EXPONENTIAL MIXING VIA ADDITIVE COMBINATORICS

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Abstract. We prove that the geodesic flow on a geometrically finite locally symmetric space of negative curvature is exponentially mixing with respect to the Bowen-Margulis-Sullivan measure. The approach is based on constructing a suitable anisotropic Banach space on which the infinitesimal generator of the flow admits an essential spectral gap. A key step in the proof involves estimating certain oscillatory integrals against the Patterson-Sullivan measure. For this purpose, we prove a general result of independent interest asserting that the Fourier transform of measures on $\mathbb{R}^d$ that do not concentrate near proper affine hyperplanes enjoy polynomial decay outside of a sparse set of frequencies. As an intermediate step, we show that the L^q-dimension ($1 < q \leq \infty$) of iterated self-convolutions of such measures tend towards that of the ambient space. Our analysis also yields that the Laplace transform of the correlation function of smooth observables extends meromorphically to the entire complex plane in the convex cocompact case and to a strip of explicit size beyond the imaginary axis in the case the manifold admits cusps.

1. Introduction

1.1. **Exponential mixing and Pollicott-Ruelle resonances.** Let X be the unit tangent bundle of a quotient of a real, complex, quaternionic, or a Cayley hyperbolic space by a discrete, geometrically finite, non-elementary group of isometries Γ. Denote by g_t the geodesic flow on X and by $\mathrm{m}^{\mathrm{BMS}}$ the Bowen-Margulis-Sullivan probability measure of maximal entropy for g_t. Let δ_Γ be the critical exponent of Γ. We refer the reader to Section 2 for definitions. The following is the main result of this article in its simplest form.

Theorem 1.1. *The geodesic flow on X is exponentially mixing with respect to $\mathrm{m}^{\mathrm{BMS}}$. More precisely, there exists $\sigma_0 = \sigma_0(X) > 0$ such that for all $f \in C_c^3(X)$, $g \in C_c^2(X)$ and $t \geq 0$,*

$$\int_X f \circ g_t \cdot g \; d\mathrm{m}^{\mathrm{BMS}} = \int_X f \; d\mathrm{m}^{\mathrm{BMS}} \int_X g \; d\mathrm{m}^{\mathrm{BMS}} + \|f\|_{C^3} \, O_g\left(e^{-\sigma_0 t}\right).$$

The dependence on g in the implicit constant is through its C^2-norm and the injectivity radius of its support.

The results also hold for functions with unbounded support and controlled growth in the cusp; cf. Section 9. Theorem 1.1 follows immediately from the following more precise result showing that the correlation function admits a finite resonance expansion.

Theorem 1.2. *There exists $\sigma > 0$ such that the following holds. There exist complex numbers $\{\lambda_i\}_{i=1}^N$ in the strip $\{-\sigma < \mathrm{Re}(z) < 0\}$, a finite rank projector Π, and a matrix Q with eigenvalues λ_i acting on the range of Π such that for all $f \in C_c^3(X)$, $g \in C_c^2(X)$ and $t \geq 0$, we have*

$$\int_X f \circ g_t \cdot g \; d\mathrm{m}^{\mathrm{BMS}} = \int_X f \; d\mathrm{m}^{\mathrm{BMS}} \int g \; d\mathrm{m}^{\mathrm{BMS}} + \int_X g \cdot e^{tQ}\Pi(f) \; d\mathrm{m}^{\mathrm{BMS}} + \|f\|_{C^3} \, O_g\left(e^{-\sigma t}\right).$$

The dependence on g in the implicit constant is through its C^2-norm and the injectivity radius of its support.

The "eigenvalues" λ_i above are known as *Pollicott-Ruelle resonances.* Theorem 1.1 follows from the above result by taking σ_0 to be the absolute value of the largest real part of the λ_i's. Indeed, the norm of the matrix e^{tQ} is bounded by $e^{-t\sigma_0}$. The reader is referred to Section 9.1 for a more precise discussion of the Banach spaces on which these operators live.

Given two bounded functions f and g on X, the associated correlation function is defined by

$$\rho_{f,g}(t) := \int_X f \circ g_t \cdot g \, d\mathrm{m}^{\mathrm{BMS}}, \qquad t \in \mathbb{R}.$$

Its (one-sided) Laplace transform is defined for any $z \in \mathbb{C}$ with positive real part $\mathrm{Re}(z)$ as follows:

$$\hat{\rho}_{f,g}(z) := \int_0^\infty e^{-zt} \rho_{f,g}(t) \, dt.$$

Theorem 1.2 implies that for suitably smooth f and g, $\hat{\rho}_{f,g}$ admits a meromorphic continuation to the half plane $\mathrm{Re}(z) > -\sigma$ with the only possible poles at $\{\lambda_i\}$.

Remark 1.3. The constant σ in Theorem 1.2 depends only on non-concentration parameters of Patterson-Sullivan measures near proper generalized sub-spheres of the boundary at infinity; cf. Corollary 12.2 for details. In particular, Theorem 1.2 implies that σ does not change if we replace Γ with a finite index subgroup. For example, in the case of geometrically finite hyperbolic surfaces, σ can be taken a non-decreasing function of the critical exponent δ_Γ if Γ is convex cocompact and of the quantity $2\delta_\Gamma - 1$ for cusped surfaces. The interested reader is referred to [MN20, MN21] for recent developments on a closely related problem yielding uniform resonance free regions for the Laplacian operator on random covers of convex cocompact hyperbolic surfaces.

Our analysis also yields the following result. Let δ_Γ denote the critical exponent of Γ and define

$$\sigma(\Gamma) := \begin{cases} \infty, & \text{if } \Gamma \text{ is convex cocompact,} \\ \min\{\delta_\Gamma, 2\delta_\Gamma - k_{\max}, k_{\min}\}, & \text{otherwise,} \end{cases} \tag{1.1}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively; cf. Section 3.1 for the definition of the rank of a cusp.

Theorem 1.4. *Let $k \in \mathbb{N}$. For all $f, g \in C_c^{k+2}(X)$, $\hat{\rho}_{f,g}$ is analytic in the half plane* $\mathrm{Re}(z) > 0$ *and admits a meromorphic continuation to the half plane:*

$$\mathrm{Re}(z) > -\min\{k, \sigma(\Gamma)/2\},$$

with 0 *being the only pole on the imaginary axis. In particular, when Γ is convex cocompact and $f, g \in C_c^\infty(X)$, $\hat{\rho}_{f,g}$ admits a meromorphic extension to the entire complex plane.*

Theorem 1.4 is deduced from an analogous result on the meromorphic continuation of the family of resolvent operators $z \mapsto R(z)$,

$$R(z) := \int_0^\infty e^{-zt} \mathcal{L}_t \, dt : C_c^k(X) \to C^k(X), \tag{1.2}$$

defined initially for z with large enough $\mathrm{Re}(z)$, where $\mathcal{L}_t$ is the transfer operator given by $f \mapsto f \circ g_t$; cf. Theorem 6.4 for a precise statement. Analogous results regarding resolvents were obtained for Anosov flows in [GLP13] and Axiom A flows in [DG16, DG18] leading to a resolution of a conjecture of Smale on the meromorphic continuation of the Ruelle zeta function; cf. [Sma67]. We refer the reader to [GLP13] for a discussion of the history of the latter problem.

1.2. **L^q-flattening of measures on $\mathbb{R}^d$ under convolution.** The key new ingredient in our proof of Theorem 1.1 is the statement that the conditional measures of the BMS measure along the strong unstable foliation enjoy polynomial Fourier decay outside of a very sparse set of frequencies; cf. Corollary 1.8.

The key step in the proof is an L^q-flattening result for convolutions of measures on $\mathbb{R}^d$ of independent interest. Roughly speaking, it states that the L^q-dimension (Def. 1.5) of a measure μ improves under iterated self-convolutions unless μ is concentrated near proper affine hyperplanes in

$\mathbb{R}^d$ at almost every scale. The proof of this result provided in Section 11 can be read independently of the rest of the article.

We formulate here a special case of our results under the following non-concentration condition and refer the reader to Definition 11.1 for a much weaker condition under which these results hold.

We need some notation before stating the result. Let $\mathcal{D}_k$ denote the dyadic partition of $\mathbb{R}^d$ by translates of the cube $2^{-k}[0,1)^d$ by $2^{-k}\mathbb{Z}^d$. We recall the notion of L^q-dimension of measures.

Definition 1.5. For $q > 1$, the *L^q-dimension* of a Borel probability measure μ on $\mathbb{R}^d$, denoted $\dim_q \mu$, is defined to be

$$\dim_q \mu := \liminf_{k\to\infty} \frac{-\log_2 \sum_{P\in\mathcal{D}_k} \mu(P)^q}{(q-1)k}.$$

The *Frostman exponent* of μ, denoted $\dim_\infty \mu$, is defined to be

$$\dim_\infty \mu := \liminf_{k\to\infty} \frac{\log_2 \max_{P\in\mathcal{D}_k} \mu(P)}{-k}.$$

We say that Borel measure μ on $\mathbb{R}^d$ is *uniformly affinely non-concentrated* if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ so that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and for all $x \in \mathbb{R}^d$, $0 < r \leq 1$, and every affine hyperplane $W < \mathbb{R}^d$, we have

$$\mu(W^{(\varepsilon r)} \cap B(x,r)) \leq \delta(\varepsilon)\mu(B(x,r)), \tag{1.3}$$

where $W^{(r)}$ and $B(x,r)$ denote the r-neighborhood of V and the r-ball around x respectively.

The following is our main result on flattening under convolution with non-concentrated measures.

Theorem 1.6. *Let $1 < q < \infty$ and $\eta > 9$ be given. Then, there exists $\varepsilon = \varepsilon(q,\eta) > 0$ such that if μ is any compactly supported Borel probability measure on $\mathbb{R}^d$ which is uniformly affinely non-concentrated, then*

$$\dim_q(\mu * \nu) > \dim_q \nu + \varepsilon,$$

for every compactly supported probability measure ν on $\mathbb{R}^d$ with $\dim_q \nu \leq d - \eta$.

In particular, $\dim_\infty \mu^{*n}$ converges to d at a rate depending only on the non-concentration parameters of μ, and, hence, the same holds for $\dim_q \mu^{*n}$ for all $q > 1$.

Remark 1.7. We refer the reader to Section 11 where a more quantitative form of Theorem 1.6 is obtained under a much weaker *non-uniform* non-concentration condition; cf. Definition 11.1. This quantitative form is necessary for our applications and the weaker hypothesis is essential in the presence of cusps.

The L^2-dimension case of Theorem 1.6 has the following immediate corollary asserting that the Fourier transform of affinely non-concentrated measures enjoys polynomial decay outside of a very sparse set of frequencies.

Corollary 1.8. *Let μ be as in Theorem 1.6 and denote by $\hat{\mu}$ its Fourier transform. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $T \geq 1$,*

$$\left|\left\{\|\xi\| \leq T : |\hat{\mu}(\xi)| > T^{-\delta}\right\}\right| = O_\varepsilon(T^\varepsilon),$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Corollary 1.8 generalizes the work of Kaufman [Kau84] and Tsujii [Tsu15] for self-similar measures on $\mathbb{R}$ by different methods.

Remark 1.9. A large class of dynamically defined measures, which includes self-conformal measures, is known to be affinely non-concentrated; cf. [RS20, Proposition 4.7 and Corollary 4.9] for measures on the real line and the results surveyed in [DFSU21, Section 1.3] for measures in higher dimensions under suitable irreducibility hypotheses[1]. In particular, Theorem 1.6 applies to these measures generalizing prior known special cases for certain self-similar measures on $\mathbb{R}$ by different methods; cf. [FL09, MS18].

Theorem 1.6 was obtained for measures on the real line by Rossi and Shmerkin in [RS20] under the uniform non-concentration hypothesis above. Their work builds crucially on a 1-dimensional inverse theorem due to Shmerkin in [Shm19] which was the key ingredient in his groundbreaking solution of Furstenberg's intersection conjecture. Proposition 11.10 can be regarded as a higher dimensional substitute for Shmerkin's inverse theorem. A similar higher dimensional inverse theorem for L^q-dimension was announced by Shmerkin in his ICM survey [Shm21, Section 3.8.3].

In Section 12, we show that Corollary 1.8 applies to Patterson-Sullivan measures when X is real hyperbolic (and to certain projections of these measures in the other cases, see discussion in Section 1.5 below).

For convex cocompact hyperbolic surfaces, Bourgain and Dyatlov showed that PS measures in fact have polynomially decaying Fourier transform [BD17]. Their methods are different to ours and are based on Bourgain's sum-product estimates. Their result was extended to convex cocompact Schottky real hyperbolic 3-manifolds in [LNP21] by similar methods. These results imply Corollary 1.8 in these special cases, however Corollary 1.8 also applies to measures whose Fourier transform does not tend to 0 at infinity (e.g. the coin tossing measure on the middle 1/3 Cantor set). In forthcoming work, we apply our methods to generalize these results to hyperbolic manifolds of any dimension which are not necessarily of Schottky type.

1.3. **Exponential recurrence from the cusp.** An important ingredient in our arguments is the following exponential decay result on the measure of the set of orbits with long cusp excursions, which is of independent interest. Denote by N^+ the expanding horospherical group associated to g_t for and $t > 0$ the orbits of which give rise to the strong unstable foliation. Let N_r^+ be the r-ball around identity in N^+ (cf. Section 2.5 for the definition of the metric on N^+). We denote by $\Omega \subseteq X$ the non-wandering set for the geodesic flow; i.e. the closure of the set of its periodic orbits.

Theorem 1.10. *Let $\sigma(\Gamma)$ be as in (1.1) and let $0 < \beta < \sigma(\Gamma)/2$ be given. For every $\varepsilon > 0$, there exists a compact set $K \subseteq \Omega$ and $T_0 > 0$ such that the following holds for all $T > T_0, 0 < \theta < 1$ and $x \in \Omega$. Let χ_K be the indicator function of K. Then,*

$$\mu_x^u\left(n \in N_1^+ : \int_0^T \chi_K(g_t n x)\, dt \leq (1-\theta)T\right) \ll_{\beta,x,\varepsilon} e^{-(\beta\theta-\varepsilon)T}\mu_x^u(N_1^+).$$

The implicit constant is uniform as x varies in any fixed compact set.

The reader is referred to Theorem 7.9 for a stronger and more precise statement. Theorem 1.10 implies that the Hausdorff dimension of the set of points in $N_1^+ x$ whose forward orbit asymptotically spends all of its time in the cusp is at most $\sigma(\Gamma)/2$. This bound is not sharp and can likely be improved using a refinement of our methods. We hope to return to this problem in future work.

1.4. **Prior results.** In the case Γ is convex cocompact, Theorem 1.1 is a special case of the results of [Sto11] which extend the arguments of Dolgopyat [Dol98] to Axiom A flows under certain assumptions on the regularity of the foliations and the holonomy maps. The special case of convex cocompact hyperbolic surfaces was treated in earlier work of Naud [Nau05]. The extension to frame flows on convex cocompact manifolds was treated in [SW20, CS22].

[1]The results referenced in [DFSU21] require the open set condition, while [RS20] does not.

In the case of real hyperbolic manifolds with δ_Γ strictly greater than half the dimension of the boundary at infinity, Theorem 1.1 was obtained in [EO21], with much more precise and explicit estimates on the size of the essential spectral gap. The methods of [EO21] are unitary representation theoretic, building on the work of Lax and Phillips in [LP82], for which the restriction on the critical exponent is necessary. Earlier instances of the results of [EO21] under more stringent assumptions on the size of δ_Γ were obtained by Mohammadi and Oh in [MO15], albeit the latter results are stronger in that they in fact hold for the frame flow rather than the geodesic flow.

The case of real hyperbolic geometrically finite manifolds with cusps and arbitrary critical exponent was only recently resolved independently in [LP23] where a symbolic coding of the geodesic flow was constructed. This approach builds on extensions of Dolgopyat's method to suspension flows over shifts with infinitely many symbols; cf. [AM16, AGY06]. The extension of their result to frame flows was carried out in [LPS23].

Finally, we refer the reader to [DG16] and the references therein for a discussion of the history of the microlocal approach to the problem of spectral gaps via anisotropic Sobolev spaces.

1.5. **Outline of the argument.** The article has several parts that can be read independently of one another. For the convenience of the reader, we give a brief outline of those parts.

The first part consists of Sections 2-5. After recalling some basic facts in Section 2, we prove a key doubling result, Proposition 3.1, in Section 3 for the conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the strong unstable foliation.

In Section 4, we construct a Margulis function which shows, roughly speaking, that generic orbits with respect to $\mathrm{m}^{\mathrm{BMS}}$ are biased to return to the thick part of the manifold. In Section 5, we prove a statement on average expansion of vectors in linear representations which is essential for our construction of the Margulis function. The main difficulty in the latter result in comparison with the classical setting lies in controlling the *shape* of sublevel sets of certain polynomials in order to estimate their measure with respect to conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the unstable foliation.

The second part consists of Sections 6 and 7. In Section 6, we define anisotropic Banach spaces arising as completions of spaces of smooth functions with respect to a dynamically relevant norm and study the norm of the transfer operator as well as the resolvent in their actions on these spaces in Section 7. The proof of Theorem 1.4 is completed in Section 7. The approach of these two sections follows closely the ideas of [GL06, GL08, AG13], originating in [BKL02]. Theorem 1.10 is deduced from this analysis in Section 7.6.

The third part concerns a Dolgopyat-type estimate which is a key technical estimate in the proof of Theorems 1.1 and 1.2. Its proof occupies Section 9 with auxiliary technical results in Sections 8, 10, and 12. Readers familiar with the theory of anisotropic spaces may skip directly to Section 9, taking the results on recurrence from the cusps from previous sections as a black box.

The Dolgopyat-type estimate, obtained in Theorem 9.2, provides a contraction on the norm of resolvents with large imaginary parts. Theorems 1.1 and 1.2 are deduced from this result in Section 9.1. The principle behind Theorem 9.2, due to Dolgopyat, is to exploit the non-joint integrability of the stable and unstable foliations via certain oscillatory integral estimates; cf. [Dol98, Liv04, GLP13, GPL22, BDL18].

A major difficulty in implementing this philosophy lies in estimating these oscillatory integrals against Patterson-Sullivan measures, which are *fractal* in nature in general. In particular, we cannot argue using the standard integration by parts method in previous works on exponential mixing of SRB measures using the method of anisotropic spaces, see for instance [Liv04, GLP13, GPL22, BDL18], where the unstable conditionals are of Lebesgue class.

We deal with this difficulty using Corollary 11.5 by taking advantage of the fact that the estimate in question is an *average* over oscillatory integrals. This idea is among the main contributions of this article. We hope this method can be fruitful in establishing rates of mixing of hyperbolic flows in greater generality.

In the case of variable curvature (i.e. when X is not real hyperbolic), the action of the derivative of the geodesic flow on the strong unstable distribution is non-conformal which causes additional difficulties in the analysis, particularly in the presence of cusps. We deal with this difficulty by working with the *projection* of the unstable conditionals to the directions of slowest expansion and verify non-concentration for those projections instead. See Remark 9.15 for further discussion.

In Section 10, we obtain a linearization of the so-called temporal distance function. In Section 12, we verify the non-concentration hypotheses of Corollary 1.8 (more precisely, we verify the weaker hypothesis of Corollary 11.5) for the projection of the unstable conditionals of $\mathrm{m}^{\mathrm{BMS}}$ onto the directions with weakest expansion. This allows us to apply Corollary 11.5 towards estimating the oscillatory integrals arising in Section 9. The proof of Theorem 9.2 is completed in that section.

Finally, Section 11 is dedicated to the proof of Theorem 1.6 and Corollary 1.8. Among the key ingredients in the proof are the asymmetric Balog-Szemerédi-Gowers Lemma due to Tao and Vu (Theorem 11.6) as well as Hochman's inverse theorem for the entropy of convolutions (Theorem 11.8). This section can be read independently from the rest of the article.

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2. Preliminaries

We recall here some background and definitions on geometrically finite manifolds.

2.1. **Geometrically Finite Manifolds.** The standard reference for the material in this section is [Bow93]. Suppose G is a connected simple Lie group of real rank one. Then, G can be identified with the group of orientation preserving isometries of a real, complex, quaternionic or Cayley hyperbolic space, denoted $\mathbb{H}^d_{\mathbb{K}}$, of dimension $d \geq 2$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In the case $\mathbb{K} = \mathbb{O}$, then $d = 2$.

Fix a basepoint $o \in \mathbb{H}^d_{\mathbb{K}}$. Then, G acts transitively on $\mathbb{H}^d_{\mathbb{K}}$ and the stabilizer K of o is a maximal compact subgroup of G. We shall identify $\mathbb{H}^d_{\mathbb{K}}$ with $K\backslash G$. Denote by $A = \{g_t : t \in \mathbb{R}\}$ a one parameter subgroup of G inducing the geodesic flow on the unit tangent bundle of $\mathbb{H}^d_{\mathbb{K}}$. Let $M < K$ denote the centralizer of A inside K so that the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ may be identified with $M\backslash G$. In Hopf coordinates, we can identify $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ with $\mathbb{R} \times (\partial\mathbb{H}^d_{\mathbb{K}} \times \partial\mathbb{H}^d_{\mathbb{K}} - \Delta)$, where $\partial\mathbb{H}^d_{\mathbb{K}}$ denotes the boundary at infinity and Δ denotes the diagonal.

Let $\Gamma < G$ be an infinite discrete subgroup of G. The limit set of Γ, denoted Λ_Γ, is the set of limit points of the orbit $\Gamma \cdot o$ on $\partial\mathbb{H}^d_{\mathbb{K}}$. Note that the discreteness of Γ implies that all such limit points belong to the boundary. Moreover, this definition is independent of the choice of o in view of the negative curvature of $\mathbb{H}^d_{\mathbb{K}}$. We often use Λ to denote Λ_Γ when Γ is understood from context. We say Γ is *non-elementary* if Λ_Γ is infinite.

The *hull* of Λ_Γ, denoted $\mathrm{Hull}(\Lambda_\Gamma)$, is the smallest convex subset of $\mathbb{H}^d_{\mathbb{K}}$ containing all the geodesics joining points in Λ_Γ. The convex core of the manifold $\mathbb{H}^d_{\mathbb{K}}/\Gamma$ is the smallest convex subset containing the image of $\mathrm{Hull}(\Lambda_\Gamma)$. We say $\mathbb{H}^d_{\mathbb{K}}/\Gamma$ is *geometrically finite* (resp. *convex cocompact*) if the closed 1-neighborhood of the convex core has finite volume (resp. is compact), cf. [Bow93]. The non-wandering set for the geodesic flow is the closure of the set of vectors in the unit tangent bundle whose orbit accumulates on itself. In Hopf coordinates, this set, denoted Ω, coincides with the projection of $\mathbb{R} \times (\Lambda_\Gamma \times \Lambda_\Gamma - \Delta) \bmod \Gamma$.

A useful equivalent definition of geometric finiteness is that the limit set of Γ consists entirely of radial and bounded parabolic limit points; cf. [Bow93]. This characterization of geometric finiteness will be of importance to us and so we recall here the definitions of these objects.

A point $\xi \in \Lambda$ is said to be a *radial point* if any geodesic ray terminating at ξ returns infinitely often to a bounded subset of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$. The set of radial limit points is denoted by Λ_r.

Denote by N^+ the expanding horospherical subgroup of G associated to g_t, $t \geq 0$. A point $p \in \Lambda$ is said to be a *parabolic point* if the stabilizer of p in Γ, denoted by Γ_p, is conjugate in G to an unbounded subgroup of MN^+. A parabolic limit point p is said to be *bounded* if $(\Lambda - \{p\}/\Gamma_p)$ is compact. An equivalent charachterization is that $p \in \Lambda$ is parabolic if and only if any geodesic ray terminating at p eventually leaves every compact subset of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$. The set of parabolic limit points will be denoted by Λ_p.

Given $g \in G$, we denote by g^+ the coset of P^-g in the quotient $P^-\backslash G$, where $P^- = N^-AM$ is the stable parabolic group associated to $\{g_t : t \geq 0\}$. Similarly, g^- denotes the coset P^+g in $P^+\backslash G$. Since M is contained in $P^\pm$, such a definition makes sense for vectors in the unit tangent bundle $M\backslash G$. Geometrically, for $v \in M\backslash G$, v^+ (resp. v^-) is the forward (resp. backward) endpoint of the geodesic determined by v on the boundary of $\mathbb{H}^d_{\mathbb{K}}$. Given $x \in G/\Gamma$, we say $x^\pm$ belongs to Λ if the same holds for any representative of x in G; this notion being well-defined since Λ is Γ invariant.

Notation. Throughout the remainder of the article, we fix a discrete non-elementary geometrically finite group Γ of isometries of some (irreducible) rank one symmetric space $\mathbb{H}^d_{\mathbb{K}}$ and denote by X the quotient G/Γ, where G is the isometry group of $\mathbb{H}^d_{\mathbb{K}}$.

2.2. **Standard horoballs.** Since parabolic points are fixed points of elements of Γ, Λ contains only countably many such points. Moreover, Γ contains at most finitely many conjugacy classes of parabolic subgroups. This translates to the fact that Λ_p consists of finitely many Γ orbits.

Let $\{p_1, \ldots, p_s\} \subset \partial\mathbb{H}^d_{\mathbb{K}}$ be a maximal set of nonequivalent parabolic fixed points under the action of Γ. As a consequence of geometric finiteness of Γ, one can find a finite disjoint collection of *open* horoballs $H_1, \ldots, H_s \subset \mathbb{H}^d_{\mathbb{K}}$ with the following properties (cf. [Bow93]):

(a) H_i is centered on p_i, for $i = 1, \ldots, s$.
(b) $\overline{H_i}\Gamma \cap \overline{H_j}\Gamma = \emptyset$ for all $i \neq j$.
(c) For all $i \in \{1, \ldots, s\}$ and $\gamma_1, \gamma_2 \in \Gamma$

$$\overline{H_i}\gamma_1 \cap \overline{H_i}\gamma_2 \neq \emptyset \Longrightarrow \overline{H_i}\gamma_1 = \overline{H_i}\gamma_2, \gamma_1^{-1}\gamma_2 \in \Gamma_{p_i}.$$

(d) $\mathrm{Hull}(\Lambda_\Gamma) \setminus (\bigcup_{i=1}^s H_i\Gamma)$ is compact mod Γ.

2.3. **Conformal Densities and the BMS Measure.** The *critical exponent*, denoted δ_Γ, is defined to be the infimum over all real number $s \geq 0$ such that the Poincaré series

$$P_\Gamma(s, o) := \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma\cdot o)} \tag{2.1}$$

converges. We shall simply write δ for δ_Γ when Γ is understood from context. The Busemann function is defined as follows: given $x, y \in \mathbb{H}^d_{\mathbb{K}}$ and $\xi \in \partial\mathbb{H}^d_{\mathbb{K}}$, let $\gamma : [0, \infty) \to \mathbb{H}^d_{\mathbb{K}}$ denote a geodesic ray terminating at ξ and define

$$\beta_\xi(x, y) = \lim_{t\to\infty} \mathrm{dist}(x, \gamma(t)) - \mathrm{dist}(y, \gamma(t)).$$

A Γ-invariant conformal density of dimension s is a collection of Radon measures $\left\{\nu_x : x \in \mathbb{H}^d_{\mathbb{K}}\right\}$ on the boundary satisfying

$$\gamma_*\nu_x = \nu_{\gamma x}, \qquad \text{and} \qquad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-s\beta_\xi(x,y)}, \qquad \forall x, y \in \mathbb{H}^d_{\mathbb{K}}, \xi \in \partial\mathbb{H}^d_{\mathbb{K}}, \gamma \in \Gamma.$$

Given a pair of conformal densities $\{\mu_x\}$ and $\{\nu_x\}$ of dimensions s_1 and s_2 respectively, we can form a Γ invariant measure on $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$, denoted by $m^{\mu,\nu}$ as follows: for $x = (\xi_1, \xi_2, t) \in \mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$

$$dm^{\mu,\nu}(\xi_1, \xi_2, t) = e^{s_1\beta_{\xi_1}(o,x)+s_2\beta_{\xi_2}(o,x)}\, d\mu_o(\xi_1)\, d\nu_o(\xi_2)\, dt.$$

Moreover, the measure $m^{\mu,\nu}$ is invariant by the geodesic flow.

When Γ is geometrically finite and $\mathbb{K} = \mathbb{R}$, Patterson [Pat76] and Sullivan [Sul79] showed the existence of a unique (up to scaling) Γ-invariant conformal density of dimension δ_Γ, denoted $\{\mu_x^{\mathrm{PS}} : x \in \mathbb{H}^d_{\mathbb{R}}\}$. Geometric finiteness also implies that the measure $m^{\mu^{\mathrm{PS}},\mu^{\mathrm{PS}}}$ descends to a finite measure of full support on Ω and is the unique measure of maximal entropy for the geodesic flow. This measure is called the Bowen-Margulis-Sullivan (BMS for short) measure and is denoted $\mathrm{m}^{\mathrm{BMS}}$.

Since the fibers of the projection from G/Γ to $\mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}/\Gamma$ are compact and parametrized by the group M, we can lift such a measure to one G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, by taking locally the product with the Haar probability measure on M. Since M commutes with the geodesic flow, this lift is invariant under the group A. We refer the reader to [Rob03] and [PPS15] and references therein for details of the construction in much greater generality than that of $\mathbb{H}^d_{\mathbb{K}}$.

2.4. **Stable and unstable foliations and leafwise measures.** The fibers of the projection $G \to \mathrm{T}^1\mathbb{H}^d_{\mathbb{K}}$ are given by the compact group M, which is the centralizer of A inside the maximal compact group K. In particular, we may lift $\mathrm{m}^{\mathrm{BMS}}$ to a measure on G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, and given locally by the product of $\mathrm{m}^{\mathrm{BMS}}$ with the Haar probability measure on M. The leafwise measures of $\mathrm{m}^{\mathrm{BMS}}$ on N^+ orbits are given as follows:

$$d\mu_x^u(n) = e^{\delta_\Gamma \beta_{(nx)^+}(o,nx)} d\mu_o^{\mathrm{PS}}((nx)^+). \tag{2.2}$$

They satisfy the following equivariance property under the geodesic flow:

$$\mu_{g_t x}^u = e^{\delta t} \mathrm{Ad}(g_t)_* \mu_x^u. \tag{2.3}$$

Moreover, it follows readily from the definitions that for all $n \in N^+$,

$$(n)_* \mu_{nx}^u = \mu_x^u, \tag{2.4}$$

where $(n)_*\mu_{nz}^u$ is the pushforward of μ_{nz}^u under the map $u \mapsto un$ from N^+ to itself. Finally, since M normalizes N^+ and leaves $\mathrm{m}^{\mathrm{BMS}}$ invariant, this implies that these conditionals are $\mathrm{Ad}(M)$-invariant: for all $m \in M$,

$$\mu_{mx}^u = \mathrm{Ad}(m)_* \mu_x^u. \tag{2.5}$$

2.5. **Cygan metrics.** We recall the definition of the Cygan metric on N^+, denoted d_{N^+}. These metrics are right invariant under translation by N^+, and satisfy the following convenient scaling property under conjugation by g_t. For all $r > 0$, if N_r^+ denotes the ball of radius r around identity in that metric and $t \in \mathbb{R}$, then

$$\mathrm{Ad}(g_t)(N_r^+) = N_{e^t r}^+. \tag{2.6}$$

To define the metric, we need some notation which we use throughout the article. For $x \in \mathbb{K}$, denote by $\bar{x}$ its $\mathbb{K}$-conjugate and by $|x| := \sqrt{\bar{x}x}$ its modulus. This modulus extends to a norm on $\mathbb{K}^n$ by setting

$$\|u\|^2 := \sum_i |u_i|^2,$$

for $u = (u_1, \ldots, u_n) \in \mathbb{K}^n$.

We let $\mathrm{Im}\mathbb{K}$ denote those $x \in \mathbb{K}$ such that $\bar{x} = -x$. For example, $\mathrm{Im}\mathbb{K}$ is the pure imaginary numbers and the subspace spanned by the quaternions i, j and k in the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$ respectively. For $u \in \mathbb{K}$, we write $\mathrm{Re}(u) = (u + \bar{u})/2$ and $\mathrm{Im}(u) = (u - \bar{u})/2$.

The Lie algebra $\mathfrak{n}^+$ of N^+ splits under $\mathrm{Ad}(g_t)$ into eigenspaces as $\mathfrak{n}_\alpha^+ \oplus \mathfrak{n}_{2\alpha}^+$, where $\mathfrak{n}_{2\alpha}^+ = 0$ if and only if $\mathbb{K} = \mathbb{R}$. Moreover, we have the identification $\mathfrak{n}_\alpha^+ \cong \mathbb{K}^{d-1}$ and $\mathfrak{n}_{2\alpha}^+ \cong \mathrm{Im}(\mathbb{K})$ as real vector spaces; cf. [Mos73, Section 19]. We denote by $\|\cdot\|'$ the following quasi-norm on $\mathfrak{n}^+$: given $(u,s) \in \mathfrak{n}^+_\alpha \oplus \mathfrak{n}^+_{2\alpha}$,

$$\|(u,s)\|' := \left(\|u\|^4 + |s|^2\right)^{1/4}. \tag{2.7}$$

With this notation, we can define the metric as follows: the distance of $n := \exp(u,s)$ to identity is given by:

$$d_{N^+}(n, \mathrm{Id}) := \|(u,s)\|'. \tag{2.8}$$

Given $n_1, n_2 \in N^+$, we set $d_{N^+}(n_1,n_2) = d_{N^+}(n_1 n_2^{-1}, \mathrm{Id})$.

2.6. **Local stable holonomy.** In this Section, we recall the definition of (stable) holonomy maps which are essential for our arguments. We give a simplified discussion of this topic which is sufficient in our homogeneous setting homogeneous. Let $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. Since the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism near identity, we can choose the norm on the Lie algebra so that the following holds. We can find maps $p^- : N_1^+ \to P^- = N^-AM$ and $u^+ : N_2^+ \to N^+$ so that

$$nu^- = p^-(n)u^+(n), \qquad \forall n \in N_2^+. \tag{2.9}$$

Then, it follows by (2.2) that for all $n \in N_2^+$, we have

$$d\mu_y^u(u^+(n)) = e^{\delta\beta_{(nx)^+}(u^+(n)y, nx)} d\mu_x^u(n).$$

Moreover, by further scaling the metrics if necessary, we can ensure that these maps are diffeomorphisms onto their images. In particular, writing $\Phi(nx) = u^+(n)y$, we obtain the following change of variables formula: for all $f \in C(N_2^+)$,

$$\int f(n)\, d\mu_x^u(n) = \int f((u^+)^{-1}(n)) e^{-\delta\beta_{\Phi^{-1}(ny)}(ny, \Phi^{-1}(ny))}\, d\mu_y^u(n). \tag{2.10}$$

Remark 2.1. To avoid cluttering the notation with auxiliary constants, we shall assume that the N^- component of $p^-(n)$ belongs to N_2^- for all $n \in N_2^+$ whenever u^- belongs to N_1^-.

2.7. **Notational convention.** Throughout the article, given two quantities A and B, we use the Vinogradov notation $A \ll B$ to mean that there exists a constant $C \geq 1$, possibly depending on Γ and the dimension of G, such that $|A| \leq CB$. In particular, this dependence on Γ is suppressed in all of our implicit constants, except when we wish to emphasize it. The dependence on Γ may include for instance the diameter of the complement of our choice of cusp neighborhoods inside Ω and the volume of the unit neighborhood of Ω. We write $A \ll_{x,y} B$ to indicate that the implicit constant depends parameters x and y. We also write $A = O_x(B)$ to mean $A \ll_x B$.

3. Doubling Properties of Leafwise Measures

The goal of this section is to prove the following useful consequence of the global measure formula on the doubling properties of the leafwise measures. The result is an immediate consequence of Sullivan's shadow lemma in the case Γ is convex cocompact. In particular, the content of the following result is the uniformity, even in the case Ω is not compact. The argument is based on the topological transitivity of the geodesic flow when restricted to Ω.

Define the following exponents:

$$\begin{aligned} \Delta &:= \min\{\delta, 2\delta - k_{\max}, k_{\min}\}, \\ \Delta_+ &:= \max\{\delta, 2\delta - k_{\min}, k_{\max}\}. \end{aligned} \tag{3.1}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively. If Γ has no parabolic points, we set $k_{\max} = k_{\min} = \delta$, so that $\Delta = \Delta_+ = \delta$.

Proposition 3.1 (Global Doubling and Decay)**.** *For every $0 < \sigma \leq 5$, $x \in N_2^-\Omega$ and $0 < r \leq 1$, we have*

$$\mu_x^u(N_{\sigma r}^+) \ll \begin{cases} \sigma^{\Delta} \cdot \mu_x^u(N_r^+) & \forall 0 < \sigma \leq 1, 0 < r \leq 1, \\ \sigma^{\Delta_+} \cdot \mu_x^u(N_r^+) & \forall \sigma > 1, 0 < r \leq 5/\sigma. \end{cases}$$

Remark 3.2. The above proposition has very different flavor when applied with $\sigma < 1$, compared with $\sigma > 1$. In the former case, we obtain a global rate of decay of the measure of balls on the boundary, centered in the limit set. In the latter case, we obtain the so-called Federer property for our leafwise measures.

Remark 3.3. The restriction that $r \leq 5/\sigma$ in the case $\sigma > 1$ allows for a uniform implied constant. The proof shows that in fact, when $\sigma > 1$, the statement holds for any $0 < r \leq 1$, but with an implied constant depending on σ.

3.1. **Global Measure Formula.** Our basic tool in proving Proposition 3.1 is the extension of Sullivan's shadow lemma known as the global measure formula, which we recall in this section.

Given a parabolic fixed point $p \in \Lambda$, with stabilizer $\Gamma_p \subset \Gamma$, we define *the rank of p* to be twice the critical exponent of the Poincaré series $P_{\Gamma_p}(s, o)$ associated with Γ_p; cf. (2.1).

Given $\xi \in \partial\mathbb{H}_{\mathbb{K}}^d$, we let $[o\xi)$ denote the geodesic ray. For $t \in \mathbb{R}_+$, denote by $\xi(t)$ the point at distance t from o on $[o\xi)$. For $x \in \mathbb{H}_{\mathbb{K}}^d$, define the $\mathcal{O}(x)$ to be the *shadow* of unit ball $B(x, 1)$ in $\mathbb{H}_{\mathbb{K}}^d$ on the boundary as viewed from o. More precisely,

$$\mathcal{O}(x) := \left\{ \xi \in \partial\mathbb{H}_{\mathbb{K}}^d : [o\xi) \cap B(x, 1) \neq \emptyset \right\}.$$

Shadows form a convenient, dynamically defined, collection of neighborhoods of points on the boundary.

The following generalization of Sullivan's shadow lemma gives precise estimates on the measures of shadows with respect to Patterson-Sullivan measures.

Theorem 3.4 (Theorem 3.2, [Sch04])**.** *There exists $C = C(\Gamma, o) \geq 1$ such that for every $\xi \in \Lambda$ and all $t > 0$,*

$$C^{-1} \leq \frac{\mu_o^{\mathrm{PS}}(\mathcal{O}(\xi(t)))}{e^{-\delta t} e^{d(t)(k(\xi(t)) - \delta)}} \leq C,$$

where

$$d(t) = \mathrm{dist}(\xi(t), \Gamma \cdot o),$$

and $k(\xi(t))$ denotes the rank of a parabolic fixed point p if $\xi(t)$ is contained in a standard horoball centered at p and otherwise $k(\xi(t)) = \delta$.

A version of Theorem 3.4 was obtained earlier for real hyperbolic spaces in [SV95] and for complex and quaternionic hyperbolic spaces in [New03].

3.2. **Proof of Proposition 3.1.** Assume that $\sigma \leq 1$, the proof in the case $\sigma > 1$ being identical.

Fix a non-negative C^∞ bump function ψ supported inside N_1^+ and having value identically 1 on $N_{1/2}^+$. Given $\varepsilon > 0$, let $\psi_\varepsilon(n) = \psi(\mathrm{Ad}(g_{-\log\varepsilon})(n))$. Note that the condition that $\psi_\varepsilon(\mathrm{Id}) = \psi(\mathrm{Id}) = 1$ implies that for $x \in X$ with $x^+ \in \Lambda$,

$$\mu_x^u(\psi_\varepsilon) > 0, \qquad \forall \varepsilon > 0. \tag{3.2}$$

Note further that for any $r > 0$, we have that $\chi_{N_r^+} \leq \psi_r \leq \chi_{N_{2r}^+}$.

First, we establish a uniform bound over $x \in \Omega$. Consider the following function $f_\sigma : \Omega \to (0, \infty)$:

$$f_\sigma(x) = \sup_{0 < r \leq 1} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)}.$$

We claim that it suffices to prove that

$$f_\sigma(x) \ll \sigma^\Delta, \tag{3.3}$$

uniformly over all $x \in \Omega$ and $0 < \sigma \leq 1$. Indeed, fix some $0 < r \leq 1$ and $0 < \sigma \leq 1$. By enlarging our implicit constant if necessary, we may assume that $\sigma \leq 1/4$. From the above properties of ψ, we see that

$$\mu_x^u(N_{\sigma r}^+) \leq \mu_x^u(\psi_{(4\sigma)(r/2)}) \ll \sigma^\Delta \mu_x^u(\psi_{r/2}) \leq \sigma^\Delta \mu_x^u(N_r^+).$$

Hence, it remains to prove (3.3). By [Rob03, Lemme 1.16], for each given $r > 0$, the map $x \mapsto \mu_x^u(\psi_{\sigma r})/\mu_x^u(\psi_r)$ is a continuous function on Ω. Indeed, the weak-$*$ continuity of the map $x \mapsto \mu_x^u$ is the reason we work with bump functions instead of indicator functions directly. Moreover, continuity of these functions implies that f_σ is lower semi-continuous.

The crucial observation regarding f_σ is as follows. In view of (2.3), we have for $t \geq 0$,

$$f_\sigma(g_t x) = \sup_{0 < r \leq e^{-t}} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)} \leq f_\sigma(x).$$

Hence, for all $B \in \mathbb{R}$, the sub-level sets $\Omega_{<B} := \{f_\sigma < B\}$ are invariant by g_t for all $t \geq 0$. On the other hand, the restriction of the (forward) geodesic flow to Ω is topologically transitive. In particular, any invariant subset of Ω with non-empty interior must be dense in Ω. Hence, in view of the lower semi-continuity of f_σ, to prove (3.3), it suffices to show that f_σ satisfies (3.3) for all x in some open subset of Ω.

Recall we fixed a basepoint $o \in \mathbb{H}_{\mathbb{K}}^d$ belonging to the hull of the limit set. Let $x_o \in G$ denote a lift of o whose projection to G/Γ belongs to Ω. Let E denote the unit neighborhood of x_o. We show that $E \cap \Omega \subset \left\{f_\sigma \ll \sigma^\Delta\right\}$. Without loss of generality, we may further assume that $\sigma < 1/2$, by enlarging the implicit constant if necessary.

First, note that the definition of the conditional measures μ_x^u immediately gives

$$\mu_x^u|_{N_4^+} \asymp \mu_o^{\mathrm{PS}}|_{(N_4^+ \cdot x)^+}, \qquad \forall x \in E.$$

It follows that

$$\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+) \ll \mu_x^u(\psi_r) \ll \mu_o^{\mathrm{PS}}((N_{2r}^+ \cdot x)^+),$$

for all $0 \leq r \leq 2$ and $x \in E$. Hence, it will suffice to show

$$\frac{\mu_o^{\mathrm{PS}}((N_{\sigma r}^+ \cdot x)^+)}{\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+)} \ll \sigma^\Delta,$$

for all $0 < \sigma < 1$.

