in S}$ and $w_\infty = \left(w_\infty^{(\sigma)}\right)_{\sigma\in S}$, we see that

$$w_\infty^{(\sigma)} = \begin{cases} \mathrm{Id} & \sigma \in S_{\mathrm{ue}} \\ w^{(\sigma)} & \sigma \in S \smallsetminus S_{\mathrm{ue}}. \end{cases}$$

This contradicts (3.14). □

Armed with Proposition 5.3, Corollary 1.4 now follows at once from the following:

Corollary 5.4. *If ν is a finite measure on $\mathscr{X}_{d+1}^S$ and $b \in G^S$ satisfies that $b^n x_0 \to \infty$ for ν-a.e. x_0, then $b_*^n\nu \to 0$. In particular, in Case 1 of Theorem 1.3, $b_*^n\nu \to 0$ as $n \to \infty$.*

Proof. We prove the first assertion; the second assertion follows immediately in view of Proposition 5.3. For any compact $K \subset \mathscr{X}_{d+1}^S$ and for ν-a.e. $x \in \mathscr{X}_{d+1}^S$, by assumption there is $n_0 = n_0(x, K)$ such that for all $n \geq n_0$, $b^n x \notin K$. This implies that for any $\varepsilon > 0$, there is $n_0 = n_0(K)$ such that

$$\nu\left(\left\{x \in \mathscr{X}_{d+1}^S : \forall n \geq n_0,\ b^n x \notin K\right\}\right) < \varepsilon.$$

Thus any measure ν' which is an accumulation point of the sequence $(b_*^n\nu)$ satisfies $\nu'(K) \leq \varepsilon$. Since K and ε were arbitrary we have that $a_*^n\nu \to 0$. □

6. Measure Classification for an Auxiliary Random Walk and Proof of Theorem 1.3

Theorem 1.3 is derived from a detailed analysis of stationary measures for a related measure $\bar{\mu}$. We now introduce this random walk and state the corresponding measure rigidity result, Theorem 6.1 below.

Let S_{ue} be as in (1.9). Recall that $\varrho = \frac{r}{q}$ with $\gcd(r, q) = 1$, and let

$$S_{\mathrm{dt}} \stackrel{\mathrm{def}}{=} \{\infty\} \cup \{p \in S_{\mathrm{f}} : p | r\}, \qquad S_{\mathrm{tr}} \stackrel{\mathrm{def}}{=} \{p \in S_{\mathrm{f}} : \gcd(p, q) = \gcd(p, r) = 1\}, \tag{6.1}$$

so that

$$S = S_{\mathrm{ue}} \sqcup S_{\mathrm{dt}} \sqcup S_{\mathrm{tr}}. \tag{6.2}$$

In the remainder of the paper, for an algebraic group $\mathbf{J}$, we will simplify the notation (1.8) by writing $\mathrm{J}_{\mathrm{ue}} = \mathrm{J}_{S_{\mathrm{ue}}}$, $\mathrm{J}_{\mathrm{dt}} = \mathrm{J}_{S_{\mathrm{dt}}}$, $\mathrm{J}_{\mathrm{tr}} = \mathrm{J}_{S_{\mathrm{tr}}}$. Using the same probability vector $\mathbf{p}$ appearing in (1.11), we let

$$\bar{\mu} \stackrel{\mathrm{def}}{=} \sum p_i \delta_{\bar{h}_i}, \tag{6.3}$$

where $\bar{h}_i = \left(\bar{h}_i^{(\sigma)}\right)_{\sigma \in S}$ are defined by

$$\bar{h}_i^{(\sigma)} \stackrel{\mathrm{def}}{=} \begin{cases} \begin{pmatrix} \varrho \, \mathrm{Id}_d & -\mathbf{y}_i \\ \mathbf{0} & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{ue}} \\ \begin{pmatrix} \varrho \, \mathrm{Id}_d & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{dt}} \\ \mathrm{Id} & \text{if } \sigma \in S_{\mathrm{tr}}. \end{cases} \tag{6.4}$$

The subscripts dt, tr stand respectively for 'deterministic' and 'trivial'. To explain this terminology, and motivate the use of this modified random walk, note first that words in the generators $\bar{h}_i$ always remain at a uniformly bounded distance from the corresponding words in the generators h_i. As a result, the two random walks exhibit similar dynamical properties. On the other hand, the $\bar{\mu}$-random walk is easier to analyze. The directions tangent to G_{tr} remain bounded in forward and backward time under the μ-random walk, but these directions are left unchanged by the $\bar{\mu}$-random walk. Additionally, since the S_{dt}-coordinates of $\bar{h}_i$ are simultaneously diagonal, the behavior of the $\bar{\mu}$ random walk on these coordinates is deterministic, while still capturing the essence of the action of the μ-random walk.

Note that when the numerator of the contraction ratio ϱ is equal to 1, and when all the primes appearing in the denominators of the $\mathbf{y}_i$ also appear in the denominator of ϱ, we have $h_i = \bar{h}_i$ and $\mu = \bar{\mu}$. However, in the general case, these random walks are different. The following is our result about $\bar{\mu}$-stationary measures.

Theorem 6.1. *Let Φ be an irreducible carpet IFS satisfying the open set condition, let $\bar{h}_i$ be as in (6.4), let $\mathbf{p}$ be a probability vector, let $\bar{\mu}$ be as in (6.3), and let W^{st} be as in (1.10). Then for any $\bar{\mu}$-ergodic $\bar{\mu}$-stationary measure ν, one of the following holds:*

(1) $\nu\left(\left\{x \in \mathscr{X}_{d+1}^S : \mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}\right\}\right) = 1.$
(2) $\nu = m_{\mathscr{X}_{d+1}^S}.$

It would be interesting to classify all ergodic stationary measures for the random walks generated by μ and $\bar{\mu}$.

6.1. Compact and contracting extensions of random walks.

In this section, we construct a common cover of the random walks with laws μ and $\bar{\mu}$ and prove the key Proposition 6.2, which transfers measure rigidity results from one to the other. For related results on extensions of other random walks, see [SW19, §5] and [AG24, Thm. 1.1].

Let $k_i \in G^S$ be such that $k_i \bar{h}_i = h_i$. More explicitly, we have that $k_i = \left(k_i^{(\sigma)}\right)_\sigma$, where

$$k_i^{(\sigma)} \stackrel{\text{def}}{=} \begin{cases} \mathrm{Id} & \text{if } \sigma \in S_{\mathrm{ue}} \cup \{\infty\} \\ \begin{pmatrix} \mathrm{Id}_d & -\mathbf{y}_i \\ 0 & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{dt}} \smallsetminus \{\infty\} \\ \begin{pmatrix} \varrho\, \mathrm{Id}_d & -\mathbf{y}_i \\ 0 & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{tr}}. \end{cases}$$

Consider the subgroup $F = \prod_{\sigma \in S} F^{(\sigma)}$ of G^S defined as follows:

$$F^{(\sigma)} \stackrel{\text{def}}{=} \begin{cases} \{\mathrm{Id}\} & \sigma \in S_{\mathrm{ue}} \cup \{\infty\} \\ \mathbf{U}(\mathbb{Q}_\sigma) & \sigma \in S_{\mathrm{dt}} \smallsetminus \{\infty\} \\ \overline{\left\langle k_i^{(\sigma)} \right\rangle} & \sigma \in S_{\mathrm{tr}}. \end{cases}$$

Note that the group F is normalized by both h_i and $\bar{h}_i$ for all i. We shall consider an extension of our random walks on $\mathscr{X}_{d+1}^S$ to the space

$$E = \mathscr{X}_{d+1}^S \times F.$$

The random walk is generated by the following elements:

$$\bar{h}_i^F(x, f) \stackrel{\text{def}}{=} (\bar{h}_i x, k_i \bar{h}_i f \bar{h}_i^{-1}). \tag{6.5}$$

In analogy with (1.11) and (6.3), define the law of this random walk as

$$\bar{\mu}^F = \sum_i p_i \delta_{\bar{h}_i^F}.$$

We define two projections $P, \bar{P} : E \to \mathscr{X}_{d+1}^S$ by $P(x, f) = fx$ and $\bar{P}(x, f) = x$. Then, we note the following equivariance properties of these projections:

$$P\left(\bar{h}_i^F(x, f)\right) = h_i P(x, f), \qquad \bar{P}\left(\bar{h}_i^F(x, f)\right) = \bar{h}_i \bar{P}(x, f). \tag{6.6}$$

Proposition 6.2. *There exists a probability measure m_F on F such that every $\bar{\mu}^F$-stationary probability measure m on E with $\bar{P}_* m = m_{\mathscr{X}_{d+1}^S}$ is of the form $m = m_{\mathscr{X}_{d+1}^S} \otimes m_F$.*

Proof. We will deduce this result from a combination of uniqueness results for Haar measures on compact groups, stationary measures for contracting IFS's, and from the mixing property of the $\bar{\mu}$-random walk on $\mathscr{X}_{d+1}^S$.

We begin by giving a more explicit description of the action of the random walk on the F-coordinate. Let

$$Q \stackrel{\text{def}}{=} \prod_{\sigma \in S_{\mathrm{tr}}} F^{(\sigma)}.$$

Since $F^{(\sigma)}$ is compact for $\sigma \in S_{\mathrm{tr}}$, Q is a compact factor of F. For $f = \left(f^{(\sigma)}\right) \in F$ we have

$$(k_i \bar{h}_i f \bar{h}_i^{-1})^{(\sigma)} = \begin{cases} \mathrm{Id} & \text{if } \sigma \in S_{\mathrm{ue}} \cup \{\infty\} \\ \begin{pmatrix} \mathrm{Id}_d & -\mathbf{y}_i + \varrho \mathbf{x} \\ 0 & 1 \end{pmatrix} & \text{if } \sigma \in S_{\mathrm{dt}} \smallsetminus \{\infty\} \text{ and } f^{(\sigma)} = u(\mathbf{x}),\ \mathbf{x} \in \mathbb{Q}_\sigma^d \\ k_i^{(\sigma)} f^{(\sigma)} & \text{if } \sigma \in S_{\mathrm{tr}}. \end{cases}$$

In particular, if $f \in Q$ then $k_i \bar{h}_i f \bar{h}_i^{-1} = k_i^Q f \in Q$, where k_i^Q is the projection of k_i to Q.

Let m be a $\bar{\mu}^F$-stationary measure with $P_* m = m_{\mathscr{X}_{d+1}^S}$. Let $\bar{P}_0 : \mathscr{X}_{d+1}^S \times F \to \mathscr{X}_{d+1}^S \times Q$ and $\bar{P}_1 : \mathscr{X}_{d+1}^S \times Q \to \mathscr{X}_{d+1}^S$ denote the standard projections so that $\bar{P} = \bar{P}_1 \circ \bar{P}_0$. We first show that

$$(\bar{P}_0)_* m = m_{\mathscr{X}_{d+1}^S} \otimes m_Q, \tag{6.7}$$

where m_Q denotes the Haar probability measure on Q.

To this end, let Γ^Q denote the group generated by $\bar{h}_i^Q$, where $\bar{h}_i^Q$ denotes the restriction of the action of $\bar{h}_i^F$ to $\mathscr{X}_{d+1}^S \times Q$. Our claim will follow by [SW19, Proposition 5.3] upon verifying ergodicity of the action of Γ^Q with respect to $m_{\mathscr{X}_{d+1}^S} \otimes m_Q$. By the Howe-Moore theorem, since the group generated by $\bar{h}_i$ is unbounded, its action on $\mathscr{X}_{d+1}^S$ is weak-mixing with respect to $m_{\mathscr{X}_{d+1}^S}$. Moreover, since the group generated by $\left(k_i^{(\sigma)}\right)_{\sigma\in S_{\mathrm{tr}}}$ is dense in Q, its action by left translations on Q is ergodic with respect to m_Q. Hence, the action of Γ^Q on $\mathscr{X}_{d+1}^S \times Q$ is ergodic with respect to the product measure $m_{\mathscr{X}_{d+1}^S} \otimes m_Q$; cf. [Gla03, Theorem 9.23(2)]. This gives (6.7).

To conclude the proof, we define

$$\mathrm{U}_0 \overset{\mathrm{def}}{=} \mathrm{U}_{S_{\mathrm{dt}}\smallsetminus\{\infty\}} = \prod_{\sigma\in S_{\mathrm{dt}}\smallsetminus\{\infty\}} F^{(\sigma)},$$

which is a factor of F complementary to Q, and take advantage of the fact that action of the random walk on U_0 is given by contracting affine maps. Let

$$m_b \overset{\mathrm{def}}{=} \lim_{n\to\infty} \left(\bar{h}_{i_1}^F \circ \cdots \circ \bar{h}_{i_n}^F\right)_* m \tag{6.8}$$

be the limit measures as in Proposition 3.6, so that

$$m = \int m_b \,\mathrm{d}(\bar{\mu}^F)^{\mathbb{N}}(b). \tag{6.9}$$

We will show that for almost every b, there is $u_b \in \mathrm{U}_0$ such that

$$m_b = m_{\mathscr{X}_{d+1}^S} \otimes m_Q \otimes \delta_{u_b}. \tag{6.10}$$

Plugging into (6.9), we see that this implies the proposition with $m_F = m_Q \otimes \int \delta_{u_b}\,\mathrm{d}(\bar{\mu}^F)^{\mathbb{N}}(b)$.

By (6.7), the projection of m to $\mathscr{X}_{d+1}^S \times Q$ is invariant by elements of Γ^Q, and hence, by (6.8) we have that $(\bar{P}_0)_* m_b = m_{\mathscr{X}_{d+1}^S} \otimes m_Q$. Moreover, note that $|\varrho|_\sigma < 1$ for all $\sigma \in S_{\mathrm{dt}}$ by definition of the deterministic places. Hence, letting $\bar{P}^{U_0} : E \to \mathrm{U}_0$ denote the standard projection, it follows that $\left(\bar{P}^{U_0}\right)_* m_b$ is a Dirac mass at some point $u_b \in \mathrm{U}$. This implies (6.10) and concludes the proof. ☐

6.2. **Proof of Theorem 1.3 from Theorem 6.1.** Let ν be an ergodic μ-stationary measure on $\mathscr{X}_{d+1}^S$ and let ν^F be some lift of ν to E, that is a measure satisfying $P_*\nu^F = \nu$; the existence of such a lift can be constructed using a Borel section of the projection $P : E \to \mathscr{X}_{d+1}^S$. We will choose the section so that the F-coordinate takes values in Q. This choice implies that the sequence of measures $\left(\bar{\mu}^F\right)^{*n} * \nu^F$ are all supported on $\mathscr{X}_{d+1}^S \times Q$.

Let $\bar{\nu}^F$ be a weak-$*$ limit measure of the sequence of measures

$$\nu_N^F \overset{\mathrm{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \left(\bar{\mu}^F\right)^{*n} * \nu^F$$

as $N \to \infty$. Then, (6.6) implies the following properties of $\bar{\nu}^F$.

Lemma 6.3. *For all $N \geq 1$, the measures ν_N^F satisfy*

$$P_*\nu_N^F = \nu.$$

In particular, $\bar{\nu}^F$ is a $\bar{\mu}^F$-stationary probability measure. Moreover, we have that $\bar{P}_*\bar{\nu}^F$ is $\bar{\mu}$-stationary.

Proof. Since ν is μ-stationary, the P-equivariance in (6.6) implies that $P_*(\bar{\mu}^F * \nu^F) = \nu$. This implies the first claim. The second claim follows from the first assertion, Proposition 3.6, and the fact that $\mathscr{X}_{d+1}^S \times Q \to \mathscr{X}_{d+1}^S$ is a compact extension, so there is no loss of mass in taking limits. The last claim is immediate from the averaging construction of $\bar{\nu}^F$ and the $\bar{P}$-equivariance in (6.6). □

The following lemma, along with Proposition 6.2, concludes our deduction of Theorem 1.3 from Theorem 6.1.

Lemma 6.4. *Suppose that ν gives 0 mass to the set of points x with non-trivial stabilizer in W^{st}. Then, the same holds for $\bar{P}_*(\bar{\nu}^F)$. In particular, by Theorem 6.1, $\bar{P}_*(\bar{\nu}^F) = m_{\mathscr{X}_{d+1}^S}$.*

Proof. Let $\mathcal{W} \subset \mathscr{X}_{d+1}^S$ denote the set of points with non-trivial stabilizers in W^{st}. By definition of $\bar{P}$, we need to show that $\bar{\nu}^F(\mathcal{W} \times F) = 0$. First, we note that, since $F \subset W^{\mathrm{st}}$, $\mathcal{W}$ is F-invariant. It follows that $\mathcal{W} \times F = P^{-1}(\mathcal{W})$. These observations give

$$\bar{P}_*\bar{\nu}^F(\mathcal{W}) = \bar{\nu}^F(\mathcal{W} \times F) = \bar{\nu}^F(P^{-1}(\mathcal{W})) = P_*\bar{\nu}^F(\mathcal{W}).$$

By Lemma 6.3, we have that $P_*\bar{\nu}^F = \nu$. Hence, the first assertion follows by our hypothesis that $\nu(\mathcal{W}) = 0$. The second assertion follows from the first one via Theorem 6.1. □

Remark 6.5. We did not give an explicit example of a $\bar{\mu}$-stationary measure satisfying conclusion (1) in Theorem 6.1, but the preceding arguments show that such measures exist. Indeed, start with a μ-stationary measure on $\mathscr{X}_{d+1}^S$ which is supported on the solenoid $\mathscr{U}^S$, and note that the construction of $\bar{P}_*\bar{\nu}^F$ in the proof of Lemma 6.3 gives rise to such a measure.

7. Growth properties of the random walk

Following Benoist and Quint [BQ11], we need to understand the growth properties of a random walk generated by the support of $\bar{\mu}$, acting linearly via several finite dimensional linear representations of G^S. One major obstruction to running the same arguments given in [BQ11] without change is the absence of uniform expansion, which cannot be expected in the case where the Zariski closure of the group generated by the random walk is solvable. However, it was noted in joint work of David Simmons with the third-named author, that in some cases a useful analogue is true; cf. [SW19, Prop. 3.1(a)]. The goal of this section is to prove several analogous growth properties for the action of the random walk in its Adjoint representation on $\mathfrak{g}_\sigma, \sigma \in S_{\mathrm{ue}}$.