To this end, there is a constant $C_1 \geq 1$ such that the following holds; cf. [Cor90, Theorem 2.2][2]. For all $x \in E$, if $\xi = x^+$, then, the shadow $S_r = \{(nx)^+ : n \in N_r^+\}$ satisfies

$$\mathcal{O}(\xi(|\log r| + C_1)) \subseteq S_r \subseteq \mathcal{O}(\xi(|\log r| - C_1)), \qquad \forall 0 < r \leq 2. \tag{3.4}$$

Here, and throughout the rest of the proof, if $s \leq 0$, we use the convention

$$\mathcal{O}(\xi(s)) = \mathcal{O}(\xi(0)) = \partial \mathbb{H}_{\mathbb{K}}^d.$$

Fix some arbitrary $x \in E$ and let $\xi = x^+$. To simplify notation, set for any $t, r > 0$,

$$\begin{aligned} t_\sigma &:= \max\left\{|\log \sigma r| - C_1, 0\right\}, & t_r &:= |\log r| + C_1, \\ d(t) &:= \mathrm{dist}(\xi(t), \Gamma \cdot o), & k(t) &:= k(\xi(t)), \end{aligned}$$

where $k(\xi(t))$ is as in the notation of Theorem 3.4.

[2]The quoted result in [Cor90] is stated in terms of the so-called Carnot-Caratheodory metric on N^+, which enjoys the same scaling property in (2.6). In particular, this metric is equivalent to the Cygan metric in (2.8) by compactness of the unit sphere in the latter.

By further enlarging the implicit constant, we may assume for the rest of the argument that

$$-\log \sigma > 2C_1.$$

This insures that $t_\sigma \geq t_r$ and avoids some trivialities.

Let $0 < r \leq 1$ be arbitrary. We define constants $\sigma_0 := \sigma \leq \sigma_1 \leq \sigma_2 \leq \sigma_3 := 1$ as follows. If $k(t_\sigma) = \delta$ (i.e. $\xi(t_\sigma)$ is in the complement of the cusp neighborhoods), we set $\sigma_1 = \sigma$. Otherwise, we define σ_1 by the property that $\xi(|\log \sigma_1 r|)$ is the first point along the geodesic segment joining $\xi(t_\sigma)$ and $\xi(t_r)$ (traveling from the former point to the latter) meets the boundary of the horoball containing $\xi(t_\sigma)$. Similarly, if $k(t_r) = \delta$, we set $\sigma_2 = 1$. Otherwise, we define σ_2 by the property that $\xi(|\log \sigma_2 r|)$ is the first point along the same segment, now travelling from $\xi(t_r)$ towards $\xi(t_\sigma)$, which intersects the boudary of the horoball containing $\xi(t_r)$. Define

$$t_{\sigma_0} := t_\sigma, \qquad t_{\sigma_3} := t_r, \qquad t_{\sigma_i} := |\log \sigma_i r| \quad \text{for } i = 1, 2.$$

In this notation, we first observe that $k(t_{\sigma_1}) = k(t_{\sigma_2}) = \delta$. In particular, Theorem 3.4 yields

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \ll \left(\frac{\sigma_1}{\sigma_2}\right)^\delta.$$

Note further that the projection map $\mathbb{H}^d_{\mathbb{K}} \to \mathbb{H}^d_{\mathbb{K}}/\Gamma$ restricts to an (isometric) embedding on cusp horoballs. Combined with convexity of horoballs and the fact that geodesics in $\mathbb{H}^d_{\mathbb{K}}$ are unique distance minimizers, this implies that, for $i = 0, 2$, the distance between the projections of $\xi(t_{\sigma_i})$ and $\xi(t_{\sigma_{i+1}})$ to $\mathbb{H}^d_{\mathbb{K}}/\Gamma$ is equal to $|t_{\sigma_i} - t_{\sigma_{i+1}}|$. In particular, there is a constant $C_2 \geq 1$, depending only on the diameter of the complement of the cusp neighborhoods in the quotient $\mathbb{H}^d_{\mathbb{K}}$ and on the constant C_1, such that, for $i = 0, 2$, we have

$$-C_2 - \log(\sigma_i/\sigma_{i+1}) \leq d(t_{\sigma_i}) \leq -\log(\sigma_i/\sigma_{i+1}) + C_2.$$

Hence, it follows using Theorem 3.4 and the above discussion that

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \ll \left(\frac{\sigma_0}{\sigma_1}\right)^\delta e^{d(t_{\sigma_0})(k(t_{\sigma_0})-\delta)} \ll \left(\frac{\sigma_0}{\sigma_1}\right)^{2\delta - k(t_{\sigma_0})}.$$

Similarly, we obtain

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_3 r})} \ll \left(\frac{\sigma_2}{\sigma_3}\right)^\delta e^{-d(t_{\sigma_3})(k(t_{\sigma_3})-\delta)} \ll \left(\frac{\sigma_2}{\sigma_3}\right)^{k(t_{\sigma_3})}.$$

Therefore, using the following trivial identity

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma r})}{\mu_o^{\mathrm{PS}}(S_r)} = \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_r)},$$

we see that $f(x) \ll \sigma^\Delta$, where Δ is as in the statement of the proposition. As $x \in E$ was arbitrary, we find that $E \subset \left\{ f_\sigma \ll \sigma^\Delta \right\}$, thus concluding the proof in the case $\sigma \leq 1$. Note that in the case $\sigma > 1$, the constants σ_i satisfy $\sigma_i/\sigma_{i+1} \geq 1$, so that combining the 3 estimates requires taking the maximum over the exponents, yielding the bound with Δ_+ in place of Δ in this case.

Now, let $r \in (0, 1]$ and suppose $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. By [Cor90, Theorem 2.2], the analog of (3.4) holds, but with shadows from the viewpoint of x and y, in place of the fixed basepoint o. Recalling the map $n \mapsto u^+(n)$ in (2.9), one checks that this implies that this map is Lipschitz on N_1^+ with respect to the Cygan metric, with Lipschitz constant $\asymp C_1$. Moreover, the Jacobian of the change of variables associated to this map with respect to the measures μ_x^u and μ_y^u is bounded on N_1^+, independently of y and u^-; cf. (2.10) for a formula for this Jacobian. Hence, the estimates for $x \in N_2^- \Omega$ follow from their counterparts for points in Ω.

4. Margulis Functions In Infinite Volume

We construct Margulis functions on Ω which allow us to obtain quantitative recurrence estimates to compact sets. Our construction is similar to the one in [BQ11] in the case of lattices in rank 1 groups. We use geometric finiteness of Γ to establish the analogous properties more generally. The idea of Margulis functions originated in [EMM98].

Throughout this section, we assume Γ is a non-elementary, geometrically finite group containing parabolic elements. The following is the main result of this section. A similar result in the special case of quotients of $\mathrm{SL}_2(\mathbb{R})$ follows from combining Lemma 9.9 and Proposition 7.6 in [MO23].

Theorem 4.1. *Let $\Delta > 0$ denote the constant in (3.1). For every $0 < \beta < \Delta/2$, there exists a proper function $V_\beta : N_1^-\Omega \to \mathbb{R}_+$ such that the following holds. There is a constant $c \geq 1$ such that for all $x \in N_1^-\Omega$ and $t \geq 0$,*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_t n x)\, d\mu_x^u(n) \leq c e^{-\beta t} V_\beta(x) + c.$$

Our key tool in establishing Theorem 4.1 is Proposition 4.2, which is a statement regarding average expansion of vectors in linear represearntations of G. The fractal nature of the conditional measures μ_x^u poses serious difficulties in establishing this latter result.

4.1. Construction of Margulis functions.

Let $p_1, \ldots, p_d \in \Lambda$ be a maximal set of inequivalent parabolic fixed points and for each i, let Γ_i denote the stabilizer of p_i in Γ. Let $P_i < G$ denote the parabolic subgroup of G fixing p_i. Denote by U_i the unipotent radical of P_i and by A_i a maximal $\mathbb{R}$-split torus inside P_i. Then, each U_i is a maximal connected unipotent subgroup of G admitting a closed (but not necessarily compact) orbit from identity in G/Γ. As all maximal unipotent subgroups of G are conjugate, we fix elements $h_i \in G$ so that $h_i U_i h_i^{-1} = N^+$. Note further that G admits an Iwasawa decomposition of the form $G = K A_i U_i$ for each i, where K is our fixed maximal compact subgroup.

Denote by W the adjoint representation of G on its Lie algebra. The specific choice of representation is not essential for the construction, but is convenient for making some parameters more explicit. We endow W with a norm that is invariant by K.

Let $0 \neq v_0 \in W$ denote a vector that is fixed by N^+. In particular, v_0 is a highest weight vector for the diagonal group A (with respect to the ordering determined by declaring the roots in N^+ to be positive). Let $v_i = h_i v_0 / \|h_i v_0\|$. Note that each of the vectors v_i is fixed by U_i and is a weight vector for A_i. In particular, there is an additive character $\chi_i : A_i \to \mathbb{R}$ such that

$$a \cdot v_i = e^{\chi_i(a)} v_i, \qquad \forall a \in A_i. \tag{4.1}$$

We denote by A_i^+ the subsemigroup of A_i which expands U_i (i.e. the positive Weyl chamber determined by U_i). We let $\alpha_i : A_i \to \mathbb{R}$ denote the simple root of A_i in $\mathrm{Lie}(U_i)$. Then,

$$\chi_i = \chi_{\mathbb{K}} \alpha_i, \qquad \chi_{\mathbb{K}} = \begin{cases} 1, & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}. \end{cases} \tag{4.2}$$

Given $\beta > 0$, we define a function $V_\beta : G/\Gamma \to \mathbb{R}_+$ as follows:

$$V_\beta(g\Gamma) := \max_{w \in \bigcup_{i=1}^d g\Gamma \cdot v_i} \|w\|^{-\beta/\chi_{\mathbb{K}}}. \tag{4.3}$$

The fact that $V_\beta(g\Gamma)$ is indeed a maximum will follow from Lemma 4.6.

4.2. Linear expansion.

The following result is our key tool in establishing the contraction estimate on V_β in Theorem 4.1.

Proposition 4.2. *For every $0 \leq \beta < \Delta/2$, there exists $C = C(\beta) \geq 1$ so that for all $t > 0$, $x \in N_1^- \Omega$, and all non-zero vectors v in the orbit $G \cdot v_0 \subset W$, we have*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u(n) \leq C e^{-\beta t} \|v\|^{-\beta/\chi_{\mathbb{K}}}.$$

We postpone the proof of Proposition 4.2 to Section 5. Let $\pi_+ : W \to W^+$ denote the projection onto the highest weight space of g_t. The difficulty in the proof of Proposition 4.2 beyond the case $G = \mathrm{SL}_2(\mathbb{R})$ lies in controlling the *shape* of the subset of N^+ on which $\|\pi_+(n \cdot v)\|$ is small, so that we may apply the decay results from Proposition 3.1, that are valid only for balls of the form N_ε^+. We deal with this problem by using a convexity trick. A suitable analog of the above result holds for any non-trivial linear representation of G.

The following proposition establishes several geometric properties of the functions V_β which are useful in proving, and applying, Theorem 4.1. summarizes the main geometric properties of the functions V_β. This result is proved in Section 4.4.

Proposition 4.3. *Suppose V_β is as in (4.3). Then,*

(1) For every x in the unit neighborhood of Ω, we have that

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathbb{K}}/\beta}(x),$$

where $\mathrm{inj}(x)$ denotes the injectivity radius at x. In particular, V_β is proper on Ω.

(2) For all $g \in G$ and all $x \in X$,

$$\|g\|^{-\beta} V_\beta(x) \leq V_\beta(gx) \leq \left\|g^{-1}\right\|^\beta V_\beta(x).$$

(3) There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_0$.

4.3. **Proof of Theorem 4.1.** In this section, we use Proposition 4.3 to translate the linear expansion estimates in Proposition 4.2 into a contraction estimate for the functions V_β.

Let $t_0 > 0$ be be given and define

$$\omega_0 := \sup_{n \in N_1^+} \max \left\{ \|g_{t_0} n\|^{1/\chi_{\mathbb{K}}}, \left\|(g_{t_0} n)^{-1}\right\|^{1/\chi_{\mathbb{K}}} \right\},$$

where $\|\cdot\|$ denotes the operator norm of the action of G on W. Then, for all $n \in N_1^+$ and all $x \in X$, we have

$$\omega_0^{-1} V_1(x) \leq V_1(g_{t_0} n x) \leq \omega_0 V_1(x), \tag{4.4}$$

where $V_1 = V_\beta$ for $\beta = 1$.

Let ε_0 be as in Proposition 4.3(3). Suppose $x \in X$ is such that $V_1(x) \leq \omega_0/\varepsilon_0$. Then, by (4.4), for any $\beta > 0$, we have that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x) \, d\mu_x^u(n) \leq B_0 := (\omega_0^2 \varepsilon_0^{-1})^\beta. \tag{4.5}$$

Now, suppose $x \in N_1^- \Omega$ is such that $V_1(x) \geq \omega_0/\varepsilon_0$ and write $x = g\Gamma$ for some $g \in G$. Then, by Proposition 4.3(3), there exists a unique vector $v_\star \in \bigcup_i g\Gamma \cdot v_i$ satisfying $V_1(x) = \|v_\star\|^{-1/\chi_{\mathbb{K}}}$. Moreover, by (4.4), we have that $V_1(g_{t_0} n x) \geq 1/\varepsilon_0$ for all $n \in N_1^+$. And, by definition of ω_0, for all $n \in N_1^+$, $\|g_{t_0} n v_\star\|^{1/\chi_{\mathbb{K}}} \leq \varepsilon_0$. Thus, applying Proposition 4.3(3) once more, we see that $g_{t_0} n v_\star$ is the unique vector in $\bigcup_i g_{t_0} n g\Gamma \cdot v_i$ satisfying

$$V_\beta(g_{t_0} n x) = \|g_{t_0} n v_\star\|^{-1/\chi_{\mathbb{K}}}, \qquad \forall n \in N_1^+.$$

Moreover, since the vectors v_i all belong to the G-orbit of v_0, it follows that $v_\star$ also belongs to $G \cdot v_0$. Thus, we may apply Proposition 4.2 as follows. Fix some $\beta > 0$ and let $C = C(\beta) \geq 1$ be the constant in the conclusion of the proposition. Then,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} nx) d\mu_x^u = \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_{t_0} n v_\star\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u \leq C e^{-\beta t_0} \|v_\star\|^{-\beta/\chi_{\mathbb{K}}} = C e^{-\beta t_0} V_\beta(x).$$

Combining this estimate with (4.5), we obtain for any fixed t_0,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} nx) \, d\mu_x^u(n) \leq C e^{-\beta t_0} V_\beta(x) + B_0, \tag{4.6}$$

for all $x \in \Omega$. We claim that there is a constant $c_1 = c_1(\beta) > 0$ such that, if t_0 is large enough, depending on β, then

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{kt_0} nx) \, d\mu_x^u(n) \leq c_1^k e^{-\beta k t_0} V_\beta(x) + 2B_0, \tag{4.7}$$

for all $k \in \mathbb{N}$. By Proposition 4.3, this claim completes the proof since $V_\beta(g_t y) \ll V_\beta(g_{\lfloor t/t_0 \rfloor t_0} y)$, for all $t \geq 0$ and $y \in X$, with an implied constant depending only on t_0 and β.

The proof of (4.7) is by now a standard argument, with the key ingredient in carrying it out being the doubling estimate Proposition 3.1. We proceed by induction. Let $k \in \mathbb{N}$ be arbitrary and assume that (4.7) holds for such k. Let $\left\{ n_i \in \mathrm{Ad}(g_{kt_0})(N_1^+) : i \in I \right\}$ denote a finite collection of points in the support of $\mu_{g_{kt_0}x}^u$ such that $N_1^+ n_i$ covers the part of the support inside $\mathrm{Ad}(g_{kt_0}(N_1^+))$. We can find such a cover with uniformly bounded multiplicity, depending only on N^+. That is

$$\sum_{i \in I} \chi_{N_1^+ n_i}(n) \ll \chi_{\cup_i N_1^+ n_i}(n), \qquad \forall n \in N^+.$$

Let $x_i = n_i g_{kt_0} x$. By (4.6), and a change of variable, cf. (2.3) and (2.4), we obtain

$$e^{\delta k t_0} \int_{N_1^+} V_\beta(g_{(k+1)t_0} nx) \, d\mu_x^u \leq \sum_{i \in I} \int_{N_1^+} V_\beta(g_{t_0} n x_i) \, d\mu_{x_i}^u \leq \sum_{i \in I} \mu_{x_i}^u(N_1^+) \left(C e^{-\beta t_0} V_\beta(x_i) + B_0 \right).$$

It follows using Proposition 4.3 that $\mu_y^u(N_1^+) V_\beta(y) \ll \int_{N_1^+} V_\beta(ny) \, d\mu_y^u(n)$ for all $y \in X$. Hence,

$$\int_{N_1^+} V_\beta(g_{(k+1)t_0} nx) \, d\mu_x^u(n) \ll e^{-\delta k t_0} \sum_{i \in I} \int_{N_1^+} \left(C e^{-\beta t_0} V_\beta(n x_i) + B_0 \right) \, d\mu_{x_i}^u(n).$$

Note that since g_t expands N^+ by at least e^t, we have

$$\mathcal{A}_k := \mathrm{Ad}(g_{-kt_0}) \left(\bigcup_i N_1^+ n_i \right) \subseteq N_2^+.$$

Using the bounded multiplicity property of the cover, we see that, for any non-negative function φ, we have

$$\sum_{i \in I} \int_{N_1^+} \varphi(n x_i) \, d\mu_{x_i}^u = \int_{N^+} \varphi(n g_{kt_0} x) \sum_{i \in I} \chi_{N_1^+ n_i}(n) \, d\mu_{g_{kt_0}x}^u \ll \int_{\bigcup_i N_1^+ n_i} \varphi(n g_{kt_0} x) \, d\mu_{g_{kt_0}x}^u.$$

Changing variables back so the integrals take place against μ_x^u, we obtain

$$\begin{aligned} e^{-\delta k t_0} \sum_{i \in I} \int_{N_1^+} (C e^{-\beta t_0} V_\beta(n x_i) + B_0) \, d\mu_{x_i}^u &\ll \int_{\mathcal{A}_k} \left(C e^{-\beta t_0} V_\beta(g_{kt_0} nx) + B_0 \right) \, d\mu_x^u \\ &\leq C e^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} nx) \, d\mu_x^u + B_0 \mu_x^u(N_2^+). \end{aligned}$$

To apply the induction hypothesis, we again pick a cover of N_2^+ by balls of the form $N_1^+ n$, for a collection of points $n \in N_2^+$ in the support of μ_x^u. We can arrange for such a collection to have a uniformly bounded cardinality and multiplicity. By essentially repeating the above argument, and using our induction hypothesis for k, in addition to the doubling property in Proposition 3.1, we obtain

$$Ce^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} nx)\, d\mu_x^u + B_0 \mu_x^u(N_2^+) \ll (Cc_1^k e^{-\beta(k+1)t_0} V_\beta(x) + 2B_0 Ce^{-\beta t_0} + B_0)\mu_x^u(N_1^+),$$

where we also used Proposition 4.3 to ensure that $V_\beta(nx) \ll V_\beta(x)$, for all $n \in N_3^+$. Taking c_1 to be larger than the product of C with all the uniform implied constants accumulated thus far in the argument, we obtain

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{(k+1)t_0} nx)\, d\mu_x^u(n) \leq c_1^{k+1} e^{-\beta(k+1)t_0} V_\beta(x) + 2c_1 e^{-\beta t_0} B_0 + B_0.$$

Taking t_0 large enough so that $2c_1 e^{-\beta t_0} \leq 1$ completes the proof.

4.4. **Geometric properties of Margulis functions and proof of Proposition 4.3.** In this section, we give a geometric interpretation of the functions V_β which allows us to prove Proposition 4.3. Item (2) follows directly from the definitions, so we focus on the remaining properties.

The data in the definition of V_β allows us to give a linear description of cusp neighborhoods as follows. Given $g \in G$ and i, write $g = kau$ for some $k \in K$, $a \in A_i$ and $u \in U_i$. Geometrically, the size of the A component in the Iwasawa decomposition $G = KA_iU_i$ corresponds to the value of the Busemann cocycle $|\beta_{p_i}(Kg, o)|$, where Kg is the image of g in $K\backslash G$; cf. [BQ16, Remark 6.5] and the references therein for the precise statement. This has the following consequence. We can find $0 < \varepsilon_i < 1$ such that

$$\left\| \mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)} \right\| < \varepsilon_i \Longleftrightarrow Kg \in H_{p_i}, \tag{4.8}$$

where H_{p_i} is the standard horoball based at p_i in $\mathbb{H}^d_{\mathbb{K}} \cong K\backslash G$.

The functions $V_\beta(x)$ roughly measure how far into the cusp x is. More precisely, we have the following lemma.

Lemma 4.4. *The restriction of V_β to any bounded neighborhood of Ω is a proper map.*

Proof. In view of Property (2) of Proposition 4.3, it suffices to prove that V_β is proper on Ω. Now, suppose that for some sequence $g_n \in G$, we have $g_n\Gamma$ tends to infinity in Ω. Then, since Γ is geometrically finite, this implies that the injectivity radius at $g_n\Gamma$ tends to 0. Hence, after passing to a subsequence, we can find $\gamma_n \in \Gamma$ such that $g_n\gamma_n$ belongs to a single horoball among the horoballs constituting our fixed standard cusp neighborhood; cf. Section 2.2. By modifying γ_n on the right by a fixed element in Γ if necessary, we can assume that $Kg_n\gamma_n$ converges to one of the parabolic points p_i (say p_1) on the boundary of $\mathbb{H}^d_{\mathbb{K}} \cong K\backslash G$.

Moreover, geometric finiteness implies that $(\Lambda_\Gamma \setminus \{p_1\})/\Gamma_1$ is compact. Thus, by multiplying $g_n\gamma_n$ by an element of Γ_1 on the right if necessary, we may assume that $(g_n\gamma_n)^-$ belongs to a fixed compact subset of the boundary, which is disjoint from $\{p_1\}$.

Thus, for all large n, we can write $g_n\gamma_n = k_n a_n u_n$, for $k_n \in K$, $a_n \in A_i$ and $u_n \in U_i$, such that the eigenvalues of $\mathrm{Ad}(a_n)$ are bounded above; cf. (4.8). Moreover, as $(g_n\gamma_n)^-$ belongs to a compact set that is disjoint from $\{p_1\}$ and $(g_n\gamma_n)^+ \to p_1$, the set $\{u_n\}$ is bounded. To show that $V_\beta(g_n\Gamma) \to \infty$, since U_i fixes v_i and K is a compact group, it remains to show that a_n contracts v_i to 0. Since $g_n\gamma_n$ is unbounded in G while k_n and u_n remain bounded, this shows that the sequence a_n is unbounded. Upper boundedness of the eigenvalues of $\mathrm{Ad}(a_n)$ thus implies the claim. □

Remark 4.5. The above lemma is false without restricting to Ω in the case Γ has infinite covolume since the injectivity radius is not bounded above on G/Γ. Note also that this lemma is false in the case Γ is not geometrically finite, since the complement of cusp neighborhoods inside Ω is compact if and only if Γ is geometrically finite.

The next crucial property of the functions V_β is the following linear manifestation of the existence of cusp neighborhoods consisting of disjoint horoballs. This lemma implies Proposition 4.3(3).

Lemma 4.6. *There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_0$.*

Remark 4.7. The constant ε_0 roughly depends on the distance from a fixed basepoint to the cusp neighborhoods.

Proof of Lemma 4.6. Let $g \in G$ and i be given. Write $g = kau$, for some $k \in K$, $a \in A_i$ and $u \in U_i$. Since U_i fixes v_i and the norm on W is K-invariant, we have $\|g \cdot v_i\| = \|a \cdot v_i\| = e^{\chi_i(a)}$; cf. (4.1). Moreover, since W is the adjoint representation, we have

$$\left\|\mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)}\right\| \asymp e^{\chi_i(a)},$$

and the implied constant, denoted C, depends only on the norm on the Lie algebra.

Let $0 < \varepsilon_i < 1$ be the constants in (4.8) and define $\varepsilon_0 := \min_i \varepsilon_i / C$. Let $x = g\Gamma \in G/\Gamma$. Suppose that there are vectors $\gamma_1, \gamma_2 \in \Gamma$ and vectors v_{i_1}, v_{i_2} in our finite fixed collection of vectors v_i such that $\left\|g\gamma_j \cdot v_{i_j}\right\| < \varepsilon_0$ for $j = 1, 2$. Then, the above discussion, combined with the choice of ε_i in (4.8), imply that $Kg\gamma_j$ belongs to the standard horoball H_j in $\mathbb{H}^d_{\mathbb{K}}$ based at p_{i_j}. However, this implies that the two standard horoballs $H_1\gamma_1^{-1}$ and $H_2\gamma_2^{-1}$ intersect non-trivially. By choice of these standard horoballs, this implies that the two horoballs $H_j\gamma_j^{-1}$ are the same and that the two parabolic points p_{i_j} are equivalent under Γ. In particular, the two vectors v_{i_1}, v_{i_2} are in fact the same vector, call it v_{i_0}. It also follows that $\gamma_1^{-1}\gamma_2$ sends H to itself and fixes the parabolic point it is based at. Thus, $\gamma_1^{-1}\gamma_2$ fixes v_{i_0} by definition. But, then, we get that

$$g\gamma_2 \cdot v_{i_0} = g\gamma_1(\gamma_1^{-1}\gamma_2) \cdot v_{i_0} = g\gamma_1 \cdot v_{i_0}.$$

This proves uniqueness of the vector in $\bigcup_i g\Gamma \cdot v_i$ with length less than ε_0, if it exists, and concludes the proof. $\square$

Finally, we verify Proposition 4.3 (1) relating the injectivity radius to V_β.

Lemma 4.8. *For all x in the unit neighborhood of Ω, we have*

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathbb{K}}/\beta}(x),$$

where $\chi_{\mathbb{K}}$ is given in (4.2).

Proof. Let $x \in \Omega$ and set $\tilde{x}_0 = Kx$. Let $x_0 \in K\backslash G \cong \mathbb{H}^d_{\mathbb{K}}$ denote a lift of $\tilde{x}_0$. Then, x_0 belongs to the hull of the limit set of Γ; cf. Section 2.

Since $\mathrm{inj}(\cdot)^{-1}$ and V_β are uniformly bounded above and below on the complement of the cusp neighborhoods inside Ω, it suffices to prove the lemma under the assumption that x_0 belongs to some standard horoball H based at a parabolic fixed point p. We may also assume that the lift x_0 is chosen so that p is one of our fixed finite set of inequivalent parabolic points $\{p_i\}$.

Geometric finiteness of Γ implies that there is a compact subset $\mathcal{K}_p$ of $\partial\mathbb{H}^d_{\mathbb{K}} \setminus \{p\}$, depending only on the stabilizer Γ_p in Γ, with the following property. Every point in the hull of the limit set is equivalent, under Γ_p, to a point on the set of geodesics joining p to points in $\mathcal{K}_p$. Thus, after adjusting x_0 by an element of Γ_p if necessary, we may assume that x_0 belongs to this set. In particular, we can find $g \in G$ so that $x_0 = Kg$ and g can be written as kau in the Iwasawa decomposition associated to p, for some $k \in K, a \in A_p$, and $u \in U_p$[3] with the property that $\mathrm{Ad}(a)$ is contracting on U_p and u is of uniformly bounded size.

[3]The groups A_p and U_p were defined at the beginning of the section.

Note that it suffices to prove the statement assuming the injectivity radius of x is smaller than $1/3$, while the distance of x_0 to the boundary of the cusp horoball H_p is at least 1. Now, let $\gamma \in \Gamma$ be a non-trivial element such that $x_0\gamma$ is at distance at most $1/2$ from x_0. Then, this implies that both x_0 and $x_0\gamma$ belong to H_p. In particular, the standard horoballs H_p and $H_p\gamma$ intersect non-trivially, and hence must be the same. It follows that γ belongs to Γ_p.

Let M_p denote the centralizer of A_p inside K. Since Γ_p is a subgroup of M_pU_p, we can find v in the Lie algebra of M_pU_p so that $\gamma = \exp(v)$. In view of the discreteness of Γ, we have that $\|v\| \gg 1$. Since the exponential map is close to an isometry near the origin, we see that

$$\mathrm{dist}(g\gamma g^{-1}, \mathrm{Id}) \asymp \|\mathrm{Ad}(au)(v)\| \geq e^{\chi_{\mathbb{K}}\alpha(a)} \|\mathrm{Ad}(u)(v)\|,$$

where $\chi_{\mathbb{K}}$ is given in (4.2) and we used K-invariance of the norm. Here, α is the simple root of A_p in the Lie algebra of U_p and $e^{\chi_{\mathbb{K}}\alpha(a)}$ is the smallest eigenvalue of $\mathrm{Ad}(a)$ on the Lie algebra of the parabolic group stabilizing p. Note that since x_0 belongs to H_p, $\alpha(a)$ is strictly negative.

Recalling that u belongs to a uniformly bounded neighborhood of identity in G and that $\|v\| \gg 1$, it follows that $\mathrm{dist}(g\gamma g^{-1}, \mathrm{Id}) \gg e^{\chi_{\mathbb{K}}\alpha(a)}$. Since γ was arbitrary, this shows that the injectivity radius at x satisfies the same lower bound.

Finally, let $v_p \in \{v_i\}$ denote the vector fixed by U_p. Using the above Iwasawa decomposition, we see that $V_\beta^{1/\beta}(x) \geq \|av_p\|^{-1/\chi_{\mathbb{K}}} = e^{-\chi_p(a)/\chi_{\mathbb{K}}}$, where χ_p is the character on A_p determined by v_p, cf. (4.1). This concludes the proof in view of (4.2) and the fact that $\chi_p = \chi_{\mathbb{K}}\alpha$. $\square$

5. Shadow Lemmas, Convexity, and Linear Expansion

The goal of this section is to prove Proposition 4.2 estimating the average rate of expansion of vectors with respect to leafwise measures. This completes the proof of Theorem 4.1.

5.1. **Proof of Proposition 4.2.** We may assume without loss of generality that $\|v\| = 1$. Let W^+ denote the highest weight subspace of W for $A_+ = \{g_t : t > 0\}$. Denote by π_+ the projection from W onto W^+. In our choice of representation W, the eigenvalue of A_+ in W^+ is $e^{\chi_{\mathbb{K}}t}$, , where $\chi_{\mathbb{K}}$ is given in (4.2). It follows that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u(n) \leq e^{-\beta t} \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|\pi_+(n\cdot v)\|^{-\beta/\chi_{\mathbb{K}}} \, d\mu_x^u(n).$$

Hence, it suffices to show that, for a suitable choice of β, the integral on the right side is uniformly bounded, independently of v and x (but possibly depending on β).

For simplicity, set $\beta_{\mathbb{K}} = \beta/\chi_{\mathbb{K}}$. A simple application of Fubini's Theorem yields

$$\int_{N_1^+} \|\pi_+(n\cdot v)\|^{-\beta_{\mathbb{K}}} \, d\mu_x^u(n) = \int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n\cdot v)\|^{\beta_{\mathbb{K}}} \leq t^{-1}\Big) \, dt.$$

For $v \in W$, we define a polynomial map on N^+ by $n \mapsto p_v(n) := \|\pi_+(n\cdot v)\|^2$ and set

$$S(v,\varepsilon) := \left\{n \in N^+ : p_v(n) \leq \varepsilon\right\}.$$

To apply Proposition 3.1, we wish to efficiently estimate the radius of a ball in N^+ containing the sublevel sets $S\big(v, t^{-2/\beta_{\mathbb{K}}}\big) \cap N_1^+$. We have the following claim.

Claim 5.1. There exists a constant $C_0 > 0$, such that, for all $\varepsilon > 0$, the diameter of $S(v,\varepsilon) \cap N_1^+$ is at most $C_0\varepsilon^{1/4\chi_{\mathbb{K}}}$.

Let us show how to conclude the proof assuming this claim. By estimating the integral over $[0,1]$ trivially, we obtain

$$\int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n\cdot v)\|^{\beta_{\mathbb{K}}} \leq t^{-1}\Big) \, dt \leq \mu_x^u(N_1^+) + \int_1^\infty \mu_x^u\Big(S\big(v, t^{-2/\beta_{\mathbb{K}}}\big) \cap N_1^+\Big) \, dt. \tag{5.1}$$

Claim 5.1 implies that if $\mu_x^u\left(S(v,\varepsilon)\cap N_1^+\right) > 0$ for some $\varepsilon > 0$, then $S(v,\varepsilon)\cap N_1^+$ is contained in a ball of radius $2C_0\varepsilon^{1/4\chi_{\mathbb{K}}}$, centered at a point in the support of the measure $\mu_x^u|_{N_1^+}$. Recalling that $\beta_{\mathbb{K}} = \beta/\chi_{\mathbb{K}}$, we thus obtain

$$\int_1^\infty \mu_x^u\left(S\left(v, t^{-2/\beta_{\mathbb{K}}}\right)\cap N_1^+\right)\,dt \leq \int_1^\infty \sup_{n\in\mathrm{supp}(\mu_x^u)\cap N_1^+} \mu_x^u\left(B_{N^+}\left(n, 2C_0 t^{-1/2\beta}\right)\right)\,dt, \tag{5.2}$$

where for $n\in N^+$ and $r>0$, $B_{N^+}(n,r)$ denotes the ball of radius r centered at n.

To estimate the integral on the right side of (5.2), we use the doubling results in Proposition 3.1. Note that if $n\in\mathrm{supp}(\mu_x^u)$, then nx belongs to the limit set Λ_Γ. Since $x\in N_1^-\Omega$ by assumption, this implies that nx belongs to $N_2^-\Omega$ for all $n\in N_1^+$ in the support of μ_x^u; cf. Remark 2.1. Hence, changing variables using (2.4) and applying Proposition 3.1, we obtain for all $n\in\mathrm{supp}(\mu_x^u)\cap N_1^+$,

$$\mu_x^u\left(B_{N^+}\left(n, 2C_0t^{-1/2\beta}\right)\right) = \mu_{nx}^u\left(B_{N^+}\left(\mathrm{Id}, 2C_0t^{-1/2\beta}\right)\right) \ll t^{-\Delta/2\beta}\mu_{nx}^u(N_1^+).$$

Moreover, for $n\in N_1^+$, we have, again by Proposition 3.1, that

$$\mu_{nx}^u(N_1^+) \leq \mu_x^u(N_2^+) \ll \mu_x^u(N_1^+).$$

Put together, this gives

$$\int_1^\infty \sup_{n\in\mathrm{supp}(\mu_x^u)\cap N_1^+} \mu_x^u\left(B_{N^+}\left(n, 2C_0t^{-1/2\beta}\right)\right)\,dt \ll \mu_x^u(N_1^+)\int_1^\infty t^{-\Delta/2\beta}\,dt.$$

The integral on the right side above converges whenever $\beta < \Delta/2$, which concludes the proof.

5.2. **Prelimiary facts.** We begin by recalling the Bruhat decomposition of G. Denote by P^- the subgroup MAN^- of G.

Proposition 5.2 (Theorem 5.15, [BT65])**.** *Let $w\in G$ denote a non-trivial Weyl "element" satisfying $wg_tw^{-1} = g_{-t}$. Then,*

$$G = P^-N^+\bigsqcup P^-w. \tag{5.3}$$

We shall need the following result, which is yet another reflection in linear representations of G of the fact that G has real rank 1.

Proposition 5.3. *Let V be a normed finite dimensional representation of G, and $v_0\in V$ be any highest weight vector for g_t ($t>0$) with weight $e^{\lambda t}$ for some $\lambda\geq 0$. Let v be any vector in the orbit $G\cdot v_0$ and define*

$$G(v, V^{<\lambda}(g_t)) = \left\{g\in G : \lim_{t\to\infty}\frac{\log\|g_tgv\|}{t} < \lambda\right\}.$$

Then, there exists $g_v\in G$ such that

$$G(v, V^{<\lambda}(g_t)) \subseteq P^-g_v.$$

Proof. Let $h\in G$ be such that $v = hv_0$ and let $g\in G(v, V^{<\lambda}(g_t))$. By the Bruhat decomposition, either $gh = pn$ for some $p\in P^-$ and $n\in N^+$, or $gh = pw$ for some $p\in P^-$ and w being the long Weyl "element". Suppose we are in the first case, and note that N^+ fixes v_0 since it is a highest weight vector for g_t. Moreover, $\mathrm{Ad}(g_t)(p)$ converges to some element in G as t tends to ∞. Since $g_tgv = e^{\lambda t}\mathrm{Ad}(g_t)(p)v_0$, we see that $\log\|g_tgv\|/t\to\lambda$ as t tends to ∞, thus contradicting the assumption that g belongs to $G(v, V^{<\lambda}(g_t))$. Hence, gh must belong to P^-w. This implies the conclusion by taking $g_v := wh^{-1}$.

∎

The following immediate corollary is the form we use this result in our arguments.

Corollary 5.4. *Let the notation be as in Proposition 5.3. Then, $N^+ \cap G(v, W^{0-}(g_t))$ contains at most one point.*

Proof. Recall the Bruhat decomposition of G in Proposition 5.2. Let $g_v \in G$ be as in Proposition 5.3 and suppose that $n_0 \in P^- g_v \cap N^+$. Let $p_0 \in P^-$ be such that $n_0 = p_0 g_v$.

First, assume $g_v = p_v n_v$ for some $p_v \in P^-$ and $n_v \in N^+$. Then, $n_0 = p_0 p_v n_v$. Then, $n_0 n_v^{-1} \in P^- \cap N^+ = \{\mathrm{Id}\}$. In particular, $n_0 = n_v$, and the claim follows in this case.

Now assume that $g_v = p_v w$ for some $p_v \in P^-$, so that $n_0 = p_0 p_v w \in P^- w \cap N^+$. This is a contradiction, since the latter intersection is empty as follows from the Bruhat decomposition. □

5.3. **Convexity and Proof of Claim 5.1.** Let $B_1 \subset \mathrm{Lie}(N^+)$ denote a compact convex set whose image under the exponential map contains N_1^+ and denote by B_2 a compact set containing B_1 in its interior.

Define $\mathfrak{n}_1^+$ to be the unit sphere in the Lie algebra $\mathfrak{n}^+$ of N^+ in the following sense:

$$\mathfrak{n}_1^+ := \left\{ u \in \mathfrak{n}^+ : d_{N^+}(\exp(u), \mathrm{Id}) = 1 \right\},$$

where d_{N^+} is the Cygan metric on N^+; cf. Section 2.5. Given $u, b \in \mathfrak{n}^+$, define a line $\ell_{u,b} : \mathbb{R} \to \mathfrak{n}^+$ as follows:

$$\ell_{u,b}(t) := tu + b,$$

and denote by $\mathcal{L}$ the space of all such lines $\ell_{u,b}$ such that $u \in \mathfrak{n}_1^+$. We endow $\mathcal{L}$ with the topology inherited from its natural identification with its $\mathfrak{n}_1^+ \times \mathfrak{n}^+$. Then, the subset $\mathcal{L}(B_1)$ of all such lines such that b belongs to the compact set B_1 is compact in $\mathcal{L}$.

Recall that a vector $v \in W$ is said to be unstable if the closure of the orbit $G \cdot v$ contains 0. Highest weight vectors are examples of unstable vectors. Let $\mathcal{N}$ denote the null cone of G in W, i.e., the closed cone consisting of all unstable vectors. Let $\mathcal{N}_1 \subset \mathcal{N}$ denote the compact set of unit norm unstable vectors. Note that, for any $v \in \mathcal{N}$, the restriction of p_v to any $\ell \in \mathcal{L}$ is a polynomial in t of degree at most that of p_v. We note further that the function

$$\rho(v, \ell) := \sup \{ p_v(\ell(t)) : \ell(t) \in B_2 \}$$

is continuous and non-negative on the compact space $\mathcal{N}_1 \times \mathcal{L}(B_1)$. We claim that

$$\rho_\star := \inf \{ \rho(v, \ell) : (v, \ell) \in \mathcal{N}_1 \times \mathcal{L}(B_1) \}$$

is strictly positive. Indeed, by continuity and compactness, it suffices to show that ρ is non-vanishing. Suppose not and let (v, ℓ) be such that $\rho(v, \ell) = 0$. Since B_1 is contained in the interior of B_2, the intersection

$$I(\ell) := \{ t \in \mathbb{R} : \ell(t) \in B_2 \}$$

is an interval (by convexity of B_2) with non-empty interior. Since $p_v(\ell(\cdot))$ is a polynomial vanishing on a set of non-empty interior, this implies it vanishes identically. On the other hand, Corollary 5.4 shows that p_v has at most 1 zero in all of $\mathfrak{n}^+$, a contradiction.

Positivity of $\rho_\star$ has the following consequence. Our choice of the representation W implies that the degree of the polynomial p_v is at most $4\chi_{\mathbb{K}}$, where $\chi_{\mathbb{K}}$ is given in (4.2). This can be shown by direct calculation in this case.[4] By the so-called (C, α)-good property (cf. [Kle10, Proposition 3.2]), we have for all $\varepsilon > 0$

$$|\{ t \in I(\ell) : p_v(\ell(t)) \leq \varepsilon \}| \leq C_d \, (\varepsilon / \rho_\star)^{1/4\chi_{\mathbb{K}}} \, |I(\ell)|,$$

where $C_d > 0$ is a constant depending only on the degree of p_v, and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

[4] In general, such a degree can be calculated from the largest eigenvalue of g_t in W; for instance by restricting the representation to suitable subalgebras of the Lie algebra of G that are isomoprhic to $\mathfrak{sl}_2(\mathbb{R})$ and using the explicit description of $\mathfrak{sl}_2(\mathbb{R})$ representations.

To use this estimate, we first note that the length of the intervals $I(\ell)$ is uniformly bounded over $\mathcal{L}(B_1)$. Indeed, suppose for some $u = (u_\alpha, u_{2\alpha}), b \in \mathfrak{n}^+$ and $\ell = \ell_{u,b} \in \mathcal{L}(B_1)$, $I(\ell)$ has endpoints $t_1 < t_2$ so that the points $\ell(t_i)$ belong to the boundary of B_2. Recall that the Lie algebra $\mathfrak{n}^+$ of N^+ decomposes into g_t eigenspaces as $\mathfrak{n}^+_\alpha \oplus \mathfrak{n}^+_{2\alpha}$, where $\mathfrak{n}^+_{2\alpha} = 0$ if and only if $\mathbb{K} = \mathbb{R}$. Set $x_1 = \ell(t_1)$ and $x_2 = \ell(t_2)$. Since N^+ is a nilpotent group of step at most 2, the Campbell-Baker-Hausdorff formula implies that $\exp(x_2)\exp(-x_1) = \exp(Z)$, where $Z \in \mathfrak{n}^+$ is given by

$$Z = x_2 - x_1 + \frac{1}{2}[x_2, -x_1] = (t_2 - t_1)u + \frac{1}{2}(t_2 - t_1)[b, u].$$

Note that since $\mathfrak{n}^+_{2\alpha}$ is the center of $\mathfrak{n}^+$, $[b, u] = [b, u_\alpha]$ belongs to $\mathfrak{n}^+_{2\alpha}$. Hence, we have by (2.8) that

$$d_{N^+}(\exp(x_1), \exp(x_2)) = \left((t_2 - t_1)^4 \|u_\alpha\|^4 + (t_2^2 - t_1^2)^2 \left\| u_{2\alpha} + \frac{1}{2}[b, u] \right\|^2 \right)^{1/4}.$$

Since $\exp(u)$ is at distance 1 from identity, at least one of $\|u_\alpha\|$ and $\|u_{2\alpha}\|$ is bounded below by 10^{-1}. Moreover, we can find a constant $\theta \in (0, 10^{-2})$ so that for all $b \in B_1$ and all $y_\alpha \in \mathfrak{n}^+_\alpha$ with $\|y_\alpha\| \leq \theta$ such that $\|[b, y_\alpha]\| \leq 10^{-2}$. Together this implies that

$$\min\left\{ t_2 - t_1, (t_2^2 - t_1^2)^{1/2} \right\} \ll \operatorname{diam}(B_1),$$

where $\operatorname{diam}(B_1)$ denotes the diameter of B_1. This proves that $|I(\ell)| = t_2 - t_1 \ll 1$, where the implicit constant depends only on the choice of B_1. We have thus shown that

$$|\{t \in I(\ell) : p_v(\ell(t)) \leq \varepsilon\}| \ll \varepsilon^{1/4\chi_{\mathbb{K}}}. \tag{5.4}$$

We now use our assumption that v belongs to the G orbit of a highest weight vector v_0. Since v_0 is a highest weight vector, it is fixed by N^+. Hence, the Bruhat decomposition, cf. (5.3) with the roles of P^- and P^+ reversed, implies that the orbit $G \cdot v_0$ can be written as

$$G \cdot v_0 = P^+ \cdot v_0 \bigsqcup P^+ w \cdot v_0,$$

where w is the long Weyl "element". Recall that $P^+ = N^+ MA$, where M is the centralizer of $A = \{g_t\}$ in the maximal compact group K. In particular, M preserves eigenspaces of A and normalizes N^+. Recall further that the norm on W is chosen to be K-invariant.