Throughout this section, we fix an irreducible carpet-IFS Φ with common rational contraction ratio $\varrho > 0$. Given a finite list of indices $i_1, \ldots, i_n$ in $\{1, \ldots, k\}$, following [SW19] we write

$$\bar{h}_1^n \stackrel{\mathrm{def}}{=} \bar{h}_{i_n} \circ \cdots \circ \bar{h}_{i_1} \quad \text{and} \quad \bar{h}_n^1 \stackrel{\mathrm{def}}{=} \bar{h}_{i_1} \circ \cdots \circ \bar{h}_{i_n}, \tag{7.1}$$

where the $\bar{h}_i$ are as in §6.

Further, for $\sigma \in S$ and $x \in \mathbb{Q}$, $|x|_\sigma$ denotes the σ-adic absolute value of x, $\mathfrak{g}_\sigma$ denotes the Lie algebra of $\mathbf{G}_\sigma$, where we view $\mathfrak{g}_\sigma$ as a vector space over $\mathbb{Q}_\sigma$, equipped with the σ-adic norm. We denote the Lie algebra of G^S by $\mathfrak{g}$, that is,

$$\mathfrak{g} = \bigoplus_{\sigma \in S} \mathfrak{g}_\sigma.$$

7.1. Action on the Lie algebra.

In this subsection, for $h = (h_\sigma)_{\sigma \in S}$, $\|h\|_\sigma$ denotes the operator norm of the action of $\mathrm{Ad}(h_\sigma)$ on $\mathfrak{g}_\sigma$. The following Proposition implies that the norm of each of the random walk elements in each place is dictated by the growth of the scalar contraction ratios.

Proposition 7.1. *There is a constant $C > 1$ such that for any $n \in \mathbb{N}$, the following holds. Denoting by $A_n \asymp_C B_n$ the inequalities $C^{-1}A_n \le B_n \le CA_n$, for any word $(i_1, \ldots, i_n)$ of length n, for all*

$\sigma \in S$ *we have*

$$\left\|\bar{h}_1^n\right\|_\sigma \asymp_C \begin{cases} |\varrho|_\sigma^n, & \sigma \in S_{\mathrm{ue}}, \\ |\varrho|_\sigma^{-n}, & \sigma \in S_{\mathrm{dt}}. \end{cases} \tag{7.2}$$

Proof. The element $\begin{pmatrix} \varrho^n \mathrm{Id}_d & 0 \\ 0 & 1 \end{pmatrix}$ acts on $\mathfrak{g}_\sigma$ diagonally, with three eigenvalues $\varrho^n, 1, \varrho^{-n}$. Thus in the case $\sigma \in S_{\mathrm{dt}}$, the statement follows from (6.4).

For $\sigma \in S_{\mathrm{ue}}$, we first show that the left-hand side of (7.2) dominates the right-hand side. Indeed, let $v = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)$, for some $X \in \mathbb{Q}_\sigma^d$ with $\|X\|_\sigma = 1$. Then $\mathrm{Ad}\left(\bar{h}_1^n\right)(v) = \varrho^n v$, and hence $\left\|\bar{h}_1^n\right\|_\sigma \geq |\varrho|_\sigma^n$.

For the opposite inequality, denote by

$$\mathbf{y} = f_{i_1} \circ \cdots \circ f_{i_n}(0)$$

the translation vector of the map $f_{i_1} \circ \cdots \circ f_{i_n}$. Then by a straightforward induction using (1.1) and (6.4) we have

$$\left(\bar{h}_1^n\right)_\sigma = \begin{pmatrix} \varrho^n \mathrm{Id}_d & -\mathbf{y} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varrho^n \mathrm{Id}_d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_d & -\varrho^{-n}\mathbf{y} \\ 0 & 1 \end{pmatrix}.$$

Since the operator norm is sub-multiplicative, for an upper bound it suffices to give separate upper bounds for the two elements in this product. As we saw in the case $\sigma \in S_{\mathrm{dt}}$, the operator norm of the first element $\begin{pmatrix} \varrho^n \mathrm{Id}_d & 0 \\ 0 & 1 \end{pmatrix}$ is $|\varrho|_\sigma^n$. Again by a straightforward induction we see that we can express each of the coefficients of $\varrho^{-n}\mathbf{y}$ as a sum $\sum_{i=0}^n b_i \varrho^{-i}$, where the b_i are contained in the set of coordinates of the vectors $\mathbf{y}_1, \ldots, \mathbf{y}_k$, which is a finite set. By the ultrametric property of the σ-adic absolute value and the fact that $|\varrho|_\sigma \geq 1$, we deduce that the operator norm of the second element $\begin{pmatrix} \mathrm{Id}_d & -\varrho^{-n}\mathbf{y} \\ 0 & 1 \end{pmatrix}$ is bounded, independently of n. This completes the proof. □

Let B, β be as in (3.9). As in [BQ11], to simplify notation we identify the index set $\{1, \ldots, k\}$ with the random walk elements $\{\bar{h}_1, \ldots, \bar{h}_k\}$. In particular, for $b \in B$ we let

$$b_1^n \stackrel{\mathrm{def}}{=} \bar{h}_1^n \quad \text{and} \quad b_n^1 \stackrel{\mathrm{def}}{=} \bar{h}_n^1$$

be the elements given by (7.1) corresponding to the n-prefix $(i_1, \ldots, i_n)$ of b. For $\sigma \in S$ we also let $\mathfrak{u}_\sigma$ denote the Lie algebra of the group $\mathbf{U}(\mathbb{Q}_\sigma)$ (see (1.10)). Let $\mathbb{P}(\mathfrak{g})$ be the projective space over $\mathfrak{g}$, let d be some metric on $\mathbb{P}(\mathfrak{g})$ inducing the topology, and for two subsets $A, A' \subset \mathbb{P}(\mathfrak{g})$, we let $\mathrm{dist}(A, A') \stackrel{\mathrm{def}}{=} \inf\{d(a, a') : a \in A, a' \in A'\}$.

The following lemma provides a weaker version of the aforementioned results of [SW19] which suffices for our purposes (cf. [BQ11, Cor. 5.5] and [SW19, Prop. 3.1]).

Lemma 7.2. *For all $\delta > 0$ there are $C > 1$ and $m_0 \in \mathbb{N}$ such that for all $\sigma \in S_{\mathrm{ue}}$ and all non-zero $v_\sigma \in \mathfrak{g}_\sigma$ we have*

$$\beta\left(\left\{b \in B \colon \forall m \geq m_0, \quad \|\mathrm{Ad}(b_1^m)\|_\sigma \|v_\sigma\|_\sigma \leq C \|\mathrm{Ad}(b_1^m) v_\sigma\|_\sigma\right\}\right) \geq 1 - \delta.$$

Moreover, for every $\delta > 0$ and $\eta > 0$, there is $m_1 \in \mathbb{N}$ such that for all $\sigma \in S_{\mathrm{ue}}$, for any $v \in \mathfrak{g}_\sigma \smallsetminus \{0\}$,

$$\beta\left(\{b \in B : \forall m \geq m_1, \quad \mathrm{dist}\left(\mathrm{Ad}(b_1^m) v, \mathfrak{u}_\sigma\right) < \eta\}\right) > 1 - \delta$$

(where we identify $v, \mathfrak{u}_\sigma$ with their image in $\mathbb{P}(\mathfrak{g})$).

Proof. This follows from [SW19, Sections 3 & 6]. By the argument of [SW19, Section 6.1], assumptions I, II, III are satisfied for the restriction of the random walk to $\mathfrak{g}_\sigma$, with $W = \mathfrak{u}_\sigma$. Now the desired estimates follow from [SW19, Prop. 3.1]. Note that in [SW19], V is a real vector space, but the arguments given there are valid in vector spaces over $\mathbb{Q}_\sigma$. Note that the arguments in [SW19] use the Oseledec theorem; for a p-adic version of the Oseledec theorem, see [Rag79]. □

7.2. **Action on the exterior powers of the Lie algebra.** Let $\mathbf{P}$ be the normalizer of $\mathbf{U}$ and let $\mathfrak{p}_\sigma$ be the Lie algebra of $P(\mathbb{Q}_\sigma)$. Note that $\dim \mathbf{U} = d$ and $\dim \mathbf{P} = d^2 + d$. For $r \in \{1, \dots, \dim \mathbf{P}\}$, we let $V^{\wedge r} = V^{\wedge r}(\sigma) \stackrel{\text{def}}{=} \bigwedge^r \mathfrak{g}_\sigma$. There is an action of the elements of the random walk on $V^{\wedge r}$ and $\mathbb{P}(V^{\wedge r})$ through the r^{th} exterior power. We denote these actions respectively by

$$v \mapsto \bar{h}v \quad \text{and} \quad [v] \mapsto \bar{h}[v], \qquad \text{where } v \in V^{\wedge r} \smallsetminus \{0\},\ \bar{h} \in \operatorname{supp} \bar{\mu}^{\otimes k}.$$

Lemma 7.3. *For any $\sigma \in S_{\mathrm{ue}}$ and any $r \in \{1, \dots, \dim \mathbf{P}\}$ we define a nonzero subspace $W^{(r)} = W^{(r)}(\sigma)$ in $V^{\wedge r}(\sigma)$ as follows:*

$$W^{(r)} \stackrel{\text{def}}{=} \begin{cases} \operatorname{span}(u_1 \wedge \cdots \wedge u_r : u_i \in \mathfrak{u}_\sigma) & \text{if } r \in \{1, \dots, d\} \\ \operatorname{span}(u_1 \wedge \cdots \wedge u_d \wedge v_{d+1} \wedge \cdots \wedge v_r : u_i \in \mathfrak{u}_\sigma, v_j \in \mathfrak{p}_\sigma) & \text{if } r \in \{d+1, \dots, d^2+d\}. \end{cases}$$

Then we have:

(i) The subspaces $W^{(r)}$ are $\bar{h}_i$-invariant for each r and i;
(ii) For every $v \in V^{\wedge r} \smallsetminus \{0\}$ and β-a.e. b, $\|b_1^n v\| \stackrel{n\to\infty}{\longrightarrow} \infty$, and any accumulation point of the sequence $(b_1^n[v])_{n\in\mathbb{N}}$ is contained in $\mathbb{P}\left(W^{(r)}\right)$.

Sketch of Proof. Assertion (i) is a straightforward computation which is left to the reader. The second assertion follows from the results of [SW19]. Before outlining the proof, we caution the reader not to confuse $W^{(r)}$ with $W^{\wedge r}$; for $r > 1$, what we denote here by $W^{(r)}$ is properly contained in the space denoted by $W^{\wedge r}$ in [SW19]. For the proof, we replace V with $V^{\wedge r}$ and consider our random walk elements as contained in $\mathrm{GL}(V^{\wedge r})$. With this modification, what we denote here by $W^{(r)}$ corresponds to the subspace of $V^{\wedge r}$ which in [SW19] was denoted by $W^{\wedge 1}$. The reader can now check that the arguments of [SW19, §6] imply that the random walk on $V^{\wedge r}$ satisfies properties (I), (II), and (III), and thus the results of [SW19, §3] imply assertion (ii). □

Corollary 7.4. *For $\sigma \in S_{\mathrm{ue}}$ and $r \in \{1, \dots, \dim \mathbf{P}\}$, and for the $\bar{\mu}$-random walk on $V^{\wedge r}$, there are no stationary measures besides the Dirac mass on $\{0\}$.*

Sketch of proof. By Lemma 7.3 (ii), the norm of v increases under most random trajectories for the $\bar{\mu}$ random walk. Consider the probability space $B \times V^{\wedge d}$, equipped with the transfer map $(b, v) \mapsto (Tb, b_1 v)$. A stationary measure on $V^{\wedge d}$ gives rise to an invariant probability measure on $B \times V^{\wedge d}$, but from growth of trajectories one sees that the function $(b, v) \mapsto \|v\|$ almost surely goes to infinity under repeated application of the transfer map. This contradicts Poincaré recurrence. See [SW19, Prop. 3.2 & Prop. 3.7] for more details. □

7.3. **The effect of changing prefixes.** Finally, we record the following lemma which measures the effect of changing the prefix of a long random walk trajectory. We introduce the notation $\|\gamma_\sigma\|_{\mathrm{op}}$ to denote the operator norm, with respect to the Euclidean metric, of the linear operator γ_σ acting linearly on $\mathbb{Q}_\sigma^{d+1}$ (this should not be confused with the notation $\|\gamma\|_\sigma$ used in Proposition 7.1 and Lemma 7.2).

Lemma 7.5. *Given $a = (a_1, \dots)$ and $b = (b_1, \dots)$ in B, and given $n \in \mathbb{N}$, let*

$$\gamma(a, n, b) \stackrel{\text{def}}{=} a_1^n \circ \left(b_n^1\right)^{-1} = \bar{h}_{a_n} \cdots \bar{h}_{a_1} \cdot \left(\bar{h}_{b_n}\right)^{-1} \cdots \left(\bar{h}_{b_1}\right)^{-1}.$$

Let $\sigma \in S_{\mathrm{ue}}$ and

$$G_{a,b,\sigma}(n) \stackrel{\text{def}}{=} \|\gamma(a, n, b)_\sigma\|_{\mathrm{op}} \qquad \text{and} \qquad G_{\max,b,\sigma}(n) \stackrel{\text{def}}{=} \max_{a' \in B} G_{a',b}(n).$$

Then

(1) For $\sigma \notin S_{\mathrm{ue}}$ we have $\gamma(a, n, b)_\sigma = \mathrm{Id}$.

(2) *There is $c_1 > 1$ such that for any $\sigma \in S_{\mathrm{ue}}$, any $b \in B$ and any $n \in \mathbb{N}$,*

$$\frac{1}{c_1} \le \frac{G_{\max,b,\sigma}(n)}{|\varrho|_\sigma^n} \le c_1.$$

In particular $G_{\max,b,\sigma}(n) \overset{n\to\infty}{\longrightarrow} \infty$.

(3) *For any $\alpha > 0$ there is $c_2 > 0$ such that for all b and all $n \in \mathbb{N}$, there is a set $B_0 = B_0(b,n) \subset B$ with $\beta(B_0) \ge 1-\alpha$, such that for all $a \in B_0$ and any $\sigma \in S_{\mathrm{ue}}$, we have*

$$G_{a,b,\sigma}(n) \ge c_2\, G_{\max,b,\sigma}(n). \tag{7.3}$$

Proof. Assertion (1) is clear from (6.4). For $\sigma \in S_{\mathrm{ue}}$ write $(\bar{h}_i)_\sigma = u(-\mathbf{y}_i)g_0$, where $u : \mathbb{Q}_\sigma^d \to \mathbf{U}(\mathbb{Q}_\sigma)$ is the map as in (1.2), and $g_0 \overset{\text{def}}{=} \operatorname{diag}(\varrho, \dots, \varrho, 1)$. We have a commutation relation $g_0\, u(\mathbf{y}) = u(\varrho \mathbf{y})\, g_0$. Carrying out a matrix multiplication, and using the commutation relation to move all the diagonal matrices to one side, we get that

$$\begin{aligned}(\gamma(a,n,b))_\sigma &= \begin{cases} u(-\mathbf{y}_{a_m})g_0 \cdots u(-\mathbf{y}_{a_1})g_0 g_0^{-1}u(\mathbf{y}_{b_n}) \cdots g_0^{-1}u(\mathbf{y}_{b_1}) & \sigma \in S_{\mathrm{ue}} \\ \mathrm{Id} & \sigma \notin S_{\mathrm{ue}} \end{cases} \\ &= \begin{cases} u(\mathbf{y}_0) & \sigma \in S_{\mathrm{ue}} \\ \mathrm{Id} & \sigma \notin S_{\mathrm{ue}}, \end{cases}\end{aligned}$$

where $\mathbf{y}_0 = \mathbf{y}_0(a,n,b)$ is given by

$$\mathbf{y}_0 = \sum_{i=1}^n \varrho^{i-1}\mathbf{y}_{b_i} - \sum_{i=1}^n \varrho^{n-i}\mathbf{y}_{a_i} = \sum_{j=1}^n \varrho^{n-j}(\mathbf{y}_{b_{n-j}} - \mathbf{y}_{a_j}). \tag{7.4}$$

Since the $\mathbf{y}_i$ are contained in a finite set, the size of $G_{a,b,\sigma}(n)$ is comparable to $\max(1, \|\mathbf{y}_0\|_\sigma)$, i.e., to $|\varrho|_\sigma^r$ where $r = r(a,b,n)$ is the largest power $n-j$ appearing in (7.4) with a nonzero coefficient. It is clear from (7.4) that $r(a,b,n) \le n-1$ for all a. Also, for each b, n, we can choose a_1 so that $\mathbf{y}_{b_{n-1}} - \mathbf{y}_{a_1}$ is nonzero, and this is the coefficient of ϱ^{n-1}. This implies that for any sequence starting with a_1 we have $r(a,b,n) = n-1$. This proves (2).

To prove (3), given α, let ℓ be large enough so that any cylinder set in B defined by specifying one prefix of length ℓ has β-measure less than α; namely, we choose $\ell > \alpha / \log(\max_i p_i)$. Arguing as in the proof of (2), we see that the only way to have $r(a,b,n) < n-\ell$ is to have $\tau_{a_j} = \tau_{b_{n-j}}$ for $j \le \ell$, and this means that the first ℓ digits of a are determined by the last ℓ digits of b. We define B_0 to be the complement of this prefix set of length ℓ corresponding to b, and the statement follows. □

8. Preparations For Exponential Drift: Non-atomicity of limit measures

Following [BQ11], the first key step in running the exponential drift argument is to show that, given a stationary measure ν, the limit measures ν_b are non-atomic almost surely. This property however fails for our random walks. Indeed, in case $\mu = \bar{\mu}$ the stationary measure $\nu = \hat{\nu}_0$ does have atomic limit measures, as can be seen from the proof of Proposition 4.1. In the general case, it can be shown that the property fails due to the deterministic forward-contracting space $\mathfrak{u}_{\mathrm{dt}} = \mathrm{Lie}(\mathrm{U}_{\mathrm{dt}})$. Nevertheless, in this section we will show that non-atomicity of limit measures (in a strong form) does hold under the hypotheses of Theorem 8.1 below.