First, we consider the case $v \in P^+ w \cdot v_0$ and has unit norm. For $v' \in W$, we write $[v']$ for its image in the projective space $\mathbb{P}(W)$. Then, since $w \cdot v_0$ is a joint weight vector of A, we see that the image of $P^+ w \cdot v_0$ in $\mathbb{P}(W)$ has the form $N^+ M \cdot [w \cdot v_0]$. Setting $v_1 := w \cdot v_0$, we see that

$$S(nm \cdot v_1, \varepsilon) = S(mv_1, \varepsilon) \cdot n^{-1} = \operatorname{Ad}(m^{-1})(S(v_1, \varepsilon)) \cdot n^{-1}, \tag{5.5}$$

where we implicitly used the fact that M commutes with the projection π_+ and preserves the norm on W. Since the metric on N^+ is right invariant under translations by N^+ and is invariant under $\operatorname{Ad}(M)$, the above identity implies that it suffices to estimate the diameter of $S(v_1, \varepsilon) \cap N_1^+$ in the case $v \in P^+ w \cdot v_0$. Similarly, in the case $v \in P^+ \cdot v_0$, it suffices to estimate the diameter of $S(v_0, \varepsilon) \cap N_1^+$.

Let $\tilde{S}(v, \varepsilon) = \log S(v, \varepsilon)$ denote the pre-image of $S(v, \varepsilon)$ in the Lie algebra $\mathfrak{n}^+$ of N^+ under the exponential map. By Corollary 5.4, for any non-zero $v \in \mathcal{N}$, either $S(v, \varepsilon)$ is empty for all small enough ε, or there is a unique global minimizer of $p_v(\cdot)$ on N^+, at which p_v vanishes. In either case, for any given $v \in \mathcal{N} \setminus \{0\}$ in the null cone, the set $\tilde{S}(v, \varepsilon)$ is convex for all small enough $\varepsilon > 0$, depending on v. Let $s_0 > 0$ be such that $\tilde{S}(v, \varepsilon)$ is convex for $v \in \{v_0, v_1\}$ and for all $0 \leq \varepsilon \leq s_0$.

Fix some $v \in \{v_0, v_1\}$ and $\varepsilon \in [0, s_0]$. Suppose that $x_1 \neq x_2 \in \tilde{S}(v, \varepsilon) \cap B_1$. Let r denote the distance $d_{N^+}(x_1, x_2)$. Let $u' = x_2 - x_1$, $u = u'/r$ and $b = x_1$. Set $\ell = \ell_{u,b}$ and note that $\ell_{u,b}(0) = x_1$ and $\ell_{u,b}(r) = x_2$. Since B_1 is convex, the set $\tilde{S}(v, \varepsilon) \cap B_1$ is also convex. Hence, the entire interval

$(0,r)$ belongs to the set on the left side of (5.4) and, hence, that $r \ll \varepsilon^{1/4\chi_{\mathbb{K}}}$. Since x_1 and x_2 were arbitrary, this shows that the diameter of $\tilde{S}(v,\varepsilon)\cap B_1$ is $O(\varepsilon^{1/4\chi_{\mathbb{K}}})$ as desired.

6. Anisotropic Banach Spaces and Transfer Operators

In this section, we define the Banach spaces on which the transfer operator and resolvent associated to the geodesic flow have good spectral properties.

The transfer operator, denoted $\mathcal{L}_t$, acts on continuous functions as follows: for a continuous function f, let

$$\mathcal{L}_t f := f \circ g_t. \tag{6.1}$$

For $z \in \mathbb{C}$, the resolvent $R(z) : C_c(X) \to C(X)$ is defined formally as follows:

$$R(z)f := \int_0^\infty e^{-zt}\mathcal{L}_t f \; dt.$$

If Γ is not convex cocompact, we fix a choice of $\beta > 0$ so that Theorem 4.1 holds and set $V = V_\beta$. If Γ is convex cocompact, we take $V = V_\beta \equiv 1$ and we may take β as large as we like in this case. Note that the conclusion of Theorem 4.1 holds trivially with this choice of V. In particular, we shall use its conclusion throughout the argument regardless of whether Γ admits cusps.

Denote by $C_c^{k+1}(X)^M$ the subspace of $C_c^{k+1}(X)$ consisting of M-invariant functions, where M is the centralizer of the geodesic flow inside the maximal compact group K. In particular, $C_c^{k+1}(X)^M$ is naturally identified with the space of C_c^{k+1} functions on the unit tangent bundle of $\mathbb{H}^d_{\mathbb{K}}/\Gamma$; cf. Section 2. The following is the main result of this section.

Theorem 6.1 (Essential Spectral Gap)**.** *Let $k \in \mathbb{N}$ be given. Then, there exists a seminorm $\|\cdot\|_k$ on $C_c^{k+1}(X)^M$, non-vanishing on functions whose support meets Ω, and such that for every $z \in \mathbb{C}$, with $\mathrm{Re}(z) > 0$, the resolvent $R(z)$ extends to a bounded operator on the completion of $C_c^{k+1}(X)^M$ with respect to $\|\cdot\|_k$ and having spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ is bounded above by $1/(\mathrm{Re}(z)+\sigma_0)$, where*

$$\sigma_0 := \min\{k, \beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0 = k$.

By the completion of a topological vector space V with respect to a seminorm $\|\cdot\|$, we mean the Banach space obtained by completing the quotient topological vector space V/W with respect to the induced norm, where W is the kernel of $\|\cdot\|$.

The proof of Theorem 6.1 occupies Sections 6 and 7.

6.1. Anisotropic Banach Spaces.

We construct a Banach space of functions on X containing C^∞ functions satisfying Theorem 6.1.

Given $r \in \mathbb{N}$, let $\mathcal{V}_r^-$ denote the space of all C^r vector fields on N^+ pointing in the direction of the Lie algebra $\mathfrak{n}^-$ of N^- and having norm at most 1. More precisely, $\mathcal{V}_r^-$ consists of all C^r maps $v : N^+ \to \mathfrak{n}^-$, with C^r norm at most 1. Similarly, we denote by $\mathcal{V}_r^0$ the set of C^r vector fields $v : N^+ \to \mathfrak{a} := \mathrm{Lie}(A)$, with C^r norm at most 1. Note that if $\omega \in \mathfrak{a}$ is the vector generating the flow g_t, i.e. $g_t = \exp(t\omega)$, then each $v \in \mathcal{V}_r^0$ is of the form $v(n) = \phi(n)\omega$, for some $\phi \in C^r(N^+)$ such that $\|\phi\|_{C^r(N^+)} \leq 1$. Define

$$\mathcal{V}_r = \mathcal{V}_r^- \cup \mathcal{V}_r^0.$$

For $v \in \mathcal{V}$, denote by L_v the differential operator on $C^1(X)$ given by differentiation with respect to the vector field generated by v. Hence, for $\varphi \in C^1(G/\Gamma)$,

$$L_v\varphi(x) = \lim_{s\to 0} \frac{\varphi(\exp(sv)x) - \varphi(x)}{s}.$$

For each $k \in \mathbb{N}$, we define a norm on $C^k(N^+)$ functions as follows. Letting $\mathcal{V}^+$ be the unit ball in the Lie algebra of N^+, $0 \leq \ell \leq k$, and $\phi \in C^k(N^+)$, we define $c_\ell(\phi)$ to be the supremum of

$|L_{v_1}\cdots L_{v_\ell}(\phi)|$ over N^+ and all tuples $(v_1,\dots,v_\ell)\in(\mathcal{V}^+)^\ell$. We define $\|\phi\|_{C^k}$ to be $\sum_{\ell=0}^k 2^{-\ell}c_\ell(\phi)$. One then checks that for all $\phi_1,\phi_2\in C^k(N^+)$, we have

$$\|\phi_1\phi_2\|_{C^k}\leq\|\phi_1\|_{C^k}\|\phi_2\|_{C^k}. \tag{6.2}$$

Following [GL06, GL08], we define a norm on $C_c^{k+1}(X)$ as follows. Given $f\in C_c^{k+1}(X)$, k,ℓ non-negative integers, $\gamma=(\gamma_1,\dots,\gamma_\ell)\in\mathcal{V}_{k+\ell}^\ell$ (i.e. ℓ tuple of $C^{k+\ell}$ vector fields) and $x\in X$, define

$$e_{k,\ell,\gamma}(f;x):=\frac{1}{V(x)}\sup\frac{1}{\mu_x^u(N_1^+)}\left|\int_{N_1^+}\phi(n)L_{\gamma_1}\cdots L_{\gamma_\ell}(f)(g_snx)\,d\mu_x^u(n)\right|, \tag{6.3}$$

where the supremum is taken over all $s\in[0,1]$ and all functions $\phi\in C^{k+\ell}(N_1^+)$ which are compactly supported in the interior of N_1^+ and having $\|\phi\|_{C^{k+\ell}(N_1^+)}\leq 1$.

For $\gamma\in\mathcal{V}_{k+\ell+1}^\ell$, we define $e'_{k,\ell,\gamma}(f;x)$ analogously to $e_{k,\ell,\gamma}(f;x)$, but where we take $s=0$ and take the supremum over $\phi\in C^{k+\ell+1}(N_{1/10}^+)$ instead[5] of $C^{k+\ell}(N_1^+)$. Given $r>0$, set

$$\Omega_r^-:=N_r^-\Omega. \tag{6.4}$$

We define

$$e_{k,\ell,\gamma}(f):=\sup_{x\in\Omega_1^-}e_{k,\ell,\gamma}(f;x),\qquad e_{k,\ell}(f)=\sup_{\gamma\in\mathcal{V}_{k+\ell}^\ell}e_{k,\ell,\gamma}(f). \tag{6.5}$$

Finally, we define $\|f\|_k$ and $\|f\|'_k$ by

$$\|f\|_k:=\max_{0\leq\ell\leq k}e_{k,\ell}(f),\qquad \|f\|'_k:=\max_{0\leq\ell\leq k-1}\sup_{\gamma\in\mathcal{V}_{k+\ell+1}^\ell,x\in\Omega_{1/2}^-}e'_{k,\ell,\gamma}(f;x). \tag{6.6}$$

Note that the (semi-)norm $\|f\|'_k$ is weaker than $\|f\|_k$ since we are using more regular test functions and vector fields, and we are testing fewer derivatives of f.

Remark 6.2. Since the suprema in the definition of $\|\cdot\|_k$ are restricted to points on Ω_1^-, $\|\cdot\|_k$ defines a seminorm on $C_c^{k+1}(X)^M$. Moreover, since Ω_1^- is invariant by g_t for all $t\geq 0$, the kernel of this seminorm, denoted W_k, is invariant by $\mathcal{L}_t$. The seminorm $\|\cdot\|_k$ induces a norm on the quotient $C_c^{k+1}(X)^M/W_k$, which we continue to denote $\|\cdot\|_k$.

Definition 6.3. We denote by $\mathcal{B}_k$ the Banach space given by the completion of the quotient $C_c^{k+1}(X)^M/W_k$ with respect to the norm $\|\cdot\|_k$, where $C_c^{k+1}(X)^M$ denotes the subspace consisting of M-invariant functions.

Note that since $\|\cdot\|'_k$ is dominated by $\|\cdot\|_k$, $\|\cdot\|'_k$ descends to a (semi-)norm on $C_c^{k+1}(X)^M/W_k$ and extends to a (semi-)norm on $\mathcal{B}_k$, again denoted $\|\cdot\|'_k$.

The following is a reformulation of Theorem 6.1 in the above setup.

Theorem 6.4. *For all $z\in\mathbb{C}$, with $\mathrm{Re}(z)>0$, and for all $k\in\mathbb{N}$, the operator $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ acting on $\mathcal{B}_k$ is bounded above by $1/(\mathrm{Re}(z)+\sigma_0)$, where*

$$\sigma_0:=\min\{k,\beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0=k$.

[5] The restriction on the supports allows us to handle non-smooth conditional measures; cf. proof of Prop. 6.6.

6.2. **Hennion's Theorem and Compact Embedding.** Our key tool in estimating the essential spectral radius is the following refinement of Hennion's Theorem, based on Nussbaum's formula.

Theorem 6.5 (cf. [Hen93] and Lemma 2.2 in [BGK07])**.** *Suppose that $\mathcal{B}$ is a Banach space with norm $\|\cdot\|$ and that $\|\cdot\|'$ is a seminorm on $\mathcal{B}$ so that the unit ball in $(\mathcal{B}, \|\cdot\|)$ is relatively compact in $\|\cdot\|'$. Suppose R is a bounded operator on $\mathcal{B}$ such that for some $n \in \mathbb{N}$, there exist constants $r > 0$ and $C > 0$ satisfying*

$$\|R^n v\| \leq r^n \|v\| + C \|v\|', \tag{6.7}$$

for all $v \in \mathcal{B}$. Then, the essential spectral radius of R is at most r.

In this Section, we show, roughly speaking, that the inclusion of $\mathcal{B}_k$ into $\mathcal{B}'_k$ is a compact operator; Proposition 6.6.

Proposition 6.6. *Let $K \subseteq X$ be such that*

$$\sup \{V(x) : x \in K\} < \infty.$$

Then, every sequence $f_n \in C_c^{k+1}(X)^M$, such that f_n is supported in K and has $\|f_n\|_k \leq 1$ for all n, admits a Cauchy subsequence in $\|\cdot\|'_k$.

6.3. **Proof of Proposition 6.6.** We adapt the arguments in [GL06,GL08] with the main difference being that we bypass the step involving integration by parts over N^+ since our conditionals μ_x^u need not be smooth in general. The idea is to show that since all directions in the tangent space of X are accounted for in the definition of $\|\cdot\|_k$ (differentiation along the weak stable directions and integration in the unstable directions), one can estimate $\|\cdot\|'_k$ using finitely many coefficients $e_k(f; x_i)$. More precisely, we first show that there exists $C \geq 1$ so that for all sufficiently small $\varepsilon > 0$, there exists a finite set $\Xi \subset \Omega$ so that for all $f \in C_c^{k+1}(X)^M$, which is supported in K,

$$\|f\|'_k \leq C\varepsilon \|f\|_k + C \sup \int_{N_1^+} \phi L_{v_1} \cdots L_{v_\ell} f \, d\mu_{x_i}^u, \tag{6.8}$$

where the supremum is over all $0 \leq \ell \leq k-1$, all $(v_1, \ldots, v_\ell) \in \mathcal{V}^\ell_{k+\ell+1}$, all functions $\phi \in C^{k+\ell+1}(N_2^+)$ with $\|\phi\|_{C^{k+\ell+1}} \leq 1$ and all $x_i \in \Xi$.

First, we show how (6.8) completes the proof. Let $f_n \in C_c^{k+1}(K)$ be as in the statement. Let $\varepsilon > 0$ be small enough so that (6.8) holds. Since $C^{k+\ell+1}(N_2^+)$ is compactly included inside $C^{k+\ell}(N_2^+)$, we can find a finite collection $\{\phi_j : j\} \subset C^{k+\ell}(N_2^+)$ which is ε dense in the unit ball of $C^{k+\ell+1}(N_2^+)$. Similarly, we can find a finite collection of vector fields $\{(v_1^m, \ldots, v_\ell^m) : m\} \subset \mathcal{V}^\ell_{k+\ell}$ which is ε dense in $\mathcal{V}^\ell_{k+\ell+1}$ in the $C^{k+\ell+1}$ topology. Then, we can find a subsequence, also denoted f_n, so that the finitely many quantities

$$\left\{ \int_{N_1^+} \phi_j L_{v_1^m} \cdots L_{v_\ell^m} f_n \, d\mu_{x_i}^u : i, j, m \right\}$$

converge. Together with (6.8), this implies that

$$\|f_{n_1} - f_{n_2}\|'_k \ll \varepsilon,$$

for all large enough n_1, n_2, where we used the fact that $\|f_n\|_k \leq 1$ for all n. As ε was arbitrary, one can extract a Cauchy subequence by a standard diagonal argument. Thus, it remains to prove (6.8).

Fix some $f \in C_c^{k+1}(X)^M$ which is supported inside K. Let an arbitrary tuple $\gamma = (v_1, \ldots, v_\ell) \in \mathcal{V}^\ell_{k+\ell+1}$ be given and set

$$\psi = L_{v_1} \cdots L_{v_\ell} f.$$

Let $\phi \in C^{k+\ell+1}(N^+_{1/10})$ and write $Q = N^+_{1/10}$. To estimate $e'_{k,\ell,\gamma}(f;z)$ using the right side of (6.8), we need to estimate integrals of the form

$$\frac{1}{V(z)} \frac{1}{\mu_z^u\left(N_1^+\right)} \int_{N_1^+} \phi(n)\psi(nz)\, d\mu_z^u(n), \tag{6.9}$$

for all $z \in \Omega^-_{1/2}$.

Denote by $\rho : X \to [0,1]$ a smooth function which is identically one on the 1-neighborhood Ω^1 of Ω and vanishes outside its 2-neighborhood. Note that if f is supported outside of Ω^1, then the integral in (6.9) vanishes for all z and the estimate follows. The same reasoning implies that

$$\|\rho f\|_k = \|f\|_k, \qquad \|\rho f\|'_k = \|f\|'_k.$$

Hence, we may assume that f is supported inside the intersection of K with Ω^1. In particular, for the remainder of the argument, we may replace K with (the closure of) its intersection with Ω^1.

This discussion has the important consequence that we may assume that K is a compact set in light of Proposition 4.3. Let K_1 denote the 1-neighborhood of K and fix some $z \in K_1 \cap \Omega^-_{1/2}$. By shrinking ε, we may assume it is smaller than the injectivity radius of K_1. Hence, we can find a finite cover $B_1, \ldots, B_M$ of $K_1 \cap \Omega^-_{1/2}$ with flow boxes of radius ε and with centers $\Xi := \{x_i\} \subset \Omega^-_{1/2}$.

Step 1: We first handle the case where z belongs to the same unstable manifold as one of the x_i's. Note that we may assume that Q intersects the support of μ_z^u non-trivially, since otherwise the integral in question is 0. Let $u \in Q$ be one point in this intersection and let $x = uz$. Thus, by (2.4), we get

$$\int_{N_1^+} \phi(n)\psi(nz)\, d\mu_z^u(n) = \int_Q \phi(n)\psi(nz)\, d\mu_z^u(n) = \int_{Qu^{-1}} \phi(nu)\psi(nx)\, d\mu_x^u(n).$$

Let $\phi_u(n) := \phi(nu)$. Then, ϕ_u is supported inside Qu^{-1}. Moreover, since $u \in Q$, $Q_u := Qu^{-1}$ is a ball of radius $1/10$ containing the identity element. Hence, $Qu^{-1} \subset N_1^+$ and, thus,

$$\int_{Q_u} \phi(nu)\psi(nx)\, d\mu_x^u(n) = \int_{N_1^+} \phi_u(n)\psi(nx)\, d\mu_x^u(n).$$

Fix some $\varepsilon > 0$. We may assume that $\varepsilon < 1/10$. Note that x belongs to the 1-neighborhood of K. Then, $x = u_2^{-1}x_i$ for some i and some $u_2 \in N_\varepsilon^+$, by our assumption in this step that z belongs to the unstable manifold of one of the x_i's. By repeating the above argument with z, u, x, Q and ϕ replaced with x, u_2, x_i, Q_u and ϕ_u respectively, we obtain

$$\int_{N_1^+} \phi_u(n)\psi(nx)\, d\mu_x^u(n) = \int_{Q_u u_2^{-1}} \phi_u(nu_2)\psi(nx_i)\, d\mu_{x_i}^u(n).$$

Note that Q_u is contained in the ball of radius $1/5$ centered around identity. Since $u_2 \in N_\varepsilon^+$ and $\varepsilon < 1/10$, we see that $Q_u u_2^{-1} \subset N_1^+$. It follows that

$$\int_{N_1^+} \phi_u(n)\psi(nx_i)\, d\mu_{x_i}^u(n) = \int_{N_1^+} \phi_{u_2 u}(n)\psi(nx_i)\, d\mu_{x_i}^u(n),$$

where $\phi_{u_2u}(n) = \phi_u(nu_2) = \phi(nu_2u)$. The function ϕ_{u_2u} satisfies $\|\phi_{u_2u}\|_{C^{k+\ell+1}} = \|\phi\|_{C^{k+\ell+1}} \leq 1$. Finally, let $\varphi_1, \varphi_2 : N^+ \to [0,1]$ be non-negative bump C^0 functions where $\varphi_1 \equiv 1$ on N_1^+ and while φ_2 is equal to 1 at identity and its support is contained inside N_1^+. Since $y \mapsto \mu_y^u(\varphi_i)$ is continuous for $i = 1, 2$, by [Rob03, Lemme 1.16], and is non-zero on Ω_1^-, we can find, by compactness of K_1, a constant $C \geq 1$, depending only on K (and the choice of φ_1, φ_2), such that

$$1/C \leq \mu_y^u\left(N_1^+\right) \leq C, \qquad \forall y \in K_1 \cap \Omega_1^-. \tag{6.10}$$

Hence, recalling that $\psi = L_{v_1} \cdots L_{v_\ell} f$ and that $V(z) \gg 1$, we conclude that the integral in (6.9) is bounded by the second term in (6.8).

Step 2: We reduce to the case where z is contained in the unstable manifolds of the x_i's. Let i be such that $z \in B_i$. Set $z_1 = z$ and let $z_0 \in (N_\varepsilon^+ \cdot x_i)$ be the unique point in the intersection of $N_\varepsilon^+ \cdot x_i$ with the local weak stable leaf of z_1 inside B_i. Let $p_1^- \in P^- := MAN^-$ be an element of the ε neighborhood of identity P_ε^- in P^- such that $z_1 = p_1^- z_0$.

We will estimate the integral in (6.9) using integrals at z_0. The idea is to perform weak stable holonomy between the local strong unstable leaves of z_0 and z_1. To this end, we need some notation. Let $Y \in \mathfrak{p}^-$ be such that $p_1^- = \exp(Y)$ and set

$$p_t^- = \exp(tY), \qquad z_t = p_t^- z_0,$$

for $t \in [0,1]$. Let us also consider the following maps $u_t^+ : N_1^+ \to N^+$ and $\tilde{p}_t^- : N_1^+ \to P^-$ defined by the following commutation relations

$$np_t^- = \tilde{p}_t^-(n) u_t^+(n), \qquad \forall n \in N_1^+.$$

Recall we are given a test function $\phi \in C^{k+\ell+1}(N_{1/10}^+)$. We can rewrite the integral we wish to estimate as follows:

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u(n) = \int_{N_1^+} \phi(n)\psi(np_1^- z_0)\, d\mu_{z_1}^u(n) = \int \phi(n)\psi(\tilde{p}_1^-(n)u_1^+(n)z_0)\, d\mu_{z_1}^u(n).$$

Let $U_t^+ \subset N^+$ denote the image of u_t^+. Note that if ε is small enough, $U_t^+ \subseteq N_2^+$ for all $t \in [0,1]$. We may further assume that ε is small enough so that the map u_t^+ is invertible on U_t^+ for all $t \in [0,1]$ and write $\phi_t := \phi \circ (u_t^+)^{-1}$. For simplicity, set

$$p_t^-(n) := \tilde{p}_t^-((u_t^+)^{-1}(n)).$$

Write $m_t(n) \in M$ and $b_t^-(n) \in AN^-$ for the components of $p_t^-(n)$ along M and AN^- respectively so that

$$p_t^-(n) = m_t(n) b_t^-(n).$$

We denote by J_t the Radon-Nikodym derivative of the pushforward of $\mu_{z_1}^u$ by u_t^+ with respect to $\mu_{z_t}^u$; cf. (2.10) for an explicit formula. Thus, changing variables using $n \mapsto u_1^+(n)$, and using the M-invariance of f, we obtain

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u = \int \phi_1(n)\psi(p_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u = \int \phi_1(n)\tilde{\psi}_1(b_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u,$$

where $\tilde{\psi}$ is given by

$$\tilde{\psi}_t := L_{\tilde{v}_1^t} \cdots L_{\tilde{v}_\ell^t} f, \qquad \tilde{v}_i(n) := \mathrm{Ad}(m_t((u_t^+)^{-1}(n)))(v_i((u_t^+)^{-1}(n))).$$

Here, we recall that $\mathrm{Ad}(M)$ commutes with A and normalizes N^- so that $\tilde{v}_i^t$ is a vector field with the same target as v_i.

Let $\mathfrak{b}^-$ denote the Lie algebra of AN^- and denote by $\tilde{w}_t' : U_t^+ \times [0,1] \to \mathfrak{b}^-$ the vector field tangent to the paths defined by b_t^-. More explicitly, $\tilde{w}_t'$ is given by the projection of tY to $\mathfrak{b}^-$. Denote $\tilde{w}_t(n) := \mathrm{Ad}(m_t(n))(\tilde{w}_t'(n))$. Then, using the M-invariance of f as above once more, we can write

$$\psi(b_1^-(n)nz_0) - \psi(nz_0)) = \int_0^1 \frac{\partial}{\partial t}\tilde{\psi}_t(b_t^-(n)nz_0)\, dt = \int_0^1 L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)\, dt.$$

To simplify notation, let us set $w_t = \tilde{w}_t \circ u_t^+$, and

$$F_t := L_{\tilde{v}_1^t \circ u_t^+} \cdots L_{\tilde{v}_\ell^t \circ u_t^+} f.$$

Using a reverse change of variables, we obtain for every $t \in [0,1]$ that

$$\int \phi_1(n) L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u = \int (\phi_1 J_1)\circ u_t^+(n) L_{w_t}(F_t)(\tilde{p}_t^-(n)u_t^+(n)z_0)J_t^{-1}(n)\, d\mu_{z_t}^u$$
$$= \int (\phi_1 J_1)\circ u_t^+(n)\cdot L_{w_t}(F_t)(nz_t)\cdot J_t^{-1}(n)\, d\mu_{z_t}^u(n),$$

where we used the identities $\tilde{p}_t^-(n)u_t^+(n) = np_t^-$ and $z_t = p_t^- z_0$. Let us write

$$\Phi_t(n) := (\phi_1 J_1)\circ u_t^+(n)\cdot J_t^{-1}(n),$$

which we view as a test function[6]. Hence, the last integral above amounts to integrating $\ell+1$ weak stable derivatives of f against a $C^{k+\ell}$ function. Moreover, since ϕ is supported in $N_{1/10}^+$, we may assume that ε is small enough so that Φ_t is supported in N_1^+ for all $t \in [0,1]$, and meets the requirements on the test functions in the definition of $\|f\|_k$. Since $z = z_1$ belongs to $\Omega_{1/2}^-$ by assumption, we may further shrink ε if necessary so that the points z_t all[7] belong to Ω_1^-. Thus, decomposing w_t into its A and N^- components, and noting that $\|w_t\| \ll \varepsilon$, we obtain the estimate

$$\int \Phi_t(n)\cdot L_{w_t}(F_t)(nz_t)\, d\mu_{z_t}^u(n) \ll \varepsilon \|f\|_k V(z_t)\mu_{z_t}^u(N_1^+). \tag{6.11}$$

To complete the argument, note that the integral we wish to estimate satisfies

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u = \int (\phi_1 J_1)(n)\psi(nz_0)\, d\mu_{z_0}^u + \int_0^1 \int \Phi_t(n)\cdot L_{w_t}(F_t)(nz_t)\, d\mu_{z_t}^u(n)\, dt. \tag{6.12}$$

Moreover, recall that z_0 belongs to the same unstable manifold as some $x_i \in \Xi$. Additionally, since ϕ is supported in $N_{1/10}^+$, by taking ε small enough, we may assume that ϕ_1 is supported inside $N_{1/5}^+$. Hence, arguing similarly to Step 1, viewing $\phi_1 J_1$ as a test function, we can estimate the first term on the right side above using the right side of (6.8).

The second term in (6.12) is also bounded by the right side of (6.8), in view of (6.11). Here we are using that $y \mapsto \mu_y^u(N_1^+)$ and $y \mapsto V(y)$ are uniformly bounded as y varies in the compact set K_1; cf. (6.10). This completes the proof of (6.8) in all cases, since ϕ and z were arbitrary.

7. The Essential Spectral Radius of Resolvents

In this section, we study the operator norm of the transfer operators $\mathcal{L}_t$ and the resolvents $R(z)$ on the Banach spaces constructed in the previous section. These estimates constitute the proof of Theorem 6.1. With these results in hand, we deduce Theorem 1.4 at the end of the section.

7.1. Strong continuity of transfer operators.

Recall that a collection of measurable subsets $\{B_i\}$ of a space Y are said to have *intersection multiplicity* bounded by a constant $C \geq 1$ if for all i, the number of sets B_j in the collection that intersect B_i non-trivially is at most C. In this case, one has

$$\sum_i \chi_{B_i}(y) \leq C\chi_{\cup_i B_i}(y), \qquad \forall y \in Y.$$

The following lemma implies that the operators $\mathcal{L}_t$ are uniformly bounded on $\mathcal{B}_k$ for $t \geq 0$.

Lemma 7.1. *For every $k, \ell \in \mathbb{N}\cup\{0\}$, $\gamma \in \mathcal{V}_{k+\ell}^\ell$, $t \geq 0$, and $x \in \Omega_1^-$,*

$$e_{k,\ell,\gamma}(\mathcal{L}_t f; x) \ll_\beta e^{-\varepsilon(\gamma)t} e_{k,\ell,\gamma}(f)(e^{-\beta t} + 1/V(x)),$$

where $\varepsilon(\gamma) \geq 0$ is the number of stable derivatives determined by γ. In partitcular, $\varepsilon(\gamma) = 0$ if only if $\ell = 0$ or all components of γ point in the flow direction.

[6] The Jacobians are smooth maps as they are given in terms of Busemann functions; cf. (2.10).

[7] This type of estimate is the reason we use stable thickenings Ω_r^- of Ω in the definition of the norm instead of Ω.

Proof. Fix some $x \in \Omega$ and $\gamma = (v_1, \dots, v_\ell) \in \mathcal{V}_{k+\ell}^{\ell}$. Since the Lie algebra of N^- has the orthogonal decomposition $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, where α is the simple positive root in $\mathfrak{g}$ with respect to g_t, we have that g_t contracts the norm of each stable vector $v \in \mathcal{V}_{k+\ell}^-$ by at least e^{-t}. It follows that for all $v \in \mathcal{V}_{k+\ell}^-$ and $w \in \mathcal{V}_{k+\ell}^0$,

$$L_v(\mathcal{L}_t f)(x) = \|v_t\| \, L_{\bar{v}_t}(f)(g_t x), \qquad L_w(\mathcal{L}_t f)(x) = L_w(f)(g_t x), \tag{7.1}$$

for all $f \in C^{k+1}(X)^M$, where $v_t = \mathrm{Ad}(g_t)(v)$ and $\bar{v}_t = v_t / \|v_t\|$. Moreover, we have

$$\|v_t\| \leq e^{-t} \|v\| = e^{-t} \|v\| .$$

Let ϕ be a test function and $\psi \in C(X)^M$. Using (2.3) to change variables, we get

$$\int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu_x^u(n) = e^{-\delta t} \int_{\mathrm{Ad}(g_t)(N_1^+)} \phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu_{g_t x}^u(n).$$

Let $\{\rho_i : i \in I\}$ be a partition of unity of $\mathrm{Ad}(g_t)(N_1^+)$ so that each ρ_i is non-negative, C^∞, and supported inside some ball of radius 1 centered inside $\mathrm{Ad}(g_t)(N_1^+)$. Such a partition of unity can be chosen so that the supports of ρ_i have a uniformly bounded multiplicity[8], depending only on N^+. Denote by $I(\Lambda)$ the subset of indices $i \in I$ such that there is $n_i \in N^+$ in the support of the measure $\mu_{g_t x}^u$ with the property that the support of ρ_i is contained in $N_1^+ \cdot n_i$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu_{g_t x}^u$ is the 0 measure. Then, we obtain

$$\int_{\mathrm{Ad}(g_t)(N_1^+)} \phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu_{g_t x}^u(n) = \sum_{i \in I(\Lambda)} \int_{N_1^+ \cdot n_i} \rho_i(n)\phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu_{g_t x}^u(n).$$

Setting $x_i = n_i g_t x$ and changing variables using (2.4), we obtain

$$\int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu_x^u(n) = e^{-\delta t} \sum_{i \in I(\Lambda)} \int_{N_1^+} \rho_i(n n_i)\phi(g_{-t} n n_i g_t)\psi(n x_i) \, d\mu_{x_i}^u(n). \tag{7.2}$$

The bounded multiplicity of the partition of unity implies that the balls $N_1^+ \cdot n_i$ have intersection multiplicity bounded by a constant C_0, depending only on N^+. Enlarging C_0 if necessary, we may also choose ρ_i so that $\|\rho_i\|_{C^{k+\ell}} \leq C_0$. In particular, C_0 is independent of t and x.

For each i, let $\bar{\phi}_i(n) = \rho_i(n n_i)\phi(g_{-t} n n_i g_t)$. Since ρ_i is chosen to be supported inside $N_1^+ n_i$, then $\bar{\phi}_i$ is supported inside N_1^+. Moreover, since ρ_i is C^∞, $\bar{\phi}_i$ is of the same differentiability class as ϕ. Since conjugation by g_{-t} contracts N^+, we see that $\|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} \leq \|\phi\|_{C^{k+\ell}} \leq 1$ (note that the supremum norm of $\phi \circ \mathrm{Ad}(g_{-t})$ does not decrease, and hence we do not gain from this contraction). Hence, since $\|\rho_i\|_{C^{k+\ell}} \leq C_0$, (6.2) implies that $\|\bar{\phi}_i\|_{C^{k+\ell}} \leq C_0$.

First, let us suppose that $t \geq 1$. Then, using Remark 2.1, since $x \in N_1^- \Omega$, one checks that x_i belongs to $N_1^- \Omega$ as well for all i. Applying (7.2) with $\psi = L_{v_1} \cdots L_{v_\ell} f$, we obtain

$$\begin{aligned} \int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu_x^u &= e^{-\delta t} \sum_{i \in I(\Lambda)} \int_{N_1^+} \bar{\phi}_i(n)\psi(n x_i) \, d\mu_{x_i}^u \\ &\leq C_0 e_{k,\ell,\gamma}(f) \, \|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} \, e^{-\delta t} \sum_{i \in I(\Lambda)} \mu_{x_i}^u(N_1^+) V(x_i). \end{aligned} \tag{7.3}$$

[8]Note that the analog of the classical Besicovitch covering theorem fails to hold for N^+ with the Cygan metric when N^+ is not abelian; cf. [KR95, pg. 17]. Instead, such a partition of unity can be constructed using the Vitali covering lemma with the aid of the right invariance of the Haar measure. To obtain a uniform bound on the multiplicity here and throughout, it is important that such an argument is applied to balls with uniformly comparable radii.

By the log Lipschitz property of V provided by Proposition 4.3, and by enlarging C_0 if necessary, we have $V(x_i) \leq C_0 V(nx_i)$ for all $n \in N_1^+$. It follows that

$$\sum_{i \in I(\Lambda)} \mu_{x_i}^u(N_1^+) V(x_i) \leq C_0 \sum_{i \in I(\Lambda)} \int_{N_1^+} V(nx_i)\, d\mu_{x_i}^u(n).$$

Recall that the balls $N_1^+ \cdot n_i$ have intersection multiplicity at most C_0. Moreover, since the support of ρ_i is contained inside $\mathrm{Ad}(g_t)(N_1^+)$, the balls $N_1^+ n_i$ are all contained in $N_2^+ \mathrm{Ad}(g_t)(N_1^+)$. Hence, applying the equivariance properties (2.3) and (2.4) once more yields

$$\sum_{i \in I(\Lambda)} \int_{N_1^+} V(nx_i)\, d\mu_{x_i}^u(n) \leq C_0 \int_{N_2^+ \mathrm{Ad}(g_t)(N_1^+)} V(ng_t x)\, d\mu_{g_t x}^u(n) \leq C_0 e^{\delta t} \int_{N_3^+} V(g_t nx)\, d\mu_x^u(n).$$

Here, we used the positivity of V and that $\mathrm{Ad}(g_{-t})(N_2^+)N_1^+ \subseteq N_3^+$. Combined with (7.2) and the contraction estimate on V, Theorem 4.1, it follows that

$$\int_{N_1^+} \phi(n)\psi(g_t nx)\, d\mu_x^u \leq C_0^3 (ce^{-\beta t} V(x) + c) \mu_x^u(N_3^+) e_{k,\ell,\gamma}(f),$$

for a constant $c \geq 1$ depending on β. By Proposition 3.1, we have $\mu_x^u(N_3^+) \leq C_1 \mu_x^u(N_1^+)$, for a uniform constant $C_1 \geq 1$, which is independent of x. This estimate concludes the proof in view of (7.1).

Now, let $s \in [0,1]$ and $t \geq 0$. If $t + s \geq 1$, then the above argument applied with $t+s$ in place of t implies that

$$\left| \int_{N_1^+} \phi(n)\psi(g_{t+s} nx)\, d\mu_x^u \right| \ll_\beta e^{-\varepsilon(\gamma)t} e_{k,\ell,\gamma}(f)(e^{-\beta t} V(x) + 1) \mu_x^u(N_1^+),$$

as desired. Otherwise, if $t + s < 1$, then by definition of $e_{k,\ell,\gamma}$, we have that

$$\left| \int_{N_1^+} \phi(n)\psi(g_{t+s} nx)\, d\mu_x^u \right| \leq e_{k,\ell,\gamma}(f) V(x) \mu_x^u(N_1^+).$$

Since t is at most 1 in this case, the conclusion of the lemma follows in this case as well.

□

As a corollary, we deduce the following strong continuity statement which implies that the infinitesimal generator of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. When restricted to $C_c^{k+1}(X)^M$, this generator is nothing but the differentiation operator in the flow direction. This strong continuity is also important in applying the results of [But16a] to deduce exponential mixing from our spectral bounds on the resolvent in Section 9.

Corollary 7.2. *The semigroup $\{\mathcal{L}_t : t \geq 0\}$ is strongly continuous; i.e. for all $f \in \mathcal{B}_k$,*

$$\lim_{t \downarrow 0} \|\mathcal{L}_t f - f\|_k = 0.$$

Proof. For all $f \in C_c^{k+1}(X)^M$, one easily checks that since $V(\cdot) \gg 1$ on any bounded neighborhood of Ω, then

$$\|\mathcal{L}_t f - f\|_k \ll \sup_{0 \leq s \leq 1} \|\mathcal{L}_{t+s} f - \mathcal{L}_s f\|_{C^k(X)}.$$

Moreover, since f belongs to C^{k+1}, the right side above tends to 0 as $t \to 0$ by the mean value theorem.

Now, let f be a general element of $\mathcal{B}_k$ and suppose that $\|\mathcal{L}_t f - f\|_k \nrightarrow 0$. Then, there is $t_n \to 0$ such that $\|\mathcal{L}_{t_n} f - f\|_k \to c \neq 0$. For every $j \in \mathbb{N}$, let $f_j \in C_c^{k+1}(X)^M$ be such that $\|f - f_j\|_k < 1/j$. For each j, let n_j be large enough such that $\left\|\mathcal{L}_{t_{n_j}} f_j - f_j\right\|_k < 1/j$. Then,

$$\left\|\mathcal{L}_{t_{n_j}} f - f\right\|_k \leq \left\|\mathcal{L}_{t_{n_j}} f - \mathcal{L}_{t_{n_j}} f_j\right\|_k + \left\|\mathcal{L}_{t_{n_j}} - f_j\right\|_k + \|f_j - f\|_k .$$

The last two terms on the right side are each bounded by $1/j$ by construction. By Lemma 7.1, we also have that $\left\|\mathcal{L}_{t_{n_j}} f - \mathcal{L}_{t_{n_j}} f_j\right\|_k$ is $O(\|f - f_j\|_k)$. It follows that $\left\|\mathcal{L}_{t_{n_j}} f - f\right\|_k \ll 1/j \to 0$, which contradicts the hypothesis that $\|\mathcal{L}_{t_n} f - f\|_k \to c \neq 0$. $\square$

7.2. **Towards a Lasota-Yorke inequality for the resolvent.** Recall that for all $n \in \mathbb{N}$,

$$R(z)^n = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t \, dt, \tag{7.4}$$

as follows by induction on n. The following corollary is immediate from Lemma 7.1 and the fact that

$$\left| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \, dt \right| \leq \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-\mathrm{Re}(z)t} \, dt = 1/\mathrm{Re}(z)^n, \tag{7.5}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Corollary 7.3. *For all $n, k, \ell \in \mathbb{N} \cup \{0\}$, $f \in C_c^{k+1}(X)^M$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we have*

$$e_{k,\ell}(R(z)^n f; x) \ll_\beta e_{k,\ell}(f) \left(\frac{1}{(\mathrm{Re}(z) + \beta)^n} + \frac{V(x)^{-1}}{\mathrm{Re}(z)^n} \right) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n.$$

In particular, $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$.

Note that Lemma 7.1 does not provide contraction in the part of the norm that accounts for the flow direction. In particular, the estimate in this lemma is not sufficient to control the essential spectral radius of the resolvent. The following lemma provides the first step towards a Lasota-Yorke inequality for resolvents for the coefficients $e_{k,\ell}$ when $\ell < k$. The idea, based on regularization of test functions, is due to [GL06]. The doubling estimates on conditional measures in Proposition 3.1 are crucial for carrying out the argument.

Lemma 7.4. *For all $t \geq 2$ and $0 \leq \ell < k$, we have*

$$e_{k,\ell}(\mathcal{L}_t f) \ll_{k,\beta} e^{-kt} e_{k,\ell}(f) + e'_{k,\ell}(f).$$

Proof. Fix some $0 \leq \ell < k$. Let $x \in \Omega_1^-$ and $\phi \in C^{k+\ell}(N_1^+)$. Let $(v_i)_i \in \mathcal{V}_{k+\ell}^\ell$ and set $F = L_{v_1} \cdots L_{v_\ell} f$. We wish to estimate the following:

$$\sup_{0 \leq s \leq 1} \int_{N_1^+} \phi(n) F(g_{t+s} n x) \, d\mu_x^u.$$

To simplify notation, we prove the desired estimate for $s = 0$, the general case being essentially identical.

Let $\varepsilon > 0$ to be determined and choose ψ_ε to be a C^∞ bump function supported inside N_ε^+ and satisfying $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$. Define the following regularization of ϕ

$$\mathcal{M}_\varepsilon(\phi)(n) = \frac{\int_{N^+} \phi(un) \psi_\varepsilon(u) \, du}{\int_{N^+} \psi_\varepsilon(u) \, du},$$

where du denotes the right-invariant Haar measure on N^+. Recall the definition of the coefficients c_r above (6.2). Let $0 \le m < k+\ell$ and $(w_j) \in (\mathcal{V}^+)^m$. Then,

$$\begin{aligned} |L_{w_1} \cdots L_{w_m}(\phi - \mathcal{M}_\varepsilon(\phi))(n)| &\le \frac{\int |L_{w_1} \cdots L_{w_m}(\phi)(n) - L_{w_1} \cdots L_{w_m}(\phi)(un)| \psi_\varepsilon(u)\, du}{\int \psi_\varepsilon(u)\, du} \\ &\ll c_{m+1}(\phi) \frac{\int \mathrm{dist}(n, un) \psi_\varepsilon(u)\, du}{\int \psi_\varepsilon(u)\, du}. \end{aligned}$$

Now, note that if $\psi_\varepsilon(u) \neq 0$, then $\mathrm{dist}(u, \mathrm{Id}) \le \varepsilon$. Hence, right invariance of the metric on N^+ implies that $c_m(\phi - \mathcal{M}_\varepsilon(\phi)) \ll \varepsilon c_{m+1}(\phi)$.

Moreover, we have that $c_m(\mathcal{M}_\varepsilon(\phi)) \le c_m(\phi)$ for all $0 \le m \le k+\ell$. It follows that $c_{k+\ell}(\phi - \mathcal{M}_\varepsilon(\phi)) \le 2c_{k+\ell}(\phi)$. Finally, given $(w_i) \in (\mathcal{V}^+)^{k+\ell+1}$, integration by parts gives

$$L_{w_1} \cdots L_{w_{k+\ell+1}}(\mathcal{M}_\varepsilon(\phi))(n) = \frac{-\int_{N^+} L_{w_2} \cdots L_{w_{k+\ell+1}}(\phi)(un) \cdot L_{w_1}(\psi_\varepsilon)(u)\, du}{\int_{N^+} \psi_\varepsilon(u)\, du}.$$

In particular, since $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$, we get $c_{k+\ell+1}(\mathcal{M}_\varepsilon(\phi)) \ll \varepsilon^{-1} c_{k+\ell}(\phi)$. Since g_t expands N^+ by at least e^t, this discussion shows that for any $t \ge 0$, if $\|\phi\|_{C^{k+\ell}} \le 1$, then

$$\begin{aligned} \|(\phi - \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} &\ll \varepsilon \sum_{m=0}^{k+\ell-1} \frac{e^{-mt}}{2^m} + \frac{e^{-(k+\ell)t}}{2^{k+\ell}}, \\ \|\mathcal{M}_\varepsilon(\phi) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} &\ll \sum_{m=0}^{k+\ell} \frac{e^{-mt}}{2^m} + \frac{\varepsilon^{-1} e^{-(k+\ell+1)t}}{2^{k+\ell+1}}. \end{aligned} \tag{7.6}$$

Set $\mathcal{A}_t = \mathrm{Ad}(g_t)(N_1^+)$. Then, taking $\varepsilon = e^{-kt}$, we obtain

$$\begin{aligned} \int_{N_1^+} \phi(n) F(g_t n x)\, d\mu_x^u &= \int \phi(n) F(g_t n x)\, d\mu_x^u \\ &= \int (\phi - \mathcal{M}_\varepsilon(\phi))(n) F(g_t n x)\, d\mu_x^u + \int \mathcal{M}_\varepsilon(\phi)(n) F(g_t n x)\, d\mu_x^u. \end{aligned} \tag{7.7}$$

To estimate the second term, we recall that the test functions for the weak norm were required to be supported inside $N_{1/10}^+$. On the other hand, the support of $\mathcal{M}_\varepsilon(\phi)$ may be larger, but still inside $N_{1+\varepsilon}^+$. To remedy this issue, we pick a partition of unity $\{\rho_i : i \in I\}$ of N_2^+, so that each ρ_i is smooth, non-negative, and supported inside some ball of radius $1/20$. We also require that $\|\rho_i\|_{C^{k+\ell+1}} \ll_k 1$. We can find such a partition of unity with bounded cardinality and multiplicity, depending only on N^+ (through its dimension and metric).

Similarly to Lemma 7.1, we denote by $I(\Lambda) \subseteq I$, the subset of those indices i such that there is some $n_i \in N^+$ in the support of of μ_x^u so that the support of ρ_i is contained inside $N_{1/10}^+ \cdot n_i$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu_x^u$ is the 0 measure.

Now, observe that the functions $n \mapsto \rho_i(nn_i)\mathcal{M}_\varepsilon(\phi)(nn_i)$ are supported inside $N_{1/10}^+$. Thus, writing $x_i = n_i g_1 x$, using a change of variable, and arguing as in the proof of Lemma 7.1, cf. (7.3), we obtain

$$\begin{aligned} \int \mathcal{M}_\varepsilon(\phi)(n) F(g_t n x)\, d\mu_x^u &= e^{-\delta} \sum_{i \in I(\Lambda)} \int (\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-1})(n) F(g_{t-1} n g_1 x)\, d\mu_{g_1 x}^u \\ &\ll e'_{k,\ell}(f) \cdot \sum_{i \in I(\Lambda)} \|(\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} \cdot V(x_i) \mu_{x_i}^u(N_1^+). \end{aligned}$$

The point of replacing x with $g_1 x$ is that since x belongs to $N_1^- \Omega$, $g_1 x$ belongs to $N_{1/2}^- \Omega$, which satisfies the requirement on the basepoints in the definition of the weak norm.

Note that the bounded multiplicity property of the partition of unity, together with the doubling property in Proposition 3.1, imply that

$$\sum_{i\in I}\mu^u_{x_i}(N_1^+)\ll\mu^u_x(N_3^+)\ll\mu^u_x(N_1^+).$$

Moreover, combining the Leibniz estimate (6.2) with (7.6), we see that the $C^{k+\ell+1}$ norm of $(\rho_i\mathcal{M}_\varepsilon(\phi))\circ\mathrm{Ad}(g_{-t})$ is $O_k(1)$. Hence, by properties of the height function V in Proposition 4.3, it follows that

$$\int\mathcal{M}_\varepsilon(\phi)(n)F(g_tnx)\,d\mu^u_x\ll_k e'_{k,\ell}(f)V(x)\mu^u_x(N_1^+).$$

Using a completely analogous argument to handle the issues of the support of the test function, we can estimate the first term in (7.7) as follows:

$$\frac{1}{V(x)\mu^u_x(N_1^+)}\int_{N_1^+}(\phi-\mathcal{M}_\varepsilon(\phi))(n)F(g_tnx)\,d\mu^u_x\ll_k e^{-kt}e_{k,\ell}(f).$$

Since $(v_i)\in\mathcal{V}^\ell_{k+\ell}$, $x\in\Omega_1^-$ and $\phi\in C^{k+\ell}(N_1^+)$ were all arbitrary, this completes the proof. □

It remains to estimate the coefficients $e_{k,k}$. First, the following estimate in the case all the derivatives point in the stable direction follows immediately from Lemma 7.1.

Lemma 7.5. *For all $\gamma=(v_i)\in(\mathcal{V}^-_{2k})^k$, we have*

$$e_{k,k,\gamma}(R(z)^nf)\ll_\beta\frac{1}{(\mathrm{Re}(z)+k)^n}e_{k,k}(f).$$

Proof. Indeed, Lemma 7.1 shows that

$$e_{k,k,\gamma}(\mathcal{L}_tf)\ll e^{-kt}e_{k,k}(f).$$

Moreover, induction and integration by parts give $|\int_0^\infty t^{n-1}e^{-(z+k)t}/(n-1)!dt|\leq 1/(\mathrm{Re}(z)+k)^n$. This completes the proof. □

To give improved estimates on the the coefficient $e_{k,k,\gamma}$ in the case some of the components of γ point in the flow direction, the idea (cf. [AG13, Lem. 8.4] and [GLP13, Lem 4.5]) is to take advantage of the fact that the resolvent is defined by integration in the flow direction, which provides additional smoothing. This is leveraged through integration by parts to estimate the coefficient $e_{k,k}$ by $e_{k,k-1}$.

To see how such estimate can be turned into a gain on the norm of the resolvents, following [AG13], we define the following equivalent norms to $\|\cdot\|_k$. First, let us define the following coefficients:

$$e_{k,\ell,s}:=\begin{cases}e_{k,\ell} & 0\leq\ell<k,\\ \sup_{\gamma\in(\mathcal{V}^-_{2k})^k}e_{k,k,\gamma} & \ell=k,\end{cases},\qquad e_{k,k,\omega}:=\sup_{\gamma\in\mathcal{V}^k_{2k}\setminus(\mathcal{V}^-_{2k})^k}e_{k,k,\gamma}.$$

Given $B\geq1$, define

$$\|f\|_{k,B,s}:=\sum_{\ell=0}^k\frac{e_{k,\ell,s}(f)}{B^\ell},\qquad\|f\|_{k,B,\omega}:=\frac{e_{k,k,\omega}(f)}{B^k}.$$

Finally, we set

$$\|f\|_{k,B}:=\|f\|_{k,B,s}+\|f\|_{k,B,\omega}\,.\tag{7.8}$$

Lemma 7.6. *Let $n,k\in\mathbb{N}$ and $z\in\mathbb{C}$ with $\mathrm{Re}(z)>0$ be given. Then, if B is large enough, depending on n,k,β and z, we obtain for all $f\in C_c^{k+1}(X)^M$ that*

$$\|R(z)^nf\|_{k,B,\omega}\leq\frac{1}{(\mathrm{Re}(z)+k+1)^n}\|f\|_{k,B}\,.$$

Proof. Fix an integer $n \geq 0$. We wish to estimate integrals of the form

$$\int_{N_1^+} \phi(u) L_{v_1} \cdots L_{v_k}\left(\int_0^\infty \frac{t^n e^{-zt}}{n!} \mathcal{L}_{t+s} f \, dt \right)(ux) \, d\mu_x^u(u) = \int_{N_1^+} \phi(u) \int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} \cdots L_{v_k} (\mathcal{L}_{t+s} f)(ux) \, dt \, d\mu_x^u(u),$$

with $0 \leq s \leq 1$ and at least one of the v_i pointing in the flow direction.

First, let us consider the case v_k points in the flow direction. Then, $v_k(u) = \psi_k(u)\omega$, where ω is the vector field generating the geodesic flow, for some function ψ_k in the unit ball of $C^{2k}(N^+)$. Hence, for a fixed $u \in N_1^+$, integration by parts in t, along with the fact that f is bounded, yields

$$\begin{aligned} &\int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} L_{v_2} \cdots L_{v_k} (\mathcal{L}_{t+s} f)(ux) \, dt \\ &= \psi_k(u) z \int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_{t+s} f)(ux) \, dt - \psi_k(u) \int_0^\infty \frac{t^{n-1} e^{-zt}}{(n-1)!} L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_{t+s} f)(ux) \, dt \\ &= \psi_k(u) z L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_s R(z)^{n+1} f)(ux) - \psi_k(u) L_{v_1} \cdots L_{v_{k-1}} (\mathcal{L}_s R^n(z) f)(ux). \end{aligned}$$

Recall by Lemma 7.1 that $e_{k,\ell}(R(z)^n f) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n$ for all $n \in \mathbb{N}$; cf. Corollary 7.3. It follows that

$$e_{k,k,\gamma}(R(z)^{n+1} f) \leq e_{k,k-1}(R(z)^n f) + |z| e_{k,k-1}(R(z)^{n+1} f) \ll_\beta \left(\frac{\mathrm{Re}(z) + |z|}{\mathrm{Re}(z)^{n+1}} \right) e_{k,k-1}(f).$$

In the case v_k points in the stable direction instead, we note that $L_v L_w = L_w L_v + L_{[v,w]}$ for any two vector fields v and w, where $[v,w]$ is their Lie bracket. In particular, we can write $L_{v_1} \cdots L_{v_k}$ as a sum of at most k terms involving $k-1$ derivatives in addition to one term of the form $L_{w_1} \cdots L_{w_k}$, where w_k points in the flow direction. Each of the terms with one fewer derivative can be bounded by $e_{k,k-1}(R(z)^{n+1} f) \ll_\beta e_{k,k-1}(f)/\mathrm{Re}(z)^{n+1}$, while the term with k derivatives is controlled as in the previous case. Hence, taking the supremum over $\gamma \in \mathcal{V}_{2k}^k \setminus (\mathcal{V}_{2k}^-)^k$ and choosing B to be large enough, we obtain the conclusion. $\square$

7.3. **Decomposition of the transfer operator according to recurrence of orbits.** In order to make use of the compact embedding result in Proposition 6.6, we need to localize our functions to a fixed compact set. This is done with the help of the Margulis function V. In this section, we introduce some notation and prove certain preliminary estimates for that purpose.

Recall the notation in Theorem 4.1. Let $T_0 \geq 1$ be a constant large enough so that $e^{\beta T_0} > 1$. We will enlarge T_0 over the course of the argument to absorb various auxiliary uniform constants. Define V_0 by

$$V_0 = e^{3\beta T_0}. \tag{7.9}$$

Let $\rho_{V_0} \in C_c^\infty(X)$ be a non-negative M-invariant function satisfying $\rho_{V_0} \equiv 1$ on the unit neighborhood of $\{x \in X : V(x) \leq V_0\}$ and $\rho_{V_0} \equiv 0$ on $\{V > 2V_0\}$. Moreover, we require that $\rho_{V_0} \leq 1$. Note that since T_0 is at least 1, we can choose ρ_{V_0} so that its C^{2k} norm is independent of T_0.

Let $\psi_1 = \rho_{V_0}$ and $\psi_2 = 1 - \psi_1$. Then, we can write

$$\mathcal{L}_{T_0} f = \tilde{\mathcal{L}}_1 f + \tilde{\mathcal{L}}_2 f,$$

where $\tilde{\mathcal{L}}_i f = \mathcal{L}_{T_0}(\psi_i f)$, for $i \in \{1,2\}$. It follows that for all $j \in \mathbb{N}$, we have

$$\mathcal{L}_{jT_0} f = \sum_{\varpi \in \{1,2\}^j} \tilde{\mathcal{L}}_{\varpi_1} \cdots \tilde{\mathcal{L}}_{\varpi_j} f = \sum_{\varpi \in \{1,2\}^j} \mathcal{L}_{jT_0}(\psi_\varpi f), \qquad \psi_\varpi = \prod_{i=1}^j \psi_{\varpi_i} \circ g_{-(j-i)T_0}. \tag{7.10}$$

Note that if $\varpi_i = 1$ for some $1 \leq i \leq j$, then, by Proposition 4.3, we have

$$\sup_{x \in \mathrm{supp}(\psi_\varpi)} V(x) \leq e^{\beta I_\varpi T_0} V_0, \qquad I_\varpi = j - \max\{1 \leq i \leq j : \varpi_i = 1\}. \tag{7.11}$$

For simplicity, let us write

$$f_\varpi := \psi_\varpi f.$$

The following lemma estimates the effect of multiplying by a fixed smooth function such as ψ_ϖ.

Lemma 7.7. *Let $\psi \in C^{2k}(X)$ be given. Then, if $B \geq 1$ is large enough, depending on k and $\|\psi\|_{C^{2k}}$, we have*

$$\|\psi f\|_{k,B,s} \leq \|f\|_{k,B,s}.$$

Proof. Given $0 \leq \ell \leq k$ and $0 \leq s \leq 1$, we wish to estimate integrals of the form

$$\int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\psi f)(g_s n x) \, d\mu_x^u(n).$$

The term $L_{v_1} \cdots L_{v_\ell}(\psi f)$ can be written as a sum of ℓ terms, each consisting of a product of i derivatives of ψ by $\ell - i$ derivatives of f, for $0 \leq i \leq \ell$. Viewing the product of ϕ by i derivatives of ψ as a $C^{k+\ell-i}$ test function, and using (6.2) to bound the $C^{k+\ell-i}$ norm of such a product, we obtain a bound of the form

$$e_{k,\ell,s}(\psi f) \leq \|\psi\|_{C^{2k}} \sum_{i=0}^{\ell} e_{k,i,s}(f).$$

Hence, given $B \geq 1$, we obtain

$$\|f\|_{k,B,s} = \sum_{\ell=0}^{k} \frac{1}{B^\ell} e_{k,\ell}(\psi f) \leq \|\psi\|_{C^{2k}} \sum_{\ell=0}^{k} \frac{1}{B^\ell} \sum_{i=0}^{\ell} e_{k,i,s}(f) \leq \|\psi\|_{C^{2k}} \sum_{\ell=0}^{k} \frac{k-\ell}{B} \frac{e_{k,\ell,s}(f)}{B^\ell}.$$

Thus, the conclusion follows as soon as B is large enough, depending only on k and $\|\psi\|_{C^{2k}}$. $\square$

The above lemma allows us to estimate the norms of the operators $\tilde{\mathcal{L}}_i$, for $i = 1, 2$ as follows.

Lemma 7.8. *If $B \geq 1$ is large enough, depending on k and $\|\rho_{V_0}\|_{C^{2k}}$, we obtain*

$$\left\| \tilde{\mathcal{L}}_1 f \right\|_{k,B,s} \ll_\beta \|f\|_{k,B,s}, \qquad \left\| \tilde{\mathcal{L}}_2 f \right\|_{k,B,s} \ll_\beta e^{-\beta T_0} \|f\|_{k,B,s}.$$

Proof. The first inequality follows by Lemmas 7.1 and 7.7, since $\|\psi_i\|_{C^k} \ll 1$ for $i = 1, 2$. The second inequality follows similarly since

$$\psi_2(g_{T_0} n x) \neq 0 \Longrightarrow V(g_{T_0} n x) \geq V_0.$$

By Proposition 4.3, this in turn implies that, whenever $\psi_2(g_{T_0} n x) \neq 0$ for some $n \in N_1^+$, then $V(x) \gg e^{\beta T_0}$, by choice of V_0. $\square$

7.4. **Proof of Theorems 6.1 and 6.4.** Theorem 6.1 follows at once from 6.4. Theorem 6.4 will follow upon verifying the hypotheses of Theorem 6.5. The boundedness assertion follows by Corollary 7.3. It remains to estimate the essential spectral radius of the resolvent $R(z)$.

Write $z = a + ib \in \mathbb{C}$. Fix some parameter $0 < \theta < 1$ and define

$$\sigma := \min\{k, \beta\theta\}.$$

Let $0 < \epsilon < \sigma/5$ be given. We show that for a suitable choice of r and B, the following Lasota-Yorke inequality holds:

$$\left\| R(z)^{r+1} f \right\|_{k,B} \leq \frac{\|f\|_{k,B}}{(a + \sigma - 3\epsilon)^{r+1}} + C'_{k,r,z,\beta} \|\Psi_r f\|'_k, \tag{7.12}$$

where $C'_{k,r,z,\beta} \geq 1$ is a constant depending on k, r and z, while Ψ_r is a compactly supported smooth function on X, and whose support depends on r.

First, we show how (7.12) implies the result. Hennion's Theorem, Theorem 6.5, applied with the norm $\|\cdot\| = \|\cdot\|_{k,B}$ and the semi-norm $\|\cdot\|' = \|\Psi_r \bullet\|'_k$, implies that the essential spectral radius of $R(z)$, with respect to the norm $\|\cdot\|_{k,B}$, is at most $1/(a+\sigma-3\epsilon)$. Equivalence of the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,B}$ implies that the same estimate also holds for the essential spectral radius $\rho_{ess}(R(z))$ with respect to $\|\cdot\|_k$. Note that the compact embedding requirement follows by Proposition 6.6 again by equivalence of the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,B}$. Since $\epsilon > 0$ was arbitrary, this shows that $\rho_{ess}(R(z)) \leq 1/(a+\sigma)$. Finally, as $0 < \theta < 1$ was arbitrary, we obtain that

$$\rho_{ess}(R(z)) \leq \frac{1}{\mathrm{Re}(z) + \sigma_0},$$

completing the proof.

To show (7.12), let an integer $r \geq 0$ be given and $J_r \in \mathbb{N}$ to be determined. Using (7.10) and a change of variable, we obtain

$$\begin{aligned} R(z)^{r+1} f &= \int_0^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \\ &= \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt + \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt + \sum_{j=1}^{J_r} \int_{jT_0}^{(j+1)T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt. \end{aligned}$$

First, by Lemma 7.6, if B is large enough, depending on r, k and z, we obtain

$$\left\| R(z)^{r+1}(z) f \right\|_{k,B,\omega} \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

It remains to estimate $\left\| R(z)^{r+1} f \right\|_{k,B,s}$. Note that $\int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq T_0^{r+1}/r!$. Hence, taking r large enough, depending on k, a, β and T_0, and using Lemma 7.1, we obtain for any $B \geq 1$,

$$\left\| \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \right\|_{k,B,s} \ll_\beta \|f\|_{k,B} \int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

Similarly, taking J_r to be large enough, depending on k, a, β, and r, we obtain for any $B \geq 1$,

$$\left\| \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \right\|_{k,B,s} \ll_\beta \|f\|_{k,B} \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-at}}{r!} \; dt \leq \frac{1}{(a+k+1)^{r+1}} \|f\|_{k,B}.$$

To estimate the remaining term in $R(z)^{r+1} f$, let $1 \leq j \leq J_r$ and $\varpi = (\varpi_i)_i \in \{1,2\}^j$ be given. Let θ_ϖ denote the number of indices i such that $\varpi_i = 2$. Then, taking B large enough, depending on k and $C^{2k}(\psi_\varpi)$, it follows from Lemma 7.1 and induction on Lemma 7.8 that

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \leq C_0 \|\mathcal{L}_{jT_0}(\psi_\varpi f)\|_{k,B,s} \leq C_0^{j+1} e^{-\beta\theta_\varpi jT_0} \|f\|_{k,B,s}, \tag{7.13}$$

where we take $C_0 \geq 1$ to be larger than the implied uniform constant in Lemma 7.8 and the implied constant in Lemma 7.1. Suppose $\theta_\varpi \geq \theta$. Then, by taking T_0 to be large enough, we obtain

$$\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \leq e^{-(\beta\theta-\epsilon)jT_0} \|f\|_{k,B,s}.$$

On the other hand, if $\theta_\varpi < \theta$, we apply Lemma 7.4 to obtain for all $0 \leq \ell < k$,

$$e_{k,\ell}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_{k,\beta} e^{-(t+jT_0)k} e_{k,\ell}(\psi_\varpi f) + e'_{k,\ell}(\psi_\varpi f),$$

where we may assume that T_0 is at least 2 so that the same holds for $t + jT_0$, thus verifying the hypothesis of the lemma. Moreover, we note that (7.11), implies that ψ_ϖ is supported inside a sublevel set of V, depending only on θ and J_r. Let Ψ_r denote a smooth bump function on X which is identically 1 on the union of the (finitely many) supports of ψ_ϖ as ϖ ranges over tuples in $\{1,2\}^j$

with $\theta_\varpi < \theta$ and for $1 \leq j \leq J_r$. Note that for any such ϖ, arguing as in the proof of Lemma 7.7, we obtain

$$e'_{k,\ell}(\psi_\varpi f) = e'_{k,\ell}(\psi_\varpi \Psi_r f) \ll_k \|\Psi_r f\|'_k .$$

For the coefficient $e_{k,k}$, Lemma 7.5 shows that for any $\gamma \in (\mathcal{V}_{2k}^-)^k$, we have

$$e_{k,k,\gamma}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_\beta e^{-(t+jT_0)k} e_{k,k}(\psi_\varpi f).$$

Combining these estimates, and using Lemma 7.7, we obtain

$$\begin{aligned} \|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} &\leq C_0 e^{-(\sigma-\epsilon)jT_0} \|\psi_\varpi f\|_{k,B,s} + C_{k,r,z,\beta} \|\Psi_r f\|'_k \\ &\leq e^{-(\sigma-2\epsilon)jT_0} \|\psi_\varpi f\|_{k,B,s} + C_{k,r,z,\beta} \|\Psi_r f\|'_k , \end{aligned}$$

where we enlarge the constant C_0 as necessary to subsume the implied constants and the constant $C_{k,r,z,\beta} \geq 1$ is large enough, depending on B, so the above inequality holds. The inequality on the second line follows by taking T_0 large enough depending on C_0 and ϵ.

Putting the above estimates together, we obtain

$$\begin{aligned} \left\| \sum_{j=1}^{J_r} \int_{jT_0}^{(j+1)T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \, dt \right\|_{k,B,s} &\leq \sum_{j=1}^{J_r} e^{-ajT_0} \sum_{\varpi \in \{1,2\}^j} \int_0^{T_0} \frac{(t+jT_0)^r e^{-at}}{r!} \|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} \, dt \\ &\leq \|f\|_{k,B,s} \sum_{j=1}^{J_r} e^{-(a+\sigma-2\epsilon)jT_0} \int_0^{T_0} \frac{(t+jT_0)^r e^{-at}}{r!} dt \\ &+ C_{k,r,z,\beta} \|\Psi_r f\|'_k \sum_{j=1}^{J_r} 2^j e^{-ajT_0} \int_0^{T_0} \frac{(t+jT_0)^r e^{-at}}{r!} \, dt \\ &\leq e^{(\sigma-2\epsilon)T_0} \|f\|_{k,B,s} \int_1^{J_r} \frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!} \, dt + C'_{k,r,z,\beta} \|\Psi_r f\|'_k , \end{aligned}$$

where we take $C'_{k,r,z,\beta} \geq 1$ to be a constant large enough so that the last inequality holds.

Next, we note that

$$\int_1^{J_r} \frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!} \, dt \leq \int_0^\infty \frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!} \, dt = \frac{1}{(a+\sigma-2\epsilon)^{r+1}}.$$

Thus, taking r to be large enough depending on a and T_0, and combining the estimates on $\left\|R(z)^{r+1} f\right\|_{k,B,\omega}$ and $\left\|R(z)^{r+1} f\right\|_{k,B,s}$, we obtain (7.12) as desired.

7.5. **Proof of Theorem 1.4.** Recall the notation in the statement of the theorem. We note that switching the order of integration in the definition of the Laplace transform shows that

$$\hat{\rho}_{f,g}(z) = \int R(z)(f) g \, dm^{\mathrm{BMS}}, \qquad \mathrm{Re}(z) > 0.$$

In particular, the poles of $\hat{\rho}_{f,g}$ are contained in those of the resolvent $R(z)$.

On the other hand, Corollary 7.2 implies that the infinitesimal generator $\mathfrak{X}$ of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. Moreover, $R(z)$ coincides with the resolvent operator $(\mathfrak{X} - z\mathrm{Id})^{-1}$ associated to $\mathfrak{X}$, whenever z belongs to the resolvent set (complement of the spectrum) of $\mathfrak{X}$.

We further note that the spectra of $\mathfrak{X}$ and $R(z)$ are related by the formula $\sigma(\mathfrak{X}) = z - 1/\sigma(R(z))$. In particular, by Theorem 6.4, in the half plane $\mathrm{Re}(z) > -\sigma_0$, the poles of $R(z)$ coincide with the eigenvalues of $\mathfrak{X}$. In view of this relationship between the spectra, the fact that the imaginary axis does not contain any poles for the resolvent, apart from 0, follows from the mixing property of the geodesic flow with respect to $\mathrm{m}^{\mathrm{BMS}}$. We refer the reader to Lemma 9.7 for a proof of this assertion[9].

Finally, we note that in the case Γ has cusps, β was an arbitrary constant in $(0, \Delta/2)$, so that we may take σ_0 in the conclusion of Theorem 6.4 to be the minimum of k and $\Delta/2$ in this case. This completes the proof of Theorem 1.4.

7.6. **Exponential recurrence from the cusp and Proof of Theorem 1.10.** As a corollary of our analysis, we obtain the following stronger form of Theorem 1.10 regarding the exponential decay of the measure of orbits spending a large proportion of their time in the cusp. This result is crucial to our arguments in later sections. The deduction of Theorem 1.10 in its continuous time formulation from the following result follows using Proposition 4.3 and is left to the reader.

Theorem 7.9. *For every $\varepsilon > 0$, there exists $r_0 \asymp_\beta 1/\varepsilon$ such that the following holds for all $m \in \mathbb{N}, r \geq r_0, 0 < \theta < 1$ and $x \in N_1^- \Omega$. Let $H = e^{4\beta r_0}$, and let χ_H be the indicator function of the set $\{x : V(x) > H\}$. Then,*

$$\mu_x^u \left(n \in N_1^+ : \sum_{1 \leq \ell \leq m} \chi_H(g_{r\ell} n x) > \theta m \right) \leq e^{-(\beta\theta - \varepsilon)m} V(x) \mu_x^u(N_1^+).$$

Proof. The argument is very similar to the proof of the estimate (7.13), with small modifications allowing for the height H to be independent of the step size r. This subtle difference from (7.13) will be important later in the proof of Corollary 12.2.

Let $r_0 \geq 1$ to be chosen later in the argument depending on ε and β and set $V_0 = e^{2\beta r_0}$. As before, let $\rho_{V_0} : X \to [0,1]$ denote a smooth compactly supported function which is identically 1 on $\{V \leq V_0\}$ and vanishing outside $\{V > 2V_0\}$. Let $\psi = 1 - \rho_{V_0}$, and set

$$\tilde{\psi} = \psi \cdot \psi \circ g_{-r_0}.$$

Then, roughly speaking, $\tilde{\psi}$ is the indicator function of the set of points which land in the cusp in two successive steps of size r_0. Let $r \geq r_0$ and define the following operators:

$$\tilde{\mathcal{L}}_1(f) := \mathcal{L}_r f, \qquad \tilde{\mathcal{L}}_2(f) = \mathcal{L}_r(\tilde{\psi} f).$$

Then, given $m \in \mathbb{N}$ and $\varpi \in \{1,2\}^m$, let $\mathcal{L}_\varpi = \tilde{\mathcal{L}}_{\varpi_1} \circ \cdots \circ \tilde{\mathcal{L}}_{\varpi_m}$. Then, note that

$$\mathcal{L}_\varpi(f) = \mathcal{L}_{mr}(\tilde{\psi}_\varpi f), \qquad \text{where} \quad \tilde{\psi}_\varpi = \prod_{\ell : \varpi_\ell = 2} \tilde{\psi} \circ g_{(\ell - k)r}.$$

Similarly to Lemma 7.8, Lemma 7.1 implies the bounds

$$e_{1,0}(\tilde{\mathcal{L}}_1 f) \ll_\beta e_{1,0}(f), \qquad e_{1,0}(\tilde{\mathcal{L}}_2 f) \ll_\beta e^{-\beta r_0} e_{1,0}(f). \tag{7.14}$$

Note that the argument in Lemma 7.8 only guarantees the second bound for the coefficient $e_{1,0}$ since $\tilde{\psi}$ involves composition with g_{-r_0} which scales its stable derivatives by powers of e^{r_0}.

Let $C_\beta \geq 1$ be a constant large enough so that $V(g_t y) \geq e^{-\beta|t|} V(y)/C_\beta$ for all $y \in X$ and $t \in \mathbb{R}$. Such a constant exists by Proposition 4.3. By enlarging r_0 if necessary, we may assume that $e^{\beta r_0} \geq 2C_\beta$. Let $H = e^{3\beta r_0}$ and define

$$E_\varpi = \left\{ n \in N_1^+ : V(g_{\ell r} n x) > H \Leftrightarrow \varpi_\ell = 2, \text{ for all } \ell = 1, \dots, m \right\}.$$

Then, for all $n \in N_1^+$,

$$\tilde{\psi}_\varpi(g_{mr} n x) \geq \mathbb{1}_{E_\varpi}(n). \tag{7.15}$$

[9] Lemma 9.7 is obtained for a slightly different norm but the proof is identical.

Denote by θ_ϖ the proportion of indices ℓ for which $\varpi_\ell = 2$. Then, we see that

$$\left\{ n \in N_1^+ : \sum_{1 \le \ell \le m} \chi_H(g_{r\ell} nx) > \theta m \right\} \subseteq \bigcup_{\varpi : \theta_\varpi > \theta} E_\varpi.$$

We wish to apply (7.14) with f the constant function on X. One checks that this f belongs to the space $\mathcal{B}_1$ and $e_{1,0}(f) \ll 1$. Let $C_1 \ge 1$ denote a constant larger than $e_{1,0}(f)$ and the two implicit constants in (7.14). Then, applying (7.14) iteratively m times, and using (7.15), we obtain

$$\mu_x^u(E_\varpi) \le e_{1,0}(\mathcal{L}_\varpi(f)) \le C_1^m e^{-\beta\theta_\varpi m r_0} V(x) \mu_x^u(N_1^+) e_{1,0}(f) \le C_1^{m+1} e^{-\beta\theta_\varpi m r_0} V(x) \mu_x^u(N_1^+).$$

Since there are at most 2^m choices of ϖ, the result follows by taking r_0 large enough so that $(2C_1)^{1/r_0} \le e^\varepsilon$. □

8. Fractal Mollifiers

In this section, we introduce certain mollification operators on smooth functions on X. These operators have the advantage that, roughly speaking, their Lipschitz norms are dominated by the norms introduced in (6.6). This property is very convenient in the estimates carried out in Section 9. The idea of using mollifiers to handle analogous steps is due to [BL12].

8.1. **Definition and regularity of mollifiers.** Fix a non-negative C^∞ bump function ψ supported inside $N_{1/2}^+$ and having value identically 1 on $N_{1/4}^+$. We also choose ψ to be symmetric and $\mathrm{Ad}(M)$-invariant, i.e.

$$\psi(n) = \psi(n^{-1}), \qquad \psi(mnm^{-1}) = \psi(n), \quad \forall n \in N^+, m \in M. \tag{8.1}$$

Given $\varepsilon > 0$, define $\mathbb{M}_\varepsilon : C(X) \to C(X)$ be the operator defined by

$$\mathbb{M}_\varepsilon(f)(x) = \int \frac{\psi_\varepsilon(n)}{\int \psi_\varepsilon \, d\mu_{nx}^u} f(nx) \, d\mu_x^u(n), \qquad \psi_\varepsilon(n) = \psi(\mathrm{Ad}(g_{-\log\varepsilon})(n)). \tag{8.2}$$

Note that ψ_ε is supported inside $N_{\varepsilon/2}^+$.

Remark 8.1. The condition that $\psi_\varepsilon(\mathrm{Id}) = \psi(\mathrm{Id}) = 1$ implies that for $x \in X$ with $x^+ \in \Lambda_\Gamma$,

$$\mu_x^u(\psi_\varepsilon) > 0, \qquad \forall \varepsilon > 0. \tag{8.3}$$

In particular, since the conditional measures μ_x^u are supported on points nx with $(nx)^+ \in \Lambda_\Gamma$, the mollifier $\mathbb{M}_\varepsilon(f)$ is a well-defined function on all of X. That $\mathbb{M}_\varepsilon(f)$ is continuous follows by continuity of the map $x \mapsto \mu_x^u$ in the weak-$*$ topology; cf. [Rob03, Lemme 1.16].

Remark 8.2. We note that $\mathrm{Ad}(M)$-invariance of ψ_ε and the conditional measures μ_x^u (cf. (2.5)) implies that $\mathbb{M}_\varepsilon(f)$ is M-invariant whenever f is.

To simplify arguments related to the regularity of the function $n \mapsto \psi_\varepsilon(n)/\mu_{nx}^u(\psi_\varepsilon)$, we introduce the following slightly stronger version of the norm $\|\cdot\|_1$ which suffices for our purposes.

Let $C^{k,\alpha}(N_1^+)$ denote the space of C^k-functions ϕ on N_1^+, all of whose derivatives of order k are α-Hölder continuous functions on N_1^+. We endow this space with the standard norm denoted $\|\phi\|_{C^{k,\alpha}}$. We define coefficients $e_{1,0}^\star(f)$ and $e_{1,1}^\star(f)$, similarly to the coefficients $e_{1,0}$ and $e_{1,1}$ respectively in (6.3) and (6.5), but where, in both coefficients, the supremum is taken over all test functions $\phi \in C^{0,1}(N_1^+)$ with $\|\phi\|_{C^{0,1}} \le 1$, instead of $C^1(N_1^+)$ and $C^2(N_1^+)$. Using these definitions, we introduce the following seminorm on $C_c^2(X)$:

$$\|f\|_1^\star = e_{1,0}^\star(f) + e_{1,1}^\star(f). \tag{8.4}$$

We denote by $\mathcal{B}_\star$ the Banach space completion of the quotient space $C_c^2(X)^M$ of M-invariant compactly supported C^2-functions by the kernel of the seminorm $\|\cdot\|^\star$ with respect to the induced norm on the quotient.

The first result asserts that $\mathbb{M}_\varepsilon(f)$ is a good approximation of f.

Proposition 8.3. *For all $0 < \varepsilon \le 1/10$, and $t \ge 1$, we have*

$$e^\star_{1,0}(\mathcal{L}_t(f - \mathbb{M}_\varepsilon(f))) \ll (\varepsilon + 1)e^{-t} e^\star_{1,0}(f).$$

In light of this statement, we will in fact only use $\mathbb{M}_\varepsilon$ with $\varepsilon = 1/10$. However, for clarity, we state and prove the remaining results for a general value of ε.

The following results estimate the regularity of mollifiers. Recall the constant $\Delta_+ \ge 0$ in (3.1). The first result is an estimate of L^∞ type.

Proposition 8.4. *For all $0 < \varepsilon \le 1$ and $x \in N_1^-\Omega$, we have*

$$|\mathbb{M}_\varepsilon(f)(x)| \ll \varepsilon^{-\Delta_+ - 1} e^\star_{1,0}(f) V(x).$$

Finally, we need the following Lipschitz estimate on mollifiers along the stable direction. Recall the stable parabolic group $P^- = N^- AM$ parametrizing the weak stable manifolds of g_t.

Proposition 8.5. *For all $0 < \varepsilon \le 1/10$, $p^- \in P^-$, and $x \in X$ so that x belongs to $N^-_{3/4}\Omega$ and p^- is of the form $u^- g_t m$ for $u^- \in N^-_{1/10}$, $|t| \le 1/10$ and $m \in M$, we have that*

$$|\mathbb{M}_\varepsilon(f)(p^- x) - \mathbb{M}_\varepsilon(f)(x)| \ll \mathrm{dist}(p^-, \mathrm{Id})\varepsilon^{-\Delta_+ - 2} \cdot \|f\|_1^\star V(x).$$

The above results are straightforward in the case of smooth mollifiers, however some care is required in our case due to the fractal nature of the conditionals and (possible) non-compactness of Ω. This is in part the reason for the non-standard shape of the chosen mollifier. The proofs of the above results are rather technical and can be skipped on a first reading.

8.2. **Preliminary estimates.** We begin by providing some tameness estimates for our mollifiers. The first lemma extends the applicability of Proposition 3.1 to points that are near, but not necessarily in, Ω.

Lemma 8.6. *For all $x \in N_1^-\Omega$, and $0 < \varepsilon \le 1$, we have*

$$\frac{\mu^u_{nx}(N^+_{5\varepsilon})}{\mu^u_{nx}(\psi_\varepsilon)} \ll 1,$$

uniformly over $n \in N_1^+$ in the $(\varepsilon/10)$-neighborhood of the support of μ^u_x.

Proof. Since $\psi_\varepsilon \equiv 1$, $\mu^u_{nx}(\psi_\varepsilon) \ge \mu^u_{nx}(N^+_{\varepsilon/4})$. Let u be in the support of μ^u_x, which is at distance $\varepsilon/10$ from n. In particular, $ux \in N_2^-\Omega$ by Remark 2.1. Hence, using a change of variables and Proposition 3.1, we obtain

$$\frac{\mu^u_{nx}(N^+_{5\varepsilon})}{\mu^u_{nx}(\psi_\varepsilon)} \le \frac{\mu^u_{nx}(N^+_{5\varepsilon})}{\mu^u_{nx}(N^+_{\varepsilon/4})} \le \frac{\mu^u_{ux}(N^+_{5\varepsilon} \cdot (nu^{-1}))}{\mu^u_{ux}(N^+_{\varepsilon/4} \cdot (nu^{-1}))} \le \frac{\mu^u_{ux}(N^+_{6\varepsilon})}{\mu^u_{ux}(N^+_{\varepsilon/8})} \ll 1.$$

∎

The next statement is roughly a Lipschitz estimate on conditional measures.

Lemma 8.7. *For all $0 < \varepsilon \le 1$ and $x \in N_1^-\Omega$, we have the following. For all $n_1, n_2 \in N_1^+$ with $d_{N^+}(n_1, n_2) \le \varepsilon/2$, we have*

$$\left| \frac{1}{\mu^u_{n_1 x}(\psi_\varepsilon)} - \frac{1}{\mu^u_{n_2 x}(\psi_\varepsilon)} \right| \ll \frac{\varepsilon^{-1} d_{N^+}(n_1, n_2)}{\mu^u_{n_2 x}(\psi_\varepsilon)},$$

provided n_1 is at distance at most $\varepsilon/10$ from the support of μ^u_x.

Proof. Let $\sigma = n_1 n_2^{-1}$. Since ψ_ε is supported inside $N^+_{\varepsilon/2}$. we have by the symmetry of ψ in (8.1) and the right invariance of the metric d_{N^+} on N^+ that

$$|\mu^u_{n_1x}(\psi_\varepsilon) - \mu^u_{n_2x}(\psi_\varepsilon)| \leq \int |\psi_\varepsilon(n) - \psi_\varepsilon(n\sigma)| \, d\mu^u_{n_1x}(n) = \int |\psi_\varepsilon(n^{-1}) - \psi_\varepsilon(\sigma^{-1}n^{-1})| \, d\mu^u_{n_1x}(n)$$
$$\ll \|\psi_\varepsilon\|_{C^1} \, d_{N^+}(n_1, n_2)\mu^u_{n_1x}(N^+_\varepsilon),$$

where on the last line we used the fact that the integrands are non-zero only on the union $N^+_{\varepsilon/2} \cup N^+_{\varepsilon/2}\sigma \subseteq N^+_\varepsilon$. Moreover, Lemma 8.6 implies that $\mu^u_{n_1x}(N^+_\varepsilon)/\mu^u_{n_1x}(\psi_\varepsilon) \ll 1$. The conclusion follows since $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$. $\square$

8.3. **Regularity of mollifiers and proof of Proposition 8.3.** Let $\varphi \in C^{0,1}(N^+_1)$ be a test function and let $x \in N^-_1\Omega$. Set $\varphi_t = \varphi \circ \mathrm{Ad}(g_{-t})$ and $x_t = g_t x$. Then, using (2.3) to change variables, we obtain

$$\int \varphi(n)\mathbb{M}_\varepsilon(f)(g_t nx) \, d\mu^u_x(n) = e^{-\delta t} \int \varphi_t(n)\mathbb{M}_\varepsilon(f)(nx_t) \, d\mu^u_{x_t}(n).$$

We can rewrite the integral on the right side in a convenient form using the following series of formal manipulations. Let $\psi_{\varepsilon,y}(n) = \psi_\varepsilon(n)/\mu^u_y(\psi_\varepsilon)$. First, using the definition of $\mathbb{M}_\varepsilon$ and (2.4) to change variables, we get

$$\int \varphi_t(n)\mathbb{M}_\varepsilon(f)(nx_t) \, d\mu^u_{x_t}(n) = \int \varphi_t(n) \int \psi_{\varepsilon,n'nx_t}(n')f(n'nx_t) \, d\mu^u_{nx_t}(n') \, d\mu^u_{x_t}(n)$$
$$= \int \varphi_t(n) \int \psi_{\varepsilon,n'x_t}(n'n^{-1})f(n'x_t) \, d\mu^u_{x_t}(n') \, d\mu^u_{x_t}(n).$$

Next, using Fubini's Theorem and the symmetry of ψ_ε provided by (8.1), we get

$$\int \varphi_t(n)\mathbb{M}_\varepsilon(f)(nx_t) \, d\mu^u_{x_t}(n) = \int \left(\int \varphi_t(n)\psi_{\varepsilon,n'x_t}(n'n^{-1}) \, d\mu^u_{x_t}(n) \right) f(n'x_t) \, d\mu^u_{x_t}(n')$$
$$= \int \left(\int \varphi_t(n)\psi_{\varepsilon,n'x_t}(n(n')^{-1}) \, d\mu^u_{x_t}(n) \right) f(n'x_t) \, d\mu^u_{x_t}(n').$$

Finally, we obtain the desired convenient form of the integral upon changing variables using (2.4) once more to get

$$\int \varphi_t(n)\mathbb{M}_\varepsilon(f)(nx_t) \, d\mu^u_{x_t}(n) = \int \left(\int \varphi_t(nn')\psi_{\varepsilon,n'x_t}(n) \, d\mu^u_{n'x_t}(n) \right) f(n'x_t) \, d\mu^u_{x_t}(n').$$

It is thus natural to define the following function:

$$\Phi_{\varepsilon,x,t}(n') := \int \varphi_t(nn')\psi_{\varepsilon,n'x_t}(n) \, d\mu^u_{n'x_t}(n) = \frac{\int \varphi_t(nn')\psi_\varepsilon(n) \, d\mu^u_{n'x_t}(n)}{\mu^u_{n'x_t}(\psi_\varepsilon)}.$$

Note that, since φ_t and ψ_ε are supported in $N^+_{e^t}$ and $N^+_{\varepsilon/2}$ respectively, $\Phi_{\varepsilon,x,t}$ is supported inside $N^+_{e^t+\varepsilon/2} \subset N^+_{2e^t}$.