In order to state the main result of this section, we introduce some notation. Let B^X, β^X be as in (3.13) and Proposition 3.6. Define Z to be the subgroup of G^S commuting with all of the $\bar{h}_i$. By Proposition 3.8, the elements $\left(z^{(\sigma)}\right)_{\sigma\in S}$ of Z satisfy $z^{(\sigma)} = \mathrm{Id}$ if $\sigma \in S_{\mathrm{ue}}$, $z^{(\sigma)} = \begin{pmatrix} A^{(\sigma)} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$ for some invertible $A^{(\sigma)}$ if $\sigma \in S_{\mathrm{dt}}$, and with no restrictions on $z^{(\sigma)}$ for $\sigma \in S_{\mathrm{tr}}$. Let $\mathbf{P} \overset{\text{def}}{=} \begin{pmatrix} * & * \\ \mathbf{0} & * \end{pmatrix}$ be the normalizer of the group $\mathbf{U}$ in (1.2), let S_{dt} and S_{tr} be as in (6.1), and let

$$H^{\mathrm{fne}} \overset{\text{def}}{=} \mathrm{P}_{S_{\mathrm{dt}}} \times \mathrm{G}_{S_{\mathrm{tr}}}. \tag{8.1}$$

The superscript 'fne' stands for 'forward non-expanding'; indeed, for the linear random walk consider in §7, none of the vectors in the Lie algebra $\mathfrak{h}_{\rm fne}$ of $H^{\rm fne}$ expand under any of the elements $\bar{h}_i$.

Then from (1.10) we have

$$Z \subset H^{\rm fne} \subset W^{\rm st}, \tag{8.2}$$

and from (6.4) that for all i,

$$\bar{h}_i H^{\rm fne} \bar{h}_i^{-1} = H^{\rm fne}. \tag{8.3}$$

Theorem 8.1. *Let $\Phi, \bar{h}_i, \mathbf{p}$ and $\bar{\mu}$ be as in Theorem 6.1. Let ν be an ergodic $\bar{\mu}$-stationary measure. Suppose that $\nu(H^{\rm fne}x) = 0$ for every $x \in \mathscr{X}_{d+1}^S$. Then $\nu_b(Zx) = 0$ for β^X-almost every $(b,x) \in B^X$.*

The remainder of this section is devoted to the proof of Theorem 8.1.

8.1. **Prefix ergodic theorem.** We will need a pointwise ergodic theorem specifically geared to the symbolic space. In order to state it we introduce some additional notation and terminology. Let B^* denote the set of finite words in the alphabet $\{1, \ldots, k\}$, and for $a \in B^*$ let $[a] \subset B$ denote the cylinder set of words in B whose initial word is a. A *complete prefix set* is a finite subset $P \subset B^*$ such that

$$\{[a] \colon a \in P\}$$

is a partition of B. Let $\mathrm{len}(a)$ denote the length of the word $a \in B^*$, and for $a \in B^*$ and $b \in B$ we let $ab \in B$ be the infinite word obtained by concatenation.

Theorem 8.2. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of complete prefix sets such that*

$$\min\{\mathrm{len}(a) \colon a \in P_n\} \xrightarrow{n\to\infty} \infty.$$

Then

$$\forall f \in L^\infty(B,\beta), \quad \sum_{a \in P_n} f(ab)\beta([a]) \xrightarrow{n\to\infty} \int_B f \,\mathrm{d}\beta \quad \beta\text{-a.e.}$$

We give the proof of Theorem 8.2 in §8.5. We will use the following useful consequence.

Corollary 8.3. *Let $f \in L^\infty(B,\beta)$. Then*

$$\lim_{n\to\infty} \sum_{\mathrm{len}(a)=n} f(ab)\beta([a]) = \int_B f \,\mathrm{d}\beta(b) \quad \beta\text{-a.e.}$$

8.2. **Consequences of concentration on the centralizer.** The following lemma provides a very useful consequence of the condition that $\nu_b(Zx)$ is positive for a positive measure set of pairs (b,x).

Lemma 8.4 ([BQ13, Prop. 7.8 and Lem. 7.9])**.** *Let ν be an ergodic $\bar{\mu}$-stationary probability measure on $\mathscr{X}_{d+1}^S$, and let ν_b be the system of limit measures as in (3.10). Suppose that*

$$\beta^X\left(\left\{(b,x) \in B^X : \nu_b(Zx) > 0\right\}\right) > 0. \tag{8.4}$$

Then there exists a compact subgroup $Z_0 \subset Z$ such that

(i) Z_0 preserves the measure ν;
(ii) The Z_0-action on $\mathscr{X}_{d+1}^S$ is free;
(iii) for a full measure set of (b,x) in B^X we have $\nu_b(Z_0 x) > 0$;
(iv) ν_b is Z_0-invariant almost surely;
(v) there is a bounded neighborhood W of 0 in $\mathfrak{g}$ such that the Lie algebra $\mathfrak{z}_0$ of Z_0 satisfies that the restriction of the exponential map to $W \cap \mathfrak{z}_0$ is well-defined and a homeomorphism onto its image.

Proof. First we show that the Z-action on $\mathscr{X}_{d+1}^S$ is free, and hence so is the action of any subgroup of Z. Let $x \in \mathscr{X}_{d+1}^S$ and let $Z(x)$ denote its stabilizer in Z. Since the S_{ue}-coordinates of h_i and $\bar{h}_i$ agree, Proposition 3.8 implies that the projection of elements of $Z(x)$ to the S_{ue}-coordinates is trivial, and we have from Proposition 3.7 that $Z(x)$ is a trivial group.

By (8.4), and since ν is ergodic, we have $\nu_b(Zx) > 0$ for β^X-almost every pair (b, x). Let $Z_1 \subset Z$ be the subgroup consisting of elements of Z which preserve ν, and let $E \subset \mathscr{X}_{d+1}^S$ be the set of 'typical points' for the random walk, in the sense of [BQ13, Lemma 3.7]. Then E is Z_1-invariant, $\nu(E) = 1$ and $zE \cap E = \varnothing$ for $z \in Z \smallsetminus Z_1$. By the argument in [BQ13, Proof of Prop. 7.8], we get that $\nu_b(Z_1 x) > 0$ for β^X-almost every (b, x). Since Z_1 commutes with the random walk and preserves ν, we have from (3.10) that Z_1 preserves almost every ν_b.

We now show that Z_1 is compact. For this, note that for a.e. b, the Z_1-invariant measure ν_b is supported on countably many Z_1-orbits of positive measure. For any $x \in \mathscr{X}_{d+1}^S$, let $Z_1(x)$ denote the stabilizer of x in Z_1. As we have seen, $Z_1(x) = \{\mathrm{Id}\}$. On the other hand, the orbit map $Z_1/Z_1(x) \to Z_1 x$ is a Borel isomorphism, and whenever $\nu_b(Z_1 x) > 0$, the finite Z_1-invariant measure $\nu_b|_{Z_1 x}$ induces a finite Z_1-invariant measure on $Z_1/Z_1(x)$. This implies that there are $x \in \mathscr{X}_{d+1}^S$ for which $Z_1(x)$ is a lattice in Z_1. Thus Z_1 admits the trivial group as a lattice, so is compact.

We now let $Z_0 \subset Z_1$ be a subgroup satisfying (v). To see that such a subgroup exists, see e.g. [Rat98, §3]. Since Z_1 is compact, Z_0 is a subgroup of finite index, and we claim that it satisfies the required conclusions. Indeed, properties (i) and (iv) hold for Z_1 and thus hold for Z_0. Moreover, since each Z_1-orbit is a finite union of Z_0-orbits, there is a positive measure subset of (b, x) for which $\nu_b(Z_0 x) > 0$. Since Z_0 commutes with the random walk, the set of (b, x) satisfying this property is invariant for the random walk, and by ergodicity, (iii) holds for Z_0. ☐

8.3. **Notation for the Proof of Theorem 8.1.** In the proof we will argue by contradiction, we now introduce some notation that arises when assuming that (8.4) holds. Let $Z_0 \subset Z$ be a subgroup satisfying the conclusions of Lemma 8.4. Then, since $Z_0 \subset Z$, the random walk acts on the quotient space

$$X' \stackrel{\mathrm{def}}{=} {}_{Z_0}\backslash \mathscr{X}_{d+1}^S.$$

Since Z_0 is compact, the quotient topology on X' is Hausdorff, locally compact, and second countable, and there is a Borel section $\tau : X' \to \mathscr{X}_{d+1}^S$; that is, τ satisfies $\mathrm{Id}_{X'} = \pi_{X'} \circ \tau$, where $\pi_{X'} : \mathscr{X}_{d+1}^S \to X'$ is the projection.

We fix an $\mathrm{Ad}(Z_0)$-invariant norm on each $\mathfrak{g}_\sigma$, and use it to define a metric dist_{G^S} on G^S which is both right-invariant and left Z_0-invariant. For the real place this involves defining a suitable Riemannian metric on $\mathrm{G}_{\mathbb{R}}$, for finite places the construction is explained in [Rüh16, §3], and for a general vector $v = \sum_\sigma v^{(\sigma)}$ with $v^{(\sigma)} \in \mathfrak{g}_\sigma$, the norm on $\mathfrak{g}$ is given by $\|v\| = \max_\sigma \|v^{(\sigma)}\|_\sigma$ where $\|\cdot\|_\sigma$ is the norm on $\mathfrak{g}_\sigma$. The metric dist_{G^S} induces a metric $\mathrm{dist}_{\mathscr{X}_{d+1}^S}$ on $\mathscr{X}_{d+1}^S$, and we use it to define

$$\mathrm{dist}_{X'}\left(x'_1, x'_2\right) = \inf\left\{\mathrm{dist}_{\mathscr{X}_{d+1}^S}(y_1, y_2) : y_i \in Z_0 x_i,\ i = 1, 2\right\}, \quad \text{where } x'_i = Z_0 x_i.$$

Since Z_0 is compact and since $\mathrm{dist}_{\mathscr{X}_{d+1}^S}$ is Z_0-invariant, this can also be written as

$$\mathrm{dist}_{X'}\left(x'_1, x'_2\right) = \min\left\{\mathrm{dist}_{\mathscr{X}_{d+1}^S}(x_1, z_0 x_2) : z_0 \in Z_0\right\}.$$

In the sequel, when discussing balls and distances between points in the spaces $G^S, \mathscr{X}_{d+1}^S, X'$, or norms of vectors in $\mathfrak{g}$, we will always have in mind the metrics arising from this norm on $\mathfrak{g}$. When confusion is unavoidable we will simplify notation by omitting subscripts, writing 'dist' for each of these metrics.

The pushforward ν' of ν to the quotient X' is an ergodic stationary measure with the property that the limit measures ν'_b have atoms for almost every b. By ergodicity of ν' (see [BQ11, Lemma 3.11] for a similar argument), for almost every $b \in B$, the limit measure ν'_b is in fact a purely atomic uniform measure supported by a finite subset of X' depending on b. Moreover, the cardinality of the support of ν'_b is constant almost surely and we denote it by N_0.

Let X_0 denote the collection of subsets $\Sigma \subset X'$ of cardinality N_0. Given two elements $\Sigma_1, \Sigma_2 \in X_0$, we define a metric on X_0 by

$$\mathrm{dist}_{X_0}(\Sigma_1, \Sigma_2) = \max_{x_1 \in \Sigma_1} \min_{x_2 \in \Sigma_2} \mathrm{dist}(x_1, x_2) + \max_{x_2 \in \Sigma_2} \min_{x_1 \in \Sigma_1} \mathrm{dist}(x_1, x_2).$$

The diagonal action of the group elements of the random walk on the product space $(X')^{N_0}$ induces an action on X_0. Similarly, since the group H^{fne} is normalized by Z_0, its left multiplication action on $\mathscr{X}^S_{d+1}$ induces an action on the space X' and hence on the space X_0.

Let $\mathfrak{z}$, $\mathfrak{g}_{\mathrm{dt}}$, $\mathfrak{g}_{\mathrm{ue}}$ and $\mathfrak{h}_{\mathrm{fne}}$ denote respectively the Lie algebras of $Z, \mathrm{G}_{S_{\mathrm{dt}}}, \mathrm{G}_{S_{\mathrm{ue}}}$ and H^{fne}. Let $\mathfrak{u}^-_{\mathrm{dt}} \subset \mathfrak{g}_{\mathrm{dt}}$ be the Lie algebra contracted by the restriction of the $\mathrm{Ad}(\bar{h}_i^{-1})$ to $\mathfrak{g}_{\mathrm{dt}}$, and denote

$$V_{\mathrm{ex}} \stackrel{\mathrm{def}}{=} \mathfrak{u}^-_{\mathrm{dt}} \oplus \mathfrak{g}_{\mathrm{ue}}.$$

The letters 'ex' stand for 'expanding'; indeed, under the linear random walk, all the nonzero vectors in $\mathfrak{u}^-_{\mathrm{dt}}$ expand exponentially, and the vectors in $\bigoplus_{\sigma \in S_{\mathrm{ue}}} \mathfrak{g}_\sigma$ expand for most infinite random walk paths by the results of §7. As suggested by this terminology, V_{ex} is a complementary subspace to $\mathfrak{h}_{\mathrm{fne}}$.

Let $\mathfrak{z}_0$ be the Lie algebra of Z_0, let $\mathfrak{w}$ be an $\mathrm{Ad}(Z_0)$-invariant complementary subspace to $\mathfrak{z}_0$ inside $\mathfrak{z}$, and define $\mathfrak{z}_0^\perp \subset \mathfrak{g}$ by

$$\mathfrak{z}_0^\perp \stackrel{\mathrm{def}}{=} \mathfrak{w} \oplus \mathfrak{u}_{\mathrm{dt}} \oplus V_{\mathrm{ex}}.$$

Then $\mathfrak{z}_0^\perp$ is a complementary subspace to $\mathfrak{z}_0$ in $\mathfrak{g}$, and the subspaces $\mathfrak{h}_{\mathrm{fne}}, \mathfrak{z}_0^\perp, \mathfrak{w}, \mathfrak{u}_{\mathrm{dt}}, \mathfrak{u}^-_{\mathrm{dt}}, V_{\mathrm{ex}}$ and $\mathfrak{z}_0$ are all invariant under the linear random walk.

Every vector $v_{\mathrm{dt}} \in \mathfrak{g}_{\mathrm{dt}}$ can be written uniquely as $v_{\mathrm{dt}} = v_{\mathrm{dt,fne}} + v_{\mathrm{dt,ex}}$, where $v_{\mathrm{dt,fne}} \in \mathfrak{g}_{\mathrm{dt}} \cap \mathfrak{h}_{\mathrm{fne}}$ and $v_{\mathrm{dt,ex}} \in \mathfrak{g}_{\mathrm{dt}} \cap V_{\mathrm{ex}}$. Given a vector $v \in \mathfrak{g}$, we define $v_{\mathrm{ex}} = \left(v_{\mathrm{ex}}^{(\sigma)}\right)_{\sigma \in S} \in \mathfrak{g}$ by

$$v_{\mathrm{ex}}^{(\sigma)} \stackrel{\mathrm{def}}{=} \begin{cases} v_{\mathrm{dt,ex}} & \text{if } \sigma \in S_{\mathrm{dt}}, \\ v^{(\sigma)} & \text{otherwise,} \end{cases}$$

and define

$$v_{\mathrm{fne}} = v - v_{\mathrm{ex}}.$$

From the definition of the norm, we have that for any $v \in \mathfrak{h}_{\mathrm{fne}}$ and any i, $\left\|\mathrm{Ad}\left(\bar{h}_i\right) v\right\| \leq \|v\|$. Also, since V_{ex} is invariant under $\mathrm{Ad}(Z_0)$, after adjusting the inner product defining $\|\cdot\|$, we may also assume that for any $v \in \mathfrak{g}$ we have $\|v\| \leq \|v_{\mathrm{ex}}\| + \|v_{\mathrm{fne}}\|$.

8.4. **Proof of Theorem 8.1.** We first give an overview of the proof. We will assume (8.4) and obtain a contradiction. Let X', N_0 and X_0 be as in §8.3. Define

$$\kappa \colon B \to X_0, \qquad \kappa(b) \stackrel{\mathrm{def}}{=} \operatorname{supp} \nu'_b \tag{8.5}$$

(more precisely, the right hand side of (8.5) is a well-defined measurable map on a subset of B of full measure, but we will ignore nullsets and continue to denote this subset by B). Let $\Delta \subset X_0^2$ denote the diagonal. Since $\nu(H^{\mathrm{fne}} x) = 0$ for every x and by (8.2), ν' is not a Dirac mass on X_0, but the random walk pushes it toward the Dirac mass ν'_b supported on $\kappa(b)$. In particular, for a.e. b, off-diagonal points in X_0^2 get pushed toward Δ by b_n^1.

We will show that on a certain neighborhood $\mathcal{U}$ of a compact subset of Δ, the action of the random walk on X_0^2 is essentially given by $\mathrm{Ad} \oplus \mathrm{Ad}$ on $\mathfrak{g}^2$. This step is made somewhat complicated by the fact that we have to take a quotient by the action of the compact group Z_0. The reader may wish to first consider the simpler case in which Z_0 is trivial.

Given this relation between the random walk on X_0^2, and the adjoint action, we recall from §7 that vectors tend to grow under most elements of the random walk. Using this, we will show that for many random walk paths of controlled length, points in $\mathcal{U} \smallsetminus \Delta$ get pushed away from Δ, which contradicts convergence to the diagonal. A complication in the argument is that the linear coordinates on $\mathcal{U}$ need to make sense on a large enough set of words so that expansion can be exploited. The prefix ergodic theorem will be useful for dealing with this issue.

We proceed to the details. In order to make the logic more transparent, we will break up the argument into steps.

Step 1. Setting up constants, defining the neighborhood $\mathcal{U}$, and formulating the goal. Let $T : B \to B$ be the left-shift. By Proposition 3.6, and since B^* is countable, for β-a.e. $b \in B$ we have

$$\kappa(Tb) = b_1^{-1} \cdot \kappa(b), \quad \text{and } \kappa(ab) = a_n^1 \cdot \kappa(b) \text{ for any } a \in B^*, \quad \text{where } n = \operatorname{len}(a). \tag{8.6}$$

Let $\varepsilon \in \left(0, \frac{1}{6}\right)$. By Lusin's theorem, we can find a compact set $K_1 \subset B$ such that $\kappa|_{K_1}$ is continuous, satisfies properties (8.6), and such that $\beta(K_1) > 1 - \varepsilon$. Let $f = \mathbf{1}_{K_1}$. Given $n \in \mathbb{N}$, define the function $f_n : B \to [0, 1]$ by

$$f_n(b) = \sum_{a \in (\operatorname{supp} \bar{\mu})^n} f(ab) \beta\big([a]\big).$$

By Corollary 8.3, there exists $n_0 \in \mathbb{N}$ such that the set

$$\beta\big(E(n_0)\big) > 1 - \frac{\varepsilon}{2}, \quad \text{where} \quad E(n_0) = \{b \in B : \forall n \geq n_0, \quad f_n(b) > 1 - 2\varepsilon\}.$$

Hence there exists a compact set

$$K_2 \subset E(n_0) \quad \text{such that} \quad \beta(K_2) > 1 - \varepsilon.$$

We define

$$K_3 \stackrel{\text{def}}{=} \kappa(K_1),$$

a compact subset of X_0. Recall that the *injectivity radius at* $x \in \mathscr{X}_{d+1}^S$ is the maximal r such that the restriction of the map $G^S \to \mathscr{X}_{d+1}^S$, $g \mapsto gx$ to the open ball around the identity of radius r, is injective. Given $\Sigma \in X_0$, i.e., a collection of N_0 orbits for the group Z_0, we use the same letter Σ to denote the subset of $\mathscr{X}_{d+1}^S$ comprised by these orbits, and denote by r_Σ and d_Σ respectively the minimal injectivity radius of a point in Σ, and the minimum of the pairwise distances between elements of Σ. Both of the numbers r_Σ, d_Σ depend continuously on Σ. Hence the numbers

$$r(K_3) \stackrel{\text{def}}{=} \inf\{r_\Sigma : \Sigma \in K_3\} \quad \text{and} \quad d(K_3) \stackrel{\text{def}}{=} \inf\{d_\Sigma : \Sigma \in K_3\} \tag{8.7}$$

are both positive. Choose $\iota > 0$ small enough so that

$$\iota < \min\{r(K_3), d(K_3), 1\},$$

and so that there is a neighborhood W of 0 in $\mathfrak{g}$ such that

$$\exp|_W : W \to B(\mathrm{Id}, \iota) \subset G^S \tag{8.8}$$

is well-defined and is a homeomorphism. Let $C_W > 1$ be a bi-Lipschitz constant for $\exp|_W$, that is,

$$\forall w_1, w_2 \in W, \qquad \frac{\operatorname{dist}(\exp(w_1, w_2))}{C_W} \leq \|w_1 - w_2\| \leq C_W \operatorname{dist}(\exp(w_1, w_2));$$

the fact that exp is locally bi-Lipschitz follows from the construction of the metric dist. By item (v) of Lemma 8.4, by making W smaller we can also assume that the map $\exp|_{W \cap \mathfrak{z}_0}$ is a homeomorphism onto its image, which is open in Z_0. By making W and ι even smaller we can find an open subset $W' \subset W$ containing 0, and a constant $C_{W'} > 1$, such that the two maps $(W' \cap \mathfrak{z}_0) \times (W' \cap \mathfrak{z}_0^\perp) \to G^S$,

$$(z_0, z_0^\perp) \mapsto \exp(z_0) \exp(z_0^\perp), \quad \text{and } (z_0, z_0^\perp) \mapsto \exp(z_0^\perp) \exp(z_0) \tag{8.9}$$

are both bi-Lipschitz homeomorphisms onto their image, and this image contains $B(\mathrm{Id},\iota)$ and is contained in $\exp(\mathfrak{z}_0\cap W)\exp(\mathfrak{z}_0^\perp\cap W')\cap\exp(\mathfrak{z}_0^\perp\cap W)\exp(\mathfrak{z}_0\cap W')$. Finally, by making W even smaller, and using the fact that the Z_0-action on $\mathscr{X}^S_{d+1}$ is free, we can assume that if $x_1\in\exp(W)K_3$ and $x_2=z_0x_1$ for $z_0\in Z_0$, then

$$\frac{1}{C_W}\operatorname{dist}_{\mathscr{X}^S_{d+1}}(x_1,x_2)\le\operatorname{dist}_{G^S}(\mathrm{Id},z_0)\le C_W\operatorname{dist}_{\mathscr{X}^S_{d+1}}(x_1,x_2).$$

Let

$$\delta\overset{\mathrm{def}}{=}\frac{\varepsilon}{N_0\,|S|},\tag{8.10}$$

and let $m_0\in\mathbb{N}$ and $C>1$ be constants (depending on δ) for which the conclusions of Proposition 7.1 and Lemma 7.2 hold.

For $\sigma\in S$ we let $\lambda_\sigma\overset{\mathrm{def}}{=}|\log|\varrho|_\sigma|$, and set $\lambda_{\max}\overset{\mathrm{def}}{=}\max\{\lambda_\sigma:\sigma\in S\}$. We choose L_1 satisfying

$$L_1>\frac{C\,C_W\,|S|}{\iota}.\tag{8.11}$$

Now we choose L_2 large enough so that for all $\sigma\in S$,

$$(m_0+n_0)\log(L_2)-\frac{1}{\lambda_\sigma}\log(C_W\,L_1)-1\ge m_0+n_0,\tag{8.12}$$

and so that if we define

$$r\overset{\mathrm{def}}{=}\iota\cdot L_2^{-(m_0+n_0)\lambda_{\max}},\tag{8.13}$$

then

$$r<\frac{1}{C^2\,C_W\,e^{\lambda_{\max}}\,L_1}.\tag{8.14}$$

We define

$$\Delta(K_3)\overset{\mathrm{def}}{=}\{(\Sigma,\Sigma)\colon\Sigma\in K_3\}.$$

Finally we define $\mathcal{U}$ to be the r-neighborhood of $\Delta(K_3)$. Using uniform continuity of $\kappa|_{K_1}$, let $n_1\in\mathbb{N}$ be such that for all sequences $b,b'\in K_1$ which agree on a prefix of length at least n_1, we have $\big(\kappa(b),\kappa(b')\big)\in\mathcal{U}$.

Our goal is to find finite words $\tilde a,a\in B^*$ and $b,\bar b\in B$, such that the following hold:

(I) $\tilde a ab\in K_1$ and $\tilde a a\bar b\in K_1$.
(II) $\operatorname{len}(a)\ge n_1$.
(III) The pair $(\kappa(\tilde a ab),\kappa(\tilde a a\bar b))$ is outside $\mathcal{U}$.

To see that this gives a contradiction, note that items (I) and (II) and the definition of n_1 imply that $(\kappa(\tilde a ab),\kappa(\tilde a a\bar b))\in\mathcal{U}$. This contradicts item (III).

Step 2. Linearizing the action of the random walk near the diagonal. By definition of ι, we have that for any pair $\mathbf{\Sigma}=(\Sigma_1,\Sigma_2)$ for which there is $\Sigma\in K_3$ with $\operatorname{dist}(\Sigma_i,\Sigma)<\iota/2$ for $i=1,2$, for any $x_1\in\Sigma_1$ there is exactly one element $x_2\in\Sigma_2$ such that $\operatorname{dist}(x_1,x_2)<r$. In particular this holds for $\mathbf{\Sigma}\in\mathcal{U}$. We denote this element x_2 by $\varphi_{\mathbf{\Sigma}}(x_1)$, so that $\varphi_{\mathbf{\Sigma}}\colon\Sigma_1\to\Sigma_2$ is a bijection. It is easy to see that the map $\varphi_{\mathbf{\Sigma}}$ depends continuously on $\mathbf{\Sigma}$.

We would like to estimate the displacement $\operatorname{dist}(x',\varphi_{\mathbf{\Sigma}}(x'))$ in terms of the adjoint action on $\mathfrak{g}$. To this end, note that if $x'\in\Sigma_1$ then we can write $x'=Z_0x$ for some $x\in\mathscr{X}^S_{d+1}$, and by the choice of W, there is a unique $\tilde v=\tilde v(x)\in\mathfrak{z}_0^\perp\cap W$ such that $\varphi_{\mathbf{\Sigma}}(x')=Z_0\exp(\tilde v)x$. As suggested by the notation, this choice of $\tilde v$ depends on the choice of $x\in\pi_{X'}^{-1}(x')$. However, we have that if $x'=Z_0x_1=Z_0x_2$ then $\|\tilde v(x_1)\|=\|\tilde v(x_2)\|$; indeed, if $x_2=z_0x_1$ for some $z_0\in Z_0$ then by the fact that $\mathfrak{z}_0^\perp$ is $\operatorname{Ad}(Z_0)$-invariant we see that $\tilde v(x_2)=\operatorname{Ad}(z_0)\tilde v(x_1)$, and $\|\tilde v(x_1)\|=\|\tilde v(x_2)\|$ since the norm is $\operatorname{Ad}(Z_0)$-invariant. Using the section $\tau:X'\to\mathscr{X}^S_{d+1}$, we define

$$v_{\mathbf{\Sigma}}\colon\Sigma_1\to\mathfrak{z}_0^\perp,\qquad v_{\mathbf{\Sigma}}(x')\overset{\mathrm{def}}{=}\tilde v(\tau(x')),\tag{8.15}$$

and we have that $\|v_{\mathbf{\Sigma}}(x')\|$ does not depend on the choice of the section.

Note further that

$$\forall \mathbf{\Sigma} \in \mathcal{U}, \quad \forall x' \in \Sigma_1, \quad \|v_{\mathbf{\Sigma}}(x')\| \le C_W\, r. \tag{8.16}$$

Using the fact that $\mathfrak{z}_0^{\perp}$ is invariant under the adjoint action of the random walk, the reader can now verify the following statement:

Suppose $n \in \mathbb{N}$, $\bar{h} = \bar{h}_{i_1} \circ \cdots \circ \bar{h}_{i_n}$ and $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2) \in \mathcal{U}$ satisfy that $\bar{h}(\Sigma_1) \in K_3$ and

$$\|\mathrm{Ad}(\bar{h}) v_{\mathbf{\Sigma}}(x')\| < \frac{\iota}{C_W} \qquad \text{for all } x' \in \Sigma_1. \tag{8.17}$$

Then

$$\frac{\mathrm{dist}\left(\bar{h}(\Sigma_1), \bar{h}(\Sigma_2)\right)}{C_W} \le \max_{x' \in \Sigma_1} \|\mathrm{Ad}(\bar{h}) v_{\mathbf{\Sigma}}(x')\| \le C_W\, \mathrm{dist}\left(\bar{h}(\Sigma_1), \bar{h}(\Sigma_2)\right). \tag{8.18}$$

Step 3. Choosing $a, b, \bar{b}$. By (8.2) we have that $Z_0 \subset H^{\mathrm{fne}}$ and thus the partition of X into H^{fne}-orbits induces a well-defined partition of X', which we will continue to refer to as H^{fne}-orbits and denote by $H^{\mathrm{fne}}x'$ (although H^{fne} might not act on X'). Our assumption is that these H^{fne}-orbits are of zero measure with respect to ν'. On the other hand, the supports $\kappa(b)$ of the limit measures ν'_b are finite sets of Z_0-orbits, and again by (8.2), if $\kappa(b)$ intersects an H^{fne}-orbit, the measure ν'_b assigns this orbit positive measure. This implies via Proposition 3.6 that for any fixed $b \in B$, for any $x' \in \kappa(b)$ and for β-a.e. $\bar{b}$ we have

$$\kappa(\bar{b}) \cap H^{\mathrm{fne}}x' = \varnothing. \tag{8.19}$$

Hence we can find $b, \bar{b} \in B$ such that for all $x' \in \kappa(b)$, we have (8.19), and the points $b, \bar{b}$ are generic for $\mathbf{1}_{K_1 \cap K_2}$ in the sense of the prefix ergodic theorem. The latter condition means that for both $c = b$ and $c = \bar{b}$, we have

$$\lim_{n\to\infty} \sum_{a \in (\mathrm{supp}\,\bar{\mu})^n} \mathbf{1}_{K_1 \cap K_2}(ac)\beta([a]) = \beta(K_1 \cap K_2). \tag{8.20}$$

By (8.20), since K_1 and K_2 both have measure at least $1 - \varepsilon$, there is $k \ge n_1$ such that

$$\sum_{a \in (\mathrm{supp}\,\bar{\mu})^k} \mathbf{1}_{K_1 \cap K_2}(ac)\beta([a]) > 1 - 3\varepsilon \quad (c = b, \bar{b}). \tag{8.21}$$

Since $1 - 3\varepsilon > 1/2$, we can find $a \in (\mathrm{supp}\,\bar{\mu})^k$ such that

$$ab, a\bar{b} \in K_1 \cap K_2. \tag{8.22}$$

By (8.3) and (8.19) we still have

$$\text{for all } x' \in \kappa(ab), \qquad \kappa(a\bar{b}) \cap H^{\mathrm{fne}}x' = \varnothing. \tag{8.23}$$

The choice of k ensures that the word a satisfies (II).

Step 4. Choosing the word $\tilde{a}$. We now choose $\tilde{a}$. Let

$$\mathbf{\Sigma} = (\Sigma_1, \Sigma_2) \quad \text{where} \quad \Sigma_1 \stackrel{\mathrm{def}}{=} \kappa(ab), \quad \Sigma_2 \stackrel{\mathrm{def}}{=} \kappa(a\bar{b}). \tag{8.24}$$

By (8.22) and the definition of n_1, we have that $\mathbf{\Sigma} \in \mathcal{U}$. It follows from (8.2) and (8.23) that for any $x' \in \Sigma_1$, $v_{\mathbf{\Sigma}}(x') \notin \mathfrak{h}_{\mathrm{fne}}$. In the notation introduced in §8.3 above, this means that $v_{\mathbf{\Sigma}}(x')_{\mathrm{ex}} \neq 0$. Given $\sigma \in S$, and $x' \in \Sigma_1$, let

$$\alpha_\sigma(x') \stackrel{\mathrm{def}}{=} \left\|v_{\mathbf{\Sigma}}(x')_{\mathrm{ex}}\right\|_\sigma \qquad \text{and} \qquad S_{\mathrm{good}}(x') \stackrel{\mathrm{def}}{=} \{\sigma \in S \colon \alpha_\sigma(x') > 0\}. \tag{8.25}$$

Now set

$$n(x') \stackrel{\mathrm{def}}{=} \min_{\sigma \in S_{\mathrm{good}}(x')} \left\lfloor \frac{1}{\lambda_\sigma} \left(\log\left(\frac{1}{\alpha_\sigma(x')}\right) - \log(L_1) \right) \right\rfloor.$$

This choice implies that for all $\sigma \in S_{\mathrm{good}}(x')$, we have

$$e^{\lambda_\sigma n(x')}\alpha_\sigma(x') \le \frac{1}{L_1}, \tag{8.26}$$

and there is $\sigma \in S_{\mathrm{good}}(x')$ (the one for which the minimum is attained) satisfying

$$\frac{1}{\lambda_\sigma}\log\left(\frac{1}{\alpha_\sigma(x')}\right) \le n(x') + 1 + \frac{1}{\lambda_\sigma}\log(L_1) \quad \text{and} \quad e^{\lambda_\sigma n(x')}\alpha_\sigma(x') \ge \frac{1}{e^{\lambda_\sigma}L_1}. \tag{8.27}$$

Our choices ensure

$$n(x') \ge m_0 + n_0. \tag{8.28}$$

Indeed, since $\mathbf{\Sigma} \in \mathcal{U}$, for all σ we have

$$\frac{1}{\lambda_\sigma}\left(\log\left(\frac{1}{\alpha_\sigma(x')}\right) + \log(C_W)\right) \overset{(8.16)\&(8.25)}{\ge} -\frac{1}{\lambda_\sigma}\log r \overset{(8.13)}{\ge} (m_0+n_0)\log L_2.$$

Now (8.28) follows by (8.12) and (8.27).

Let

$$n \overset{\mathrm{def}}{=} \min_{x'\in\Sigma_1} n(x'). \tag{8.29}$$

With this choice of n, let

$$P_n \overset{\mathrm{def}}{=} \{\tilde{a} \in (\operatorname{supp}\bar{\mu})^n\colon \text{ for } c = b, \bar{b}, \quad \tilde{a}ac \in K_1\}, \qquad \mathcal{P}_n \overset{\mathrm{def}}{=} \bigcup_{\tilde{a}\in P_n}[\tilde{a}].$$

Since $ab, a\bar{b} \in K_2$ by (8.22), and $n \ge n_0$ by (8.28) and (8.29), the definition of n_0 ensures

$$\beta(\mathcal{P}_n) \ge 1 - 2\varepsilon. \tag{8.30}$$

Next define

$$\Xi_n(x') \overset{\mathrm{def}}{=} \left\{(\tilde{a}_1,\dots,\tilde{a}_n) \in (\operatorname{supp}\bar{\mu})^n : \forall \sigma \in S_{\mathrm{good}}(x'),\ \|\tilde{a}_n^1\|_\sigma \|v_{\mathbf{\Sigma}}(x')_{\mathrm{ex}}\|_\sigma \le C\|\tilde{a}_n^1 \cdot v_{\mathbf{\Sigma}}(x')_{\mathrm{ex}}\|_\sigma\right\}.$$

Note that the Bernoulli measure has a symmetry property $\beta([a_1^n]) = \beta([a_n^1])$. Thus using Lemma 7.2, and since $n \ge m_0$, we have

$$\beta(\Xi_n(x')) \ge 1 - \delta \quad \text{for each } x' \in \Sigma_1.$$

The choice (8.10) now implies that $\beta\big(\bigcap_{x'\in\Sigma_1}\Xi_n(x')\big) > 1-\varepsilon$. Combining with (8.30) we have

$$\beta\left(P_n \cap \bigcap_{x'\in\Sigma_1}\Xi_n(x')\right) \ge 1 - 3\varepsilon > 0.$$

In particular, the above intersection is non-empty. Fix a word $\tilde{a}$ in this intersection. The definition of P_n now ensures that property (I) holds.