We wish to estimate integrals of the form

$$\int (\varphi_{t+s}(n') - \Phi_{\varepsilon,x,t+s}(n')f(n'x_{t+s}) \, d\mu^u_{x_{t+s}}(n'),$$

for arbitrary $t \geq 1$, $s \in [0,1]$, basepoints x and test functions φ. First, we note that it suffices to estimate the integrals when $s = 0$ since t is at least 1 by assumption. We proceed by essentially regarding $\varphi_t - \Phi_{\varepsilon,x,t}$ itself as a test function. Note that $\Phi_{\varepsilon,x,t}$ may not be well-defined for arbirary $n' \in N^+$, since $\mu^u_{n'x_t}(\psi_\varepsilon)$ could be 0 for those n' with $(n'x_t)^+ \notin \Lambda_\Gamma$. However, $\Phi_{\varepsilon,x,t}$ is well-defined on the $(\varepsilon/4)$-neighborhood of the support of $\mu^u_{x_t}$ by definition of ψ_ε.

For this reason, let $\theta_\varepsilon : N^+ \to [0,1]$ be a smooth bump function which is identically 1 on the $(\varepsilon/100)$-neighborhood of the support of the measure $\mu^u_{x_t}$ and vanishes outside of its $(\varepsilon/50)$-neighborhood. We can choose such a function to satisfy

$$\|\theta_\varepsilon\|_{C^0} \leq 1, \qquad \|\theta_\varepsilon\|_{C^1} \ll \varepsilon^{-1},$$

for instance by convolving (with respect to the Haar measure) the indicator function of the $(\varepsilon/100)$-neighborhood of the support with $\psi_{\varepsilon/200}$. Then, we observe that

$$\int (\varphi_t - \Phi_{\varepsilon,x,t})(n') f(n'x_t)\, d\mu^u_{x_t}(n') = \int ((\varphi_t - \Phi_{\varepsilon,x,t})\theta_\varepsilon)(n') f(n'x_t)\, d\mu^u_{x_t}(n').$$

The upshot is that $\vartheta := (\varphi_t - \Phi_{\varepsilon,x,t})\theta_\varepsilon$ is a well-defined function on N^+. Thus, arguing exactly as in the proof of Lemma 7.1, the conclusion of the proposition will follow as soon as we estimate the norm $\|\vartheta\|_{C^{0,1}}$; cf. (7.3).

We begin by estimating the C^0 norm $\|\vartheta\|_{C^0}$. Let $n' \in N^+$ be an arbitrary point in the support of θ_ε. Note that

$$\varphi_t(n') = \frac{\int \varphi_t(n')\psi_\varepsilon(n)\, d\mu^u_{n'x_t}(n)}{\mu^u_{n'x_t}(\psi_\varepsilon)}, \tag{8.5}$$

and, hence,

$$|\varphi_t(n') - \Phi_{\varepsilon,x,t}(n')| \leq \frac{\int |\varphi_t(n') - \varphi_t(nn')|\psi_\varepsilon(n)\, d\mu^u_{n'x_t}(n)}{\mu^u_{n'x_t}(\psi_\varepsilon)}.$$

We further observe that if $\psi_\varepsilon(n) \neq 0$ for some $n \in N^+$, then n is at distance at most $\varepsilon/2$ from identity. Moreover, since $\|\varphi\|_{C^{0,1}} \leq 1$ and g_t expands N^+ by at least e^t, the Lipschitz constant of φ_t is at most e^{-t}. Hence, using the right invariance of the metric on N^+, for any such n, $|\varphi_t(n') - \varphi_t(nn')| \leq e^{-t}\varepsilon/2$. As n' was arbitrary and $|\theta_\varepsilon(n')| \leq 1$, it follows that $\|\vartheta\|_{C^0} \leq e^{-t}\varepsilon/2$.

It remains to estimate the Lipschitz constant of ϑ. Let $n_1, n_2 \in N_1^+$ be arbitrary points in the support of θ_ε. Then, note that since $\|\theta_\eta\|_{C^0} \leq 1$ and $\|\theta_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$, we have

$$\begin{aligned}
&|\vartheta(n_1) - \vartheta(n_2)| \\
&\ll |(\varphi_t - \Phi_{\varepsilon,x,t})(n_1) - (\varphi_t - \Phi_{\varepsilon,x,t})(n_2)| + |(\varphi_t - \Phi_{\varepsilon,x,t})(n_1)|\, \|\theta_\varepsilon\|_{C^1}\, d_{N^+}(n_1,n_2) \\
&\ll |(\varphi_t - \Phi_{\varepsilon,x,t})(n_1) - (\varphi_t - \Phi_{\varepsilon,x,t})(n_2)| + e^{-t} d_{N^+}(n_1,n_2).
\end{aligned}$$

Let $\sigma = n_1 n_2^{-1}$. Using (8.5) and a change of variable, we have

$$\begin{aligned}
(\varphi_t - \Phi_{\varepsilon,x,t})(n_1) - &(\varphi_t - \Phi_{\varepsilon,x,t})(n_2) \\
&= \frac{\int (\varphi_t(n_1) - \varphi_t(nn_1))(\psi_\varepsilon(n) - \psi_\varepsilon(n\sigma))\, d\mu^u_{n_1x_t}(n)}{\mu^u_{n_1x_t}(\psi_\varepsilon)} \\
&\quad + \int (\varphi_t(n_2) - \varphi_t(nn_2))\psi_\varepsilon(n)\, d\mu^u_{n_2x_t}(n) \times \left(\frac{1}{\mu^u_{n_1x_t}(\psi_\varepsilon)} - \frac{1}{\mu^u_{n_2x_t}(\psi_\varepsilon)} \right).
\end{aligned}$$

In estimating the Lipschitz constant, without loss of generality, we may assume that the distance between n_1 and n_2 is at most $\varepsilon/2$. Hence, arguing as before and using Lemma 8.7, we obtain the following estimate on the second term:

$$\begin{aligned}
\int (\varphi_t(n_2) - \varphi_t(nn_2))\psi_\varepsilon(n)\, d\mu^u_{n_2x_t}(n) &\times \left(\frac{1}{\mu^u_{n_1x_t}(\psi_\varepsilon)} - \frac{1}{\mu^u_{n_2x_t}(\psi_\varepsilon)} \right) \\
&\ll \varepsilon e^{-t} \mu^u_{n_2x_t}(\psi_\varepsilon) \frac{\varepsilon^{-1} d_{N^+}(n_1,n_2)}{\mu^u_{n_2x_t}(\psi_\varepsilon)} \leq e^{-t} d_{N^+}(n_1,n_2).
\end{aligned}$$

To estimate the first term, note that symmetry of ψ_ε (cf. (8.1)) implies that $|\psi_\varepsilon(n) - \psi_\varepsilon(n\sigma)|$ is $O(\varepsilon^{-1} d_{N^+}(n_1, n_2))$. Moreover, since $\sigma = n_1 n_2^{-1}$ and the support of ψ_ε are contained in $N^+_{\varepsilon/2}$, the function $\psi_\varepsilon(n) - \psi_\varepsilon(n\sigma)$ is supported inside N^+_ε. Hence,

$$\int (\varphi_t(n_1) - \varphi_t(nn_1))(\psi_\varepsilon(n) - \psi_\varepsilon(n\sigma))\, d\mu^u_{n_1 x_t}(n) \ll \varepsilon e^{-t} \times \varepsilon^{-1} d_{N^+}(n_1, n_2) \times \mu^u_{n_1 x_t}(N^+_\varepsilon).$$

Combined with Lemma 8.6 and the estimate on the second term, this shows that

$$|\vartheta(n_1) - \vartheta(n_2)| \ll e^{-t} d_{N^+}(n_1, n_2),$$

thus completing the proof.

8.4. **Pointwise estimates and proof of Proposition 8.4.** As in the proof of Proposition 8.3, let $\theta_\varepsilon : N^+ \to [0,1]$ denote a smooth function that is identicaly 1 on the $(\varepsilon/100)$-neighborhood of the support of μ^u_x and vanishing outside its $(\varepsilon/50)$-neighborhood. We again note that we can find such θ_ε with $\|\theta_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$. Set $\Psi(n) = \psi_\varepsilon(n)\mu^u_x(N^+_1)/\mu^u_{nx}(\psi_\varepsilon)$ and note that (2.4) implies that the function $\Psi\theta_\varepsilon$ belongs to $C^0(N^+_1)$. Moreover, we have that

$$\begin{aligned}\mathbb{M}_\varepsilon(f)(x) &= \frac{1}{\mu^u_x(N^+_1)} \int \Psi(n) f(nx)\, d\mu^u_x = \frac{1}{\mu^u_x(N^+_1)} \int (\Psi\theta_\varepsilon)(n) f(nx)\, d\mu^u_x \\ &\ll e^\star_{1,0}(f) V(x) \left\|\Psi\theta_\varepsilon\right\|_{C^{0,1}}.\end{aligned}$$

Hence, the result follows once we estimate the norm $\|\Psi\theta_\varepsilon\|_{C^{0,1}}$. We begin by proving that $\|\Psi\theta_\varepsilon\|_{C^0}$ is $O(\varepsilon^{-\Delta_+})$. As a first step, we show that

$$\frac{\mu^u_x(N^+_1)}{\mu^u_{nx}(\psi_\varepsilon)} \ll \varepsilon^{-\Delta_+}, \qquad \forall n \in N^+, \psi_\varepsilon(n)\theta_\varepsilon(n) \neq 0. \tag{8.6}$$

Since $\|\theta_\varepsilon \psi_\varepsilon\|_{C^0} \leq 1$, this will show that $\|\Psi\theta_\varepsilon\|_{C^0} \ll \varepsilon^{-\Delta_+}$.

Fix some n with $\psi_\varepsilon(n)\theta_\varepsilon(n) \neq 0$. Then, we can find u in the $\varepsilon/2$ ball around identity in N^+ such that ux belongs to $N^-_2\Omega$ (cf. Remark 2.1) and u is at distance at most $10^{-2}\varepsilon$ from n. Since $\psi_\varepsilon \equiv 1$ on $N^+_{\varepsilon/4}$, we have by (2.4) that

$$\mu^u_{nx}(\psi_\varepsilon) \geq \mu^u_{nx}(N^+_{\varepsilon/4}) = \mu^u_{ux}(N^+_{\varepsilon/4} \cdot (nu^{-1})) \geq \mu^u_{ux}(N^+_{\varepsilon/10}). \tag{8.7}$$

Similarly, we have that

$$\mu^u_x(N^+_1) \leq \mu^u_{ux}(N^+_2).$$

Let $k \in \mathbb{N}$ be the smallest integer such that $2^{-k} \leq \varepsilon/4$. Applying Proposition 3.1 with $\sigma = 2^{k+1}$ and $r = 2^{-k}$, since $ux \in \Omega$, we obtain

$$\mu^u_{ux}(N^+_2) = \mu^u_{ux}(N^+_{2^{k+1}2^{-k}}) \ll 2^{(k+1)\Delta_+} \mu^u_{ux}(N^+_{2^{-k}}) \ll \varepsilon^{-\Delta_+} \mu^u_{ux}(N^+_{\varepsilon/4}).$$

Together with (8.7), this concludes the proof of (8.6).

Next, we estimate the Lipschitz norm of $\Psi\theta_\varepsilon$ as a function on N^+_1. Let $n_1, n_2 \in N^+$ be such that $n_1 n_2^{-1} \in N^+_{\varepsilon/10}$, and $(\theta_\varepsilon\psi_\varepsilon)(n_i) \neq 0$ for $i = 1, 2$. Then, Lemma 8.7 and (8.6) imply that

$$\begin{aligned}|\Psi(n_1) - \Psi(n_2)| &\leq \mu^u_x(N^+_1) \left(\left| \frac{1}{\mu^u_{n_1 x}(\psi_\varepsilon)} - \frac{1}{\mu^u_{n_2 x}(\psi_\varepsilon)} \right| + \frac{|\psi_\varepsilon(n_1) - \psi_\varepsilon(n_2)|}{\mu^u_{n_2 x}(\psi_\varepsilon)} \right) \\ &\ll \varepsilon^{-1} d_{N^+}(n_1, n_2) \frac{\mu^u_x(N^+_1)}{\mu^u_{n_2 x}(\psi_\varepsilon)} \ll \varepsilon^{-\Delta_+ - 1} d_{N^+}(n_1, n_2).\end{aligned}$$

Since $\|\theta_\varepsilon\|_{C^0} \leq 1$ and $\|\theta_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$, this shows that the Lipschitz norm of $\Psi\theta_\varepsilon$ is at most $\varepsilon^{-\Delta_+ - 1}$ and concludes the proof.

8.5. **Weak stable derivatives and proof of Proposition 8.5.** The idea of the proof is based on performing local stable holonomy between the strong unstable disks $N_1^+ \cdot x$ and $N_1^+ \cdot p^- x$ and proceeding exactly as in the proof of Proposition 6.6. The main ingredient is an estimate on the regularity of the test functions arising from composing $\psi_\varepsilon(n)/\mu^u_{nx}(\psi_\varepsilon)$ with holonomy maps from x to intermediate points between x and $p^- x$ along the weak stable manifold. We omit the details of the proof since it follows by elaborating the same ideas in the proof of Proposition 8.3. We only remark that for $p^- = u^- g_t m$ as in the statement, letting w in the Lie algebra of N^- be so that $u^- = \exp(w)$, then for all $r \in \mathbb{R}$ with $|r| \leq |t|$ and all $s \in [0,1]$, one checks that the points $\exp(sw) g_r m x$ all belong to $N_1^- \Omega$. This is relevant in ensuring that the basepoints arising over the course of carrying out the analogous estimate to (6.11) all satisfy the requirement on basepoints for the norm $\|\cdot\|_1^\star$.

9. Spectral gap for resolvents with large imaginary parts

In this Section, we establish the key estimate in the proof of Theorems 1.1 and 1.2. The estimates in Sections 6 and 7 allow us to show that there is a half plane $\{\mathrm{Re}(z) > -\eta\}$, for a suitable $\eta > 0$, containing at most countably many isolated eigenvalues for the generator of the geodesic flow. To show exponential mixing, it remains to rule out the accumulation of such eigenvalues on the imaginary axis as their imaginary part tends to ∞.

Remark 9.1. Throughout the rest of this section, if X has cusps, we require the Margulis function $V = V_\beta$ in the definition of all the norms we use to have

$$\beta = \Delta/4 \tag{9.1}$$

in the notation of Theorem 4.1. In particular, the contraction estimate in Theorem 4.1 holds with V^p in place of V for all $1 \leq p \leq 2$. Recall that the constant Δ is given in (3.1).

Similarly to (7.8), we define for $B \neq 0$ an equivalent norm to $\|\cdot\|_1^\star$ defined in (8.4) as follows:

$$\|f\|_B^\star := e_{1,0}^\star(f) + \frac{e_{1,1}^\star(f)}{B}. \tag{9.2}$$

The following result is one of the main technical contributions of this article.

Theorem 9.2. *There exist constants $b_\star \geq 1$, and $\varkappa, a_\star, \sigma_\star > 0$, such that the following holds. For all $z = a_\star + ib \in \mathbb{C}$ with $|b| \geq b_\star$ and for $m = \lceil \log |b| \rceil$, we have that*

$$e_{1,0}^\star(R(z)^m f) \leq C_\Gamma \frac{\|f\|_{1,B}^\star}{(a_\star + \sigma_\star)^m},$$

where $C_\Gamma \geq 1$ is a constant depending only on the fundamental group Γ and $B = |b|^{1+\varkappa}$.

Remark 9.3. The constants $b_\star, \varkappa, a_\star$, and $\sigma_\star$ depend only on non-concentration parameters of the Patterson-Sullivan measure near proper subvarieties of the boundary at infinity. For geometrically finite surfaces, these parameters are nothing but the critical exponent δ in the convex cocompact case and the quantity $2\delta - 1$ in the cusped case; cf. Definition 11.1 for the precise definition of non-concentration and Corollary 12.2 where this non-concentration is established. This non-concentration property is used to apply the results of Section 11 in the proof of Theorem 9.2.

9.1. **Proof of Theorems 1.1 and 1.2.** We show here the deduction of the exponential mixing assertion from Theorem 9.2 using the results in [But16a, But16b].

Recall the Banach space $\mathcal{B}_\star$ defined below (8.4) and the weak norm $\|\cdot\|_1'$ defined in (6.6). The link between the norms we introduced and decay of correlations is furnished in the following lemma.

Lemma 9.4. *For all $f,\varphi\in C_c^2(X)^M$, we have that*

$$\int f\cdot\varphi\, d\mathrm{m}^{\mathrm{BMS}} \ll_\varphi \|f\|_1',$$

where the implied constant depends on $\|\varphi\|_{C^2}$ and the injectivity radius of its support.

Proof. Using a partition of unity, we may assume φ is supported inside a flow box. The implied constant then depends on the number of elements of the partition of unity needed to cover the support of φ. Inside each such flow box, the measure $\mathrm{m}^{\mathrm{BMS}}$ admits a disintegration in terms of the conditional measures μ_x^u averaged against a suitable measure on the transversal to the strong unstable foliation. Thus, the lemma follows by definition of the norm by viewing the restriction of φ to each local unstable leaf as a test function. □

In particular, this lemma implies that decay of correlations (for mean 0 functions) would follow at once if we verify that $\|\mathcal{L}_t f\|_1'$ decays in t with a suitable rate. It is shown in [But16a][10] that such decay follows from suitable spectral bounds on the resolvent. We list here the results that verify the hypotheses of [But16a].

We take $\|\cdot\|_1'$ to be the weak norm $\|\cdot\|_{\mathcal{A}}$ in the notation of [But16a], while we take $\|\cdot\|_1^\star$ as the strong norm $\|\cdot\|_{\mathcal{B}}$. The strong continuity of the semigroup $\mathcal{L}_t$ is provided by Corollary 7.2, while Theorem 6.4 verifies [But16a, Assumption 2][11]. The following lemma verifies the weak Lipschitz property in [But16a, Assumption 1].

Lemma 9.5. *For all $t\geq 0$,*

$$\|\mathcal{L}_t f - f\|_1' \ll t\,\|f\|_1^\star.$$

Proof. Recall that the norm $\|\cdot\|_1'$ only involves the coefficient $e_{1,0}'$; cf. (6.6). Let $x\in N_1^-\Omega$ and $t\geq 0$. Then, given any test function ϕ for $e_{1,0}'$, we have that

$$\int_{N_1^+}\phi(n)(f(g_t nx)-f(nx))\,d\mu_x^u = \int_0^t\int_{N_1^+}\phi(n)L_\omega f(g_r nx)\,d\mu_x^u dr,$$

where L_ω denotes the derivative with respect to the vector field generating the geodesic flow. Hence, Lemma 7.1 implies that

$$\left|\int_{N_1^+}\phi(n)(f(g_t nx)-f(nx))\,d\mu_x^u\right| \leq V(x)\mu_x^u(N_1^+)\int_0^t e_{1,1}^\star(\mathcal{L}_r f)\,dr \ll tV(x)\mu_x^u(N_1^+)e_{1,1}^\star(f),$$

where $e_{1,1}^\star$ is the coefficient defined above (8.4). This completes the proof since x and ϕ are arbitrary. □

Finally, the following corollary verifies [But16a, Assumption 3A], thus completing the proof of Theorems 1.1 and 1.2.

Corollary 9.6. *Let the notation be as in Theorem 9.2. Then, there exist constants $c_\star,\lambda_\star>0$, such that the following holds. For all $z=a_\star+ib\in\mathbb{C}$ and for $q=\lceil c_\star\log|b|\rceil$, we have the following bound on the operator norm of $R(z)$:*

$$\|R(z)^q\|_1^\star \leq \frac{1}{(a_\star+\lambda_\star)^q},$$

whenever $|b|\geq b_\Gamma$, where $b_\Gamma\geq 1$ is a constant depending on Γ.

[10] See also the erratum [But16b].

[11] Corollary 7.2 and Theorem 6.4 are obtained for the norms $\|\cdot\|_k$, $k\geq 1$, however the proof extends readily to the norm $\|\cdot\|_1^\star$ taking $\|\cdot\|_1'$ as its associated norm.

Proof. First, we verify the corollary for the norm $\|\cdot\|_B^\star$. Let $e_{1,1,b}^\star$ be the scaled seminorm $e_{1,1}^\star/|b|^{1+\varkappa}$. Note that the arguments of Lemmas 7.5 and 7.6 imply that for $z = a_\star + ib$ with $|b| \geq a_\star$, we have

$$e_{1,1,b}^\star(R(z)^m f) \leq C_\Gamma \frac{\|f\|_B^\star (a_\star + |z|)}{a_\star^m b^{1+\varkappa}} \leq \frac{3C_\Gamma \|f\|_B^\star}{a_\star^m |b|^\varkappa},$$

for some constant $C_\Gamma \geq 1$ depending only on Γ, where we used the fact that $a_\star + |z| \leq 3|b|$.

Moreover, if $m = \lceil \log |b| \rceil \geq 3/2$, we have that $|b|^\varkappa \geq e^{\varkappa m/2} \geq (1 + \varkappa/2)^m$ and hence $a_\star^m |b|^\varkappa$ is at least $(a_\star + \varkappa/2)^m$. It follows that, for all $f \in \mathcal{B}_\star$, we have

$$e_{1,1,b}^\star(R(z)^m f) \leq \frac{3C_\Gamma \|f\|_B^\star}{(a_\star + \varkappa/2)^m}.$$

This estimate, combined with the estimate in Theorem 9.2 implies that whenever $|b| \geq b_\star$,

$$\|R(z)^m\|_B^\star \ll_\Gamma (a_\star + \sigma_1)^{-m},$$

where $\sigma_1 > 0$ is the minimum of $\sigma_\star$ and $\varkappa/2$. In particular, if $|b|$ is large enough, depending on Γ, we can absorb the implied constant in the estimate above to obtain

$$\|R(z)^m\|_B^\star \leq (a_\star + \sigma_1/2)^{-m}.$$

Let $p \in \mathbb{N}$ be a large integer to be chosen shortly. To obtain the claimed estimate for the norm $\|\cdot\|_1^\star$, note that for any f in the Banach space $\mathcal{B}_\star$, since $\|\cdot\|_B^\star \leq \|\cdot\|_1^\star \leq B \|\cdot\|_B^\star = |b|^{1+\varkappa} \|\cdot\|_B^\star$, iterating the above estimate yields

$$\left\|R(z)^{2pm} f\right\|_1^\star \leq B \left\|R(z)^{2pm} f\right\|_B^\star \leq \frac{B \|R(z)^{pm} f\|_B^\star}{(a_\star + \sigma_1/2)^{pm}} \leq \frac{B \|f\|_1^\star}{(a_\star + \sigma_1/2)^{2pm}}.$$

Since $m = \lceil \log |b| \rceil$, choosing p large enough, depending only on $a_\star$ and σ_1, we can ensure that $B/(a_\star + \sigma_1/2)^{pm} \leq 1/a_\star^{pm}$. In particular, taking $\lambda_\star$ to be the positive solution of the quadratic polynomial $x \mapsto x^2 + 2a_\star x - a_\star \sigma_1/2$, we obtain the desired estimate with $c_\star = 2p$. ☐

Let $\mathfrak{X}$ denote the generator of the semigroup $\mathcal{L}_t$ acting on $\mathcal{B}_\star$ (which exists by Corollary 7.2). In light of the above results, we obtain by [But16a, But16b, Theorem 1] the following decomposition of the transfer operator $\mathcal{L}_t$. There are complex numbers $\{\lambda_i\}_{i=1}^N$ with $\mathrm{Re}(\lambda_i) \leq 0$, finite rank projectors $\Pi_i : \mathcal{B}_\star \to \mathcal{B}_\star$, bounded operators $\mathcal{N}_i$, and a one-parameter semigroup $\mathcal{P}_t$ of bounded operators on $\mathcal{B}_\star$ such that

$$\mathcal{L}_t = \mathcal{P}_t + \sum_{i=1}^N e^{t\lambda_i} e^{t\mathcal{N}_i} \Pi_i.$$

Moreover, for a suitable $\sigma > 0$ depending only on $\lambda_\star$ in Corollary 9.6 and σ_0 given by Theorem 6.4, we have that

$$\|\mathcal{P}_t f\|_1' \ll e^{-\sigma t} \|\mathfrak{X} f\|_1^\star,$$

for all $t \geq 0$ and $f \in \mathcal{B}_\star$. Finally, we have $\mathcal{N}_i^{d_i} = 0$ for some $d_i \in \mathbb{N}$, and $\Pi_i \Pi_j = \delta_{ij}$, $\Pi_i \mathcal{P}_t = 0$, and $\mathcal{N}_i \Pi_i = \Pi_i \mathcal{N}_i = \mathcal{N}_i$ for all i, j, t.

Thus, letting

$$Q = \sum_{\mathrm{Re}(\lambda_i) \neq 0}^N \lambda_i \mathcal{N}_i, \qquad \Pi = \sum_{\mathrm{Re}(\lambda_i) \neq 0} \Pi_i,$$

this concludes the proofs of Theorem 1.1 and 1.2 once we show that the only eigenvalue λ_i on the imaginary axis is 0 and that its associated nilpotent operator $\mathcal{N}_i$ vanishes. This is proved in the following lemma.

Lemma 9.7. *The intersection of the spectrum of $\mathfrak{X}$ with the imaginary axis consists only of the eigenvalue* 0 *which has algebraic multiplicity one.*

Proof. In what follows, we endow elements φ of $C_c^2(X)$ with the norm $\|\varphi\|'_{C^2}$ given by multiplying the C^2-norm of φ with a suitable power of the reciprocal of the injectivity radius of its support so that $\|\varphi\|'_{C^2}$ dominates the implicit constant depending on φ in Lemma 9.4. Such power exists by the proof of the lemma. The dual space $C_c^2(X)^*$ is endowed with the corresponding strong dual norm.

Let $\Phi : \mathcal{B}_\star \to C_c^2(X)^*$ denote the linear map which extends the mapping $f \mapsto (\varphi \mapsto \int f\varphi \, d\mathrm{m}^{\mathrm{BMS}})$ from $C_c^2(X)^M$ to the dual space $C_c^2(X)^*$. The fact that this mapping extends continuously to $\mathcal{B}_\star$ follows by Lemma 9.4. We claim that Φ is injective. This claim is routine in the absence of cusps, and we briefly outline why it also holds in general.

To prove this claim, note first that the coefficients $e^\star_{1,0}(\cdot;x)$ and $e^\star_{1,1}(\cdot;x)$ extend from C_c^2 to define seminorms on $\mathcal{B}_\star$. In particular, given any $f \in \mathcal{B}_\star$ and $f_n \in C_c^2(X)^M$ tending to f in $\mathcal{B}_\star$, we have $e^\star_{1,\ell}(f;x) = \lim_{n\to\infty} e^\star_{1,\ell}(f_n;x)$ for $\ell = 0,1$ and for every $x \in N_1^-\Omega$. Since the coefficient $e^\star_{1,\ell}(f)$ is defined by taking a supremum over x, it follows that we can find a sequence $x_n \in N_1^-\Omega$ such that $e^\star_{1,\ell}(f_n;x_n)$ converges to $e^\star_{1,\ell}(f)$. In particular, we obtain the following inequality which serves to exchange the order of taking limits and suprema

$$e^\star_{1,\ell}(f) \leq \sup_{x\in N_1^-\Omega} \lim_{n\to\infty} e^\star_{1,\ell}(f_n;x). \tag{9.3}$$

Now, suppose $f \in \mathcal{B}_\star$ is in the kernel of Φ and let $f_n \in C_c^2(X)^M$ be a sequence of functions converging to f. By continuity, $\Phi(f_n)$ tends to 0 in $C_c^2(X)^*$. One then checks that this implies that for every fixed $x \in N_1^-\Omega$, we have that $e^\star_{1,\ell}(f_n;x) \to 0$ as $n \to \infty$ for $\ell = 0,1$. Combined with (9.3), this shows that $\|f\|_1^\star = 0$, and hence Φ is injective as claimed.

We now show that this injectivity implies the claim of the lemma. Via the relationship between the spectra of $\mathfrak{X}$ and the resolvents (cf. Section 7.5), we obtain by Theorem 6.4 that the intersection of the spectrum $\sigma(\mathfrak{X})$ with the imaginary axis consists of a discrete set of eigenvalues. Similarly, finiteness of the multiplicities of each of these eigenvalues is a consequence of quasi-compactness of the resolvent.

Let $b \in \mathbb{R}$ be such that ib is one such eigenvalue with eigenvector $0 \neq f \in \mathcal{B}_\star$ and note that this implies that $\mathcal{L}_t f = e^{ibt} f$. We show that $\Phi(f)$ is a multiple of the measure $\mathrm{m}^{\mathrm{BMS}}$. This implies that $b = 0$ by injectivity since $\mathrm{m}^{\mathrm{BMS}}$ is the image of the constant function 1 under Φ. To do so, we use the fact that the geodesic flow is mixing[12] with respect to $\mathrm{m}^{\mathrm{BMS}}$ by work of Rudolph [Rud82] and Babillot [Bab02]. Let $\varphi \in C_c^2(X)$ be arbitrary and let $\theta_n = \int f_n \, d\mathrm{m}^{\mathrm{BMS}}$ and $\xi = \int \varphi \, d\mathrm{m}^{\mathrm{BMS}}$. Then, for every $t \geq 0$ and $n \in \mathbb{N}$, we have

$$|\Phi(f)(\varphi) - \theta_n\xi| \leq \left|\Phi(f)(\varphi) - \int \varphi \mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right| + \left|\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}} - \theta_n\xi\right|. \tag{9.4}$$

By mixing, for every fixed n, the second term can be made arbitrarily small by taking t large enough. Moreover, since $\Phi(f) = e^{-ibt}\Phi(\mathcal{L}_t f)$, the first term is bounded by

$$\left|e^{-ibt}\Phi(\mathcal{L}_t f)(\varphi) - e^{-ibt}\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right| + |e^{-ibt} - 1| \left|\int \varphi\mathcal{L}_t f_n \, d\mathrm{m}^{\mathrm{BMS}}\right|. \tag{9.5}$$

The first term in (9.5) is equal to $|\Phi(\mathcal{L}_t(f - f_n)(\varphi)|$, which is $O_\varphi(\|f - f_n\|_1^\star)$ in view of Lemmas 9.4 and 7.1. Similarly, since f_n converges to f in $\mathcal{B}_\star$, the second term is $O_\varphi(|e^{-ibt} - 1| \, \|f\|_1^\star)$. To bound this term, note that one can find arbitrarily large t so that e^{ibt} is arbitrarily close to 1.

Therefore, using a diagonal argument, this implies that we can find a sequence $t(n)$ tending to infinity so that the upper bound in (9.4) tends to 0 with n. If $\xi \neq 0$, the above argument implies

[12]We refer the reader to [BDL18, Corollary 5.4] for this deduction using only ergodicity of the flow.

that θ_n is $O_\varphi(\Phi(f)(\varphi))$ and hence converges (along a subsequence) to some $\theta \in \mathbb{R}$. In particular, the values of $\Phi(f)$ and $\theta \mathrm{m}^{\mathrm{BMS}}$ agree on φ in this case. If $\xi = 0$, then the above argument shows that $\Phi(f)(\varphi) = 0$ so that the same conclusion also holds.

The assertion on the algebraic multiplicity, which in particular involves ruling out the presence of Jordan blocks, is standard and can be deduced from quasi-compactness of the resolvent and the bound on its norm given in Corollary 7.3 following very similar lines to [BDL18, Corollary 5.4] to which we refer the interested reader for details. □

9.2. **Proof of Theorem 9.2.** The remainder of this section is dedicated to the proof of Theorem 9.2. Let $a \in (0, 2]$ to be determined. We assume that $z = a + ib$ with $b > 0$, the other case being identical. For the convenience of the reader, we summarize the notation used in this section in Table 1.

Time partition. Let $p : \mathbb{R} \to [0, 1]$ be a smooth bump function supported in $(-1, 1)$ with the property that

$$\sum_{j\in\mathbb{Z}} p(t - j) = 1, \qquad \forall t \in \mathbb{R}. \tag{9.6}$$

Let $m \in \mathbb{N}$ and $T_0 > 0$ be parameters to be specified later. Changing variables, we obtain

$$\begin{aligned} R(z)^m &= \int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!}\mathcal{L}_t \, dt \\ &= \int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!} p(t/T_0)\mathcal{L}_t \, dt + \sum_{j=0}^\infty \frac{((j+2)T_0)^{m-1}e^{-zjT_0}}{(m-1)!} \int_\mathbb{R} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0} \, dt, \end{aligned} \tag{9.7}$$

where we define p_j as follows:

$$p_j(t) := \left(\frac{jT_0 + t}{(j+2)T_0}\right)^{m-1} p\left(\frac{t - T_0}{T_0}\right). \tag{9.8}$$

Note that p_j is supported in the interval $(0, 2T_0)$ for all $j \geq 0$.

Contribution of pre-mixing times. We also discard the first few terms in the sum over j. Let $J_0 \in \mathbb{N}$ be a parameter to be specified later. By the triangle inequality for the seminorm $e^\star_{1,0}$ and Lemma 7.1, we have

$$\begin{aligned} &e^\star_{1,0}\left(\sum_{j=0}^{J_0} \frac{((j+2)T_0)^{m-1}e^{-zjT_0}}{(m-1)!} \int_\mathbb{R} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0} f \, dt\right) \\ &\leq \int_0^{(J_0+2)T_0} \frac{t^{m-1}e^{-at}}{(m-1)!} e^\star_{1,0}(\mathcal{L}_t f) dt \ll \frac{((J_0+2)T_0)^m e^\star_{1,0}(f)}{(m-1)!}. \end{aligned}$$

We will choose

$$m = \lceil \log b \rceil. \tag{9.9}$$

Hence, since $a \leq 2$ by assumption, when b is large enough[13], we get

$$e^\star_{1,0}\left(\sum_{j=0}^{J_0} \frac{((j+2)T_0)^{m-1}e^{-zjT_0}}{(m-1)!} \int_\mathbb{R} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0} f \, dt\right) \ll \frac{e^\star_{1,0}(f)}{(a+1)^m}. \tag{9.10}$$

A similar argument also shows that

$$e^\star_{1,0}\left(\int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!} p(t/T_0)\mathcal{L}_t f \, dt\right) \ll \frac{e^\star_{1,0}(f)}{(a+1)^m},$$

[13]Over the course of the proof, b will be assumed large depending on all the parameters we choose in the argument.

Notation	Definition
$\delta = \delta_\Gamma$	critical exponent
$\Lambda = \Lambda_\Gamma$, $\Omega = \Omega_\Gamma$	limit set and non-wandering set
Δ, Δ_+	(3.1)
β	Remark 9.1
m	$\lceil \log b \rceil$
T_0	time discretization
p_j	partition of time variable (9.8)
J_0	initial segment of resolvent (9.10)
j	summand index in resolvent (9.11)
ε_1	small parameter to absorb implicit consts (9.16)
f_ϖ	$\psi_\varpi f$ (9.14)
α	proportion of time in cusp (9.17)
K_j	fixed compact set (9.18)
ι_j	inj radius of K_j
$p_{j,w}$	(9.22)
w	discretization of $(0, 2T_0)$ (9.22)
g_j^w	g_{w+jT_0} (9.24)
$\mathbb{M}$	mollifier (9.24)
F	$\mathbb{M}(f_\varpi)$ (9.24)
$\mathcal{P}_j^0$	flow boxes meeting $N_1^-\Omega$ (9.28)
D	volume entropy (9.19)
γ	1/2
g^γ	amount of time we flow $g_{(w+jT_0)/2}$ (9.30)
x_j	$g^\gamma x$ (9.34)
$N_1^+(j)$	neighborhood of N_1^+ (9.35)
F_γ	$\mathcal{L}_{(w+jT_0)/2} F$ (9.36)
y_ρ	center of flow box B_ρ (9.38)
T_ρ	transversal to strong unstable in B_ρ (9.38)
$I_{\rho,j}$	indexes unstable leaves landing in B_ρ at time $(w + jT_0)/2$
W_ℓ	ℓ^{th} unstable piece in B_ρ
$x_{\rho,\ell}$	center of W_ℓ (9.39)
$s_{\rho,\ell}$	return time to compact for $x_{\rho,\ell}$ (9.41)
W_ρ	local unstable leaf of y_ρ (9.50)
τ_ℓ	the temporal distance function (9.51)
$\phi_{\rho,\ell}$	test function after change of variables (9.51)
$J\Phi_\ell$	Jacobian of stable holonomy (9.51)
$\varkappa$	(9.55)
J_ρ	support of integration in t (9.56)
A_i	cusp adapted partition (9.59)
t_i, r_i, y_ρ^i	cusp-adapted partition parameters (9.60)
$w_{k,\ell}^i$	frequencies (9.64)
$C_{\rho,j,i}/S_{\rho,j,i}$	close/separated pairs of unstable disks (9.66)
κ	Proposition 9.13
ε_2, λ	Theorem 9.16
A	$D + 2\Delta_+ + 1$ (9.75)

TABLE 1. Summary of notation in the proof of Theorem 9.2.

where we used the fact that $p(t/T_0)$ is supported in $(-T_0, T_0)$. Thus, we may assume for the remainder of the section that

$$j > J_0. \tag{9.11}$$

Let $0 < \varepsilon_1 \ll 1$ be a small parameter to be chosen later. The advantage of taking J_0 large is that it allows us to give a reasonable estimate on the sum of the errors of each term in (9.7). Indeed, taking J_0 large enough so that $2/J_0 \leq \varepsilon_1$, in view of (7.5), we have that

$$\sum_{j=J_0+1}^{\infty} \frac{((j+2)T_0)^{m-1} e^{-ajT_0}}{(m-1)!} \leq e^{2aT_0} \left(1 + \frac{2}{J_0}\right)^m \int_0^\infty \frac{t^{m-1} e^{-at}}{(m-1)!}\, dt = e^{2aT_0} \left(\frac{1+\varepsilon_1}{a}\right)^m. \tag{9.12}$$

We will take J_0 large enough (independently of b) so that the loss of a factor of $1+\varepsilon_1$ does not exceed the gains we make over the course of the proof.

Contribution of non-recurrent orbits. We will estimate the contribution of each term in the sum over j in (9.7) individually.

In Section 7.3, we defined a the decomposition of the operator $\mathcal{L}_{T_0}$ using a given height $V_0 \geq 1$; cf. (7.10). In particular, we can rewrite the j^{th} of the sum in (9.7) as follows:

$$\int_{\mathbb{R}} p_j(t) e^{-zt} \mathcal{L}_{t+jT_0} f \, dt = \sum_{\varpi \in \{1,2\}^j} \int_{\mathbb{R}} p_j(t) e^{-zt} \mathcal{L}_{t+jT_0}(\psi_\varpi f)\, dt. \tag{9.13}$$

We estimate the contribution of each ϖ separately. Fix some $\varpi \in \{1,2\}^j$, and for convenience, set

$$f_\varpi := \psi_\varpi f. \tag{9.14}$$

We will frequently use the estimates

$$e^\star_{1,0}(f_\varpi) \ll e^\star_{1,0}(f), \qquad \|f_\varpi\|_1^\star \ll \|f\|_1^\star, \tag{9.15}$$

which follow by Lemma 7.7.

In this subsection, we handle the contribution of the terms corresponding to trajectories which spend a large proportion of their time at height larger than V_0. More precisely, let $\alpha \geq 0$ be a small parameter to be chosen at the end of the argument, and suppose $\varpi \in \{1,2\}^j$ is such that

$$\#\{1 \leq i \leq j : \varpi_i = 2\} \geq \alpha j.$$

First, it follows by Lemma 7.8 and induction that

$$e^\star_{1,0}(\mathcal{L}_{t+jT_0}(f_\varpi)) \leq C_0 e^\star_{1,0}(\mathcal{L}_{jT_0}(f_\varpi)) \leq C_0^{j+1} e^{-\beta\alpha j T_0} e^\star_{1,0}(f_\varpi),$$

where we take $C_0 \geq 1$ to be a constant larger than the implicit constant in that lemma. In what follows, we assume ε_1 is smaller than $\beta\alpha/10$. We take T_0 to be large enough, depending on ε_1 and C_0, we may assume that $C_0 \leq e^{\varepsilon_1 T_0}$. Hence, by (9.15), we obtain

$$e^\star_{1,0}(\mathcal{L}_{t+jT_0}(f_\varpi)) \ll e^{-(\beta\alpha - \varepsilon_1) j T_0} e^\star_{1,0}(f).$$

Finally, in light of (9.12), since there are at most 2^j such words ϖ, taking T_0 large enough so that $2^{\varepsilon_1} \leq e^{\varepsilon_1 T_0}$ and summing the above errors over j, we obtain an error term of the form

$$e^{2aT_0} e^\star_{1,0}(f) \left(\frac{1+\varepsilon_1}{a + \beta\alpha - 2\varepsilon_1}\right)^m \leq e^\star_{1,0}(f) \left(\frac{1+2\varepsilon_1}{a+\beta\alpha - 2\varepsilon_1}\right)^m \leq \frac{e^\star_{1,0}(f)}{(a+\beta\alpha/2)^m}, \tag{9.16}$$

where the first inequality can be ensured to hold by taking b large enough in view of (9.9) and the second inequality holds whenever ε_1 is small enough.

Contribution of recurrent orbits. The remainder of the section, is dedicated to estimating the contribution of orbits that spend a definite proportion of time in the thick part, i.e. the terms where ϖ satisfies

$$\#\{1 \leq i \leq j : \varpi_i = 2\} < \alpha j. \tag{9.17}$$

Let us define

$$K_j := \left\{ y \in X : V(y) \leq e^{(2\beta\alpha j + 3\beta)T_0} \right\}, \qquad \iota_j := \min\{1, \mathrm{inj}(K_j)\}. \tag{9.18}$$

We note that Proposition 4.3 implies that

$$\iota_j^{-1} \ll e^{(4\alpha j + 6)T_0}, \tag{9.19}$$

where we used the fact that $\chi_{\mathbb{K}} \leq 2$; cf. (4.2).

Recalling the definition of ψ_ϖ in (7.11), we have that ψ_ϖ is supported on the points of height at most $e^{\beta\alpha j T_0} V_0$. Hence, the support of f_ϖ satisfies

$$\mathrm{supp}(f_\varpi) \subseteq K_j. \tag{9.20}$$

Let $x \in N_1^-\Omega$ be arbitrary. The same argument in the proof of (9.16) shows that if $V(x) \geq e^{\beta\alpha j T_0}$, then Lemma 7.1 implies that we obtain a gain of $e^{-(\beta\alpha - \varepsilon_1)jT_0} e^{\star}_{1,0}(f)$. Thus, we may assume for the remainder of the section that

$$V(x) \leq e^{\beta\alpha j T_0}. \tag{9.21}$$

Fix some suitable test function ϕ for $e^{\star}_{1,0}$. In particular, ϕ has $C^{0,1}(N^+)$ norm at most 1. The integrals we wish to estimate take the form

$$\begin{aligned} &\int_{N_1^+} \phi(n) \int_{\mathbb{R}} p_j(t) e^{-zt} \mathcal{L}_{t+jT_0}(f_\varpi)(g_s n x)\, dt d\mu_x^u(n) \\ &\qquad = \int_{\mathbb{R}} e^{-zt} \int_{N_1^+} p_j(t)\phi(n) f_\varpi(g_{s+t+jT_0} n x)\, d\mu_x^u(n) dt, \end{aligned}$$

for all $s \in [0,1]$. We again only provide the estimate in the case $s = 0$ to simplify notation, the general case being essentially identical.

Recall that p_j is supported in the interval $(0, 2T_0)$. In particular, the extra t in $\mathcal{L}_{t+jT_0}$ could be rather large, which will ruin certain trivial estimates later. To remedy this, recall the partition of unity of $\mathbb{R}$ given in (9.6) and set

$$p_{j,w}(t) := p_j(t+w)p(t), \qquad \forall w \in \mathbb{Z}. \tag{9.22}$$

Using a change of variable, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} e^{-zt} \int_{N_1^+} p_j(t)\phi(n) f_\varpi(g_{t+jT_0} n x)\, d\mu_x^u(n) dt \\ &\qquad = \sum_{w\in\mathbb{Z}} e^{-zw} \int_{\mathbb{R}} e^{-zt} \int_{N_1^+} p_{j,w}(t)\phi(n) f_\varpi(g_{t+w+jT_0} n x)\, d\mu_x^u(n) dt. \end{aligned} \tag{9.23}$$

Note the above sum is supported on $0 \leq w \ll T_0$, and the support of each integral in t is now $(-1,1)$. For the remainder of the section, we fix some $w \in \mathbb{Z}$ in that support.

Approximation with mollifiers. Let $\mathbb{M} := \mathbb{M}_{1/10}$, where for $\varepsilon > 0$, $\mathbb{M}_\varepsilon$ denotes the mollifier defined in Section 8. To simplify notation, we set

$$g_j^w := g_{w+jT_0}, \qquad F := \mathbb{M}(f_\varpi). \tag{9.24}$$

Since $\phi \in C^{0,1}(N_1^+)$ with $\|\phi\|_{C^{0,1}} \leq 1$, it follows by Proposition 8.3 and (9.15) that

$$\left| \int_{N_1^+} \phi(n) \mathcal{L}_t(f_\varpi - F)(g_j^w n x)\, d\mu_x^u \right| \ll e^{-(t+w+jT_0)} e^{\star}_{1,0}(f) V(x) \mu_x^u(N_1^+).$$

Arguing as in (9.16), summing the above errors over j, we get an error term of the form

$$e^{\star}_{1,0}(f)V(x)\mu^u_x(N_1^+) \times e^{2aT_0}T_0\left(\frac{1+\varepsilon_1}{(a+1-\varepsilon_1)}\right)^m \leq \frac{e^{\star}_{1,0}(f)V(x)\mu^u_x(N_1^+)}{(a+1/2)^m}, \tag{9.25}$$

where we again assume that b is large enough and ε_1 is small enough so that the above inequality holds.

Hence, we may replace f_ϖ with F in (9.23). We will frequently use the following observation. Writing $F = F - f_\varpi + f_\varpi$ and using Proposition 8.3 and (9.15), we have that

$$e^{\star}_{1,0}(F) \ll e^{\star}_{1,0}(f). \tag{9.26}$$

Partitions of unity and flow boxes. We let $\mathcal{P}_j$ denote a partition of unity of the unit neighborhood of K_j so that each $\rho \in \mathcal{P}_j$ is M-invariant and supported inside a flow box B_ρ of radius $\iota_j/10$. With the aid of the Vitali covering lemma, we can arrange for the collection $\{B_\rho\}$ to have a uniformly bounded multiplicity, depending only on the dimension of G. We can choose such a partition of unity so that for all $\rho \in \mathcal{P}_j$,

$$\|\rho\|_{C^1} \ll \iota_j^{-1}. \tag{9.27}$$

We also introduce the following subcollection of $\mathcal{P}_j$:

$$\mathcal{P}_j^0 := \left\{\rho \in \mathcal{P}_j : B_\rho \cap N^-_{1/2}\Omega \neq \emptyset\right\}. \tag{9.28}$$

Note that the cardinality of $\mathcal{P}_j^0$ is controlled in terms of the injectivity radius ι_j in (9.18). Indeed, since Γ is geometrically finite, the unit neighborhood of Ω has finite volume. Moreover, the flow boxes B_ρ with $\rho \in \mathcal{P}_j^0$ are all contained in such a unit neighborhood and have uniformly bounded multiplicity; cf. (9.28). Finally, each B_ρ has radius at least ι_j for all $\rho \in \mathcal{P}_j$. Thus, letting $D \in \mathbb{N}$ be such that the Lebesgue measure of B_ρ is $\asymp \iota_j^D$, we see that

$$\#\mathcal{P}_j^0 \ll_\Gamma \iota_j^{-(2D+1)}. \tag{9.29}$$

Note that the dimension of X is $2D+1+\dim(M)$, however the bound above involves $2D+1$ only since each flow box is M-invariant.

Localizing away from the cusp. We begin by restricting the support of the integral away from the cusp. Define the following smoothed cusp indicator function $\zeta_j : X \to [0,1]$:

$$\zeta_j(y) := 1 - \sum_{\rho\in\mathcal{P}_j} \rho(y).$$

Let

$$\gamma = 1/2, \qquad g^\gamma := g_{\gamma(w+jT_0)}. \tag{9.30}$$

It will be convenient to take T_0 large enough so that

$$(1-\gamma)(w+jT_0) = \gamma(w+jT_0) \geq 4. \tag{9.31}$$

First, we note that Proposition 8.4 implies

$$|\mathcal{L}_t F(g_j^w nx)| \ll e^{\star}_{1,0}(f_\varpi)\mathcal{L}_t V(g_j^w nx).$$

Note that by definition, ζ_j is supported outside of the sublevel set K_j in (9.18). Hence, the Cauchy-Schwarz inequality yields

$$\left|\int_{N_1^+} \zeta_j(g^\gamma nx)\mathcal{L}_t V(g_j^w nx)\, d\mu^u_x\right|^2 \leq \mu^u_x\left(n \in N_1^+ : V(g^\gamma nx) > e^{2\beta\alpha jT_0}\right) \times \int_{N_1^+} \mathcal{L}_t V^2(g_j^w nx)\, d\mu^u_x,$$

where we used the fact that $|\phi|$ is bounded by 1 and ζ_j is non-negative. Recall that we are assuming that V^2 satisfies the Margulis inequality in Theorem 4.1; cf. Remark 9.1. Hence, by Theorem 4.1 and Chebychev's inequality, we obtain

$$\left| \int_{N_1^+} \phi(n)\zeta_j(g^\gamma nx)\mathcal{L}_t F(g_j^w nx)\, d\mu_x^u \right| \ll e_{1,0}^\star(f_\varpi)\mu_x^u(N_1^+)V^{3/2}(x)e^{-\beta\alpha jT_0}.$$

Using the bound on $V(x)$ in (9.21) along with (9.15), we thus obtain

$$\begin{aligned}&\int_{N_1^+} \phi(n)\mathcal{L}_t F(g_j^w nx)\, d\mu_x^u(n)\\ &= \sum_{\rho\in\mathcal{P}_j} \int_{N_1^+} \phi(n)\rho(g^\gamma nx)\mathcal{L}_t F(g_j^w nx)\, d\mu_x^u + O\left(e_{1,0}^\star(f)\mu_x^u(N_1^+)V(x)e^{-\beta\alpha jT_0/2}\right).\end{aligned} \tag{9.32}$$

As before, using (9.12) and taking b large enough and ε_1 small enough, we see that the sum of the above error terms over j gives an error term of the form

$$O\left(\frac{e_{1,0}^\star(f)\mu_x^u(N_1^+)V(x)}{(a+\beta\alpha/4)^m}\right). \tag{9.33}$$

Saturation and post-localization. Our next step is to partition the integral over N_1^+ into pieces according to the flow box they land in under flowing by g^γ. To simplify notation, we write

$$x_j := g^\gamma x. \tag{9.34}$$

We denote by $N_1^+(j)$ a neighborhood of N_1^+ defined by the property that the intersection

$$B_\rho \cap (\mathrm{Ad}(g^\gamma)(N_1^+(j))\cdot x_j)$$

consists entirely of full local strong unstable leaves in B_ρ. We note that since $\mathrm{Ad}(g^\gamma)$ expands N^+ and B_ρ has radius < 1, $N_1^+(j)$ is contained inside N_2^+. Since ϕ is supported inside N_1^+, we have

$$\chi_{N_1^+}(n)\phi(n) = \chi_{N_1^+(j)}(n)\phi(n), \qquad \forall n\in N^+. \tag{9.35}$$

For simplicity, we set

$$\varphi_j(n) := \phi(\mathrm{Ad}(g^\gamma)^{-1}n), \qquad \mathcal{A}_j := \mathrm{Ad}(g^\gamma)(N_1^+(j)).$$

For $\rho\in\mathcal{P}$, we let $\mathcal{W}_{\rho,j}$ denote the collection of connected components of the set

$$\{n\in\mathcal{A}_j : nx_j\in B_\rho\}.$$

Moreover, since $x\in N_1^-\Omega$, we see that the the restriction of the support of μ_x^u to N_1^+ consists of points $n\in N^+$ with $nx\in N_2^-\Omega$; cf. Remark 2.1. In view of (9.31), this implies that the non-zero summands in the right side of (9.32) necessarily correspond to those ρ in $\mathcal{P}_j^0$.

To simplify notation, let

$$F_\gamma := \mathcal{L}_{(1-\gamma)(w+jT_0)}(F). \tag{9.36}$$

In view of (9.35), changing variables using (2.3) yields

$$\begin{aligned}&\sum_{\rho\in\mathcal{P}_j}\int_{N_1^+} \phi(n)\rho(g^\gamma nx)F(g_{t+w+jT_0}nx)\, d\mu_x^u(n)\\ &\qquad = e^{-\delta\gamma(w+jT_0)} \sum_{\rho\in\mathcal{P}_j^0, W\in\mathcal{W}_{\rho,j}} \int_{n\in W} \varphi_j(n)\rho(nx_j)F_\gamma(g_t nx_j)\, d\mu_{x_j}^u(n).\end{aligned} \tag{9.37}$$

Transversals. We fix a system of transversals $\{T_\rho\}$ to the strong unstable foliation inside the boxes B_ρ. Since B_ρ meets $N_{1/2}^-\Omega$ for all $\rho \in \mathcal{P}_j^0$, we take y_ρ in the intersection $B_\rho \cap N_{1/2}^-\Omega$. In this notation, we can find neighborhoods of identity $P_\rho^- \subset P^- = MAN^-$ and $N_\rho^+ \subset N^+$ such that

$$B_\rho = N_\rho^+ P_\rho^- \cdot y_\rho, \qquad T_\rho = P_\rho^- \cdot y_\rho. \tag{9.38}$$

We also let M_ρ, A_ρ, and N_ρ^- be neighborhoods of identity in M, A and N^- respectively so that $P_\rho^- = M_\rho A_\rho N_\rho^-$.

Centering the integrals. It will be convenient to center all the integrals in (9.37) so that their basepoints belong to the transversals T_ρ of the respective flow box B_ρ; cf. (9.38).

Let $I_{\rho,j}$ denote an index set for $\mathcal{W}_{\rho,j}$. For $W \in \mathcal{W}_{\rho,j}$ with index $\ell \in I_{\rho,j}$, let $n_{\rho,\ell} \in W$, $m_{\rho,\ell} \in M_\rho$, $n_{\rho,\ell}^- \in N_\rho^-$, and $t_{\rho,\ell} \in (-\iota_j, \iota_j)$ be such that

$$x_{\rho,\ell} := m_{\rho,\ell} g_{-t_{\rho,\ell}} n_{\rho,\ell} \cdot x_j = n_{\rho,\ell}^- \cdot y_\rho \in T_\rho. \tag{9.39}$$

Note that since x belongs to $N_1^-\Omega$, we have that

$$x_{\rho,\ell} \in N_1^-\Omega, \tag{9.40}$$

cf. (9.31) and Remark 2.1. Moreover, if we let $u_\ell = \mathrm{Ad}((g^\gamma)^{-1})(n_{\rho,\ell}) \in N_1^+(j)$, then in light of the restriction on ϖ in (9.17), we may and will assume that there is $s_{\rho,\ell} > 0$ such that

$$\gamma(w + jT_0) \leq s_{\rho,\ell} \leq \gamma(w + jT_0) + \alpha j T_0, \qquad V(g_{s_{\rho,\ell}} u_\ell x) \ll_{T_0} 1. \tag{9.41}$$

Indeed, the support of f_ϖ is restricted to those points $n \in N^+$ whose g_t orbit spends at most α-proportion of $t \in [0, jT_0]$ outside the set $\{V \leq V_0\}$.

Regularity of test functions. For each such ℓ and W, let us denote $W_\ell = \mathrm{Ad}(m_{\rho,\ell} g_{t_{\rho,\ell}})(W n_{\rho,\ell}^{-1})$ and

$$\widetilde{\phi}_{\rho,\ell}(t,n) := p_{j,w}(t - t_{\rho,\ell}) \cdot e^{z t_{\rho,\ell}} \cdot \phi(\mathrm{Ad}(m_{\rho,\ell} g^\gamma g_{-t_{\rho,\ell}})^{-1}(n n_{\rho,\ell})) \cdot \rho(g_{t_{\rho,\ell}} n x_{\rho,\ell}). \tag{9.42}$$

Note that $\widetilde{\phi}_{\rho,\ell}$ has bounded support in the t direction and (9.27) implies

$$\left\| \widetilde{\phi}_{\rho,\ell} \right\|_{C^0(\mathbb{R}\times N^+)} \leq 1, \qquad \left\| \widetilde{\phi}_{\rho,\ell}(t,\cdot) \right\|_{C^{0,1}(N^+)} \ll \iota_j^{-1}, \tag{9.43}$$

for all $t \in \mathbb{R}$. Moreover, recalling (9.8), we see that

$$\left\| \widetilde{\phi}_{\rho,\ell} \right\|_{C^{0,1}(\mathbb{R}\times N^+)} \ll \iota_j^{-1} m. \tag{9.44}$$

Changing variables using (2.3) and (2.4), we can rewrite the integral in t of the right side of (9.37) as follows:

$$\begin{aligned} e^{-\delta\gamma(w+jT_0)} &\int_{\mathbb{R}} e^{-zt} p_{j,w}(t) \sum_{\rho \in \mathcal{P}_j^0, W \in \mathcal{W}_{\rho,j}} \int_{n \in W} \varphi_j(n) \rho(n x_j) F_\gamma(g_t n x_j) \, d\mu_{x_j}^u(n) dt \\ &= e^{-\delta\gamma(w+jT_0)} \sum_{\rho \in \mathcal{P}_j^0} \sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n \in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n) F_\gamma(g_{t+t_{\rho,\ell}} n x_{\rho,\ell}) \, d\mu_{x_{\rho,\ell}}^u(n) dt, \end{aligned} \tag{9.45}$$

where we also used M-invariance of F_γ; cf. Remark 8.2.

Mass estimates. We record here certain counting estimates which will allow us to sum error terms in later estimates over $\mathcal{P}_j^0$. Note that by definition of $N_1^+(j)$, we have $\bigcup_{\rho\in\mathcal{P}_j, W\in\mathcal{W}_{\rho,j}} W \subseteq \mathcal{A}_j$. Thus, using the log-Lipschitz and contraction properties of V, it follows that

$$\begin{aligned}\sum_{\rho\in\mathcal{P}_j^0,\ell\in I_{\rho,j}} \mu^u_{x_{\rho,\ell}}(W_\ell)V(x_{\rho,\ell}) &\ll \int_{\mathcal{A}_j} V(nx_j)\, d\mu^u_{x_j}(n) \\ &= e^{\delta\gamma(w+jT_0)} \int_{N_1^+(j)} V(g_j^w nx)\, d\mu^u_x(n) \ll e^{\delta\gamma(w+jT_0)}\mu^u_x(N_1^+)V(x), \quad (9.46)\end{aligned}$$

where we used the fact that $|t_{\rho,\ell}| < 1$ and the last inequality follows by Proposition 3.1 since $N_1^+(j) \subseteq N_2^+$. We also used the fact that the partition of unity $\mathcal{P}_j^0$ has uniformly bounded multiplicity.

Remark 9.8. We note the exact same argument as above gives

$$\sum_{\rho\in\mathcal{P}_j^0,\ell\in I_{\rho,j}} \mu^u_{x_{\rho,\ell}}(W_\ell)V^2(x_{\rho,\ell}) \ll e^{\delta\gamma(w+jT_0)}\mu^u_x(N_1^+)V^2(x), \quad (9.47)$$

in view of our choice of V at the beginning of the section; cf. Remark 9.1.

We shall also need the following weighted number of flow boxes parametrized by $\mathcal{P}_j^0$. For each $\rho\in\mathcal{P}_j^0$, we fix some $\ell_\rho \in I_{\rho,j}$. If $I_{\rho,j}$ is empty, we set $x_{\rho,\ell_\rho} = y_\rho$ and $W_{\ell_\rho} = \emptyset$. This lemma is only relevant in the case Γ contains parabolic elements, since otherwise the estimate in (9.29) suffices.

Lemma 9.9. *Recall the constant D in (9.29). Then, we have*

$$\sum_{\rho\in\mathcal{P}_j^0} \mu^u_{x_{\rho,\ell_\rho}}(W_{\ell_\rho}) \ll \iota_j^{-(2D+1)} e^{\delta\alpha jT_0}.$$

Proof. Fix some ρ with $I_{\rho,j} \neq \emptyset$, and write $\ell = \ell_\rho$. Then, recall that $W_\ell = \mathrm{Ad}(m_{\rho,\ell}g_{t_{\rho,\ell}})(Wn_{\rho,\ell}^{-1})$ and that $Wn_{\rho,\ell}^{-1} = N^+_{\iota_j} \subseteq \mathcal{A}_j \subset N^+$. Recalling (9.39), we have that

$$x_{\rho,\ell} = m_{\rho,\ell}g_{-t_{\rho,\ell}}n_{\rho,\ell}\cdot x_j = m_{\rho,\ell}g_{-t_{\rho,\ell}}g^\gamma u_\ell \cdot x,$$

where $u_\ell = \mathrm{Ad}((g^\gamma)^{-1})(n_{\rho,\ell})$. In other words, u_ℓ is nothing but the point on the unstable disk through x whose forward orbit at time $\gamma(w+jT_0)$ lands on the weak stable disk through y_ρ. Note that u_ℓ belongs to $N_1^+(j) \subset N_2^+$, since $n_{\rho,\ell}$ belongs to $\mathcal{A}_j$.

Arguing as in (9.41), using the restriction on ϖ in (9.17), we can find s' between $\gamma(w+jT_0)-\alpha jT_0$ and $\gamma(w+jT_0)$ so that $V(g_{s'}u_\ell x) \ll_{T_0} 1$. Hence, changing variables using (2.3), (2.4), and (2.5), we get

$$\begin{aligned}e^{-\delta t_{\rho,\ell}}\mu^u_{x_{\rho,\ell}}(W_\ell) = \mu^u_{n_{\rho,\ell}x_j}(Wn_{\rho,\ell}^{-1}) &= e^{\delta\gamma(w+jT_0)}\mu^u_{u_\ell x}\left(N^+_{e^{-\gamma(w+jT_0)}\iota_j}\right) \\ &= e^{\delta(\gamma(w+jT_0)-s')}\mu^u_{g_{s'}u_\ell x}\left(N^+_{e^{s'-\gamma(w+jT_0)}\iota_j}\right).\end{aligned}$$

Since the height of $g_{s'}u_\ell x$ is $O_{T_0}(1)$ and $r := e^{s'-\gamma(w+jT_0)}\iota_j \leq 1$, the measure $\mu^u_{g_{s'}u_\ell x}\left(N_r^+\right)$ is $O_{T_0}(1)$ by definition of the conditional measures in (2.2). Hence, our choice of s' implies

$$\mu^u_{x_{\rho,\ell}}(W_\ell) \ll_{T_0} e^{\delta\alpha jT_0}.$$

The lemma follows by combining these estimates with (9.29). $\square$

The point of the above lemma is that the sum in question has, in general, much fewer terms than the sum in (9.46).

Stable holonomy. Fix some $\rho \in \mathcal{P}_j^0$. Recall the points $y_\rho \in T_\rho$ and $n_{\rho,\ell}^- \in N_\rho^-$ satisfying (9.39). The product map $M \times N^- \times A \times N^+ \to G$ is a diffeomorphism on a ball of radius 1 around identity; cf. Section 2.6. Hence, given $\ell \in I_{\rho,j}$, we can define maps $\tilde{u}_\ell$, $\tilde{\tau}_\ell$, m_ℓ and $\tilde{u}_\ell^-$ from W_ℓ to N^+, $\mathbb{R}$, M and N^- respectively by the following formula

$$g_{t+t_{\rho,\ell}} n n_{\rho,\ell}^- = g_{t+t_{\rho,\ell}} m_\ell(n) \tilde{u}_\ell^-(n) g_{\tilde{\tau}_\ell(n)} \tilde{u}_\ell(n) = m_\ell(n) \tilde{u}_\ell^-(t,n) g_{t+t_{\rho,\ell}+\tilde{\tau}_\ell(n)} \tilde{u}_\ell(n), \tag{9.48}$$

where we set $\tilde{u}_\ell^-(t,n) = \mathrm{Ad}(g_{t+t_{\rho,\ell}})(\tilde{u}_\ell^-(n))$. We define the following change of variable map:

$$\Phi_\ell : \mathbb{R} \times W_\ell \to \mathbb{R} \times N^+, \qquad \Phi_\ell(t,n) = (t + \tilde{\tau}_\ell(n), \tilde{u}_\ell(n)). \tag{9.49}$$

We suppress the dependence on ρ and j to ease notation. Then, Φ_ℓ induces a map between the weak unstable manifolds of $x_{\rho,\ell}$ and y_ρ, also denoted Φ_ℓ, and defined by

$$\Phi_\ell(g_t n x_{\rho,\ell}) = g_{t+\tilde{\tau}_\ell(n)} \tilde{u}_\ell(n) y_\rho.$$

In particular, this induced map coincides with the local strong stable holonomy map inside B_ρ.

Note that we can find a neighborhood $W_\rho \subset N^+$ of identity of radius $\asymp \iota_j$ such that

$$\Phi_\ell(\mathbb{R} \times W_\ell) \subseteq \mathbb{R} \times W_\rho, \tag{9.50}$$

for all $\ell \in I_{\rho,j}$. Moreover, by shrinking the radius ι_j of the flow boxes by an absolute amount (depending only on the metric on G) if necessary, we may assume that all the maps Φ_ℓ in (9.49) are invertible on $\mathbb{R} \times W_\rho$. Hence, we can define the following:

$$\begin{aligned} \tau_\ell(n) &= \tilde{\tau}_\ell(\tilde{u}_\ell^{-1}(n)) + t_{\rho,\ell} \in \mathbb{R}, \qquad u_\ell^-(t,n) = \tilde{u}_\ell^-(t - \tau_\ell(n), \tilde{u}_\ell^{-1}(n)) \in N^-, \\ \phi_{\rho,\ell}(t,n) &= e^{-a(t-\tau_\ell(n))} \times J\Phi_\ell(n) \times \widetilde{\phi}_{\rho,\ell}(t - \tau_\ell(n), \tilde{u}_\ell^{-1}(n)), \end{aligned} \tag{9.51}$$

and $J\Phi_\ell$ denotes the Jacobian of the change of variable Φ_ℓ; cf. (2.10).

Changing variables and using M-invariance of F_γ, we obtain

$$\begin{aligned} &\sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n \in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n) F_\gamma(g_{t+t_{\rho,\ell}} n x_{\rho,\ell}) \, d\mu^u_{x_{\rho,\ell}}(n) dt \\ &\qquad = \sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} \int_{W_\rho} e^{-ib(t-\tau_\ell(n))} \phi_{\rho,\ell}(t,n) F_\gamma(u_\ell^-(t,n) g_t n y_\rho) \, d\mu^u_{y_\rho}(n) dt. \end{aligned} \tag{9.52}$$

Stable derivatives. Our next step is to remove F_γ from the sum over ℓ in (9.52). Due to non-joint integrability of the stable and unstable foliations, our estimate involves a derivative of f in the flow direction. In particular, in view of the way we obtain contraction in the norm of flow derivatives in Lemma 7.6, this step is the most "expensive" estimate in our argument. In essence, all the prior setup was aimed at optimizing the gain in this step.

Recall the definition of F_γ in (9.36). Since y_ρ belongs to $N^-_{1/2}\Omega$ and $u_\ell^-(t,n)$ belongs to a neighborhood of identity in N^- of radius $O(\iota_j)$ (cf. (9.18)), uniformly over (t,n) in the support of our integrals, Proposition 8.5 and (9.15) yield

$$|F_\gamma(u_\ell^-(t,n) g_t n y_\rho) - F_\gamma(g_t n y_\rho)| \ll e^{-(1-\gamma)(w+jT_0)} \|f\|_1^\star \mu^u_{y_\rho}(N_1^+) V(y_\rho), \tag{9.53}$$

where we implicitly used the fact that $W_\rho \subset N_1^+$ and $|t| \leq 1$. Indeed, the additional gain is due to the fact that g_s contracts N^- by at least e^{-s} for all $s \geq 0$.

To sum the above errors over ℓ and ρ, we wish to use (9.46). We first note that Proposition 3.1 and the fact W_ρ has diameter $\asymp \iota_j$ imply that

$$\mu^u_{y_\rho}(N_1^+) \ll \iota_j^{-\Delta_+} \mu^u_{y_\rho}(W_\rho),$$

where Δ_+ is the constant in (3.1). Moreover, Propositions 3.1 and 4.3 allow us to use closeness of y_ρ and $x_{\rho,\ell}$ along with regularity of holonomy to deduce that

$$V(y_\rho)\mu^u_{y_\rho}(W_\rho) \asymp V(x_{\rho,\ell})\mu^u_{x_{\rho,\ell}}(W_\ell). \tag{9.54}$$

Here, we also use the fact that both $x_{\rho,\ell}$ and y_ρ belong to $N_1^-\Omega$; cf. (9.40).

Hence, we can use (9.46) to estimate the sum of the errors in (9.53) yielding the following estimate on the main term in (9.45):

$$e^{-\delta\gamma(w+jT_0)} \sum_{\rho\in\mathcal{P}_j^0}\sum_{\ell\in I_{\rho,j}} \int_{\mathbb{R}}\int_{W_\rho} \left(\sum_{\ell\in I_{\rho,j}} e^{-ib(t-\tau_\ell(n))}\phi_{\rho,\ell}(t,n)\right) F_\gamma(g_t n y_\rho)\, d\mu^u_{y_\rho} dt$$
$$+ O\left(e^{-(1-\gamma)(w+jT_0)} \|f\|_1^\star\, \mu^u_x(N_1^+)V(x)\iota_j^{-\Delta_+}\right),$$

where we used that the above integrands have uniformly bounded support in $\mathbb{R}\times N^+$, independently of ℓ (and ρ). Indeed, the boundedness in the $\mathbb{R}$ direction follows from that of the partition of unity p_j; cf. (9.8). We also used (9.43) to bound the C^0 norm of $\phi_{\rho,\ell}$. Summing the above error term over j and w using (9.12) and (9.19), we obtain

$$O_{T_0}\left(\frac{\|f\|_1^\star\,\mu^u_x(N_1^+)V(x)\times(1+\varepsilon_1)^m}{(a+(1-\gamma)-4\alpha\Delta_+-\varepsilon_1)^m}\right).$$

Taking α and ε_1 small enough, while taking b large, we get

$$O\left(\frac{\|f\|_1^\star\,\mu^u_x(N_1^+)V(x)}{(a+3/10)^m}\right).$$

Recall the norm $\|\cdot\|_B^\star$ defined in (9.2) and note that $\|\cdot\|_1^\star \leq B\,\|\cdot\|_B^\star$. Choosing a and $\varkappa>0$ small enough, we can ensure that $e^{1+\varkappa}/(a+3/10)$ is at most $1/(a+1/10)$. With this choice, taking $B=b^{1+\varkappa}$ yields an error term of the form:

$$O\left(\frac{\|f\|_B^\star\,\mu^u_x(N_1^+)V(x)}{(a+1/10)^m}\right). \tag{9.55}$$

Mollifiers and Cauchy-Schwarz. We are left with estimating integrals of the form:

$$\int_{\mathbb{R}\times W_\rho} \Psi_\rho(t,n)F_\gamma(g_t n y_\rho)\, d\mu^u_{y_\rho} dt, \qquad \Psi_\rho(t,n) := \sum_{\ell\in I_{\rho,j}} e^{-ib(t-\tau_\ell(n))}\phi_{\rho,\ell}(t,n). \tag{9.56}$$

We begin by giving an apriori bound on Ψ_ρ. Denote by $J_\rho\subset\mathbb{R}$ the bounded support of the integrand in t coordinate of the above integrals. Note that (9.43) and the fact that $|t|\ll 1$ imply

$$\|\phi_{\rho,\ell}\|_{L^\infty(J_\rho\times W_\rho)} \ll 1, \qquad \|\Psi_\rho\|_{L^\infty(J_\rho\times W_\rho)} \ll \#I_{\rho,j}. \tag{9.57}$$

To simplify notation, we let

$$r=(1-\gamma)(w+jT_0).$$

Note that we have that $y_\rho\in N_1^-\Omega$, $|J_\rho|\ll 1$, $r\geq 1$ and $W_\rho\subseteq N_1^+$. Hence, Proposition 8.4, along with (9.26), the definition of F_γ in (9.36) and the Cauchy-Schwarz inequality, yield

$$\left|\int_{\mathbb{R}\times W_\rho}\Psi_\rho(t,n)F_\gamma(g_t n y_\rho)\, d\mu^u_{y_\rho} dt\right|^2 \ll e^\star_{1,0}(f)^2 \int_{J_\rho\times W_\rho} |\Psi_\rho(t,n)|^2\, d\mu^u_{y_\rho}dt \int_{N_1^+} V^2(g_r n y_\rho)\, d\mu^u_{y_\rho}.$$

Thus, Remark 9.1 and the Margulis inequality for V^2 in Theorem 4.1 yield

$$\left|\int_{\mathbb{R}\times W_\rho}\Psi_\rho(t,n)F_\gamma(g_t n y_\rho)\, d\mu^u_{y_\rho} dt\right|^2 \ll e^\star_{1,0}(f)^2 V^2(y_\rho)\mu^u_{y_\rho}(N_1^+) \int_{J_\rho\times W_\rho} |\Psi_\rho(t,n)|^2\, d\mu^u_{y_\rho}dt. \tag{9.58}$$

Cusp-adapted partitions. To estimate the right side of (9.67), it will be convenient to linearize the phase functions τ_k. For this purpose, we need to pick a partition of unity of W_ρ, where the diameter of the ball supporting an element of the partition of unity is determined by a certain return time of its center to a given compact set. This is achieved in the next result.

Proposition 9.10. *For all $b \geq 1$ and $x \in N_1^-\Omega$, there exists a cover $\{A_i : i\}$ of N_1^+ and a set $\mathcal{R}_x \subseteq N_1^+$ with $\mu_x^u(N_1^+ \setminus \mathcal{R}_x) \ll b^{-\beta/2}V(x)\mu_x^u(N_1^+)$ such that for all i with $A_i \cap \mathcal{R}_x \neq \emptyset$, we have*

1. *A_i has the form $A_i = N_{r_i}^+ \cdot u_i$ for some $r_i > 0$ and $u_i \in N_1^+$.*
2. *If $t_i = -\log r_i$, then $V(g_{t_i}ux) \ll_\beta 1$ for all $u \in A_i$.*
3. *$b^{-8/10} \ll_\beta r_i \ll_\beta b^{-7/10}$.*
4. *$\sum_i \mu_x^u(A_i) \ll \mu_x^u(N_1^+)$.*

Proof. Let $r_0 \geq 1$ be the constant provided by Theorem 7.9 applied with $\varepsilon = \beta/100$. Let $m_0 = \lceil r_0^{-1} \log b \rceil$ and let $H = e^{3\beta r_0}$ be the height provided by Theorem 7.9. Then, we have

$$\mu_x^u\left(n \in N_1^+ : \sum_{1 \leq \ell \leq m_0} \chi_H(g_{\ell r_0}nx) > 99m_0/100\right) \ll_\beta b^{-\beta/2}V(x)\mu_x^u(N_1^+).$$

Denote the set on the left side in the above estimate by $\mathcal{E}_x$ and define a function $\varsigma : N_1^+ \setminus \mathcal{E}_x \to [7/10, 8/10]$ by setting $\varsigma(n)$ to be the least value of $\eta \in [7/10, 8/10]$ such that $V(g_{\eta \log b}nx) \leq H$.

Let $\mathcal{R}_x := \mathrm{supp}(\mu_x^u) \cap N_1^+ \setminus \mathcal{E}_x$ and consider its cover $\{A_u : u \in \mathcal{R}_x\}$, where each A_u is the ball around each u of radius $b^{-\varsigma(u)}$. Using the Vitali covering lemma and the uniform doubling result in Proposition 3.1, we can find a finite subcover $\{A_{u_i} : i\}$ such that $\sum_i \mu_x^u(A_{u_i}) \ll \mu_x^u(N_1^+)$. This completes the proof by taking $A_i := A_{u_i}$.

$\square$

Let $\{A_i\}$ be the cover provided by Proposition 9.10, applied with $x = y_\rho$. Since $W_\rho \subseteq N_1^+$, by discarding elements of this cover that are disjoint of W_ρ if necessary, we shall assume that each A_i intersects W_ρ non-trivially. Combining this result with (9.57), we obtain

$$\int_{J_\rho \times W_\rho} |\Psi_\rho(t,n)|^2 \, d\mu_{y_\rho}^u dt \leq \sum_i \int_{J_\rho \times A_i} |\Psi_\rho(t,n)|^2 \, d\mu_{y_\rho}^u dt + O\left(b^{-\beta/2} \# I_{\rho,j}^2 V(y_\rho)\mu_{y_\rho}^u(N_1^+)\right). \quad (9.59)$$

Linearizing the phase. We now turn to estimating the sum of oscillatory integrals in (9.59). For $k, \ell \in I_{\rho,j}$, we let

$$\psi_{k,\ell}(t,n) := \phi_{\rho,k}(t,n)\overline{\phi_{\rho,\ell}(t,n)}.$$

Expanding the square, we get

$$\sum_i \int_{J_\rho \times A_i} |\Psi_\rho(t,n)|^2 \, d\mu_{y_\rho}^u dt = \sum_i \sum_{k,\ell \in I_{\rho,j}} \int_{J_\rho \times A_i} e^{-ib(\tau_k(n) - \tau_\ell(n))} \psi_{k,\ell}(t,n) \, d\mu_{y_\rho}^u dt.$$

Using (2.3) and (2.4), we change variables in the integrals using the maps taking each A_i onto N_1^+. More precisely, recall that A_i is a ball of radius r_i around u_i such that $u_i y_\rho \in \Omega$. Letting

$$t_i = -\log r_i, \quad y_\rho^i = g_{t_i}u_i y_\rho, \quad \tau_k^i = \tau_k(\mathrm{Ad}(g_{-t_i})(n)u_i), \quad \psi_{k,\ell}^i(t,n) = \psi_{k,\ell}(t, \mathrm{Ad}(g_{-t_i})(n)u_i), \quad (9.60)$$

we can rewrite the above sum as

$$\sum_i \sum_{k,\ell \in I_{\rho,j}} \int_{J_\rho \times A_i} e^{-ib(\tau_k(n) - \tau_\ell(n))} \psi_{k,\ell}(t,n) \, d\mu_{y_\rho}^u dt$$
$$\leq \sum_i e^{-\delta t_i} \sum_{k,\ell \in I_{\rho,j}} \left| \int_{J_\rho \times N_1^+} e^{-ib(\tau_k^i(n) - \tau_\ell^i(n))} \psi_{k,\ell}^i(t,n) d\mu_{y_\rho^i}^u dt \right|. \quad (9.61)$$

We note that the radius r_i of A_i satisfies

$$b^{-8/10} \ll e^{-t_i} = r_i \ll b^{-7/10}. \tag{9.62}$$

We also recall from Proposition 9.10 that r_i was chosen so that

$$V(y_\rho^i) \ll 1, \qquad \forall i. \tag{9.63}$$

This is important for the proof of Theorem 9.16 below.

Next, we use the coordinate parametrization of N^+ by its Lie algebra $\mathfrak{n}^+ := \mathrm{Lie}(N^+)$ via the exponential map. We suppress composition with exp from our notation for simplicity and continue to denote by $\mu^u_{y^i_\rho}$ and N_1^+ their pushforward to $\mathfrak{n}^+$ by exp.

Recall from Section 2.5 the parametrization of N^- by its Lie algebra $\mathfrak{n}^- = \mathfrak{n}^-_\alpha \oplus \mathfrak{n}^-_{2\alpha}$ via the exponential map and similarly for N^+. Let $w_i = (v_i, r_i) \in \mathfrak{n}^+_\alpha \times \mathfrak{n}^+_{2\alpha}$ be such that $u_i = \exp(w_i)$, where u_i is the center of the ball A_i. Recall the notation for transverse intersection points $n^-_{\rho,k}$ in (9.39). For each $k \in I_{\rho,j}$, write

$$n^-_{\rho,k} = \exp(u_k + s_k)$$

with $u_k \in \mathfrak{n}^-_\alpha$ and $s_k \in \mathfrak{n}^-_{2\alpha}$. With this notation, we have the following formula for the temporal functions τ_k. The proof of this lemma is given in Section 10.

Lemma 9.11. *For every i, there exists a bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{n}^- \times \mathfrak{n}^+_\alpha \to \mathbb{R}$ such that the following holds. For every $k \in I_{\rho,j}$, there is a constant $c^i_k \in \mathbb{R}$ such that for all $n = \exp(v,r) \in N^+_1$ with $v \in \mathfrak{n}^+_\alpha$ and $r \in \mathfrak{n}^+_{2\alpha}$, we have that*

$$\tau^i_k(n) - \tau^i_\ell(n) = c^i_{k,\ell} + e^{-t_i}\langle u_k - u_\ell + s_k - s_\ell, v\rangle + O(e^{-2t_i}).$$

Moreover, there exists a proper linear subspace $L_i \subset \mathfrak{n}^-_\alpha$ such that for every $(u,s) \in \mathfrak{n}^-_\alpha \times \mathfrak{n}^-_{2\alpha}$, the linear functional $\langle u+s,\cdot\rangle : \mathfrak{n}^+_\alpha \to \mathbb{R}$ satisfies

$$\|\langle u+s,\cdot\rangle\| \gg \mathrm{dist}(u, L_i),$$

where $\|\langle u+s,\cdot\rangle\| := \sup_{\|v\|=1} |\langle u+s, v\rangle|$.

Remark 9.12. The proof of the lemma also shows that if X is real hyperbolic, then we can take $L_i = \{0\}$.

To simplify notation, we set

$$w^i_{k,\ell} := e^{-t_i}(u_k - u_\ell + s_k - s_\ell). \tag{9.64}$$

Note that $\mathrm{Ad}(g_{-t_i})$ contracts N^+ by at least $e^{-t_i} \ll b^{-7/10}$; cf. (9.62). In light of (9.57), this ensures that the Lipschitz norm of $\psi^i_{k,\ell}$ along N^+ is $\ll b^{-7/10}$. Moreover, we recall that $|J_\rho| \ll 1$. Then, we can estimate the right side of (9.61) as follows:

$$\begin{aligned}\sum_i e^{-\delta t_i} \sum_{k,\ell \in I_{\rho,j}} &\left| \int_{J_\rho \times N_1^+} e^{-ib(\tau^i_k(n) - \tau^i_\ell(n))} \psi^i_{k,\ell}(t,n)\, d\mu^u_{y^i_\rho}\, dt \right| \\ &\ll \sum_i e^{-\delta t_i} \sum_{k,\ell \in I_{\rho,j}} \left| \int_{N_1^+} e^{-ib\langle w^i_{k,\ell}, v\rangle} d\mu^u_{y^i_\rho} \right| + b^{-4/10} \mu^u_{y_\rho}(N_1^+) \# I^2_{\rho,j}. \end{aligned} \tag{9.65}$$

Excluding close pairs of unstable manifolds. Let $L_i \subset \mathfrak{n}_\alpha^-$ denote the subspace provided by Lemma 9.11. Denote by $C_{\rho,j,i}$ the following subset of $I_{\rho,j}^2$:

$$C_{\rho,j,i} = \left\{ (k,\ell) \in I_{\rho,j}^2 : \operatorname{dist}(u_k - u_\ell, L_i) \leq b^{-1/10} \right\}.$$

We also set

$$S_{\rho,j,i} = I_{\rho,j}^2 \setminus C_{\rho,j,i}. \tag{9.66}$$

Then, $C_{\rho,j,i}$ parametrizes pairs of unstable manifolds which are too close along the direction $L_i \oplus \mathfrak{n}_{2\alpha}^-$ in the stable foliation. Recall that $L_i \oplus \mathfrak{n}_{2\alpha}^- = \{0\}$ when X is real hyperbolic. In particular, in this case, $C_{\rho,j,i}$ simply parametrizes pairs of unstable manifolds which are too close along the stable foliation. With this notation, the sum on the right side of (9.65) can be estimated as follows:

$$\begin{aligned} &\sum_i e^{-\delta t_i} \sum_{k,\ell \in I_{\rho,j}} \left| \int_{N_1^+} e^{-ib\langle w_{k,\ell}^i, v\rangle} d\mu_{y_\rho^i}^u \right| \\ &\qquad\qquad \ll \# C_{\rho,j,i} \mu_{y_\rho}^u(W_\rho) + \sum_i e^{-\delta t_i} \sum_{(k,\ell) \in S_{\rho,j,i}} \left| \int_{N_1^+} e^{-ib\langle w_{k,\ell}^i, v\rangle} d\mu_{y_\rho^i}^u \right|. \end{aligned} \tag{9.67}$$

We estimate the first term in (9.67) via the following proposition, proved in Section 12.3, using the non-concentration properties of Patterson-Sullivan measures obtained in Section 12. We note that this non-concentration property is not needed for the proof in the constant curvature case.

Proposition 9.13. *There exists a constant $\kappa > 0$ such that for all $\ell \in I_{\rho,j}$,*

$$\# \left\{ k \in I_{\rho,j} : (k,\ell) \in C_{\rho,j,i} \right\} \ll_{T_0} (b^{-\kappa/10} + e^{-\kappa(\gamma-\alpha)(w+jT_0)}) e^{\delta\gamma(w+jT_0)}.$$

Remark 9.14. The constant κ will be provided by Theorem 11.17.

Summarizing our estimates in (9.59), (9.65), (9.67), and Proposition 9.13, we have shown that

$$\begin{aligned} &\int_{J_\rho \times W_\rho} |\Psi_\rho(t,n)|^2 \, d\mu_{y_\rho}^u dt \\ &\ll \sum_i e^{-\delta t_i} \sum_{(k,\ell) \in S_{\rho,j,i}} \left| \int_{N_1^+} e^{-ib\langle w_{k,\ell}^i, v\rangle} d\mu_{y_\rho^i}^u \right| \\ &+ \Big((b^{-\beta/2} V(y_\rho) + b^{-4/10}) \# I_{\rho,j} + (b^{-\kappa/10} + e^{-\kappa(\gamma-\alpha)(w+jT_0)}) e^{\delta\gamma(w+jT_0)} \Big) \times \# I_{\rho,j} \mu_{y_\rho}^u(N_1^+). \end{aligned} \tag{9.68}$$

9.3. **The role of additive combinatorics.**

To proceed, we wish to make use of the oscillations due to the large frequencies $bw_{k,\ell}^i$ to obtain cancellations. First, we note that Lemma 9.11 and the separation between pairs of unstable manifolds with indices in $S_{\rho,j,i}$ implies that the frequencies $bw_{k,\ell}^i$ have large size. More precisely, the linear functionals $\langle w_{k,\ell}^i, \cdot\rangle : \mathfrak{n}_\alpha^+ \to \mathbb{R}$ satisfy

$$b^{-9/10} \ll \left\| \langle w_{k,\ell}^i, \cdot \rangle \right\| \ll b^{-7/10}. \tag{9.69}$$

Let $\pi : \mathfrak{n}^+ \to \mathfrak{n}_\alpha^+$ denote the projection parallel to $\mathfrak{n}_{2\alpha}^+$ and note that the integrands on the right side of (9.68) depend only on the $\mathfrak{n}_\alpha^+$ component of the variable. To simplify notation, we let[14]

$$\nu_i := \pi_* \mu_{y_\rho^i}^u \Big|_{N_1^+}. \tag{9.70}$$

[14]Note that π is the identity map in the real hyperbolic case.

Remark 9.15. It is worth emphasizing that the linearization provided by Lemma 9.11 only depends on the unstable directions with weakest expansion under the flow. The reason we do so is that our metric on $\mathfrak{n}^+$ is not invariant by addition when X is not real hyperbolic (it is invariant by the nilpotent group operations) and our non-concentration estimates for the measures $\mu^u_\bullet$ only hold for this metric. This in particular means the results of Section 11 do not apply to these measures in this case, which is the reason we work with projections. It is possible to develop the theory in Section 11 for measures and convolutions on nilpotent groups such as N^+ to avoid working with projections, however we believe the approach we adopt here is more amenable to generalizations beyond the algebraic setting of this article.