Step 5. Verifying property (III). We need to show that for $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2)$ as in (8.24) we have $\operatorname{dist}\big(\tilde{a}_n^1\Sigma_1, \tilde{a}_n^1\Sigma_2\big) \ge r$. For this it suffices to check condition (8.17), and show that for some $x' \in \Sigma_1$,

$$\left\|\operatorname{Ad}\left(\tilde{a}_n^1\right) v_{\mathbf{\Sigma}}(x')\right\| \ge C_W\, r. \tag{8.31}$$

For (8.17), we note that for any $x' \in \Sigma_1$, the non-expanding coordinates of $v_{\mathbf{\Sigma}}(x')$ are of small norm since $\mathbf{\Sigma} \in \mathcal{U}$. For $\sigma \in S_{\mathrm{good}}(x')$ we have

$$\left\|\operatorname{Ad}(\bar{h})\right\|_{\mathrm{op}}\alpha_\sigma(x') \overset{(7.2)}{\le} C\,e^{\lambda_\sigma n}\alpha_\sigma(x') \overset{(8.26)\&(8.29)}{\le} \frac{C}{L_1} \overset{(8.11)}{\le} \frac{\iota}{C_W}.$$

Let $x' \in \Sigma_1$ be the element for which the minimum in (8.29) is attained, and let $\sigma \in S_{\mathrm{good}}(x')$ be the place for which (8.27) holds. Since $\tilde{a} \in \Xi_n(x')$, we get that

$$\begin{aligned} \left\|\mathrm{Ad}(\tilde{a}_n^1)v_{\mathbf{\Sigma}}(x')\right\| &\geq \left\|\mathrm{Ad}(\tilde{a}_n^1)v_{\mathbf{\Sigma}}(x')_\sigma\right\| \geq \frac{1}{C}\left\|\mathrm{Ad}(\tilde{a}_1^n)\right\|_\sigma \left\|v_{\mathbf{\Sigma}}(x_1)_\sigma\right\|_\sigma \\ &\overset{(7.2)}{\geq} \frac{1}{C^2}\, e^{\lambda_\sigma n}\alpha_\sigma(x') \overset{(8.27)}{\geq} \frac{1}{e^{\lambda_{\max}}C^2 L_1} \overset{(8.14)}{\geq} C_W r. \end{aligned}$$

This proves (8.31) and completes the proof. ◻

8.5. **Proof of the prefix ergodic theorem.** In this subsection we prove Theorem 8.2. For a bounded function f on B and a $A \subset B$, we define the variation of f on A by

$$\mathrm{Var}(f, A) \overset{\mathrm{def}}{=} \sup_{x\in A} f(x) - \inf_{x \in A} f(x).$$

Clearly for any measurable function f, any set A with $\beta(A) > 0$, and any $x_0 \in A$, we have

$$\left| f(x_0) - \frac{1}{\beta(A)}\int_A f \mathrm{d}\beta \right| \leq \mathrm{Var}(f, A).$$

Equip B with its standard ultra-metric, and suppose first that f is continuous. Then, since B is compact, f is uniformly continuous. Using the condition that the minimal length of a word $a \in P_n$ goes to infinity, which implies that the diameter of the corresponding cylinder set $[a]$ goes to zero, we see that there is n_0 so that for all $n > n_0$ and $a \in P_n$,

$$\mathrm{Var}(f, [a]) < \varepsilon.$$

Since the sets $\{[a] : a \in P_n\}$ are a partition of B, we have for all $n > n_0$ and any b that

$$\begin{aligned} \left| \sum_{a\in P_n} \beta([a]) f(ab) - \int f\mathrm{d}\beta \right| &= \left| \sum_{a\in P_n} \beta([a]) f(ab) - \sum_{a \in P_n} \int_{[a]} f \mathrm{d}\beta \right| \\ \leq \sum_{a\in P_n} \beta([a]) \left| f(ab) - \frac{1}{\beta([a])}\int_{[a]} f\mathrm{d}\beta \right| &< \sum_{a\in P_n} \beta([a])\, \varepsilon = \varepsilon. \end{aligned}$$

The general case, in which $f \in L^\infty(B, \beta)$, will now be proved by an approximation argument. By replacing f with a bounded function agreeing with it almost everywhere, subtracting a constant and rescaling, we may assume that

$$\int_B f \mathrm{d}\beta = 0 \quad \text{and} \quad \|f\|_\infty = 1.$$

Let $\varepsilon > 0$ and, using Lusin's theorem, let $K = K_\varepsilon$ be a compact set such that $f|_K$ is continuous and $\beta(K) > 1 - \varepsilon$. Let n_0 so that for all $n > n_0$ and all $a \in P_n$ we have

$$\mathrm{Var}(f, [a] \cap K) < \varepsilon.$$

For fixed $n > n_0$, and an element $a \in P_n$, the corresponding cylinder set $[a]$ is called a *continuity atom* or *CA* if $\frac{\beta([a]\cap K)}{\beta([a])} \geq 1 - \varepsilon^{1/2}$. If a is a continuity atom then for any $x_0 \in [a] \cap K$,

$$\begin{aligned}
&\left|\frac{1}{\beta([a])}\int_{[a]} f\,\mathrm{d}\beta - f(x_0)\right| \leq \frac{1}{\beta([a])}\left|\int_{[a]} f\,\mathrm{d}\beta - \int_{[a]\cap K} f\,\mathrm{d}\beta\right| \\
&+\left|\frac{1}{\beta([a])}\int_{[a]\cap K} f\,\mathrm{d}\beta - \frac{1}{\beta([a]\cap K)}\int_{[a]\cap K} f\,\mathrm{d}\beta\right| + \left|\frac{1}{\beta([a]\cap K)}\int_{[a]\cap K} f\,\mathrm{d}\beta - f(x_0)\right| \\
\leq&\frac{1}{\beta([a])}\int_{[a]\smallsetminus K}|f|\,\mathrm{d}\beta + \frac{\beta([a]\smallsetminus K)}{\beta([a])\cdot\beta([a]\cap K)}\int_{[a]\cap K}|f|\,\mathrm{d}\beta + \mathrm{Var}(f,[a]\cap K) \\
\leq&\frac{\beta([a]\smallsetminus K)}{\beta([a])} + \frac{\beta([a]\smallsetminus K)}{\beta([a])} + \varepsilon < 2\varepsilon^{1/2} + \varepsilon < 3\varepsilon^{1/2}.
\end{aligned} \tag{8.32}$$

We claim that

$$\beta\left(\bigcup \text{continuity atoms}\right) \geq 1 - \varepsilon^{1/2}. \tag{8.33}$$

Indeed, otherwise, denoting $K^c = B \smallsetminus K$, we have

$$\beta(K^c) \geq \beta\left(\bigcup_{[a]\text{ not a CA}} K^c \cap [a]\right) \geq \sum_{a \text{ not a CA}} \beta([a])\varepsilon^{1/2} \geq \varepsilon^{1/2}\varepsilon^{1/2} = \varepsilon,$$

and we get a contradiction to $\beta(K^c) < \varepsilon$.

We now define

$$\mathrm{Bad} = \left\{b \in B : \beta\left(\bigcup\{[a] : ab \notin K\}\right) > \varepsilon^{1/2}\right\},$$

and claim that

$$\beta(\mathrm{Bad}) \leq \varepsilon^{1/2}. \tag{8.34}$$

Indeed, denote by $\mathbf{1} = \mathbf{1}_{K^c}$ the indicator of K^c. Recall that for any prefix $[a]$, the measure $\beta|_{[a]}$ is the same as the pushforward of β by the map $b \mapsto ab$, multiplied by the scalar $\beta([a])$; this is easily verified for cylinder sets contained in $[a]$ and thus is true for all measurable subsets of $[a]$. Thus, if (8.34) is not true then

$$\begin{aligned}
\beta(K^c) &= \sum_{a\in P_n}\int_{[a]}\mathbf{1}\,\mathrm{d}\beta = \sum_{a\in P_n}\beta([a])\int_B \mathbf{1}(ab)\,\mathrm{d}\beta(b) \\
&= \int_B \sum_{a\in P_n}\beta([a])\mathbf{1}(ab)\,\mathrm{d}\beta(b) \geq \int_{\mathrm{Bad}}\sum_{a\in P_n}\beta([a])\mathbf{1}(ab)\,\mathrm{d}\beta(b) \\
&= \int_{\mathrm{Bad}}\beta\left(\bigcup\{[a] : ab\notin K\}\right)\mathrm{d}\beta(b) \geq \beta(\mathrm{Bad})\,\varepsilon^{1/2} > \varepsilon^{1/2}\cdot\varepsilon^{1/2} = \varepsilon,
\end{aligned}$$

giving a contradiction to $\beta(K^c) < \varepsilon$.

Now for $b \notin \mathrm{Bad}$ we have

$$\begin{aligned}
&\left|\sum_{a\in P_n} f(ab)\beta([a]) - \int_B f\,\mathrm{d}\beta\right| = \left|\sum_{a\in P_n} f(ab)\beta([a]) - \sum_{a\in P_n}\int_{[a]} f\,\mathrm{d}\beta\right| \\
\leq& \sum_{\substack{[a]\text{ is a CA}\\ ab\in K}}\beta([a])\left|f(ab) - \frac{1}{\beta([a])}\int_{[a]} f\,\mathrm{d}\beta\right| + \sum_{\substack{[a]\text{ not a CA}\\ \text{or } ab\notin K}}\left(\beta([a])\,|f(ab)| + \int_{[a]}|f|\,\mathrm{d}\beta\right) \\
\leq& 3\varepsilon^{1/2} + 2\beta\left(\bigcup\{[a] : [a]\text{ is not a CA}\}\right) + \beta\left(\bigcup\{[a] : ab\notin K\}\right) < 7\varepsilon^{1/2},
\end{aligned} \tag{8.35}$$

where in the last line we used (8.32), the definition of Bad, and (8.33). Since ε was arbitrary, combining (8.34) and (8.35) we get the desired conclusion.

9. Preparations for Exponential Drift: Non-alignment of Limit Measures

The goal of this section is to establish the following result regarding non-alignment of the limit measures ν_b along orbits of the group W^{st}. This will serve as crucial input for the exponential drift argument as described in § 2. We retain the notation introduced in §8.

Theorem 9.1. *Let $\Phi, \bar{h}_i, \mathbf{p}, \bar{\mu}$ and ν be as in Theorem 6.1. Suppose that*

$$\nu\left(\left\{x \in \mathscr{X}_{d+1}^S : \mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}\right\}\right) = 0.$$

Then, for β^X-almost every pair $(b, x) \in B^X$, we have

$$\nu_b(W^{\mathrm{st}} x) = 0.$$

The proof of Theorem 9.1 occupies the rest of this section. The notations and conditions of Theorem 6.1 will be assumed throughout this section.

A key ingredient in the proof of Theorem 9.1 is Theorem 8.1. The following Lemma shows that the hypotheses of Theorem 9.1 imply the hypotheses of Theorem 8.1.

Lemma 9.2. *Let ν be a $\bar{\mu}$-stationary measure such that $\nu(H^{\mathrm{fne}} x) > 0$ for some x. Then* $\mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}$.

Proof. We first claim that we can find two distinct words g_1, g_2 in the semigroup generated by $\left\{\bar{h}_i^{-1} : i = 1, \ldots, k\right\}$, of the same length, such that the orbits $H^{\mathrm{fne}} g_1 x, H^{\mathrm{fne}} g_2 x$ are the same. To see this, note that in order to show that two orbits are the same, it is enough to show that these orbits intersect nontrivially. Moreover, it follows from (6.4) and (8.1) that the elements $\bar{h}_i$ normalize H^{fne}, and hence the same is true for the group generated by the $\bar{h}_i$. Hence for every word g in the $\bar{h}_i^{-1}$ we have $g H^{\mathrm{fne}} x = H^{\mathrm{fne}} g x$. Let $\mu^{\otimes \ell}$ denote the ℓ-th convolution power of $\bar{\mu}$; this measure is supported on finitely many products of ℓ elements in $\mathrm{supp}(\bar{\mu})$. Suppose by contradiction that for all ℓ, the collection $\left\{g^{-1} H^{\mathrm{fne}} x : g \in \mathrm{supp}\left(\bar{\mu}^{\otimes \ell}\right)\right\}$ is disjoint. Consider the collection of numbers

$$\left\{\nu\left(g^{-1} H^{\mathrm{fne}} x\right) : g \in \mathrm{supp}\left(\bar{\mu}^{\otimes \ell}\right)\right\} = \left\{g_* \nu\left(H^{\mathrm{fne}} x\right) : g \in \mathrm{supp}\left(\bar{\mu}^{\otimes \ell}\right)\right\} \subset [0, 1].$$

The number of elements in this multi-set is k^ℓ. Since ν is stationary, the fixed number $\nu(H^{\mathrm{fne}} x)$ is an average of the numbers in this set. This implies that for large enough ℓ, there are more than $2(\nu(H^{\mathrm{fne}} x))^{-1}$ numbers in this set, each larger than $\nu(H^{\mathrm{fne}} x)/2$. This contradicts the fact that ν is a probability measure, and proves the claim.

The claim implies that there is $h_1 \in H^{\mathrm{fne}}$ such that $h_1 g_1 x = g_2 x$. Hence, $h \stackrel{\text{def}}{=} g_1^{-1} h_1^{-1} g_1 \in H^{\mathrm{fne}}$ satisfies that $h g_1^{-1} g_2 \in \mathrm{Stab}(x)$. We write $g^{(\mathrm{ue})}$ for the projection of $g \in G^S$ to G_{ue}. Since g_1 and g_2 are of the same length, it follows from (6.4) that

$$g_1^{-1} g_2 = (g_1^{-1} g_2)^{(\mathrm{ue})} \in \mathrm{U}_{\mathrm{ue}},$$

and this implies via (1.10) that $g_1^{-1} g_2 \in W^{\mathrm{st}}$. Since we also have $H^{\mathrm{fne}} \subset W^{\mathrm{st}}$, we have found that $h g_1^{-1} g_2$ belongs to W^{st}. By (8.1) we have that $h^{(\mathrm{ue})} = \mathrm{Id}$. Since $(g_1^{-1} g_2)^{(\mathrm{ue})} = g_1^{-1} g_2 \neq \mathrm{Id}$, we see that $h g_1^{-1} g_2 \neq \mathrm{Id}$. □

In the notation of §8.3, let $\mathfrak{u}_{\mathrm{dt}}^-$ be the subspace of vectors in $\mathfrak{g}_{\mathrm{dt}}$ which are contracted by the action of $\mathrm{Ad}\left(\bar{h}_i^{-1}\right)$ for each i, let $\mathfrak{u}_{\mathrm{ue}} = \mathrm{Lie}(\mathrm{U}_{S_{\mathrm{ue}}})$, and set

$$\mathfrak{w}^{\mathrm{bc}} \stackrel{\text{def}}{=} \mathfrak{u}_{\mathrm{ue}} \oplus \mathfrak{u}_{\mathrm{dt}}^-.$$

It is easily checked that, with the notation (7.1), we have

$$\mathfrak{w}^{\mathrm{bc}} = \left\{ x \in \mathfrak{g} : \forall b \in B, \lim_{n \to \infty} \left(\bar{h}_n^1 \right)^{-1} x = 0 \right\},$$

and

$$\mathrm{Lie}(W^{\mathrm{st}}) = \mathfrak{h}_{\mathrm{fne}} \oplus \mathfrak{w}^{\mathrm{bc}}. \tag{9.1}$$

We define

$$W^{\mathrm{bc}} \stackrel{\mathrm{def}}{=} \left\{ g \in G^S : \forall b \in B, \lim_{n \to \infty} \left(\bar{h}_n^1 \right)^{-1} g \left(\bar{h}_n^1 \right) = \mathrm{Id} \right\}.$$

The superscript 'bc' stands for 'backward contracting'. Clearly $\mathrm{Lie}(W^{\mathrm{bc}}) = \mathfrak{w}^{\mathrm{bc}}$ and W^{bc} is normalized by the random walk. Let $H^{\mathrm{fne}} W^{\mathrm{bc}}$ denote the set of pairwise products $\{hw : h \in H^{\mathrm{fne}}, w \in W^{\mathrm{bc}}\}$. Clearly $H^{\mathrm{fne}} W^{\mathrm{bc}} \subset W^{\mathrm{st}}$, and it follows from (9.1) that $H^{\mathrm{fne}} W^{\mathrm{bc}}$ contains an open neighborhood of Id in W^{st}. We have the following:

Lemma 9.3. *There is a finite set $F \subset G^S$ such that $W^{\mathrm{st}} = \bigcup_{f \in F} H^{\mathrm{fne}} W^{\mathrm{bc}} f$. In particular, if $\nu_b(H^{\mathrm{fne}} W^{\mathrm{bc}} x) = 0$ for β^X-a.e. pair (b, x), then $\nu_b(W^{\mathrm{st}} x) = 0$ for β^X-a.e. pair (b, x).*

Proof. In order to prove the first assertion we note that with respect to the partition (6.2), we have from (1.10) and (8.1) that

$$W^{\mathrm{st}} = \mathrm{U}_{S_{\mathrm{ue}}} \times \mathrm{G}_{S_{\mathrm{dt}}} \times \mathrm{G}_{S_{\mathrm{tr}}}, \qquad H^{\mathrm{fne}} = \{0\} \times \mathrm{P}_{S_{\mathrm{dt}}} \times \mathrm{G}_{S_{\mathrm{tr}}}.$$

Also W^{bc} contains the connected groups $\mathrm{U}_{S_{\mathrm{ue}}}$ and $\mathrm{U}^-_{S_{\mathrm{dt}}}$ with Lie algebras $\mathfrak{u}_{\mathrm{ue}}$ and $\mathfrak{u}^-_{\mathrm{dt}}$. Thus it suffices to prove the statement for $\mathrm{G}_{S_{\mathrm{dt}}}, \mathrm{P}_{S_{\mathrm{dt}}}, \mathrm{U}^-_{S_{\mathrm{dt}}}$ in place of $W^{\mathrm{st}}, H^{\mathrm{fne}}, W^{\mathrm{bc}}$. Note that the quotient $P_{S_{\mathrm{dt}}} \backslash \mathrm{G}_{S_{\mathrm{dt}}}$ is compact, and since $W^{\mathrm{bc}} H^{\mathrm{fne}}$ contains a neighborhood of the identity, $\mathrm{U}_{S_{\mathrm{dt}}}$ projects onto a subset of $\mathrm{P}_{S_{\mathrm{dt}}} \backslash \mathrm{G}_{S_{\mathrm{dt}}}$ with nonempty interior. Thus the cover $\{\mathrm{P}_{S_{\mathrm{dt}}} \mathrm{U}^-_{S_{\mathrm{dt}}} g : g \in \mathrm{G}_{S_{\mathrm{dt}}}\}$ has a finite sub-cover, and this proves the first assertion. The second assertion follows from the first one, by covering an orbit $W^{\mathrm{st}} x$ of positive measure for ν_b, by the finitely many sets $H^{\mathrm{fne}} W^{\mathrm{bc}} f x, \ f \in F$. ☐

Lemma 9.4. *Suppose that*

$$\nu \left(\left\{ x \in \mathscr{X}^S_{d+1} : \mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\} \right\} \right) = 0. \tag{9.2}$$

Then

$$\nu_b(H^{\mathrm{fne}} W^{\mathrm{bc}} x \smallsetminus H^{\mathrm{fne}} x) = 0$$

for almost every pair $(b, x) \in B^X$.