For $w \in \mathfrak{n}^-$, let

$$\hat{\nu}_i(w) := \int_{\mathfrak{n}^+_\alpha} e^{-i\langle w,v\rangle}\, d\nu_i(v). \tag{9.71}$$

Note that the total mass of ν_i, denoted $|\nu_i|$, is $\mu^u_{y^i_\rho}(N^+_1)$. Let $\lambda > 0$ be a small parameter to be chosen using Theorem 9.16 below. Define the following set of frequencies where $\hat{\nu}_i$ is large:

$$B(i,k,\lambda) := \left\{ \ell \in I_{\rho,j} : (k,\ell) \in S_{\rho,j,i} \text{ and } |\hat{\nu}_i(bw^i_{k,\ell})| > b^{-\lambda}|\nu_i| \right\}. \tag{9.72}$$

Then, splitting the sum over frequencies according to the size of the Fourier transform $\hat{\nu}_i$ and reversing our change variables to go back to integrating over A_i, we obtain

$$\begin{aligned}\sum_i e^{-\delta t_i} \sum_{(k,\ell)\in S_{\rho,j,i}} \int_{\mathfrak{n}^+_\alpha} e^{-ib\langle w^i_{k,\ell},v\rangle}\, d\nu_i(v) \\ \ll \left(\max_{i,k} \# B(i,k,\lambda) + b^{-\lambda}\# I_{\rho,j} \right) \# I_{\rho,j}\mu^u_{y_\rho}(N^+_1),\end{aligned} \tag{9.73}$$

where we again used the estimate $\sum_i \mu^u_{y^i_\rho}(A_i) \ll \mu^u_{y_\rho}(N^+_1)$.

The following key counting estimate for $B(i,k,\lambda)$ is deduced from Corollary 11.5. Its proof is given in Section 12.4.

Theorem 9.16. *For every $\varepsilon_2 > 0$, there exists $\lambda > 0$ such that for all i and k, we have*

$$\# B(i,k,\lambda) \ll_\varepsilon b^{\varepsilon_2} \left(b^{-\kappa/10} + e^{-\kappa(\gamma-\alpha)(w+jT_0)} \right) e^{\delta\gamma(w+jT_0)},$$

where $\kappa > 0$ is the constant provided by Proposition 9.13.

Combining estimates on oscillatory integrals. Let $\kappa > 0$ be as in Theorem 9.16. In what follows, we assume ε is chosen smaller than $\kappa/100$ and that $\lambda \leq \min\{\beta/2, 4/10, \kappa/20\}$. Let

$$Q = (b^{-\kappa/20} + b^{\varepsilon_2} e^{-\kappa(\gamma-\alpha)(w+jT_0)}) e^{\delta\gamma(w+jT_0)}.$$

Theorem 9.16, combined with (9.58), (9.68) and (9.73), yields:

$$\begin{aligned}\int_{\mathbb{R}\times W_\rho} \Psi_\rho(t,n) F_\gamma(g_t n y_\rho)\, d\mu^u_{y_\rho} dt \\ \ll e^\star_{1,0}(f) V(y_\rho) \mu^u_{y_\rho}(N^+_1) \times \left(b^{-\lambda/2} V(y_\rho)^{1/2} \# I_{\rho,j} + \sqrt{\# I_{\rho,j} \times Q} \right),\end{aligned} \tag{9.74}$$

where we used the elementary inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for any $x,y \geq 0$.

Our next goal is to estimate the sum of the above bound over ρ. Note that for all $\rho \in \mathcal{P}^0_j$, since W_ρ has radius $\asymp \iota_j$, cf. (9.18), we have by Proposition 3.1 and (9.54) that for all $\ell \in I_{\rho,j}$,

$$\mu^u_{y_\rho}(N^+_1) \ll \iota_j^{-\Delta_+} \mu^u_{y_\rho}(W_\rho) \asymp \iota_j^{-\Delta_+} \mu^u_{x_{\rho,\ell}}(W_\ell).$$

Hence, the Cauchy-Schwarz inequality yields

$$\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\mu^u_{y_\rho}(N_1^+)\sqrt{\# I_{\rho,j}} \ll \iota_j^{-\Delta_+}\left(\sum_{\rho\in\mathcal{P}_j^0}\mu^u_{y_\rho}(W_\rho)\times\sum_{\rho\in\mathcal{P}_j^0,\ell\in I_{\rho,j}} V^2(x_{\rho,\ell})\mu^u_{x_{\rho,\ell}}(W_\ell)\right)^{1/2}.$$

We estimate the first sum above using Lemma 9.9 and the second using (9.47) to get

$$\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\mu^u_{y_\rho}(N_1^+)\sqrt{\# I_{\rho,j}} \ll V(x)\mu^u_x(N_1^+)\times\iota_j^{-A/2}\times e^{(\delta(\gamma(w+jT_0)+\alpha jT_0)/2}, \tag{9.75}$$

where we set $A=2D+2\Delta_++1$. Similarly, using (9.46), and (9.54), we obtain

$$\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\mu^u_{y_\rho}(N_1^+)\# I_{\rho,j} \ll \iota_j^{-\Delta_+}\times e^{\delta\gamma(w+jT_0)}\mu^u_x(N_1^+)V(x).$$

Finally, since $y_\rho\in K_j$, we have $V(y_\rho)\ll_{T_0} e^{2\beta\alpha jT_0}$; cf. (9.18).

To simplify and combine the above bounds, recall from Remark 9.1 and (3.1) that $\beta\leq\delta/2$ so that $e^{\beta\alpha jT_0}\leq e^{\delta\alpha jT_0/2}$. We also have that $\lambda/2\leq\kappa/40$. It follows that upon combining the above estimate with (9.74) and (9.75), we obtain the following bound on the sum of the integrals in (9.74):

$$\begin{aligned}&e^{-\delta\gamma(w+jT_0)}\sum_{\rho\in\mathcal{P}_j^0}\int_{\mathbb{R}\times W_\rho}\Psi_\rho(t,n)F_\gamma(g_tny_\rho)\,d\mu^u_{y_\rho}dt\\&\ll e^\star_{1,0}(f)V(x)\mu^u_x(N_1^+)\times\iota_j^{-A/2}e^{\delta\alpha jT_0/2}\times\left(b^{-\lambda/2}+b^{\varepsilon_2/2}e^{-\kappa(\gamma-\alpha)(w+jT_0)/2)}\right),\end{aligned}$$

where we again used the inequality $\sqrt{x+y}\leq\sqrt{x}+\sqrt{y}$.

Arguing as before using (9.12) and (9.19), the sum of the above error terms over j and w yields an error term of the form

$$O_{T_0}\left(e^\star_{1,0}(f)V(x)\mu^u_x(N_1^+)(1+\varepsilon_1)^m\times\left[\frac{b^{-\lambda/2}}{(a-2\alpha A-\delta\alpha/2-\varepsilon_1)^m}+\frac{b^{\varepsilon_2/2}}{(a+\kappa(\gamma-\alpha)/2-\varepsilon_1)^m}\right]\right). \tag{9.76}$$

To simplify the above bound, recall that λ is chosen according to Theorem 9.16 and hence its size depends on ε_2, however κ is given by Proposition and is independent of ε_2. Moreover, γ,λ and κ are independent of a,α and ε_1, and we are free to choose the parameters α and ε_1 as small as needed. We also recall that $m=\lceil\log b\rceil$; cf. (9.9). As before, we will absorb the dependence on T_0 in (9.76) by taking b large enough at the cost of replacing ε_1 with $2\varepsilon_1$ in the denominators of the above expression. Hence, we may take $\alpha\leq\gamma/2$ and choose ε_1,a, and ε_2 small enough relative to $\kappa\gamma$ to ensure that

$$\frac{(1+\varepsilon_1)e^{\varepsilon_2/2}}{a+\kappa(\gamma-\alpha)/2-2\varepsilon_1}\leq\frac{1}{a+\kappa\gamma/10}.$$

Using the bound $e^{-\lambda/2}\leq 1/(1+\lambda/2)$ and taking α and ε_1 small enough depending on a,λ and A, we obtain

$$\frac{(1+\varepsilon_1)e^{-\lambda/2}}{a-2\alpha A-\delta\alpha/2-2\varepsilon_1}\leq\frac{1}{a+a\lambda/4}.$$

Hence, taking a small enough so that $a\lambda/4\leq\kappa\gamma/10$, the error term in (9.76) becomes

$$O\left(\frac{e^\star_{1,0}(f)V(x)\mu^u_x(N_1^+)}{a+a\lambda/4}\right). \tag{9.77}$$

9.4. **Parameter selection and conclusion of the proof.** In this subsection, we finish the proof of Theorem 9.2 assuming Lemma 9.11, Proposition 9.13, and Theorem 9.16.

Collecting the error terms in (9.10), (9.16), (9.25), (9.33), (9.55), and (9.77), and letting $\sigma_\star > 0$ be the minimum of all the gains in these error terms, we obtain

$$e^\star_{1,0}(R(z)^m f) \ll \frac{\|f\|^\star_B}{(a+\sigma_\star)^m}.$$

Letting C_Γ denote the implied constant, this estimate concludes the proof of Theorem 9.2.

10. The temporal function and proof of Lemma 9.11

In this section, we give an explicit formula for the so-called temporal functions $\tau_{k,\ell}$ appearing in Section 9 and prove Lemma 9.11. Our argument is Lie theoretic. We refer the reader to [Kna02, Chapter 1] for background on the material used in this section. Similar results are known more generally outside of the homogeneous setting by more dynamical/geometric arguments building on work of Katok and Burns [Kat94].

10.1. **Taylor expansion of temporal functions.** The proof of the first part of the lemma consists of establishing a formula for the so-called temporal function τ^i_k using the Campbell-Baker-Hausdorff formula and then proving the that higher order terms in the latter are $O(e^{-2t_i})$.

Fix $k \in I_{\rho,j}$ and recall the elements $n^-_{\rho,k} \in N^-$ which were defined by the displacement of the points $x_{\rho,k}$ from y_ρ along N^- inside the flow box B_ρ; cf. (9.39). We also recall the elements $u_k \in \mathfrak{n}^-_\alpha$ and $s_k \mathfrak{n}^-_{2\alpha}$ chosen so that $n^-\rho, k = \exp(u_k + s_k)$. In what follows, we set

$$X = u_k + s_k.$$

Given $Y \in \mathfrak{n}^+$, we write Y_α and $Y_{2\alpha}$ for its $\mathfrak{n}^+_\alpha$ and $\mathfrak{n}^+_{2\alpha}$ components respectively.

Let ω denote the vector in the Lie algebra $\mathfrak{g}$ of G generating the geodesic flow, i.e. $g_t = \exp(t\omega)$. Recall that M denotes the centralizer of $\{g_t : t \in \mathbb{R}\}$ in G. Then, denoting by $\mathfrak{m}$ its Lie algebra, we have the splitting $\mathfrak{g} = \mathbb{R}\cdot\omega \oplus \mathfrak{m} \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+$. For $v \in \mathfrak{g}$, let $\pi_0(v) \in \mathbb{R}$ be such that $\pi_0(v)\omega$ is the image of v under the projection $\mathfrak{g} \to \mathbb{R}\cdot\omega$ parallel to $\mathfrak{m} \oplus \mathfrak{n}^- \oplus \mathfrak{n}^+$.

Recall the vectors $w_i = v_i + r_i \in \mathfrak{n}^+$ defined above Lemma 9.11, where v_i and r_i denoted the $\mathfrak{n}^+_\alpha$ and $\mathfrak{n}^+_{2\alpha}$ components of w_i respectively. We also recall the return times t_i in (9.60). For $Y \in \mathfrak{n}^+$, let $Y^i = \log(\exp(\mathrm{Ad}(g_{-t_i})(Y))\exp(w_i)) \in \mathfrak{n}^+$. In particular, Y^i takes the form

$$Y^i = (v_i + e^{-t_i}Y_\alpha) + (r_i + e^{-2t_i}Y_{2\alpha} + e^{-t_i}[Y_\alpha, v_i]/2).$$

Let $Y^i_\alpha = v_i + e^{-t_i}Y_\alpha$ and $Y^i_{2\alpha} = r_i + e^{-2t_i}Y_{2\alpha} + e^{-t_i}[Y_\alpha, v_i]/2$.

By the Campbell-Baker-Hausdorff formula[15], we have that $\exp(X)\exp(Y^i) = \exp(Z)$, where $Z = X + Y^i + [X, Y^i]/2 + \dots$ can be expressed as a sum of iterated brackets of X and Y^i. In what follows, we write $Z(Y)$ instead of Z to signify the dependence on Y. Roughly, $\tau^i_k(Y)$ is given by a certain projection of $Z(Y)$ along the flow direction and the lemma will follow from an estimate on the higher order terms in this expansion. More precisely, it follows from the definitions of the functions τ_k (cf. (9.51)) and τ^i_k (cf. (9.60)) that

$$\tau^i_k(Y) = \pi_0(Z(Y^i)). \tag{10.1}$$

Denote by $\mathfrak{w}^i_k$ the sum of the terms in the definition of $Z(Y)$ involving iterated brackets of X and $v_i + r_i$ and let $c^i_k = \pi_0(\mathfrak{w}^i_k)$. Our next step is to show that

$$\pi_0(Z(Y^i)) = c^i_k + \pi_0([u_k, e^{-t_i}Y_\alpha]/2) + \pi_0(e^{-t_i}[s_k, [Y_\alpha, v_i]]/4) + O(e^{-2t_i}).$$

[15]This formula is only valid when X and Y^i are sufficiently close to the origin. As in Remark 2.1, by scaling our metrics if necessary, we shall assume that this formula is valid whenever $\exp(Y^i) \in N^+_1$ and $\exp(X) \in N^-_1$ to simplify notation.

To this end, note that the terms in the expansion of $Z(Y^i)$ we wish to estimate are of the following shapes:

(1) $[X, e^{-2t_i}Y_{2\alpha}]$.
(2) Iterated brackets involving one copy of X and more than one copy of $Y^i - v_i - r_i$.
(3) $[s_k, e^{-t_i}Y_\alpha]$.
(4) Iterated brackets involving a single copy of $e^{-t_i}Y_\alpha$ and more than one copy of X.
(5) Iterated brackets involving a single copy of $e^{-t_i}[Y_\alpha, v_i]$ and more than one copy of X.

Since $Y^i - v_i - r_i$ has size $O(e^{-t_i})$, this implies that the sum of the terms in Items (1) and (2) has size $O(e^{-2t_i})$, and hence so does its image under π_0.

To estimate the remaining terms, recall that if $\mathfrak{g}_\beta$ and $\mathfrak{g}_\gamma$ are $\mathrm{Ad}(g_t)$ eigenspaces in the Lie algebra $\mathfrak{g}$ of G corresponding to the eigenvalues $e^{\beta t}$ and $e^{\gamma t}$ respectively, then we have the relation $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] \subseteq \mathfrak{g}_{\beta+\gamma}$. This implies that the image under π_0 of the terms in Items (3) and (4) is 0. This also immediately implies the same conclusion for the terms in Item (5) except possibly for the term $e^{-t_i}[u_k, [u_k, [Y_\alpha, v_i]]]$, which belongs to $\mathfrak{m} + \mathbb{R}\cdot\omega$.

We claim that $[u_k, [u_k, [Y_\alpha, v_i]]]$ belongs to $\mathfrak{m}$, and hence its image under π_0 is 0. Let θ denote a Cartan involution of $\mathfrak{g}$ sending ω to $-\omega$. In particular, θ fixes $\mathfrak{m}$ pointwise and sends $\mathfrak{n}^+$ onto $\mathfrak{n}^-$ respecting their decompositions into $\mathrm{Ad}(g_t)$-eigenspaces. Denote by B the Killing form on $\mathfrak{g}$ and by Z' the vector $[u_k, [u_k, [Y_\alpha, v_i]]]$.

Recall that $B(\omega,\omega) \neq 0$ and ω is orthogonal to $\mathfrak{m}$ with respect to B, i.e. $B(\omega, x') = 0$ for all $x' \in \mathfrak{m}$. Hence, it suffices to show that $B(\omega, Z') = 0$. By a slight abuse of notation, denote by α the eigenvalue of $\mathrm{ad}(\omega)$ on $\mathfrak{n}_\alpha^+$. Then, using properties of the Killing form and that $u_k \in \mathfrak{n}_\alpha^-$, we obtain

$$B(\omega, Z') = B([\omega, u_k], [u_k, [Y_\alpha, v_i]]) = -\alpha B(u_k, [u_k, [Y_\alpha, v_i]]) = -\alpha B([u_k, u_k], [Y_\alpha, v_i]) = 0.$$

Thus, taking $c^i_{k,\ell} = c^i_k - c^i_\ell$ and $\langle\cdot,\cdot\rangle : \mathfrak{n}^- \times \mathfrak{n}^+_\alpha \to \mathbb{R}$ to be the following bilinear form: for any $u \in \mathfrak{n}^-_\alpha$, $s \in \mathfrak{n}^-_{2\alpha}$, and $Y_\alpha \in \mathfrak{n}^+_\alpha$:

$$\langle u + s, Y_\alpha\rangle := \pi_0([u, Y_\alpha]/2) + \pi_0([s, [Y_\alpha, v_i]]/4), \tag{10.2}$$

completes the proof of the first part of the lemma.

10.2. **The bilinear form and orthogonal projections.** To prove the second part, fix some $(u, s) \in \mathfrak{n}^-_\alpha \times \mathfrak{n}^-_{2\alpha}$ with $u \neq 0$. First, suppose that $v_i \neq 0$. Recall that the symmetric bilinear form $Q(v, w) := -B(v, \theta(w))$ is positive definite and hence induces a metric on $\mathfrak{g}$. Let $\|\cdot\|'$ and dist' denote the induced norm and metric respectively. Then, the restriction of $\|\cdot\|'$ to $\mathfrak{n}^-_\alpha$ is equivalent to our chosen norm on $\mathfrak{n}^-_\alpha$ used in (2.8) (and hence the same holds for the corresponding metrics).

Let $\bar{v}_i = v_i/\|v_i\|$. Denote by $p_i : \mathfrak{n}^-_\alpha \to \mathbb{R}$ the linear functional given by $p_i(u^-) := \pi_0([u^-, \bar{v}_i])$ and let

$$L_i = \mathrm{kernel}(p_i).$$

We claim that $\theta(v_i)$ is orthogonal to L_i with respect to Q. More succinctly, we write

$$Q(\theta(v_i), L_i) = 0. \tag{10.3}$$

Let us first show how this claim implies the lemma. Note that since $\theta(v_i)$ belongs to $\mathfrak{n}^-_\alpha$, non-degeneracy of Q and (10.3) imply that L_i is a proper subspace of $\mathfrak{n}^-_\alpha$. Moreover, we observe that (10.3) implies that $\langle u + s, \bar{v}_i\rangle$ is given by an orthogonal projection with respect to Q in the following sense. Note that orthogonality of $\mathfrak{m}$ and ω implies that

$$\langle u + s, \bar{v}_i\rangle = Q([u, \bar{v}_i], \omega)/(\|\omega\|')^2 = B([u, \bar{v}_i], \omega)/(\|\omega\|')^2,$$

where we used the fact that $\theta(\omega) = -\omega$ in the second equality. Moreover, from invariance of the Killing form by Lie brackets and the fact that $[\omega, v_i] = \alpha v_i$, we obtain

$$B([u, \bar{v}_i], \omega) = B(u, [\bar{v}_i, \omega]) = -\alpha B(u, \bar{v}_i) = \alpha Q(u, \theta(\bar{v}_i)),$$

where we also used that $\theta^2 = \mathrm{Id}$ for the last equality. Thus, in light of the orthogonality given in (10.3), we see that

$$|\langle u+s,\bar{v}_i\rangle| \asymp |Q(u,\theta(\bar{v}_i))| \gg \mathrm{dist}'(u,L_i) \gg \mathrm{dist}(u,L_i),$$

where we used the equivalence of the restriction of the two norms $\|\cdot\|$ and $\|\cdot\|'$ to $\mathfrak{n}_\alpha^-$ as noted above. This completes the deduction of the second part of the lemma from (10.3) in the case $v_i \neq 0$.

To prove (10.3), we need the following observations. Let $x \in L_i$ be arbitrary. Then, we note that $[x, v_i]$ belongs to $\mathfrak{m}$ by definition of L_i. Moreover, arguing as above using the relationship between π_0 and the Killing, we see that

$$p_i(\theta(v_i)) = \alpha(\|\theta(v_i)\|' / \|\omega\|')^2 \neq 0.$$

In particular, $[\theta(v_i), v_i]$ is non-zero and belongs to $\mathbb{R}\cdot\omega$. Finally, note that ω is orthogonal to $\mathfrak{m}$ with respect to Q. Indeed, given $y \in \mathfrak{m}$, we have that

$$Q(y,\omega) = B(y,\omega) = B(\theta(y),\theta(\omega)) = B(y,\theta(\omega)) = Q(y,\omega),$$

where we used the fact that θ fixes $\mathfrak{m}$ pointwise and $\theta(\omega) = -\omega$. This implies that $(y,\omega) = 0$ for all $y \in \mathfrak{m}$ as claimed.

Now, let $c \neq 0$ be such that $[\theta(v_i), v_i] = c\omega$. Then, invariance of the Killing form implies

$$0 = B([\theta(v_i), v_i],[x,v_i]) = cB([\omega,x],v_i) = c\alpha B(x,v_i) = c\alpha Q(x,v_i),$$

where we again used that $[\omega, v_i] = \alpha v_i$ and $\theta^2 = \mathrm{Id}$. As x was arbitrary, this implies (10.3).

If $v_i = 0$, then a similar argument shows that $\langle u+s,\theta(u)\rangle$ has size $\asymp \|u\|$. Hence, the lemma follows in this case with $L_i = \{0\}$. This concludes the proof.

Finally, we note that if X is real hyperbolic, then $\mathfrak{n}_{2\alpha}^- = \{0\} = \mathfrak{n}_{2\alpha}^+$ and (10.2) simplifies to be $\langle u, Y_\alpha\rangle = \pi_0([u,Y_\alpha]/2)$. In particular, taking $Y_\alpha = u/\|u\|$ and arguing as above shows that $\|\langle u,\cdot\rangle\| \gg \|u\| = \mathrm{dist}(u,0)$. Hence, L_i can be taken to be $\{0\}$ in this case.

11. Dimension Increase Under Iterated Convolutions

The goal of this section is to prove that measures that do not concentrate near hyperplanes in $\mathbb{R}^d$ become smoother under iterated self-convolutions in the sense of quantitative increase in their L^2-dimension; cf. Theorem 11.4 below. This result immediately implies Theorem 1.6. As a corollary, we deduce that the Fourier transforms of such measures enjoy polynomial decay outside of a very sparse set of frequencies; cf. Corollary 11.5.

Corollary 11.5 is the key ingredient in the proof of Theorem 9.16 where it is applied to (projections of) conditional measures of the BMS measure. Moreover, the proof of Proposition 9.13 in the case of cusped non-real hyperbolic manifolds requires a polynomial non-concentration estimate near hyperplanes which we deduce from Theorem 11.4; cf. Theorem 11.17.

11.1. Non-uniform affine non-concentration.

We begin by introducing our non-concentration hypothesis, which allows for an exceptional set of points and scales where concentration may happen.

Definition 11.1. Let positive functions λ, φ, and C on $(0,1]$ be given. We say that a Borel probability measure μ on $\mathbb{R}^d$ is (λ,φ,C)-*affinely non-concentrated at almost every scale* if for every $0<\varepsilon,\theta\le 1$, the following holds for all $k\in\mathbb{N}$ and $r \ge C(\theta)$:

(1) $\varphi(x)$ tends to 0 as x tends to 0.
(2) There is an exceptional set $\mathcal{E} = \mathcal{E}(k,\varepsilon,\theta,r) \subset \mathbb{R}^d$ with $\mu(\mathcal{E}) \le C(\theta)2^{-\lambda(\theta)k}$.
(3) For every $x \in \mathrm{supp}(\mu)\setminus\mathcal{E}$, there is a set of good scales $\mathcal{N}(x) \subseteq [0,k]\cap\mathbb{N}$ with $\#\mathcal{N}(x) \ge (1-\theta)k$.

(4) For every $x \in \operatorname{supp}(\mu) \setminus \mathcal{E}$, every affine hyperplane $W < \mathbb{R}^d$ and every $\ell \in \mathcal{N}(x)$, we have

$$\mu(W^{(\varepsilon 2^{-r\ell})} \cap B(x, 2^{-r\ell})) \leq (\varphi(\theta) + C(\theta)\varphi(\varepsilon))\mu(B(x, 2^{-r\ell})), \tag{11.1}$$

where $W^{(\rho)}$ and $B(x,\rho)$ denote the ρ-neighborhood of W and the ρ-ball around x respectively for any $\rho > 0$.

We say μ is affinely non-concentrated at almost every scale when λ, φ and C are understood from context. We say μ is *affinely non-concentrated at almost every scale up to scale* k_0 if μ satisfies the above conditions only for $k \leq k_0$.

This definition says that μ sees strong non-concentration near hyperplanes happening at nearly all scales outside of a small exceptional set, however the size of the exceptional set is allowed to depend on the strength and frequency of non-concentration.

Remark 11.2. Definition 11.1 is the non-concentration property we are able to verify for projections of the measures μ_x^u appearing in the proof of Theorem 9.2; cf. (9.70) . For purposes of following the arguments in this section however, there is no harm in considering the example $\lambda(x) = \beta x$ for some $\beta > 0$ and the stronger bound

$$\mu(W^{(\varepsilon 2^{-r\ell})} \cap B(x, 2^{-r\ell})) \leq C(\theta)\varphi(\varepsilon)\mu(B(x, 2^{-r\ell})),$$

in place of (11.1). In fact, the above bound holds for the measures μ_x^u themselves as can be deduced from the proof of Corollary 12.2.

For $k \in \mathbb{N}$, let

$$\Lambda_k := 2^{-k}\mathbb{Z}^d,$$

and let $\mathcal{D}_k$ be the dyadic partition of $\mathbb{R}^d$ given by translates of $2^{-k}[0,1)^d$ by Λ_k. For $x \in \mathbb{R}^d$, we denote by $\mathcal{D}_k(x)$ the unique element of $\mathcal{D}_k$ containing x. For a Borel probability measure ν, we define $\nu_k \in \operatorname{Prob}(\Lambda_k)$ to be the scale-k discretization of ν, i.e.

$$\nu_k = \sum_{\lambda \in \Lambda_k} \nu(\mathcal{D}_k(\lambda))\delta_\lambda. \tag{11.2}$$

For any $\mu \in \operatorname{Prob}(\Lambda_k)$ and $0 < q < \infty$, we set

$$\|\mu\|_q := \left(\sum_{\lambda \in \Lambda_k} \mu(\lambda)^q \right)^{1/q}.$$

The *convolution* $\mu * \nu$ of two probability measures μ and ν on $\mathbb{R}^d$ is defined by

$$\mu * \nu(A) = \int\int 1_A(x+y)\, d\mu(x)\, d\nu(y),$$

for all Borel sets $A \subseteq \mathbb{R}^d$.

The following lemma allows us to pass between measures and their discretizations.

Lemma 11.3. *Let μ and ν be Borel probability measures on $\mathbb{R}^d$.*

(1) If μ is (λ,φ,C)-affinely non-concentrated at almost every scale, then, there is $a \geq 1$ such that for every $k \in \mathbb{N}$, μ_k is $(\lambda, a\varphi, aC)$-affinely non-concentrated up to scale k.
(2) For all $q > 1$ and $k \in \mathbb{N}$, we have $\|(\mu * \nu)_k\|_q \asymp_{q,d} \|\mu_k * \nu_k\|_q$.

The lemma is a consequence of the fact that a ball of radius r with $2^{-k-1} < r \leq 2^{-k}$, $k \in \mathbb{Z}$, can be covered with $O_d(1)$ elements of $\mathcal{D}_k$ and we omit the details.

With this notation, we can now state our quantitative results.

Theorem 11.4. *Let λ, φ and C be given. For every $\varepsilon > 0$, there exist natural numbers n and k_0 such that for every (λ, φ, C)-affinely non-concentrated Borel probability measure μ supported inside a ball of radius 2^m around the origin in $\mathbb{R}^d$ and for every $k \geq k_0$, we have*

$$\|\mu_k^{*n}\|_2^2 \ll 2^{2dm(n-1)-(d-\varepsilon)k},$$

with implicit constant depending only on d and the non-concentration parameters of μ. In particular, for all $P \in \mathcal{D}_k$, we have

$$\mu_k^{*(n+1)}(P) \ll_d 2^{dmn-(d-\varepsilon)k/2}.$$

The following is a more precise version of Corollary 1.8.

Corollary 11.5. *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$ such that μ is affinely non-concentrated at almost every scale. Then, for every $\varepsilon > 0$, there is $\lambda > 0$ such that for every $T \geq 1$, the set*

$$\left\{ w \in \mathbb{R}^d : \|w\| \leq T \text{ and } |\hat{\mu}(w)| \geq T^{-\lambda} \right\}$$

can be covered by $O_\varepsilon(T^\varepsilon)$ balls of radius 1, where $\hat{\mu}$ denotes the Fourier transform of μ. The implicit constant depends only on ε and on the diameter of the support of μ and its non-concentration parameters.

11.2. **Asymmetric Balog-Szemerédi-Gowers Lemma.** The following is the asymmetric version of the Balog-Szemrédi-Gowers Lemma due to Tao and Vu. Throughout the section, for a finite set $A \subset \mathbb{R}^d$, we denote by $|A|$ its cardinality.

Theorem 11.6 (Corollary 2.36, [TV06])**.** *Let $A, B \subset \mathbb{R}^d$ be finite sets such that $\|1_A * 1_B\|_2^2 \geq 2\alpha |A||B|^2$ and $|A| \leq L|B|$ for some $0 < \alpha \leq 1$ and $L \geq 1$. Let $\varepsilon' > 0$ be given. Then, there exist sets $A' \subseteq A$ and $B' \subseteq B$ such that*

(1) A' and B' are sufficiently dense: $|A'| \gg_{\varepsilon'} \alpha^{O_{\varepsilon'}(1)} L^{-\varepsilon'} |A|$ and $|B'| \gg_{\varepsilon'} \alpha^{O_{\varepsilon'}(1)} L^{-\varepsilon'} |B|$.
(2) A' is approximately invariant by B': $|A' + B'| \ll_{\varepsilon'} \alpha^{-O_{\varepsilon'}(1)} L^{\varepsilon'} |A'|$.

Remark 11.7. The quoted result is stated in terms of the additive energy $E(A, B)$ in *loc. cit.*, which is nothing but $\|1_A * 1_B\|_2^2$.

In order to be able to bring our affine non-concentration hypothesis into play, we will need to convert the approximate additive invariance provided by the Balog-Szemerédi-Gowers Lemma into exact additive obstructions to flattening under convolution, i.e. affine subspaces. Our key tool for this step is Hochman's inverse entropy theorem for convolutions of measures, stated in the next subsection.

11.3. **Hochman's inverse theorem for entropy.** We need some notation before stating the result. For a Borel probability measure ν on N^+, the entropy $H_k(\nu)$ of ν at scale k is defined to be

$$H_k(\nu) := -\frac{1}{k} \sum_{P \in \mathcal{D}_k} \nu(P) \log_2 \nu(P).$$

By concavity of log and Jensen's inequality, we have the following elementary inequality

$$H_k(\nu) \leq \frac{\log_2 \# \{P \in \mathcal{D}_k : \nu(P) \neq 0\}}{k}. \tag{11.3}$$

It also follows from Jensen's inequality that the above inequality becomes equality if and only if ν gives equal weights to the elements P of $\mathcal{D}_k$ with $\nu(P) \neq 0$.

Given a Borel probability measure ν on $\mathbb{R}^d$ and $z \in \mathbb{R}^d$ with $\nu(\mathcal{D}_k(z)) > 0$, we define the component measure $\nu^{z,k}$ by

$$\int f \, d\nu^{z,k} := \frac{1}{\nu(\mathcal{D}_k(z))} \int_{\mathcal{D}_k(z)} f(T(y)) \, d\nu(y),$$

where $T : \mathcal{D}_k(z) \to \mathcal{D}_0(\mathbf{0})$ is the map given by composing scaling by 2^k with translation by the element of Λ_k sending $\mathcal{D}_k(z)$ to $\mathcal{D}_k(0)$.

Given a Borel subset $\mathcal{P} \subseteq \mathrm{Prob}(N^+)$ and $k \in \mathbb{N}$, we define

$$\mathbb{P}_{0\le i\le k}(\nu^{z,i} \in \mathcal{P}) := \frac{1}{k+1} \sum_{i=0}^{k} \int 1_{\mathcal{P}}(\nu^{z,i}) \, d\nu(z).$$

Given a linear subspace $0 \le V \le \mathbb{R}^d$, $\varepsilon > 0$ and a probability measure ν, we say that ν is (V,ε)-*concentrated* if there is a translate L of V such that $\nu(L^{(\varepsilon)}) > 1 - \varepsilon$. We say that ν is (V,ε,m)-*saturated* for a given $m \in \mathbb{N}$ if

$$H_m(\nu) \ge H_m(\pi_W \nu) + \dim V - \varepsilon, \tag{11.4}$$

where $W = V^\perp$ and $\pi_W \nu$ is the pushforward of ν under the orthogonal projection to W.

Theorem 11.8 (Theorem 2.8, [Hoc15])**.** *For every $\varepsilon, R > 0$ and $r \in \mathbb{N}$, there are $\sigma > 0$ and $m_0, k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ and all Borel probability measures ν and μ on $[-R,R]^d$ satisfying*

$$H_{kr}(\mu * \nu) < H_{kr}(\nu) + \sigma,$$

there exists a sequence of subspaces $0 \le V_0, \ldots, V_k \le \mathbb{R}^d$ such that

$$\mathbb{P}_{0\le i\le k}\left(\begin{array}{c} \mu^{x,ir} \text{ is } (V_i,\varepsilon) - \text{concentrated and} \\ \nu^{x,ir} \text{ is } (V_i,\varepsilon,m_0) - \text{saturated} \end{array} \right) > 1 - \varepsilon.$$

Remark 11.9. Theorem 11.8 is stated in [Hoc15] in the case $r = 1$. However, the extension to general step-size is rather routine since it roughly corresponds to working in base 2^r in place of base 2.

11.4. **Flattening of discretized measures.** The following quantitative result is the main ingredient in the proof of Theorem 11.4.

Proposition 11.10. *Let positive functions λ, φ, and C on $(0,1]$ be given. Then, for every $0 < \gamma < 1$, there exist $\eta > 0$ and $k_1, r \in \mathbb{N}$, depending on γ, λ, C, and φ, such that for all integers $k \ge k_1$ the following holds. Let μ and ν be arbitrary probability measures supported on $2^{-kr}\mathbb{Z}^d$ such that μ is (λ,φ,C)-affinely non-concentrated up to scale k and*

$$\|\nu\|_2^2 > 2^{-(1-\gamma)dkr+2dn}, \tag{11.5}$$

then

$$\|\mu * \nu\|_2 \le 2^{dm-\eta kr} \|\nu\|_2 , \tag{11.6}$$

where $m, n \in \mathbb{N}$ are such that μ is supported inside the 2^m-ball around the origin and ν is supported inside the 2^n-ball.

This proposition says that the convolution of an arbitrary measure ν with a non-concentrated measure causes ν to "spread out", i.e. leads to a quantitative reduction in the ℓ^2 norm of ν, unless $\|\nu\|_2$ is already very close to 0.

Lemma 11.11 (Effect of scaling the support). *Let μ and ν be Borel probability measures on $2^{-k}\mathbb{Z}^d$ for some k such that their supports are contained inside balls of radius 2^m and 2^n in $\mathbb{R}^d$ respectively for some $m, n \in \mathbb{N}$. Let $\tilde{\mu}$ (resp. $\tilde{\nu}$) be the measure obtained from μ (resp. ν) by composing it with the scaling map sending the 2^m-ball (resp. 2^n-ball) onto the 1-ball. Suppose that $\|\tilde{\mu} * \tilde{\nu}\|_2 \leq \delta \|\tilde{\nu}\|_2$ for some $\delta > 0$. Then, $\|\mu * \nu\|_2 \leq 2^{dm}\delta \|\nu\|_2$.*

Proof. Note that we may regard any measure τ on $2^{-k}\mathbb{Z}^d$ as an absolutely continuous measure to Lebesgue on $\mathbb{R}^d$ with density function, denoted τ, that is constant on $\mathcal{D}_k(\lambda)$ and equal to $\tau(\mathcal{D}_k(\lambda))2^{dk}$ for every $\lambda \in 2^{-k}\mathbb{Z}^d$. In particular, $\|\tau\|_2$ is equal to $\|\tau\|_{L^2(\mathbb{R}^d)}$ in this notation. We apply this observation to all the measures appearing in the lemma. The lemma now follows upon observing that for every $P \in \mathcal{D}_k$, a change of variables shows that $\mu * \nu(P) = 2^{d(m+n)}\tilde{\mu} * \tilde{\nu}(P)$ □

In light of this lemma, we may assume in the proof of Proposition 11.10 that

$$m = 0 = n, \text{ i.e. } \mu \text{ and } \nu \text{ are supported on } 2^{-k}\mathbb{Z}^d \cap [0,1)^d. \tag{11.7}$$

The remainder of this subsection is dedicated to the proof of Proposition 11.10. Let $\gamma > 0$ and $\eta > 0$ be small parameters and $r, k \in \mathbb{N}$ be a large integer to be specified over the course of the proof. We frequently assume that γ is sufficiently small so that various properties hold and the values of η, r and k will depend only γ and the non-concentration parameters. Suppose towards a contradiction that (11.5) holds but (11.6) fails.

11.4.1. *From measures to sets.* We first translate the failure of (11.6) from measures to indicator functions of certain sets using standard arguments. This allows us to apply the Balog-Szemerédi-Gowers Lemma.

Lemma 11.12 (Lemma 3.3, [Shm19]). *For every $\eta > 0$, the following holds for all large enough ℓ. Suppose that μ and ν are probability measures supported on $\Lambda_\ell \cap [0,1)^d$ such that $\|\mu * \nu\|_2$ is at least $2^{-\eta\ell}\|\nu\|_2$. Then, there exist $j, j' \leq 4\eta\ell$ such that the sets*

$$A := \left\{x \in \Lambda_\ell : 2^{-j-1}\|\nu\|_2^2 < \nu(x) \leq 2^{-j}\|\nu\|_2^2\right\}, \tag{11.8}$$

$$B := \left\{x \in \Lambda_\ell : 2^{-j'-1-d\ell} < \mu(x) \leq 2^{-j'-d\ell}\right\} \tag{11.9}$$

satisfy

(1) $\|1_A * 1_B\|_2^2 \geq 2^{-4\eta\ell}|A||B|^2$,
(2) $\|\nu|_A\|_2 \geq 2^{-2\eta\ell}\|\nu\|_2$, *and*
(3) $\mu(B) \geq 2^{-2\eta\ell}$.

Remark 11.13. The above lemma was proved in [Shm19] for measures on $\mathbb{R}$, where $d = 1$, however the short argument, based on the pigeonhole principle, goes through in the general case where one uses that

$$\#\left\{P \in \mathcal{D}_\ell : P \subset [0,1)^d\right\} = 2^{d\ell}, \qquad \forall \ell \geq 1, \tag{11.10}$$

to obtain the the bounds in the definition of the set B in place of the analogous 1-dimensional count used in *loc. cit.*

Let A and B be as in Lemma 11.12, applied with $\ell = kr$. Taking η small enough, we note that (11.5) and the definition of A imply that

$$|A| \leq 2^{4\eta kr+1+(1-\gamma)dkr} \leq 2^{(1-\gamma/2)dkr+1}. \tag{11.11}$$

11.4.2. *From ℓ^2-concentration to entropy concentration.* Lemma 11.12 enables us to apply Theorem 11.6 with $\alpha = 2^{-4\eta kr-1}$, $L = \max\{1, |A|/|B|\}$, and

$$0 < \varepsilon' < 1$$

a parameter to be chosen small enough depending on ε. Let $A' \subseteq A$ and $B' \subseteq B$ be the sets provided by Theorem 11.6. Let ν' and μ' be the uniform probability measures supported on A' and B' respectively. Combining the above estimate with (11.3), we obtain

$$H_{kr}(\mu' * \nu') \leq \frac{\log_2 |A' + B'|}{kr} \leq \frac{\log_2 |A'|}{kr} + O_{\varepsilon'}(\eta) + \log_2 L^{\varepsilon'}/kr.$$

Since ν' is the uniform measure on A', the remark following (11.3) thus implies that

$$H_{kr}(\mu' * \nu') \leq H_{kr}(\nu') + O_{\varepsilon'}(\eta) + \log_2 L^{\varepsilon'}/kr.$$

By (11.11), we have $\log_2 L^{\varepsilon'} \leq \varepsilon' \log_2 |A| \leq \varepsilon'((1-\gamma/2)dkr + 1)$.

Let

$$\varepsilon = 2^{-r},$$

and let $\sigma > 0$ and $k_0 \in \mathbb{N}$ be the parameters provided by Theorem 11.8 applied with ε, r and with $R = 1$. We shall assume that k is chosen to be larger than k_0. Hence, taking ε' small enough (depending on σ) and η small enough (depending on ε' and σ), we obtain

$$H_{kr}(\mu' * \nu') < H_{kr}(\nu') + \sigma. \tag{11.12}$$

We show that the conclusion of Theorem 11.8 is incompatible the non-concentration properties of the measure μ. Let $V_0, \dots, V_k$ be the subspaces provided by Theorem 11.8 and

$$\mathcal{S} = \left\{0 \leq i \leq k : V_i = \mathbb{R}^d\right\}.$$

We begin by showing that a significant proportion of the $V_i' s$ are proper subspaces. Intuitively, being $\mathbb{R}^d$-saturated on most scales means the measure ν is close to being absolutely continuous to Lebesgue on $\mathbb{R}^d$, in which case its ℓ^2-norm would be very close to 2^{-dk}. This would contradict (11.5).

Lemma 11.14. *If ε is chosen small enough and k large enough depending on γ, then $\#\mathcal{S} < (1-\gamma/10)k$.*

Proof. Let $\gamma_1 = \gamma/10$ and suppose that $\#\mathcal{S} \geq (1-\gamma_1)k$. Then, Theorem 11.8 and the definition of saturation (cf. (11.4)) imply that

$$\frac{1}{k+1}\sum_{i=0}^{k} \int H_{m_0}((\nu')^{z,ir})\, d\nu'(z) \geq (1-\gamma_1)(1-\varepsilon)(d-\varepsilon) = (1-\gamma_1)d - O(\varepsilon).$$

By [Hoc14, Lemma 3.4][16], this yields the following estimate on $H_{kr}(\nu')$:

$$H_{kr}(\nu') \geq (1-\gamma_1)d - O(\varepsilon) - O_r\left(\frac{m_0}{k}\right) \geq (1-\gamma_1)d - O(\varepsilon),$$

where the second inequality holds whenever k is large enough depending on r and m_0. Moreover, by the remark following (11.3), we have $H_{kr}(\nu') = \log_2 |A'|/kr \leq \log_2 |A|/kr$. Hence, we obtain that $|A| \geq 2^{((1-\gamma_1)d - O(\varepsilon))kr}$. This contradicts (11.11) when ε is small enough compared to γ. $\square$

[16]The cited result is stated for step-size $r = 1$, however its short proof extends to work for any r with minor changes.

11.4.3. *A contradiction to concentration.* Roughly speaking, our strategy is as follows. Armed with Lemma 11.14, we show that the concentration provided by Theorem 11.8 together with the non-concentration property of μ imply that B' must have a very small measure. To get a contradiction, we begin by deriving a lower bound on the measure of B' with respect to our original measure μ (not μ'). Recall the parameter ε' chosen above (11.12).

Lemma 11.15. *If η is chosen sufficiently small depending on ε', then for all sufficiently large k,*

$$\mu(B') \geq 2^{-2d\varepsilon' kr}.$$

Proof. Recall that the set B was defined in (11.8) and $B' \subseteq B$ is provided by Theorem 11.6 with $\alpha = 2^{-4\eta kr-1}$ and $L = \max\{1, |A|/|B|\}$. We calculate using Lemma 11.12 and Theorem 11.6:

$$\mu(B') = \sum_{u\in B'} \mu(u) \geq 2^{-j'-dkr-1}|B'| \gg_{\varepsilon'} 2^{-j'-dkr}2^{-O_{\varepsilon'}(\eta kr)}L^{-\varepsilon'}|B| \geq 2^{-j'-dkr}2^{-O_{\varepsilon'}(\eta kr)}|B||A|^{-\varepsilon'}.$$

By (11.11), we have that $|A|^{\varepsilon'} \ll 2^{d\varepsilon' kr}$. Moreover, Lemma 11.12 implies that

$$2^{-2\eta kr} \leq \mu(B) \leq 2^{-j'-dkr}|B|.$$

The lemma then follows once η is chosen sufficiently small depending on ε'. $\square$

Next, we define the following set of scales where the concentration provided by Theorem 11.8 gives non-trivial information:

$$\mathcal{C} := \{0,\dots,k\} \setminus \mathcal{S} = \left\{0 \leq i \leq k : V_i \lneq \mathbb{R}^d\right\}.$$

By Lemma 11.14, we know that

$$|\mathcal{C}| \geq \gamma_1 k, \qquad \gamma_1 = \gamma/10. \tag{11.13}$$

Our next goal is to transfer the concentration information provided by Theorem 11.8 for μ' to the measure μ. To do so, we convert the probabilistic concentration provided in the theorem into geometric containment into hyperplane neighborhoods.