Proof. Let $\mathbf{Z} \stackrel{\mathrm{def}}{=} B \times X \times X$ and define $R \colon \mathbf{Z} \to \mathbf{Z}$ by

$$R(b, x, x') \stackrel{\mathrm{def}}{=} \left(Tb, b_1^{-1} x, b_1^{-1} x' \right), \qquad \text{where } b \in B, \ b = (b_1, b_2, \ldots),$$

and $T : B \to B$ is the shift. We define a probability measure $\tilde{\beta}$ on $\mathbf{Z}$ by

$$\tilde{\beta} = \int_B \delta_b \otimes \nu_b \otimes \nu_b \, \mathrm{d}\beta(b). \tag{9.3}$$

It follows from the equivariance property $\nu_{h_i b} = h_{i*} \nu_b$ in Proposition 3.6 that $\nu_{Tb} = (b_1^{-1})_* \nu_b$, and this implies that $\tilde{\beta}$ is R-invariant. Suppose by contradiction that $E \subset B^X$ is a measurable set such that $\beta^X(E) > 0$ and

$$\forall (b, x) \in E \quad \nu_b(H^{\mathrm{fne}} W^{\mathrm{bc}} x \smallsetminus H^{\mathrm{fne}} x) > 0.$$

Let

$$\mathcal{Z} = \{(b, x, x') \in \mathbf{Z} \colon x' \in H^{\mathrm{fne}} W^{\mathrm{bc}} x \smallsetminus H^{\mathrm{fne}} x\}.$$

Note that $\mathcal{Z}$ is R-invariant. Indeed, suppose that $z = (b, x, x') \in \mathcal{Z}$ and write $x' = pux$ where $p \in H^{\mathrm{fne}}$ and $u \in W^{\mathrm{bc}} \smallsetminus \{\mathrm{Id}\}$. Then

$$R(z) = \left(Tb, b_1^{-1} x, \bar{b}_1 b_1^{-1} x \right), \quad \text{where } \bar{b}_1 = b_1^{-1} (pu) b_1.$$

Since the conjugation map $g \mapsto b_1^{-1} g b_1$ is a group automorphism, and the groups H^{fne}, W^{bc} are both invariant under conjugation by elements of the random walk, we have that $\bar{b}_1 \in H^{\mathrm{fne}} W^{\mathrm{bc}}$ and its W^{bc}-component $b_1^{-1} u b_1$ is nontrivial.

We claim that $\tilde{\beta}(\mathcal{Z}) > 0$. Indeed, if $\tilde{\beta}(\mathcal{Z}) = 0$ then by (9.3) and Fubini's theorem we have that for β-a.e. $b \in B$ and for ν_b-a.e. $x \in X$ we have that

$$\nu_b(H^{\mathrm{fne}} W^{\mathrm{bc}} x \smallsetminus H^{\mathrm{fne}} x) = \nu_b\big(\{x' \in X \colon x' \in H^{\mathrm{fne}} W^{\mathrm{bc}} x \smallsetminus H^{\mathrm{fne}} x\}\big) = 0;$$

but recalling (3.13), we see that this contradicts the assumption that $\beta^X(E) > 0$.

We now remove from $\mathcal{Z}$ the set of (b, x, x') for which $\mathrm{Stab}(x) \cap W^{\mathrm{st}} \neq \{\mathrm{Id}\}$. We continue to denote the resulting set by $\mathcal{Z}$ and note that by assumption (9.2), it still satisfies $\tilde{\beta}(\mathcal{Z}) > 0$. Following this modification, if $z = (b, x, x') \in \mathcal{Z}$, then there are *unique* $p \in H^{\mathrm{fne}}$ and $w \in W^{\mathrm{bc}} \smallsetminus \{\mathrm{Id}\}$ such that $x' = pwx$; indeed, if $\tilde{p} \in H^{\mathrm{fne}}$ and $\tilde{w} \in W^{\mathrm{bc}}$ also satisfy $x' = \tilde{p}\tilde{w}x$, then $w^{-1} p^{-1} \tilde{p} \tilde{w} \in \mathrm{Stab}(x) \cap W^{\mathrm{st}}$. Using this uniqueness, and denoting $p = p(z), w = w(z)$ the elements in H^{fne} for which $x' = pwx$, we obtain an almost surely well-defined map

$$\Theta \colon \mathcal{Z} \to [0, \infty), \quad \Theta(b, x, x') \stackrel{\mathrm{def}}{=} \mathrm{dist}_{\mathscr{X}_{d+1}^S}\left(x, p(z)^{-1} x'\right) \qquad (\text{where } z = (b, x, x') \in \mathcal{Z}). \tag{9.4}$$

Here $\mathrm{dist}_{\mathscr{X}_{d+1}^S}$ is the metric introduced in §8.3. Then Θ is positive on $\mathcal{Z}$.

Given $b \in B$ and $n \in \mathbb{N}$, and writing b_n^1 as in (7.1), we have

$$R^n z = \left(T^n b, (b_n^1)^{-1} x, (b_n^1)^{-1} x'\right) = \left(T^n b, (b_n^1)^{-1} x, p_{z,n} w_{z,n} (b_n^1)^{-1} x\right),$$

where

$$p_{z,n} = (b_n^1)^{-1} p(z) b_n^1, \quad w_{z,n} = (b_n^1)^{-1} w(z) b_n^1.$$

In particular, by the construction of the metrics $\mathrm{dist}_{\mathscr{X}_{d+1}^S}, \mathrm{dist}_{G^S}$, there is $C > 0$ such that for all large enough n,

$$\Theta(R^n z) = \mathrm{dist}_{\mathscr{X}_{d+1}^S}\left((b_n^1)^{-1} x, w_{z,n} (b_n^1)^{-1} x\right) \leq C \, \mathrm{dist}_{G^S}\left(\mathrm{Id}, w_{z,n}\right).$$

In particular $\Theta(R^n z) \to 0$, on a set $\mathcal{Z}$ of positive measure. This gives a contradiction to Poincaré recurrence, and completes the proof. ☐

Lemma 9.5. *Suppose that for β^X-a.e. $(b, x) \in B^X$ we have $\nu_b(Zx) = 0$ (where Z is the centralizer of the random walk as in §8). Then for β^X-a.e. $(b, x) \in B^X$ we have $\nu_b(H^{\mathrm{fne}} x) = 0$.*

Proof. Recall from (8.1) and Proposition 3.8 that $H^{\mathrm{fne}} = Z \mathrm{U}_{\mathrm{dt}}$. Let

$$E \stackrel{\mathrm{def}}{=} \Big\{(b, x) \colon \nu_b(H^{\mathrm{fne}} x) > 0\Big\} \quad \text{and} \quad F \stackrel{\mathrm{def}}{=} \left\{(b, x) \in B^X \colon \nu_b(Zx) > 0\right\}.$$

We assume $\beta^X(F) = 0$ and suppose for sake of contradiction that $\beta^X(E) > 0$. Since $\bar{\mu}$ is finitely supported, we can assume without loss of generality that for all $(b, x) \in E$, for all $k \in \mathbb{N}$, and for all $a \in \mathrm{supp}(\bar{\mu})^k$ we have $\nu_{ab} = a_* \nu_b$. Using Lusin's theorem, let $K \subset E \setminus F$ be a compact set with $\beta^X(K) > 0$ and such that the map $(b, x) \mapsto \nu_b$ is continuous on K.

Let $T^X : B^X \to B^X$ be as in (3.13). By Proposition 3.6, β^X is T-invariant and ergodic. Let $L \colon L^1(B^X, \beta^X) \to L^1(B^X, \beta^X)$ be the adjoint operator of T^X; that is, for all $f \in L^1(B^X, \beta^X)$ and for all $\varphi \in L^\infty(B^X, \beta^X)$ we have

$$\int_{B^X} f \cdot (\varphi \circ T^X) \, \mathrm{d}\beta^X = \int_{B^X} Lf \cdot \varphi \, \mathrm{d}\beta^X.$$

From the definition and by a straightforward induction, one finds that β^X-almost surely

$$L^n f(b, x) = \sum_{a \in B^*, \, \mathrm{len}(a) = n} f(ab, a_n^1 \cdot x) \, \beta([a]).$$

We recall that by the Chacon-Ornstein ergodic theorem [CO60] the averages

$$A_N^* f(b,x) = \frac{1}{N}\sum_{n=0}^{N-1} L^n f(b,x)$$

converge pointwise β^X-almost surely. We choose large compact subsets C and M of U_{dt} and Z respectively, and $\varepsilon > 0$, such that the set

$$Q' \stackrel{\mathrm{def}}{=} \{(b,x) \in K : \nu_b(MCx) > \varepsilon\}$$

has positive measure. We let $Q \subset Q'$ be compact and of positive measure. By the Chacon-Ornstein ergodic theorem and ergodicity of T^X, we know that $A_N^* \mathbf{1}_Q(b,x) \to \beta^X(Q) > 0$ for β^X-a.e. $(b,x) \in B^X$. Thus, there are $(b,x) \in Q$, a sequence $n_j \to \infty$, and a sequence of words $a(j) \in B^*$ of length n_j, such that

$$\forall j, \quad (b_j, x_j) \stackrel{\mathrm{def}}{=} \big(a(j)b, a(j)\cdot x\big) \in Q.$$

Here we view $a(j)$ as both an element of B^*, and as an element of G^S obtained by the product $(a(j))_j^1$. Passing to a subsequence, we assume that $(b_j, x_j) \stackrel{j\to\infty}{\longrightarrow} (b', x') \in Q$. We claim that

$$\nu_{b'}(Mx') > 0. \tag{9.5}$$

This will give us the desired contradiction, since $M \subset Z$ and $(b', x') \in K \subset E \smallsetminus F$.

To see (9.5), let $\mathcal{O} \subset G$ be a compact neighborhood of the identity in G^S. Using Urysohn's lemma and outer regularity of $\nu_{b'}$, we have that

$$\nu_{b'}(M\mathcal{O}x') = \inf\{\nu_{b'}(\varphi) : \varphi \in C_c(X) \text{ and } \varphi|_{M\mathcal{O}x'} = 1\}.$$

If follows from the definitions of the group U, the places S_{dt}, and the elements $\bar{h}_i$ (see (1.2), (6.1) and (6.4)) that the conjugation of elements of U_{dt} by elements of the $\bar{\mu}$ random walk is uniformly contracting, in the following sense. For any compact subset $C_1 \subset \mathrm{U}_{\mathrm{dt}}$ and any open neighborhood $\mathcal{O}_1$ of the identity in U_{dt}, there is k_0 such that for all $k \geq k_0$, for any $a \in B^*$ of length k, we have

$$a_k^1 C_1 (a_k^1)^{-1} \subset \mathcal{O}_1.$$

In particular, taking $C_1 = C$ and $\mathcal{O}_1$ so that $\mathcal{O}_1$ such that the closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}$, we have that

$$a(j)Cx = a(j)Ca(j)^{-1}x_j \subset \mathcal{O}x'$$

for all sufficiently large j. Therefore, for $\varphi \in C_c(X)$ satisfying $\varphi|_{M\mathcal{O}x'} = 1$, we have

$$\int \varphi \,\mathrm{d}\nu_{b'} \geq \limsup_{j\to\infty} \nu_{a(j)b}(M\mathcal{O}x') \geq \limsup_{j\to\infty} \nu_{a(j)b}(Ma(j)Cx) = \nu_b(MCx) > \varepsilon,$$

and therefore $\nu_{b'}(M\mathcal{O}x') \geq \varepsilon$. Now Mx' is the intersection of all sets of the form $M\mathcal{O}x'$, where $\mathcal{O}$ is a neighborhood of the identity in G^S. This yields $\nu_{b'}(Mx') \geq \varepsilon$ and proves (9.5). □

Proof of Theorem 9.1. Using Lemma 9.2 we see that under our hypotheses, we can apply Theorem 8.1. This yields

$$\text{for } \beta^X\text{-a.e. } (b,x), \quad \nu_b(Zx) = 0.$$

Hence, by Lemma 9.5,

$$\text{for } \beta^X\text{-a.e. } (b,x), \quad \nu_b(H^{\mathrm{fne}}x) = 0.$$

Combining this with Lemma 9.4, it follows that

$$\text{for } \beta^X\text{-a.e. } (b,x), \quad \nu_b(H^{\mathrm{fne}}W^{\mathrm{bc}}x) = 0.$$

Applying Lemma 9.3 we get that

$$\text{for } \beta^X\text{-a.e. } (b,x), \quad \nu_b(W^{\mathrm{st}}x) = 0.$$

□

10. Additional Invariance: The Exponential Drift

The goal of this section is to prove that, under the assumptions of Theorem 9.1, almost every limit measure decomposes as a convex combination of measures invariant under one-parameter subgroups of U_{ue}. To this end, we will employ the exponential drift argument introduced by Benoist and Quint in [BQ11]. Running the argument will require the preliminary work done in §7–§9. The following result is the main result of this section.

Theorem 10.1. *Suppose that ν is a stationary measure satisfying the hypotheses of Theorem 9.1. Then for β-a.e. $b \in B$ and for ν_b-a.e. $x \in \mathscr{X}_{d+1}^S$, there exists a non-trivial subgroup $W_{(b,x)} \subset \mathrm{U}_{\mathrm{ue}}$ generated by one-parameter subgroups and a $W_{(b,x)}$-invariant probability measure $\nu_{(b,x)}$ on $\mathscr{X}_{d+1}^S$ such that*

$$\nu_b = \int_X \nu_{(b,x)}\, \mathrm{d}\nu_b(x).$$

Moreover, for β^X-almost every $(b,x) \in B^X$ we have $W_{(b,x)} = b_1 W_{T^X(b,x)} b_1^{-1}$ and $\nu_{(b,x)} = b_{1*}\nu_{T^X(b,x)}$.

For the rest of this section we retain the notations and assumptions of Theorem 10.1. The proof relies on an analysis of leafwise measures, for which we follow [BQ11, §4]. For more information on leafwise measures, we refer the reader to [EL10]. We now recall the necessary notations and results.

Denote by $\mathcal{M}(\mathfrak{u}_{\mathrm{ue}})$ the convex cone of positive non-null Radon measures on $\mathfrak{u}_{\mathrm{ue}}$. Also let $\mathcal{M}_1(\mathfrak{u}_{\mathrm{ue}})$ be the set of rays in $\mathcal{M}(\mathfrak{u}_{\mathrm{ue}})$, i.e., the quotient space for the equivalence relation of *proportionality*, defined by

$$\sigma_1 \propto \sigma_2 \iff \exists c > 0 \text{ such that } \sigma_2 = c\sigma_1.$$

The choice of σ to denote leafwise measures is consistent with [BQ11], although we also use σ to denote elements of S; even punctilious readers should have no difficulty disambiguating these two uses. Let

$$\overline{\sigma}\colon B^X \to \mathcal{M}_1(\mathfrak{u}_{\mathrm{ue}}), \qquad (b,x) \mapsto \overline{\sigma}_{(b,x)}$$

be a family of leafwise measures for β^X with respect to the action of $\mathfrak{u}_{\mathrm{ue}}$ given by

$$\Psi\colon B^X \times \mathfrak{u}_{\mathrm{ue}} \to B^X, \qquad \Psi\big((b,x),u\big) \overset{\mathrm{def}}{=} \big(b, \exp(u)x\big). \tag{10.1}$$

For $u \in \mathfrak{u}_{\mathrm{ue}}$ we will use the notation

$$\Psi_u : B^X \to B^X, \quad \Psi_u(z) \overset{\mathrm{def}}{=} \Psi(z,u). \tag{10.2}$$

Since the Lie algebra $\mathfrak{u}_{\mathrm{ue}}$ is abelian, $\exp : \mathfrak{u}_{\mathrm{ue}} \to \mathrm{U}_{\mathrm{ue}}$ is a group isomorphism and the expression on the right-hand side of (10.1) coincides with the action of U_{ue} on $\mathscr{X}_{d+1}^S$. It will be more convenient (and also consistent with the notation of [BQ11]) to stick with exponential notation and use $\mathfrak{u}_{\mathrm{ue}}$ instead of U_{ue}.

Note that $\overline{\sigma}_{(b,x)}$ denotes an equivalence class of measures. In order to choose a concrete representative, we let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets which form an exhaustion of $\mathfrak{u}_{\mathrm{ue}}$, and we denote by $\sigma_{(b,x)}$ the element in the proportionality class $\overline{\sigma}_{(b,x)}$ satisfying $\sigma_{(b,x)}(K_{n_0}) = 1$, where n_0 is the minimal n for which $\overline{\sigma}_{(b,x)}(K_n) > 0$. Similarly, given $\sigma \in \mathcal{M}(\mathfrak{u}_{\mathrm{ue}})$, we will denote by $\overline{\sigma} \in \mathcal{M}_1(\mathfrak{u}_{\mathrm{ue}})$ its equivalence class. We will fix a conull subset $E \subset B^X$ on which σ is well-defined up to proportionality and satisfies all the characterizing properties of leafwise measures; cf. [BQ11, Prop. 4.2].

Recall that $\varrho \in \mathbb{Q}$ denote the contraction ratio of the maps in the IFS Φ, and for any $u \in \mathfrak{u}_{\mathrm{ue}}$ and any $\bar{h}_i \in \operatorname{supp}\bar{\mu}$ we have $\mathrm{Ad}_{\bar{h}_i}(u) = \varrho u$. Therefore

$$\forall u \in \mathfrak{u}_{\mathrm{ue}}, \qquad \Psi_{\varrho^{-1}u} \circ T^X = T^X \circ \Psi_u. \tag{10.3}$$

Given $a \in \mathbb{Q}_S^\times$, denote

$$\eta_a\colon \mathfrak{u}_{\mathrm{ue}} \to \mathfrak{u}_{\mathrm{ue}}, \qquad \eta_a(u) \overset{\mathrm{def}}{=} au.$$

Let $\mathcal{B}^X$ denote the Borel σ-algebra on B^X and let

$$\mathcal{Q}_\infty^X = \bigcap_{n\in\mathbb{N}} \left(T^X\right)^{-n} \mathcal{B}^X. \tag{10.4}$$

The following is an immediate consequence of the uniqueness of leafwise measures and (10.3).

Corollary 10.2 (cf. [BQ11, Cor. 6.12 and Cor. 6.13])**.** *There exists a conull subset $E \subset B^X$ such that*

$$\forall (b,x) \in E,\ \forall n \in \mathbb{N}, \quad (T^X)^n(b,x) \in E \implies \sigma_{(b,x)} \propto (\eta_{\varrho^n})_* \sigma_{(T^X)^n(b,x)}. \tag{10.5}$$

Moreover, the map $\overline{\sigma}$ is $\mathcal{Q}_\infty^X$-measurable.

Note that the right hand side of formula (10.1) involves the action of the group U_{ue} on $\mathscr{X}_{d+1}^S$, and in contrast with [BQ11], the acting group does not depend on b. Therefore for β^X-a.e. (b,x), $\bar{\sigma}(b,x)$ is the leafwise measure of ν_b with respect to the U_{ue}-action on $\mathscr{X}_{d+1}^S$.

We also introduce notation for the translation map

$$\tau\colon \mathfrak{u}_{\mathrm{ue}} \times \mathfrak{u}_{\mathrm{ue}} \to \mathfrak{u}_{\mathrm{ue}}, \qquad \tau_u(v) \stackrel{\mathrm{def}}{=} v + u.$$

We recall from [BQ11, Prop. 4.2] that (after removing a nullset), for all $z \in E$ and all $u \in \mathfrak{u}_{\mathrm{ue}}$ such that $\Psi_u(z) \in E$, we have that

$$\sigma_z \propto \tau_{u*} \sigma_{\Psi_u(z)}. \tag{10.6}$$

Given Theorem 8.1, the main step towards a proof of Theorem 10.1 is to show that there exist many generic pairs (b,x) for which the leafwise measure agrees up to proportionality with the translate by arbitrarily small non-trivial elements of $\mathfrak{u}_{\mathrm{ue}}$. The first step in the argument is the proof that there exist many nearby pairs of points whose displacement is not contained in

$$\mathfrak{w}_{\mathrm{st}} \stackrel{\mathrm{def}}{=} \mathrm{Lie}(W^{\mathrm{st}}) = \mathfrak{u}_{\mathrm{ue}} \oplus \mathfrak{g}_{\mathrm{dt}} \oplus \mathfrak{g}_{\mathrm{tr}}. \tag{10.7}$$

Lemma 10.3. *For every measurable set $L \subset B^X$ there exists a conull subset $L' \subset L$ and, for every element $(b,x) \in L'$, a sequence $(v_m)_{m\in\mathbb{N}}$ of elements of $\mathfrak{g} \smallsetminus \mathfrak{w}_{\mathrm{st}}$ such that $v_m \to 0$ as $m \to \infty$ and*

$$\forall m \in \mathbb{N}, \qquad \big(b, \exp(v_m)x\big) \in L.$$

Proof. Assume without loss of generality that $\beta^X(L) > 0$ and, using inner regularity, that L is compact. Denote by F' the projection of L to B and for any $b \in F'$ let L_b denote the fiber $\{x \in \mathscr{X}_{d+1}^S : (b,x) \in L\}$, so that

$$\beta^X(L) = \int_{F'} \nu_b(L_b)\, \mathrm{d}\beta(b).$$

Let $F \subset F'$ consist of those $b \in F'$ satisfying $\nu_b(L_b) > 0$. The intersection $\tilde{L} \subset B^X$ of the preimage of F with L is a conull subset of L. Let $b \in F$, then for ν_b-a.e. $x \in L_b$ and for every neighborhood V of 0 in $\mathfrak{g}$ for which the exponential map is well-defined,

$$\nu_b\big(L_b \cap \exp(V)x\big) > 0. \tag{10.8}$$

From Theorem 9.1 we know that for all $b \in F$ we have

$$\text{for } \nu_b\text{-a.e. } x, \quad \nu_b\big(L_b \smallsetminus W^{\mathrm{st}}x\big) > 0. \tag{10.9}$$

Then the claim holds if we take L' to be the subset of $\tilde{L}$ consisting of (b,x) for which (10.9) holds, and (10.8) holds for a countable basis of identity neighborhoods V. □

We set up some more notation. We write $(\operatorname{supp}\bar{\mu})^n$ for the words in B^* of length n. Given $(b,x) \in B^X$, define a map

$$h_{n,(b,x)}\colon (\operatorname{supp}\bar{\mu})^n \to B^X, \qquad h_{n,(b,x)}(a) \stackrel{\mathrm{def}}{=} \big(aT^n b, a_n \cdots a_1 b_n^{-1} \cdots b_1^{-1} x\big)\,;$$

i.e., for any $a \in (\operatorname{supp}\bar{\mu})^n$, and $z \in B^X$, $z' = h_{n,z}(a)$ has the same future (under T^X) as z and, in fact, $(T^X)^n(z) = (T^X)^n(z')$. The image of $h_{n,z}$ is the set of all points whose futures agree with that of z after time n. Put differently,

$$\operatorname{im}\big(h_{n,z}\big) = \big(T^X\big)^{-n}\left\{(T^X)^n(z)\right\}.$$

As in [BQ11, Prop. 2.3], using the decreasing martingale theorem, we have that for all $\varphi \in \mathrm{L}^\infty(B^X, \beta^X)$,

$$\text{for } \beta^X\text{-a.e. } z, \qquad \mathbb{E}\left(\varphi|\mathcal{Q}^X_\infty\right)(z) = \lim_{n\to\infty} \sum_{a\in(\operatorname{supp}\bar{\mu})^n} \beta([a])\,\varphi\circ h_{n,z}(a). \tag{10.10}$$

Indeed, the image of $h_{n,z}$ is the atom of z in $(T^X)^{-n}\mathcal{B}^X$ and thus one only has to check that the right hand side defines a conditional measure for β^X with respect to the sub-σ-algebra $\left(T^X\right)^{-n}\mathcal{B}^X$.

Lemma 10.4. *Let E be as in Corollary 10.2. There exists a conull subset $F \subset E$ such that for $z \in F$, for all $n \in \mathbb{N}$, and for all $a \in (\operatorname{supp}\bar{\mu})^n$ we have*

$$h_{n,z}(a) \in E \implies \sigma_z \propto \sigma_{z'}, \quad \textit{where } z' = h_{n,z}(a).$$

Proof. Formula (10.5) means that as long as $z = (b, x) \in E$, the measure σ_z can be recovered from $\sigma_{(T^X)^n z}$ for every n, by composition with the contraction map η_{ϱ^n}. Thus the conclusion of the Lemma follows from the fact $(T^X)^n(z') = (T^X)^n(z)$. □

Given $n \in \mathbb{N}$ and $c \in B^* \cup B$ such that $\operatorname{len}(c) \geq n$, following [BQ11], we use the letter R to denote the adjoint representation, so that

$$R\left(c_n^1\right) \overset{\text{def}}{=} \operatorname{Ad}_{c_1}\circ\cdots\circ\operatorname{Ad}_{c_n}, \qquad R\left(c_1^n\right) \overset{\text{def}}{=} \operatorname{Ad}_{c_n}\circ\cdots\circ\operatorname{Ad}_{c_1},$$

and set

$$F_{n,b}(a) = R\left(a_1^n\right)\circ R\left(b_n^1\right)^{-1}. \tag{10.11}$$

Note that for $\sigma \in S \smallsetminus S_{\mathrm{ue}}$, the adjoint action of $F_{n,b}(a)$ on $\mathfrak{g}_\sigma$ is trivial for any n, b, a.

Using Lemma 10.3, we can run the core of the exponential drift argument.

Proposition 10.5 (cf. [BQ11, Prop. 7.1]). *For any $\delta_0 > 0$, for β^X-almost every $z \in E$ there exist $u \in \mathfrak{u}_{\mathrm{ue}}$ and $z' \in E$ such that*

$$0 < \|u\| < \delta_0, \quad \Psi_u(z') \in E, \qquad \sigma_{\Psi_u(z')} \propto \sigma_{z'} \propto \sigma_z. \tag{10.12}$$

Proof. Let $\mathcal{Q}^X_\infty$ be the σ-algebra defined by (10.4), let $\varepsilon \in (0,1)$ be arbitrary and fix a compact subset $K \subset E$ such that $\overline{\sigma}$ is continuous on K and such that $\beta^X(K) > 1 - \varepsilon^2/2$. We denote by $\mathbf{1}_K$ the indicator function of K. Recall that the conditional expectation $\mathbb{E}\left(\mathbf{1}_K|\mathcal{Q}^X_\infty\right)(\cdot)$ is an almost everywhere defined $\mathcal{Q}^X_\infty$-measurable function on B^X which is uniquely defined up to a nullset; we fix one representative and continue to denote it by $\mathbb{E}\left(\mathbf{1}_K|\mathcal{Q}^X_\infty\right)$. We let

$$F_\varepsilon = \left\{z \in E \colon \mathbb{E}\left(\mathbf{1}_K|\mathcal{Q}^X_\infty\right)(z) > 1 - \varepsilon\right\}.$$

One computes that

$$\begin{aligned} 1 - \frac{\varepsilon^2}{2} < \beta^X(K) &\leq (1-\varepsilon)\big(1 - \beta^X(F_\varepsilon)\big) + \beta^X(F_\varepsilon) \\ &= 1 - \varepsilon + \varepsilon\beta^X(F_\varepsilon). \end{aligned}$$

Hence $\beta^X(F_\varepsilon) > 1 - \frac{\varepsilon}{2}$. Given $n \in \mathbb{N}$, let

$$\psi_n \colon B^X \longrightarrow \mathbb{R}, \qquad \psi_n(z) \overset{\text{def}}{=} \sum_{a\in(\operatorname{supp}\bar{\mu})^n} \bar{\beta}([a])\,\mathbf{1}_K\circ h_{n,z}(a).$$

Using (10.10), we fix a full measure subset $E_1 \subset E$ such that

$$\forall z \in E_1, \qquad \psi_n(z) \xrightarrow{n\to\infty} \mathbb{E}\left(\mathbf{1}_K | \mathcal{Q}_\infty^X\right)(z). \tag{10.13}$$

We let $L_1 \subset E_1$ be a compact continuity set for $\mathbb{E}\left(\mathbf{1}_K|\mathcal{Q}_\infty^X\right)$ such that $\beta^X(L_1) > 1-\varepsilon$. Using Egorov's theorem, we can additionally assume that the convergence in (10.13) is uniform on L_1, and thus there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \ \forall z \in L_1, \quad \psi_n(z) > 1-\varepsilon. \tag{10.14}$$

This implies that

$$\forall n \geq n_0, \ \forall z \in L_1, \quad \bar{\beta}\left(\bigcup \{[a] : a \in (\operatorname{supp}\bar{\mu})^n, \ h_{n,z}(a) \in K\}\right) > 1-\varepsilon.$$

Using Theorem 9.1, we fix a compact set $L \subset L_1$ such that $\beta^X(L) > 1-\varepsilon$ and

$$\forall (b,x) \in L, \quad \nu_b(W^{\mathrm{st}}x) = 0.$$

We further fix a compact subset $L' \subset L$ of measure $\beta^X(L') > 1-\varepsilon$ satisfying the conclusion of Lemma 10.3.

For what follows, we fix an element $z_0 = (b,x) \in L'$ and a sequence $(v_m)_{m\in\mathbb{N}}$ of vectors in $\mathfrak{g} \smallsetminus \mathfrak{w}_{\mathrm{st}}$ such that $\exp(v_m)$ is well-defined for all m, $v_m \to 0$ as $m \to \infty$, and

$$\forall m \in \mathbb{N} \quad z_m \in L, \text{ where } z_m \stackrel{\text{def}}{=} \big(b, \exp(v_m)x\big).$$

Using (10.14), we know that for all $m \geq 0$ and for all $n \geq n_0$ we have

$$\beta\big(\{a \in (\operatorname{supp}\bar{\mu})^n \colon h_{n,z_m}(a) \in K\}\big) > 1-\varepsilon. \tag{10.15}$$

Extending the definition in (10.2), we write $\Psi_v(b,x) = (b, \exp(v)x)$ for every $v \in \mathfrak{g}$ for which $\exp(v)$ is well-defined. Note that the map $(z,v) \mapsto \Psi_v(z)$ is continuous where defined. Recall (or see [BQ13, §5]) that if $\exp(v)$ is well-defined, then so is $\exp(\operatorname{Ad}(g)v)$ for every $g \in G^S$, with $\exp(\operatorname{Ad}(g)v) = g\exp(v)g^{-1}$. From this and using (10.11) we see that if $y \in X$ satisfies $y = \exp(v)x$, where $\exp(v)$ is well-defined, then

$$\forall a \in (\operatorname{supp}\bar{\mu})^n \qquad h_{n,(b,y)}(a) = \big(\Psi_{F_{n,b}(a)v} \circ h_{n,(b,x)}\big)(a).$$

Given $\sigma \in S$, let $v_m^{(\sigma)} \in \mathfrak{g}_\sigma$ denote the projection of v_m to $\mathfrak{g}_\sigma$. Given $m,n \in \mathbb{N}$ and $\sigma \in S$, define

$$r_{n,m,\sigma} \stackrel{\text{def}}{=} |\varrho|_\sigma^n \left\| R\left(b_n^1\right)^{-1} v_m^{(\sigma)} \right\|_\sigma \quad \text{and} \quad r_{n,m} \stackrel{\text{def}}{=} \max_{\sigma\in S} r_{n,m,\sigma}.$$

We claim that

$$\limsup_{n\to\infty} r_{n,m} = \infty. \tag{10.16}$$

To see this, let $a = \operatorname{diag}(\varrho, \ldots, \varrho, 1)$, and for each $\sigma \in S_{\mathrm{ue}}$, let

$$\mathfrak{g}_\sigma = \mathfrak{u}_\sigma \oplus \mathfrak{u}_{\sigma,1}^\perp \oplus \mathfrak{u}_{\sigma,\varrho}^\perp$$

be the decomposition of $\mathfrak{g}_\sigma$ into the eigenspaces for the adjoint action of a^{-1}, corresponding respectively to the eigenvalues $\varrho^{-1}, 1, \varrho$. Using this direct sum decomposition we write

$$v_m^{(\sigma)} = v_{m,1}^{(\sigma)} + v_{m,2}^{(\sigma)} + v_{m,3}^{(\sigma)}.$$

Since $v_m \notin \mathfrak{w}_{\mathrm{st}}$ we see from (10.7) that there is $\sigma \in S_{\mathrm{ue}}$ such that at least one of $v_{m,2}^{(\sigma)}$ and $v_{m,3}^{(\sigma)}$ is nonzero. For this choice of σ and each i we can write $\bar{h}_i^{(\sigma)} = au_i$, where u_i is in the group $U^{(\sigma)}$ defined by (3.15). The adjoint action of u_i^{-1} satisfies

$$\operatorname{Ad}_{u_i}^{-1}\left(v_{m,2}^{(\sigma)}\right) - v_{m,2}^{(\sigma)} \in \mathfrak{u} \quad \text{and} \quad \operatorname{Ad}_{u_i}^{-1}\left(v_{m,3}^{(\sigma)}\right) - v_{m,3}^{(\sigma)} \in \mathfrak{u} \oplus \mathfrak{u}_{\sigma,1}^\perp.$$

Using this one verifies that

$$\inf_n \left\| R\left(b_n^1\right)^{-1} v_m^{(\sigma)} \right\|_\sigma > 0;$$

indeed, the component in at least one of $\mathfrak{u}_{\sigma,1}^\perp, \mathfrak{u}_{\sigma,\varrho}^\perp$ is not contracted by $R\left(b_n^1\right)^{-1}$. Since $\sigma \in S_{\mathrm{ue}}$, we have $|\varrho|_\sigma > 1$. From this it follows that $|\varrho|_\sigma^n \left\| R\left(b_n^1\right)^{-1} v_m^{(\sigma)} \right\|_\sigma \to \infty$. This shows (10.16).

Let $C > 1$ be as in Lemma 7.2, let

$$C_1 \overset{\mathrm{def}}{=} \max\{|\varrho|_\sigma \colon \sigma \in S\}, \quad C_2 \overset{\mathrm{def}}{=} \max\left\{\left\|\mathrm{Ad}(\bar{h}_i^{-1})\right\|_\sigma : \sigma \in S, \, \bar{h}_i \in \operatorname{supp} \bar{\mu}\right\},$$

and

$$A' \overset{\mathrm{def}}{=} \frac{\delta_0}{2C_1C_2C}.$$

For each $n \in \mathbb{N}$ let $n_m \in \mathbb{N}$ be the minimal index for which $A' \leq r_{n_m,m}$, which is well-defined in light of (10.16). It follows from the definition of C_1 and C_2 that $r_{n+1,m} \leq C_1C_2r_{n,m}$, and hence

$$\forall m \in \mathbb{N} \quad A' \leq r_{n_m,m} \leq A'C_1C_2.$$

Since $v_m \to 0$ we have $n_m \overset{m\to\infty}{\longrightarrow} \infty$, and since the action of $F_{n,b}(a)$ on $\mathfrak{g}_\sigma$ is trivial for $\sigma \notin S_{\mathrm{ue}}$, the place σ for which the maximum in the definition of $r_{n_m,m}$ is attained, belongs to S_{ue} for all large enough m.

Given $\sigma \in S_{\mathrm{ue}}$ and $m, M \in \mathbb{N}$, let

$$\mathrm{Sh}_{m,M}^\sigma(b) \overset{\mathrm{def}}{=} \left\{ a \in B \colon \mathrm{dist}(\mathbb{Q}_\sigma F_{n_m,b}(a)v_m^{(\sigma)}, \mathfrak{u}_\sigma) \leq \frac{1}{M} \right\},$$

$$\mathrm{Eq}_m^\sigma(b) \overset{\mathrm{def}}{=} \left\{ a \in B \colon \frac{1}{C} r_{n_m,m,\sigma} \leq \left\| F_{n_m,b}(a)v_m^{(\sigma)} \right\|_\sigma \leq C r_{n_m,m,\sigma} \right\}.$$

Given $m \in \mathbb{N}$, we let $M_{m,\sigma} \in \mathbb{N} \cup \{\infty\}$ maximal such that

$$\forall M \leq M_{m,\sigma} \quad \beta\big(\mathrm{Sh}_{m,M}^\sigma(b)\big) > 1 - \varepsilon.$$

Since ε was small, using Lemmas 7.2 and 7.5, there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ we can find a word

$$a_m \in \bigcap_{\sigma \in S_{\mathrm{ue}}} \left(\mathrm{Sh}_{m,M_{m,\sigma}}^\sigma(b) \cap \mathrm{Eq}_m^\sigma(b)\right)$$

such that $h_{n_m,z_0}(a_m), h_{n_m,z_m}(a_m) \in K$. Note that

$$\text{for all } \sigma \in S_{\mathrm{ue}}, \quad M_{m,\sigma} \overset{m\to\infty}{\longrightarrow} \infty. \tag{10.17}$$

Hence, after passing to a subsequence, there is $u \in \mathfrak{g}$ such that

$$h_{n_m,z_0}(a_m) \to z' \in K, \quad h_{n_m,z_m}(a_m) \to z'' \in K, \quad \text{ and } \quad F_{n_m,b}(a_m)v_m \to u.$$

We claim that $u \in \mathfrak{u}_{\mathrm{ue}}$. Indeed, for $\sigma \notin S_{\mathrm{ue}}$ the action of $F_{n,b}(a)$ on $\mathfrak{g}_\sigma$ is trivial and hence the σ-component of $F_{n_m,b}(a_m)v_m$ is equal to $v_m^{(\sigma)}$, which goes to zero as $m \to \infty$. Furthermore, for $\sigma \in S_{\mathrm{ue}}$, it follows from the definition of $\mathrm{Sh}_{m,M}^\sigma(b)$ and from (10.17) that any limit of $F_{n_m,b}(a_m)v_m$ belongs to $\mathfrak{u}_\sigma$ (and in particular, to the domain of definition of Ψ). From the definition of $\mathrm{Eq}_m^\sigma(b)$ that for $\sigma \in S_{\mathrm{ue}}$ for which the maximum in the definition of $r_{n_m,m}$ is realized, we have

$$0 < \frac{A'}{C} \leq \|u^{(\sigma)}\| \leq \|u\| \leq A'C_1C_2C < \delta_0.$$

From the continuity of the map Ψ we have

$$z'' = \Psi_u z'.$$

Because K is a continuity set, we have that

$$\overline{\sigma}_{z'} = \overline{\sigma}_{z_0} \quad \text{ and } \overline{\sigma}_{z''} = \overline{\sigma}_{z_0}.$$

This proves (10.12). □

In what follows, given $z \in B^X$, we denote by W_z the stabilizer of $\overline{\sigma}_z$ in U_{ue}. Combining Proposition 10.5 and formula (10.6), we obtain:

Corollary 10.6. *There exists $E \subset B^X$ of full measure such that W_z is non-trivial for every $z \in E$.*

Lemma 10.7. *There exists a conull subset $E \subset B^X$ such that:*

1. *If $z = (b, x) \in E$ and $T^X z \in E$, then $W_{T^X z} = b_1^{-1} W_z b_1$ and $\overline{\sigma}_{T^X z} = \overline{(\mathrm{Ad} b_1^{-1})_* \sigma_z}$.*
2. *For every $z \in E$, we have that*
$$W_z = \prod_{\sigma \in S_{\mathrm{ue}}} W_{z,\sigma},$$
where $W_{z,\sigma} = \exp(\mathfrak{w}_{z,\sigma})$ for a Lie subalgebra $\mathfrak{w}_{z,\sigma} \subset \mathfrak{u}_\sigma$.
3. *For every $z \in E$, σ_z is W_z-invariant.*

Proof. The proof is mostly a combination of facts about closed subgroups of unipotent S-arithmetic groups and techniques used in [BQ13, §8.2]. Item (1) is an immediate consequence of the characterizing properties of leafwise measures. For item (2), we know from Corollary 10.6 that for typical $z \in B^X$ the stabilizer W_z of $\overline{\sigma}_z$ is non-trivial. Let u be a nontrivial element of W_z. Then the logarithm $v = \log u \in \mathfrak{u}_{\mathrm{ue}}$ is well-defined and non-zero, and we write $v = (v^{(\sigma)})_{\sigma \in S_{\mathrm{ue}}}$. It follows from strong approximation (see [Cas78, Ch. 3, Lem. 3.1]) that
$$\overline{\mathbb{Z}v} = \bigoplus_{\sigma \in S_{\mathrm{ue}}} \mathbb{Z}_\sigma v^{(\sigma)}.$$
In particular, for any element in W_z, each component fixes $\overline{\sigma}_z$ separately, i.e., W_z is a direct product of subgroups of U_{ue}. We denote these groups by $W_{z,\sigma}$, and note that each of the $W_{z,\sigma}$ is closed, since W_z is closed.

Let $\mathfrak{w}_{z,\sigma} \subset \mathfrak{u}_{\mathrm{ue}}$ be the Lie algebra generated by $\log(W_{z,\sigma})$ and note that, since W_z is abelian,
$$\mathfrak{w}_{z,\sigma} = \mathrm{span}_{\mathbb{Q}_\sigma} \log(W_{z,\sigma}).$$
By the preceding argument $\mathfrak{w}_{z,\sigma} = \{0\}$ if and only if $W_{z,\sigma}$ is trivial. We claim that
$$\exp(\mathfrak{w}_{z,\sigma}) = W_{z,\sigma}.$$
This certainly holds if $W_{z,\sigma}$ is trivial. By construction, $W_{z,\sigma} \subset \exp(\mathfrak{w}_{z,\sigma})$. For the opposite inclusion, we use the argument given in the proof of [BQ13, Lem. 8.3]. We suppose by contradiction that the opposite inclusion does not hold for some $z \in B^X$, and let
$$\varphi_\sigma(z) = \inf \left\{ \left\| v^{(\sigma)} \right\| : v^{(\sigma)} \in \mathfrak{w}_{z,\sigma},\, \exp(v) \notin W_{z,\sigma} \right\}.$$
Then there is a subset of B^X of positive measure on which $\varphi_\sigma > 0$. By the definition of S_{ue} we have that $|\varrho|_\sigma > 1$ for every $\sigma \in S_{\mathrm{ue}}$, and this implies that for every $b \in B$ we have that $\left\| (b_1^n)^{-1}|_{\mathfrak{u}_\sigma} \right\| \to 0$ as $n \to \infty$. By item (1) we have that $\varphi_\sigma((T^X)^n(z))$ tends to zero for β^X-a.e. z, and this contradiction to the Poincaré recurrence theorem proves the claim.

In order to prove item (3), note that, since $\mathfrak{u}_{\mathrm{ue}}$ is abelian and since the action of $\mathfrak{u}$ on Radon measures on U_{ue} is continuous, for every $\sigma \in S_{\mathrm{ue}}$ the map $\alpha_{z,\sigma} \colon \mathfrak{w}_{z,\sigma} \to \mathbb{R}$ given by
$$\forall v_\sigma \in \mathfrak{w}_{z,\sigma} \quad \tau_{v_\sigma *} \sigma_z = e^{\alpha_{z,q}(v_\sigma)} \sigma_z$$
is a continuous group homomorphism. Since $\mathfrak{w}_{z,\sigma}$ is a $\mathbb{Q}_\sigma$-vector space, $\alpha_{z,\sigma}$ is trivial. □

Proof of Theorem 10.1. For the proof we disintegrate each measure ν_b along the map that sends $z = (b, x)$ to the Lie algebra $\mathfrak{w}_z = \mathrm{Lie}(W_z)$. This is done as in Propositions 7.5 and 7.6 in [BQ11]. Note that [BQ11] treats actions of $\mathbb{R}^d$. Recall that we showed in the proof of Lemma 10.7 that if $v \in \mathfrak{u}_{\mathrm{ue}}$ satisfies that $\exp(v)$ fixes a leafwise measure, then so does $\exp(v')$ for any $v' \in \mathrm{span}_{\mathbb{Q}_\sigma}(v)$.

Since the exponential map is a bijection between $\mathfrak{u}_{\mathrm{ue}}$ and U_{ue}, all the statements in [BQ11, §4] adapt mutatis mutandi to our situation. □

11. Ratner's Theorem and Conclusion of the Proof of Theorem 6.1

In this section we complete the proof of Theorem 6.1. Let ν_z, W_z be as in the conclusion of Theorem 10.1, where $z = (b, x) \in B \times X$. Applying Ratner's theorem in the S-adic setting (see [Rat98]), and a refinement of Tomanov (see [Tom00]) we have: There is a subset $E \subset B^X$, which is T^X-invariant and satisfies $\beta^X(B^X \smallsetminus E) = 0$, such that for all $z = (b, x) \in E$, there are groups $M'(z)$ such that the following hold:

(i) $W_z \subset M'(z)$;
(ii) $M'(z)x = \overline{W_z x} = \mathrm{supp}(\nu_z)$ is a closed orbit, and ν_z is a $M'(z)$-invariant probability measure which is ergodic for the action of W_z;
(iii) For $z = (b, x) \in E$, $M'(T^X z) = b_1^{-1} M'(z) b_1$.
(iv) If g satisfies $x = g\Gamma^S$ then the group $M = M(z, g) \stackrel{\mathrm{def}}{=} g^{-1}M'(z)g$ and its Zariski closure $\mathbf{M} = \mathbf{M}(z, g)$ satisfy that $\mathbf{M}$ is $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$ and M is a finite-index subgroup of $\mathbf{M}(\mathbb{Q}_S)$.

Since $\mathbf{M}(z, g)$ is defined over $\mathbb{Q}$, it belongs to a collection which is at most countable. The conjugacy class $[\mathbf{M}(z, g)]$ of $\mathbf{M}(z, g)$ depends only on z, and we denote it by $[\mathbf{M}(z)]$. By (iii), the assignment $z \mapsto [\mathbf{M}(z)]$ is constant along T^X-orbits. By ergodicity of the stationary measure ν, the conjugacy class $[\mathbf{M}(z)]$ is the same for β^X-a.e. z. Since the collection of finite-index subgroups of $\mathbf{M}(\mathbb{Q}_S)$ is also countable, the conjugacy class of $M'(z)$ is the same for β^X-a.e. z.

Proposition 11.1. *Let $M'(z)x = \overline{W_z x}$ as above, suppose $M'(z)$ is not a unipotent group, and let $U_0 \subset M'(z)$ be a normal unipotent subgroup. Then U_0 does not act ergodically on $M'(z)x$ (with respect to the $M'(z)$-invariant measure).*

Proof. Let $g \in G^S$ so that $x = g\Gamma^S$. By a conjugation by g we can replace M' with M. That is, we can assume that x represents the identity coset and $M'(z)$ is a subgroup of finite index in $\mathbf{M}(\mathbb{Q}_S)$, for a $\mathbb{Q}$-algebraic subgroup $\mathbf{M} \subset \mathbf{G}$. Let $\mathbf{R} = \mathrm{Rad}_u(\mathbf{M})$ denote the unipotent radical of $\mathbf{M}$. Since $M'(z)$ is not unipotent, $\mathbf{R}$ is a proper subgroup of $\mathbf{M}$, and is also defined over $\mathbb{Q}$. Let $R \stackrel{\mathrm{def}}{=} M' \cap \mathbf{R}(\mathbb{Q}_S)$. Then R is normal in M', contains U_0, and the orbit Rx is closed and is strictly contained in $M'x$. Since R is normal, $\overline{U_0 m x} \subset Rmx = mRx$ is a closed proper subset of $M'x$ for any $m \in M'$. In particular, the U_0-action on $M'x$ has no dense orbits, and hence the U_0-action is not ergodic. □

We will need the following facts about the group $\mathbf{U}$ and its normalizer $\mathbf{P}$.

Proposition 11.2.
- *For any proper $\mathbb{Q}$-algebraic subgroup $\mathbf{H} \subset \mathbf{G}$, we have $\dim \mathbf{H} \le \dim \mathbf{P}$.*
- *$\mathbf{U}$ is not contained in any unipotent abelian subgroup of larger dimension.*

Proof. The first assertion can be inferred from the list of maximal subalgebras of simple Lie algebras, due to Dynkin [Dyn57]. See [Stu91] or [OVG94, p. 187]. For the second assertion, note from [BT71] that if $\mathbf{V}$ is a unipotent abelian group properly containing $\mathbf{U}$, then after applying a conjugation we can assume that $\mathbf{V}$ is a contained in the unipotent radical of a Borel subgroup of $\mathbf{G}$. However, it can be checked explicitly that $\mathbf{U}$ is maximal, as an abelian subgroup of the unipotent radical of the Borel subgroup of $\mathbf{G}$. □

For any $r_0 \in \{1, \ldots, \dim \mathbf{P}\}$ and any $\sigma \in S$ we define $V^{\wedge r_0}(\sigma)$ and $\mathbb{P}(V^{\wedge r_0}(\sigma))$ as in §7.2. Each of the elements $\bar{h}_i$ acts on each $V^{\wedge r_0}(\sigma)$ via the r_0^{th} exterior power of the Adjoint representation, and hence on $\mathbb{P}(V^{\wedge r_0}(\sigma))$. The measure $\bar{\mu}$ determines a random walk on $V^{\wedge r_0}(\sigma)$. For $\sigma \in S_{\mathrm{tr}}$ each $\bar{h}_i$ acts trivially on $V^{\wedge r_0}(\sigma)$ and this random walk is trivial. For $\sigma \in S_{\mathrm{dt}}$ each $\bar{h}_i$ acts in the same way on $V^{\wedge r_0}(\sigma)$ and this random walk is deterministic, i.e., amounts to repeated application of one map.

For each σ, the stationary measure ν on $\mathscr{X}_{d+1}^S$ gives rise to stationary measures $\bar{\nu}(\sigma)$ on $V^{\wedge r_0}(\sigma)$ and $\hat{\nu}(\sigma)$ on $V^{\wedge r_0}(\sigma)$ as follows. Write $\mathfrak{m}'_\sigma(z)$ for the Lie algebra of the projection of the Lie algebra of $M'(z)$ to the σ-component $\mathfrak{g}_\sigma$. Let $[\mathfrak{m}'_\sigma(z)] \in \mathbb{P}(V^{\wedge r})$ denote the class of $\mathfrak{m}'_\sigma(z)$, and let $\mathbf{m}'_\sigma(z)$ denote a vector in $[\mathfrak{m}'_\sigma(z)]$, whose length is equal to the volume of $M'(z)x$ with respect to the r_0-dimensional volume on $\mathscr{X}_{d+1}^S$. Note that $\mathbf{m}'_\sigma(z)$ is uniquely defined up to $\pm$. This ambiguity will not play any role in what follows.

Now define measures on $\mathbb{P}(V^{\wedge r})$ by

$$\bar{\nu}_b \overset{\text{def}}{=} \int \delta_{\mathbf{m}'_\sigma(z)} \, \mathrm{d}\nu_b(z), \quad \text{and} \quad \bar{\nu} \overset{\text{def}}{=} \int \bar{\nu}_b \, \mathrm{d}\beta(b). \tag{11.1}$$

$$\hat{\nu}_b \overset{\text{def}}{=} \int \delta_{[\mathfrak{m}'_\sigma(z)]} \, \mathrm{d}\nu_b(z), \quad \text{and} \quad \hat{\nu} \overset{\text{def}}{=} \int \hat{\nu}_b \, \mathrm{d}\beta(b). \tag{11.2}$$

Then $\bar{\nu} = \bar{\nu}(\sigma)$ (respectively $\hat{\nu} = \hat{\nu}(\sigma)$) is a $\bar{\mu}$-stationary measure on $V^{\wedge r}(\sigma)$ (respectively, on $\mathbb{P}(V^{\wedge r_0}(\sigma))$). By uniqueness of the disintegration into leafwise measures, the measures $(\bar{\nu}_b)_{b\in B}$ and $(\hat{\nu}_b)_{b\in B}$ are the systems of limit measures for $\bar{\nu}$ and $\hat{\nu}$.

Proof of Theorem 6.1. We assume that we are not in Case (2) of Theorem 6.1 and prove that we must be in Case (1). Let r be the almost-sure value of $\dim \mathbf{M}(z)$. Since we are not in Case (2), for β^X-almost every z we have $\mathbf{M} \neq \mathbf{G}$, and hence $r < \dim \mathbf{G}$. By the first item of Proposition 11.2 we have $r \leq \dim \mathbf{P} = d^2 + d$.

Assume first that $r \in \{1, \dots, d\}$. We claim that in this case, for β^X-a.e. z, the group $M'(z)$ is contained in W^{st}. Since $M'(z)$ has a finite volume orbit through x, this will imply that for β^X-a.e. (b, x), ν_b is supported on points whose stabilizer in W^{st} is nontrivial. This implies that we are in Case (1), completing the proof in this case.

In order to prove the claim, in view of (1.10) it suffices to show that for $\sigma \in S_{\mathrm{ue}}$ we have $\mathfrak{m}'_\sigma(z) \subset \mathfrak{u}_\sigma$. By item (ii) of Lemma 7.3, we have that each $\hat{\nu}_b$ is supported on $\mathbb{P}(W^{(r)}) = \mathbb{P}(\bigwedge^r \mathfrak{u}_\sigma)$. This implies that $\hat{\nu}$ is supported on $\mathbb{P}(\bigwedge^r \mathfrak{u}_\sigma)$ and $\bar{\nu}$ is supported on $\bigwedge^r \mathfrak{u}_\sigma$. The decomposition (11.1) of $\bar{\nu}_b$ into Dirac masses now shows that for β^X-a.e. z, we have $\mathfrak{m}'_\sigma(z) \subset \mathfrak{u}_\sigma$. The claim now follows via (1.10).

Now suppose $r \in \{d+1, \dots, d^2+d\}$. Let $\sigma \in S_{\mathrm{ue}}$. Consider again the measures $\bar{\nu}_b$, $\bar{\nu}$, $\hat{\nu}_b, \hat{\nu}$ defined in (11.1) and (11.2). In this case, applying item (ii) of Lemma 7.3, we have that each $\hat{\nu}_b$ is supported on $\mathbb{P}(W^{(r)})$. The definition of $W^{(r)}$ implies that for β^X-a.e. z,

$$\mathfrak{u}_\sigma \subsetneq \mathfrak{m}'_\sigma(z) \subset \mathfrak{p}_\sigma, \qquad \text{and hence } W'_z \subset \mathrm{U}_\sigma \subsetneq M'_\sigma(z) \subset \mathrm{P}_\sigma.$$

Applying Proposition 11.1 we see that $M'(z)$ is unipotent for a.e. z, and U_σ is a proper normal subgroup of $M'_\sigma(z)$. Let $C(z)$ denote the center of $M'(z)$ and denote its projection to the place $\sigma \in S_{\mathrm{ue}}$ by $C_\sigma(z)$. Since $M'(z)$ is a unipotent group, the $\dim C(z) > 0$ has positive dimension, and by standard facts about lattices in nilpotent groups (see [Rag72, Chap. 2]), the intersection of any lattice in $M'(z)$ with $C(z)$ in a lattice in $C(z)$, and in particular, contains nontrivial elements. In particular, for β^X-a.e. $z = (b, x)$, $C(z)$ contains nontrivial elements belonging to the stabilizer of x. Thus it suffices to show that $C(z) \subset W^{\mathrm{st}}$, and hence, from (1.10), that for $\sigma \in S_{\mathrm{ue}}$ we have $C_\sigma(z) \subset U_\sigma$. Suppose to the contrary that $C_\sigma(z) \not\subset U_\sigma$. Since $M'(z)$ contains U_σ as a normal subgroup, $C_\sigma(z)U_\sigma$ is an abelian unipotent group satisfying $\dim C_\sigma U_\sigma > \dim U_\sigma$. This contradicts the second item of Proposition 11.2. ☐

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