Recall that $\{\mathcal{D}_\ell : \ell \in \mathbb{N}\}$ is a refining sequence of dyadic partitions of $\mathbb{R}^d$ and $\Lambda_\ell = 2^{-\ell}\mathbb{Z}^d$. For $i \in \mathcal{C}$ and $w \in \Lambda_{ir}$, let $V_w = V_i + w$ and set

$$\rho_i = 2^{-ir}, \qquad Q_i = \bigcup_{w\in\Lambda_{ir}} V_w^{(\varepsilon\rho_i)} \cap \mathcal{D}_{ir}(w).$$

For $x \in \mathbb{R}^d$, we set

$$\mathcal{C}(x) = \{i \in \mathcal{C} : x \in Q_i\}.$$

In particular, for $x \in B'$, $\mathcal{C}(x)$ consists of scales at which x witnesses the concentration of B'.

Lemma 11.16. *If k is large enough, then the subset*

$$B'' = \left\{x \in B' : |\mathcal{C}(x)| \geq |\mathcal{C}|/2\right\} \tag{11.14}$$

satisfies

$$\mu(B'') \geq 2^{-3d\varepsilon' kr}.$$

Proof. Let $E = B' \setminus B''$. First, we give an upper bound on the measure of E with respect to μ'. Let $Q_i^c = \mathbb{R}^d \setminus Q_i$. Then, the concentration provided by Theorem 11.8 implies that

$$\int \sum_{i\in\mathcal{C}} \mathbb{1}_{Q_i^c}(x)\, d\mu'(x) < \varepsilon|\mathcal{C}|.$$

On the other hand, we have

$$\int \sum_{i\in\mathcal{C}} \mathbb{1}_{Q_i^c}(x)\, d\mu'(x) \geq \int_E \sum_{i\in\mathcal{C}} \mathbb{1}_{Q_i^c}(x)\, d\mu'(x) \geq |\mathcal{C}|\mu'(E)/2.$$

Recalling that μ' is the uniform measure on B', these inequalities show that $|B''| \geq (1-2\varepsilon)|B'|$. Hence, the assertion of the lemma follows from Lemma 11.12 by the same argument as in the proof of Lemma 11.15. $\square$

Parameter	Definition
ε	2^{-r}
ρ_i	2^{-ir}
γ_1	$\gamma/10$
γ_2	small parameter depending on γ
γ_3	$\gamma_1/2-\gamma_2$
ε'	small parameter depending on ε and γ
η	small parameter depending on ε'
m	$\lceil \gamma_3 k\rceil$

TABLE 2. Summary of parameters chosen in the proof of Proposition 11.10.

Note that the scales $\mathcal{C}(x)$ may vary with x. Similarly, the scales at which our affine non-concentration hypothesis holds also vary from point to point. To arrive at a contradiction, we partition B'' into sets where there is a fixed subset of scales of $\mathcal{C}$ at which the aforementioned phenomena hold simultaneously and find an upper bound on the measure of each piece separately.

Let B'' be as in Lemma 11.16. Let $0<\gamma_2<\gamma_1/2$ be a small parameter to be chosen depending only on γ. Recall the notation in Definition 11.1. Let $\mathcal{E}$ be the exceptional set provided by this definition for our choices of k, r, and with $\theta=\gamma_2$ and 3ε in place of ε. Let $B'''=B''\setminus\mathcal{E}$. By taking $r\geq C(\gamma_2)$, then our non-concentration hypothesis implies $\mu(\mathcal{E})\leq C(\gamma_2)2^{-\lambda(\gamma_2)k}$. Hence, taking ε' small enough depending on r and $\lambda(\gamma_2)$, we can ensure that

$$\mu(B''') \geq 2^{-3d\varepsilon' kr} - C(\gamma_2)2^{-\lambda(\gamma_2)k} \geq 2^{-k\sqrt{\varepsilon'}}, \tag{11.15}$$

for all large enough k. For $x\in B'''$, we let

$$\mathcal{G}(x) = \mathcal{C}(x)\cap\mathcal{N}(x).$$

By (11.13) and the definition of B'' in (11.14), setting $\gamma_3=\gamma_1/2-\gamma_2$, we also have

$$|\mathcal{G}(x)| \geq \gamma_3 k, \qquad \forall x\in B'''.$$

Given $\varpi\subseteq\{0,\dots,k\}$, we let

$$B'''_\varpi := \left\{x\in B''' : \varpi\subseteq\mathcal{G}(x)\right\}.$$

Then, the sets $\{B'''_\varpi : |\varpi| = \lceil\gamma_3 k\rceil\}$ provide a cover of B'''. Hence, we have that

$$\mu(B''') \leq \sum_{|\varpi|=\lceil\gamma_3 k\rceil} \mu(B'''_\varpi). \tag{11.16}$$

Fix a set $\varpi\subset[0,k]\cap\mathbb{N}$ as above for which $B'''_\varpi\neq\emptyset$ and denote by $\ell_1<\ell_2<\cdots<\ell_m$ its elements. In particular, we have

$$m := |\varpi| = \lceil\gamma_3 k\rceil. \tag{11.17}$$

To simplify notation, we set

$$F = B'''_\varpi.$$

We recall that $\Lambda_\ell(F)$ denotes those elements $v \in \Lambda_\ell = 2^{-\ell}\mathbb{Z}^d$ for which the corresponding cells $\mathcal{D}_\ell(v)$ intersect F non-trivially. Hence, we have the following basic estimate that will be allow us to proceed by induction on scales:

$$\mu(F) \leq \sum_{v\in\Lambda_{r\ell_m}(F)} \mu(\mathcal{D}_{r\ell_m}(v)) = \sum_{w\in\Lambda_{r\ell_{m-1}}(F)} \sum_{\substack{v\in\Lambda_{r\ell_m}(F)\\ \mathcal{D}_{r\ell_m}(v)\subset\mathcal{D}_{r\ell_{m-1}}(w)}} \mu(\mathcal{D}_{r\ell_m}(v)). \tag{11.18}$$

To proceed, let us summarize what our choices above entail: for every $1 \leq i \leq m$, we have

(1) For every $w \in \Lambda_{r\ell_i}(F)$, the affine subspace $V_w = V_{\ell_i} + w$ satisfies
$$F \cap \mathcal{D}_{r\ell_i}(w) \subseteq F \cap \mathcal{D}_{r\ell_i}(w) \cap V_w^{(\varepsilon\rho_{\ell_i})}.$$
(2) $V_{\ell_i} \neq \mathbb{R}^d$.
(3) F is disjoint from the exceptional set for non-concentration, i.e. $F \cap \mathcal{E}(k, 3\varepsilon, \theta) = \emptyset$.
(4) ℓ_i is a good scale for non-concentration at every point in F, i.e. $\ell_i \in \mathcal{N}(x)$ for all $x \in F$.

Observe that if $V_w^{(\varepsilon\rho_{\ell_i})}$ intersects a box $\mathcal{D}_{r\ell_{i+1}}(v)$ non-trivially, then since $\mathcal{D}_{r\ell_{i+1}}(v)$ has diameter at most $2\varepsilon\rho_{\ell_i} = 2\varepsilon 2^{-r\ell_i}$, we obtain

$$\mathcal{D}_{r\ell_{i+1}}(v) \subseteq V_w^{(3\varepsilon\rho_{\ell_i})}. \tag{11.19}$$

This containment, along with Item (1), imply that for every $1 \leq i < m$ and $w \in \Lambda_{r\ell_i}(F)$, we have that

$$\sum_{\substack{v\in\Lambda_{r\ell_{i+1}}(F)\\ \mathcal{D}_{r\ell_{i+1}}(v)\subset\mathcal{D}_{r\ell_i}(w)}} \mu(\mathcal{D}_{r\ell_{i+1}}(v)) \leq \mu(V_w^{(3\varepsilon\rho_i)} \cap \mathcal{D}_{r\ell_i}(w)).$$

Recall that μ is non-concentrated near affine subspaces in the sense of Definition 11.1. Hence, for all i and $w \in \Lambda_{r\ell_i}(F)$, Items, (2), (3) and (4) along with our non-concentration hypothesis imply that

$$\mu\left(V_w^{(3\varepsilon\rho_{r\ell_i})} \cap \mathcal{D}_{r\ell_i}(w)\right) \leq (\varphi(\gamma_2) + C(\gamma_2)\varphi(3\varepsilon))\,\mu\left(\mathcal{D}_{r\ell_i}(w)\right). \tag{11.20}$$

Combining these inequalities for $i = m-1$ with (11.18), we obtain

$$\mu(F) \leq (\varphi(\gamma_2) + C(\gamma_2)\varphi(3\varepsilon)) \sum_{w\in\Lambda_{r\ell_{m-1}}(F)} \mu(\mathcal{D}_{r\ell_{m-1}}(w)).$$

Hence, by induction, we obtain

$$\mu(F) \leq (\varphi(\gamma_2) + C(\gamma_2)\varphi(3\varepsilon))^m.$$

Recall that $F = B'''_\varpi$ and that ϖ is a subset of $\{0, \ldots, k\}$ with cardinality m (cf. (11.17)). In view of the elementary estimate $\binom{k}{m} \leq (ke/m)^m$, there are at most $(e/\gamma_3)^m$ summands in (11.16). Hence, the above estimate, combined with (11.16), implies that

$$\mu(B''') \leq \left(\frac{e\varphi(\gamma_2) + eC(\gamma_2)\varphi(3\varepsilon)}{\gamma_3}\right)^{\gamma_3 k}.$$

On the other hand, by (11.15), we have the lower bound $\mu(B''') \geq 2^{-\sqrt{\varepsilon'}k}$. In particular, we arrive at the inequality

$$2^{-\sqrt{\varepsilon'}} \leq \left(\frac{e\varphi(\gamma_2) + eC(\gamma_2)\varphi(3\varepsilon)}{\gamma_3}\right)^{\gamma_3}.$$

Recall that $\gamma_1 = \gamma/10$, γ_2 is to be chosen smaller than $\gamma_1/2$, and $\gamma_3 = \gamma_1/2 - \gamma_2$. Hence, by choosing γ_2 first to be sufficiently small relative to γ_1, then choosing ε very small, depending on γ_2, we can make the right side of the above inequality at most $1/2$ say. This gives a contradiction since $\varepsilon' < 1$.

11.5. **Proof of Theorem 11.4.** Let $\eta > 0$ and $r, k_1 \in \mathbb{N}$ be the parameter provided by Proposition 11.10. Let $n \in \mathbb{N}$ be the smallest integer such that $(n-1)\eta > (d-\varepsilon)/2$. Note that by Young's inequality, for all $a, b, k \in \mathbb{N}$, we have that

$$\left\| \mu_k^{*a} * \mu_k^{*b} \right\|_2 \leq \| \mu_k^{*a} \|_1 \left\| \mu_k^{*b} \right\|_2 = \left\| \mu_k^{*b} \right\|_2 .$$

We first observe that it suffices to prove the first assertion for multiples of r. Indeed, given any probability measure ν, $k \in \mathbb{N}$, $0 \leq s < r$, we have

$$\sum_{P \in \mathcal{D}_{kr+s}} \nu(P)^2 = \sum_{Q \in \mathcal{D}_{kr}} \sum_{P \in \mathcal{D}_{kr+s}, P \subseteq Q} \nu(P)^2 \leq 2^{dr} \sum_{Q \in \mathcal{D}_{kr}} \nu(Q)^2.$$

Let $k \geq k_1$ be given and suppose that

$$\left\| \mu_{kr}^{*\ell} \right\|_2^2 \leq 2^{2\ell dm - (d-\varepsilon)kr}, \tag{11.21}$$

for some $\ell \in \mathbb{N}$ with $1 \leq \ell \leq n$. It follows that $\| \mu_{kr}^{*n} \|_2^2 \leq 2^{2ndm - (d-\varepsilon)kr}$ as desired. Now, suppose that (11.21) fails for all $1 \leq \ell \leq n$. Then, applying Proposition 11.10 $(n-1)$-times by induction, we see that

$$\| \mu_{kr}^{*n} \|_2 \leq 2^{(n-1)(dm - \eta kr)} \| \mu_{kr} \|_2 \leq 2^{(n-1)(dm - \eta kr)},$$

where the second inequality follows since $\| \mu_{kr} \|_2 \leq 1$. On the other hand, failure of (11.21) for $\ell = n$ implies that

$$ndm - (d-\varepsilon)kr/2 < (n-1)(dm - \eta kr).$$

This gives a contradiction to our choice of n, thus proving the first assertion. The (short) deduction of the second assertion from the first can be found for instance in [MS18, Proof of Lemma 5.2].

11.6. **Proof of Theorem 1.6, Corollary 1.8, and Corollary 11.5 from Theorem 11.4.** Note that being uniformly affinely non-concentration immediately implies that μ is affinely non-concentrated at almost every (in fact at every) scale with an empty exceptional set. Hence, the second assertion of Theorem 11.4 immediately implies that $\dim_\infty \mu^{*n}$ tends to d as $n \to \infty$. The same holds for $\dim_q \mu^{*n}$ due to the inequality $\dim_q \mu \geq \dim_\infty \mu$ for all $q > 1$. Finally, the first assertion of Theorem 1.6 follows readily from Proposition 11.10; cf. [RS20, Proof of Theorem 1.1] for details of this deduction.

Corollaries 1.8 and 11.5 follows from Theorem 11.4 via the well-known relationship between L^2-dimension and Fourier transform. Namely, by [FNW02, Proof of Claim 2.8], we have[17]

$$\int_{\|\xi\| \leq 1/r} |\hat{\mu}(\xi)|^2 \, d\xi \ll_d r^{-2d} \int \mu(B(x,r))^2 \, dx.$$

for every $r > 0$ and any Borel probability measure μ on $\mathbb{R}^d$. Moreover, if $k \in \mathbb{N}$ is such that $2^{-(k+1)} < r \leq 2^{-k}$, then $B(x,r)$ can be covered by $O_d(1)$ elements of the partition $\mathcal{D}_k$. It follows that

$$\int \mu(B(x, 2^{-k}))^2 \, dx \ll_d 2^{dk} \sum_{P \in \mathcal{D}_k} \mu(P)^2 = 2^{dk} \| \mu_k \|_2^2 .$$

Hence, Corollary 11.5 follows from Theorem 11.4, Chebychev's inequality, and the fact that the Fourier transform is Lipschitz; i.e.

$$|\hat{\mu}(\xi_1) - \hat{\mu}(\xi_2)| \ll \| \xi_1 - \xi_2 \| .$$

[17]The reference [FNW02] proves this fact in the case $d = 1$, however the proof works equally well for $\mathbb{R}^d$ for any d.

11.7. **Polynomial affine non-concentration.** In this section, we show Theorem 11.4 implies quantitative non-concentration estimates near hyperplanes.

Theorem 11.17. *Suppose μ is a compactly supported Borel probability measure on $\mathbb{R}^d$ which is affinely non-concentrated at almost every scale. Then, there exist $\kappa > 0$ and $C \geq 1$, depending on the non-concentration parameters of μ and the diameter of its support, such that for all $\varepsilon > 0$ and all proper affine hyperplanes $W < \mathbb{R}^d$, we have that $\mu(W^{(\varepsilon)}) \leq C\varepsilon^\kappa$.*

We first need the following useful observation which translates polynomial non-concentration for self-convolution into a similar estimate for the original measure.

Lemma 11.18. *Let ν be a Borel probability measure, $\varepsilon, \alpha, C > 0$ be arbitrary constants, and $W < \mathbb{R}^d$ be a proper affine hyperplane. Assume that $\nu^{*2}(W^{(\varepsilon)}) \leq C\varepsilon^\alpha$. Then, $\nu(W^{(\varepsilon/2)}) \leq C\varepsilon^{\alpha/2}$.*

Proof. Note that the definition of convolution implies

$$\nu^{*2}(W^{(\varepsilon)}) = \int\int \mathbb{1}_{W^{(\varepsilon)}}(x+y)\, d\nu(x)\, d\nu(y) = \int \nu\left(W^{(\varepsilon)} - x\right)\, d\nu(x).$$

Hence, by Chebychev's inequality and our hypothesis on ν, the set

$$B = \left\{x \in \mathbb{R}^d : \nu\left(W^{(\varepsilon)} - x\right) > \varepsilon^{\alpha/2}\right\}$$

has ν measure at most $C\varepsilon^{\alpha/2}$. Hence, the conclusion of the lemma follows if $W^{(\varepsilon/2)}$ is contained inside B. Otherwise, let $x \in W^{(\varepsilon/2)} \setminus B$ and observe that $W^{(\varepsilon/2)}$ is contained inside $W^{(\varepsilon)} - x$. However, the latter set has ν measure at most $\varepsilon^{\alpha/2}$ since $x \notin B$. Hence, the lemma follows in this case as well. $\square$

We are now ready for the proof of Theorem 11.17.

Proof of Theorem 11.17. Let $m \in \mathbb{N}$ be such that the support of μ is contained in a ball of radius 2^m around the origin. By Theorem 11.4, we can find $n \in \mathbb{N}$ and $r_0 > 0$, depending only on the non-concentration parameters of μ, such that

$$\mu^{*2^n}(B(x,r)) \ll_d 2^{2^n dm} r^{d-1/2}, \tag{11.22}$$

for all $0 < r \leq r_0$ and all $x \in \mathbb{R}^d$.

Fix one such value of n once and for all. Let $\nu = \mu^{*2^n}$ and let $B \subset \mathbb{R}^d$ be a large ball containing the supports of μ^{*k} for all $0 \leq k \leq 2^n$. Now, let $0 < \varepsilon \leq 1$ and a proper affine hyperplane $W < \mathbb{R}^d$ be arbitrary. Then, note that $W^{(\varepsilon)} \cap B$ can be covered by $O_{B,d}(\varepsilon^{-(d-1)})$ balls of radius ε with multiplicity depending only on d. Then, (11.22) implies that $\nu(W^{(\varepsilon)}) \leq C'\varepsilon^{1/2}$ for a suitable constant $C' = C'(m,n,d) \geq 1$. By Lemma 11.18 and induction on n, this shows that $\mu(W^{(\varepsilon/2^n)}) \leq C'\varepsilon^\kappa$ for $\kappa = 2^{-n-1}$. Since $\varepsilon > 0$ was arbitrary, this completes the proof by taking $C = C'2^{\kappa n}$. $\square$

12. Non-concentration of Patterson-Sullivan Measures

In this section, we prove verify the non-concentration hypothesis in Corollary 11.5 for (projections of) the measures μ_x^u. This enables us to apply this corollary to prove Proposition 9.13 and Theorem 9.16 which are the remaining pieces in the proof of Theorem 9.2.

Let $\mathcal{L}'$ denote the collection of all affine hyperplanes of the Lie algebra $\mathfrak{n}^+$ and denote by $\mathcal{L}$ the set of images of elements of $\mathcal{L}'$ under the exponential map. For $\varepsilon > 0$ and $L \in \mathcal{L}$, let $L^{(\varepsilon)}$ be the ε-neighborhood of L. Recall that we fixed a choice of a Margulis function V in Remark 9.1 and define

$$t(\varepsilon) := \sup_{x \in N_1^- \Omega} \sup_{L \in \mathcal{L}} \frac{\mu_x^u(N_1^+ \cap L^{(\varepsilon)})}{V(x)\mu_x^u(N_1^+)}. \tag{12.1}$$

We also recall that Γ is a geometrically finite subgroup of $G = \mathrm{Isom}^+(\mathbb{H}^d_{\mathbb{K}})$.

Theorem 12.1. *Assume that Γ is Zariski-dense inside G. We have that $t(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

As a consequence, we verify the hypotheses of Corollary 11.5 for the measures appearing in Theorem 9.16. Namely, let $\pi : \mathfrak{n}^+ \to \mathfrak{n}^+_\alpha$ be the projection parallel to $\mathfrak{n}^+_{2\alpha}$. For $x \in N_1^- \Omega$, denote by ν_x the measure $\pi_* \mu_x^u \Big|_{N_1^+}$ normalized to be a probability measure.

Corollary 12.2. *For every $x \in N_1^- \Omega$, the measure ν_x is affinely non-concentrated at almost every scale in the sense of Definition 11.1 with parameters depending only on $V(x)$; cf. (12.6).*

12.1. **Proof of Theorem 12.1.** The key tool in our proof is the following result which is a consequence of the ergodicity of the geodesic flow. The case of real hyperbolic spaces of this result was known earlier in [FS90] by different methods.

Proposition 12.3 (Corollary 9.4, [ELO22])**.** *For all $x \in X$ and $L \in \mathcal{L}$, $\mu_x^u(L) = 0$.*

Theorem 12.1 follows from the above result and a compactness argument. Indeed, fix an arbitrary $\eta > 0$ and note that for all x with $V(x) > 1/\eta$, the inner supremum in the definition of $t(\varepsilon)$ is bounded above by η, for any choice of $\varepsilon > 0$. We now show that $t(\varepsilon) < \eta$ for all sufficiently small ε by restricting our attention to the bounded set of $x \in N_1^- \Omega$ where $V(x) \leq 1/\eta$. Suppose not and let $x_n \in N_1^- \Omega$, $L_n \in \mathcal{L}$, $\varepsilon_n > 0$ be sequences such that $V(x_n) \leq 1/\eta$, $\varepsilon_n \to 0$, and

$$\liminf_{n \to \infty} \frac{\mu^u_{x_n}(N_1^+ \cap L_n^{(\varepsilon_n)})}{\mu^u_{x_n}(N_1^+)} > 0. \tag{12.2}$$

Passing to a subsequence if necessary, we may assume $x_n \to y \in N_1^- \Omega$ and L_n converges to some $P \in \mathcal{L}$ (in the Hausdorff topology on compact sets). On the other hand, when x_n is sufficiently close to y, we can change variables using (2.3), (2.4), and (2.10) to get

$$\mu^u_{x_n}(N_1^+ \cap L_n^{(\varepsilon_n)}) = \int f_n J_n \, d\mu^u_y,$$

where J_n is the Jacobian of the change of variables and f_n is the indicator function of the image of $N_1^+ \cap L_n^{(\varepsilon_n)}$ under this change of variables. By Proposition 12.3, since L_n converges to L, f_n converges to 0 pointwise μ^u_y-almost everywhere. Additionally, J_n converges to 1 everywhere since x_n converges to y. Finally, $\mu^u_{x_n}(N_1^+)$ remains bounded away from 0 since x_n remain within a bounded set for all n. This gives a contradiction to (12.2) and concludes the proof.

12.2. **Non-concentration and proof of Corollary 12.2.** The main difficulty lies in carefully associating a set of good scales to a point in the support of the projection of μ_x^u. Our key tools are Theorem 7.9 and Theorem 12.1. To simplify notation, we let

$$\tilde{\mu}_x := \mu_x^u \Big|_{N_1^+}, \qquad \tilde{\nu}_x := \pi_* \tilde{\mu}_x,$$

where $\pi : \mathfrak{n}^+ \to \mathfrak{n}^+_\alpha$ is the projection parallel to $\mathfrak{n}^+_{2\alpha}$. In particular, $\nu_x = \tilde{\nu}_x / \mu^u_x(N_1^+)$. Let $0 < \theta, \varepsilon < 1$ be arbitrary. Let $H, r_0 = O_{\beta,\theta}(1)$ be the constants provided by Theorem 7.9 when applied with $\varepsilon = \beta\theta^2/4$ and let $r \geq r_0$. For $\ell \in \mathbb{N}$, let $t_\ell = r\ell \log 2$ and $\rho_\ell = 2^{-l}$. For $v \in \mathfrak{n}^+_\alpha$, let $\mathcal{L}_v := \pi^{-1}(v)$ and denote by $\mathcal{L}_v^{(r)}$ the r-neighborhood of $\mathcal{L}_v$. Define

$$\mathcal{E} = \left\{ v \in \mathfrak{n}^+_\alpha : \# \left\{ 1 \leq l \leq k : \tilde{\mu}_x \left(n \in \mathcal{L}_v^{(\rho_k)} : V(g_{t_\ell} n x) > H \right) > \theta \tilde{\mu}_x \left(\mathcal{L}_v^{(\rho_k)} \right) \right\} \geq \theta k \right\}.$$

Roughly speaking, $\mathcal{E}$ consists of vectors v for which a definite proportion points in the "strip" $\mathcal{L}_v^{(\rho_k)}$ above v spend a definite proportion of their time above height H. We show that this set satisfies the requirements of Definition 11.1.

To estimate the measure of $\mathcal{E}$, Let χ_H denote the indicator function of the set of points $z \in X$ with $V(z) > H$ and consider the following set

$$\tilde{\mathcal{E}} = \left\{ v \in \mathfrak{n}_\alpha^+ : \tilde{\mu}_x \left(n \in \mathcal{L}_v^{(\rho_k)} : \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx) \geq \theta^2 k/2 \right) > \theta^2 \tilde{\mu}_x \left(\mathcal{L}_v^{(\rho_k)} \right) /2 \right\}.$$

Roughly, the set $\tilde{\mathcal{E}}$ is defined by exchanging the sum over l and integration against $\tilde{\mu}_x$. We claim that

$$\mathcal{E} \subseteq \tilde{\mathcal{E}}. \tag{12.3}$$

To see this, let $v \in \mathcal{E}$ and set $\mu_l = \tilde{\mu}_x(n \in \mathcal{L}_v^{(\rho_k)} : V(g_{t_\ell} nx) > H)$. Then, we have that

$$\begin{aligned} \theta k \leq \# \left\{ 1 \leq \ell \leq k : \mu_l > \theta \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)} \right\} &< \frac{1}{\theta \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)})} \int_{\mathcal{L}_v^{(\rho_k)}} \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx)\, d\tilde{\mu}_x \\ &\leq \frac{k}{\theta \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)})} \left(\tilde{\mu}_x \left(n \in \mathcal{L}_v^{(\rho_k)} : \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx) \geq \theta^2 k/2 \right) + \theta^2 \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)})/2 \right). \end{aligned}$$

It follows that

$$\tilde{\mu}_x \left(n \in \mathcal{L}_v^{(\rho_k)} : \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx) \geq \theta^2 k/2 \right) > \theta^2 \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)})/2,$$

which in turn implies that $v \in \tilde{\mathcal{E}}$. This verifies (12.3).

We now turn to finding an upper bound on the measure of $\tilde{\mathcal{E}}$. First, we note that Fubini's Theorem and Theorem 7.9 imply

$$\begin{aligned} \int \exp \left(\frac{\beta}{2} \sum_{1 \leq l \leq k} \chi_H(g_{t_\ell} nx) \right) d\tilde{\mu}_x(n) &\leq \mu_x^u(N_1^+) + \int_1^\infty \tilde{\mu}_x \left(n : \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx) > \log t^{2/\beta} \right) dt \\ &\ll e^{\beta \theta^2 k/4} V(x) \mu_x^u(N_1^+) \left(1 + \int_1^\infty t^{-2}\, dt \right). \end{aligned}$$

To bound the integral on the left from below in terms of the measure of $\tilde{\mathcal{E}}$, let $\mathcal{B}_k$ denote a cover of $\mathfrak{n}_\alpha^+ \cong \mathbb{R}^d$ by balls of radius $\rho_k/3$ with multiplicity bounded only in terms of the dimension d. Then, we have

$$\tilde{\nu}_x(\tilde{\mathcal{E}}) \leq \sum_{B \in \mathcal{B}_k} \tilde{\nu}_x(B \cap \tilde{\mathcal{E}}) \leq \sum_{B \cap \tilde{\mathcal{E}} \neq \emptyset} \tilde{\nu}_x(B).$$

Now, given a ball $B \in \mathcal{B}_k$ which meets $\tilde{\mathcal{E}}$ and $v \in B \cap \tilde{\mathcal{E}}$, we note that the definition of $\tilde{\mathcal{E}}$ implies

$$\int_{\mathcal{L}_v^{(\rho_k)}} \exp \left(\frac{\beta}{2} \sum_{1 \leq \ell \leq k} \chi_H(g_{t_\ell} nx) \right) d\tilde{\mu}_x \geq \theta^2 e^{\beta \theta^2 k/2} \tilde{\mu}_x(\mathcal{L}_v^{(\rho_k)})/2.$$

Hence, since $\pi^{-1}(B) \subseteq \mathcal{L}_v^{(\rho_k)}$, (12.3) and the bounded multiplicity of $\mathcal{B}_k$ imply

$$\tilde{\nu}_x(\mathcal{E}) \leq \tilde{\nu}_x(\tilde{\mathcal{E}}) \ll e^{-\beta \theta^2 k/4} V(x) \mu_x^u(N_1^+)/\theta^2. \tag{12.4}$$

It remains to show that our desired non-concentration holds outside of $\mathcal{E}$. For $v \in \mathfrak{n}_\alpha^+$, define the set of scales $\mathcal{N}(v)$ as follows:

$$\mathcal{N}(v) = \left\{ 1 \leq \ell \leq k : \tilde{\mu}_x \left(n \in \mathcal{L}_v^{(\rho_k)} : V(g_{t_\ell} nx) > H \right) \leq \theta \tilde{\mu}_x \left(\mathcal{L}_v^{(\rho_k)} \right) \right\}.$$

Let $v \in \operatorname{supp}(\tilde{\nu}_x) \setminus \mathcal{E}$. By definition, we have $\#\mathcal{N}(v) \geq (1-\theta)k$. Let $\ell \in \mathcal{N}(v)$ and let $W_1 < \mathfrak{n}_\alpha^+$ be a proper hyperplane. Let $W = \pi^{-1}(W_1)$. Recall the function $t(\varepsilon)$ defined in (12.1). We wish to show that

$$\tilde{\mu}_x(W^{(\varepsilon\rho_\ell)} \cap \mathcal{L}_v^{(\rho_\ell)}) \ll (\theta + Ht(\varepsilon))\tilde{\mu}_x(\mathcal{L}_v^{(\rho_\ell)}). \tag{12.5}$$

Let $\{u_m\}$ denote a maximal $(\rho_\ell/2)$-separated subset of the set F consisting of points u in $\mathcal{L}_v^{(\rho_\ell)} \cap \operatorname{supp}(\tilde{\mu}_x)$ with $V(g_{t_\ell}ux) \leq H$. Then, $\left\{N_{\rho_\ell}^+ \cdot u_m\right\}$ is a cover of F with uniformly bounded multiplicity. Let $z_m = g_{r\ell\log 2}u_m x$ and $W_m = \operatorname{Ad}(g_{r\ell\log 2})(Wu_m^{-1})$. Denote by $\sum_m'$ the sum over those m for which the ball $N_{\rho_\ell}^+ \cdot u_m$ intersects the set $\mathcal{L}_v^{(\rho_\ell)} \cap \operatorname{supp}(\tilde{\mu}_x)$ non-trivially. Changing variables, we obtain

$$\begin{aligned}
\tilde{\mu}_x(W^{(\varepsilon\rho_\ell)} \cap \mathcal{L}_v^{(\rho_\ell)}) &\leq \tilde{\mu}_x(\mathcal{L}_v^{(\rho_\ell)} \setminus F) + 2^{-\delta r\ell} \sum_m{}' \mu_{z_m}^u(W_m^{(\varepsilon)} \cap N_1^+) \\
&\leq \theta\tilde{\mu}_x(\mathcal{L}_v^{(\rho_\ell)}) + t(\varepsilon)H2^{-\delta r\ell} \sum_m{}' \mu_{z_m}^u(N_1^+) \qquad \text{since } \ell \in \mathcal{N}(v) \\
&\ll \theta\tilde{\mu}_x(\mathcal{L}_v^{(\rho_\ell)}) + t(\varepsilon)H\mu_x^u(N_{1+\rho_\ell}^+ \cap \mathcal{L}_v^{(2\rho_\ell)}) \qquad \text{since } u_m \in F.
\end{aligned}$$

Furthermore, using a bounded multiplicity cover of $N_{1+\rho_\ell}^+ \cap \mathcal{L}_v^{(2\rho_\ell)}$ by balls of radius $4\rho_\ell$ centered inside $\mathcal{L}_v^{(\rho_\ell/2)}$ and using the doubling result in Proposition 3.1, one checks that

$$\mu_x^u(N_{1+\rho_\ell}^+ \cap \mathcal{L}_v^{(2\rho_\ell)}) \ll \mu_x^u(N_1^+ \cap \mathcal{L}_v^{(\rho_\ell)}) = \tilde{\mu}_x(\mathcal{L}_v^{(\rho_\ell)}).$$

This implies the estimate (12.5).

Let $C_1 \geq 1$ be the larger of the implicit constants in (12.4) and (12.5). These two estimates imply that ν_x satisfies Definition 11.1 by taking

$$\begin{aligned}
C(\theta) &:= C_1 V(x)H/\theta^2, \\
\varphi(\varepsilon) &:= C_1 \max\{\varepsilon, t(\varepsilon)\}, \\
\lambda(\theta) &:= \beta\theta^2 \log 2/4.
\end{aligned} \tag{12.6}$$

That $\varphi(\varepsilon)$ tends to 0 as $\varepsilon \to 0$ follows by Theorem 12.1.

12.3. **Non-concentration and proof of Proposition 9.13.** The idea of the proof is similar to that of [Liv04, Lemma 6.2], with the significant added difficulty being the non-concentration result for PS measures established in Theorem 11.17. We note however that the case of real hyperbolic manifolds is much simpler in that it does not require Theorem 11.17 and instead uses only the doubling result in Proposition 3.1.

Recall our definition of the points $x_{\rho,\ell}$ in (9.39) and of $N_1^+(j)$ in the paragraph above (9.35). For each $\ell \in I_{\rho,j}$, fix some $u_\ell \in N_1^+(j) \subseteq N_3^+$ such that

$$x_{\rho,\ell} = g^\gamma p_\ell^+ \cdot x, \qquad p_\ell^+ := m_{\rho,\ell}\, g_{t_{\rho,\ell}} u_\ell. \tag{12.7}$$

Here, we are using that the groups $A = \{g_t : t \in \mathbb{R}\}$ and M commute. Denote by P^+ the parabolic subgroup N^+AM of G. Since M is compact, $|t_{\rho,\ell}| < 1$, and $N_1^+(j)$ is contained in N_3^+, there is a uniform constant $C > 0$ such that

$$\left\{p_\ell^+ : \ell \in I_{\rho,j}\right\} \subset P_C^+, \tag{12.8}$$

where P_C^+ denotes the ball of radius C around identity in P^+.

Fix some $\ell_0 \in I_{\rho,j}$ and denote by $C_{\rho,j,i}(\ell_0)$ denote the set of indices $\ell \in I_{\rho,j}$ such that $(\ell_0, \ell) \in C_{\rho,j,i}$. Let $Z = \exp(L_i \oplus \mathfrak{n}_{2\alpha}^-) \subset N^-$. In particular[18], $Z = \{\mathrm{Id}\}$ is the trivial group in the real

[18] This is the reason Theorem 11.17 is not needed in this case.

hyperbolic case. Recalling the definition of the Cygan metric in (2.8), the definition of $C_{\rho,j,i}$ implies that

$$d_{N^-}(n^-_{\rho,\ell}(n^-_{\rho,\ell_0})^{-1}, Z) \leq b^{-1/10}.$$

Let $\epsilon := b^{-1/10}$ and denote by $Z^{(\epsilon)}$ for the ϵ-neighborhood of Z inside N^-. Let

$$\tilde{u}^-_\ell = n^-_{\rho,\ell}(n^-_{\rho,\ell_0})^{-1} \in Z^{(\epsilon)} \cap N^-_{\iota_j},$$

where we recall that the points $n^-_{\rho,\ell}$ belong to $N^-_{\iota_j/10}$ by definition of our flow boxes B_ρ; cf. paragraph preceding (9.27). Note that

$$g^\gamma p^+_\ell \cdot x = \tilde{u}^-_\ell \cdot g^\gamma p^+_{\ell_0} \cdot x, \qquad \forall \ell \in C_{\rho,j,i}(\ell_0).$$

In particular, for $t_\star := \gamma(w + jT_0)$ and $u^-_\ell = \mathrm{Ad}(g^\gamma)^{-1}(\tilde{u}^-_\ell)$, since $g^\gamma = g_{t_\star}$ (cf. (9.30)), we have that

$$p^+_\ell x = u^-_\ell \cdot p^+_{\ell_0} x \in (Z^{(e^{t_\star}\epsilon)} \cap N^-_{e^{t_\star}\iota_j}) \cdot p^+_{\ell_0} x, \qquad \forall \ell \in C_{\rho,j,i}(\ell_0). \tag{12.9}$$

Our counting estimate will follow by estimating from below the separation between the points $p^+_\ell x$, combined with a measure estimate on the sets $(Z^{(e^{t_\star}\epsilon)} \cap N^-_{e^{t_\star}\iota_j}) \cdot p^+_{\ell_0} x$.

To this end, recall the sublevel set K_j and the injectivity radius ι_j in (9.18). Recall also by (9.21) that x belongs to K_j. It follows that the injectivity radius at every point of the weak unstable ball $P^+_C \cdot x$ is $\gg \iota_j$. This implies that there is a radius r_j with $\iota_j \ll r_j \leq \iota_j$ such that for every $\ell \in C_{\rho,j,i}(\ell_0)$, the map $n^- \mapsto n^- \cdot p^+_\ell x$ is an embedding of $N^-_{r_j}$ into X and the disks

$$\left\{ N^-_{r_j} \cdot p^+_\ell x : \ell \in C_{\rho,j,i}(\ell_0) \right\}$$

are disjoint. Recalling (12.9), it follows that the disks $N^-_{r_j} \cdot u^-_\ell$ form a disjoint collection of disks inside $Z^{(e^{t_\star}\epsilon + \iota_j)} \cap N^-_{(e^{t_\star}+1)\iota_j}$. In particular,

$$\# C_{\rho,j,i}(\ell_0) \leq \frac{\mu^s_{p^+_{\ell_0} x}\left(Z^{(e^{t_\star}\epsilon+\iota_j)} \cap N^-_{(e^{t_\star}+1)\iota_j} \right)}{\min_{\ell \in C_{\rho,j,i}(\ell_0)} \mu^s_{p^+_{\ell_0} x}(N^-_{r_j} \cdot u^-_\ell)}, \tag{12.10}$$

where $\mu^s_\bullet$ denote the Patterson-Sullivan conditional measures on N^-, defined analogously to the unstable conditionals in (2.2).

To obtain good bounds on the ratio in (12.10) for a given ℓ, it will be important to change the basepoint $p^+_{\ell_0} x$ to another point with uniformly bounded height. We do so by applying the geodesic flow for a time $s_{\rho,\ell}$, comparable to $t_\star$, such that $V(g_{s_{\rho,\ell}} p^+_{\ell_0} x) \ll 1$. Fix some arbitrary $\ell \in C_{\rho,j,i}(\ell_0)$ and recall (12.7) and (12.9). Let $s_{\rho,\ell} \geq t_\star$ be as in (9.41) and set

$$y_\ell = g_{s_{\rho,\ell}} p^+_\ell x.$$

Note that our choice of u^-_ℓ implies that

$$Z^{(e^{t_\star}\epsilon+\iota_j)} \cap N^-_{(e^{t_\star}+1)\iota_j} \subseteq \left(Z^{(2e^{t_\star}\epsilon+\iota_j)} \cap N^-_{2(e^{t_\star}+1)\iota_j} \right) \cdot u^-_\ell.$$

In particular, we can use the set on the right side to estimate the numerator of (12.10). Let

$$Q := Z^{(2e^{t_\star}\epsilon+\iota_j)} \cap N^-_{2(e^{t_\star}+1)\iota_j}, \qquad Q' := \mathrm{Ad}(g_{s_{\rho,\ell}})(Q).$$

Then, changing variables using (2.4) and (2.3), we have

$$\frac{\mu^s_{p^+_{\ell_0} x}(Q \cdot u^-_\ell)}{\mu^s_{p^+_{\ell_0} x}(N^-_{r_j} \cdot u^-_\ell)} = \frac{\mu^s_{p^+_\ell x}(Q)}{\mu^s_{p^+_\ell x}(N^-_{r_j})} = \frac{\mu^s_{y_\ell}(Q')}{\mu^s_{y_\ell}(N^-_{e^{-s_{\rho,\ell}} r_j})}.$$

Hence, by Corollary 12.2 and Theorem 11.17 and (9.41), there is $\kappa > 0$ such that

$$\mu^s_{y_\ell}(Q') \ll_{T_0} (\epsilon + e^{-s_{\rho,\ell}}\iota_j)^\kappa \mu^s_{y_\ell}(N_1^+) \ll_{T_0} \epsilon^\kappa + e^{-s_{\rho,\ell}\kappa}.$$

Moreover, by the global measure formula, Theorem 3.4, since $V(y_\ell) \ll_{T_0} 1$, we have that

$$\mu^s_{y_\ell}(N^-_{e^{-s_{\rho,\ell}}r_j}) \gg_{T_0} e^{-\delta s_{\rho,\ell}} r_j^\delta \gg e^{-\delta s_{\rho,\ell}}\iota_j^\delta.$$

Here, we used [Cor90, Theorem 2.2] to relate strong stable disks of the form $N_r^- \cdot y_\ell$ to their shadows on the boundary; cf. (3.4) for a precise formulation. This concludes the proof since ℓ was arbitrary.

12.4. **Flattening and proof of Theorem 9.16.** We wish to apply Corollary 11.5. Recall that ν_i has total mass $\mu^u_{y^i_\rho}(N_1^+)$. Let

$$\mu = \frac{1}{\mu^u_{y^i_\rho}(N_1^+)}\nu_i.$$

We also recall that μ is supported on $\mathfrak{n}^+_\alpha$. We fix identifications $\mathfrak{n}^+_\alpha \cong \mathbb{K}^p \cong \mathfrak{n}^-_\alpha$ for some $p \in \mathbb{N}$; cf. Section 2.5. Note further that the restriction of the metric in (2.8) to $\mathfrak{n}^+_\alpha$ is Euclidean. In particular, we will fix a linear isomorphism of $\mathfrak{n}^+_\alpha$ and $\mathfrak{n}^-_\alpha$ with $\mathbb{R}^d$, where $d = p \dim \mathbb{K}$.

By Corollary 12.2, the measure μ is affinely non-concentrated at almost all scales in the sense of Definition 11.1. Hence, we can apply Corollary 11.5 to find $\lambda > 0$ such that, for $T = b^{4/10}$, the set

$$\mathfrak{B}(\lambda) := \left\{ w \in \mathbb{R}^d : \|w\| \leq T \text{ and } |\hat{\mu}(w)| \geq T^{-\lambda} \right\}$$

can be covered by $O_\varepsilon(T^\varepsilon)$ balls of radius 1. The result will follow once we estimate the spacing of the functionals $\langle w^i_{k,\ell}, \cdot \rangle$.

To simplify notation, let $w_\ell := \langle w^i_{k,\ell}, \cdot \rangle$. By (9.69), when b is large enough, we have that $b\,\|w_\ell\| \leq T$. In particular, we can view the set $B(i,k,\lambda)$ as a subset of $\mathfrak{B}(\lambda)$ above using the map $\ell \mapsto bw_\ell$. By Lemma 9.11, the definition of $w^i_{k,\ell}$ in (9.64), and (9.62), we have that

$$\|w_{\ell_1} - w_{\ell_2}\| \gg b^{2/10} \mathrm{dist}(u_{\ell_2} - u_{\ell_1}, L_i),$$

where L_i is a certain proper subspace of $\mathfrak{n}^-_\alpha$. In particular, by Proposition 9.13, any ball of radius 1 in $\mathbb{R}^d$ contains at most

$$O_{T_0}\left((b^{-\kappa/10} + e^{-\kappa(\gamma-\alpha)(w+jT_0)}) e^{\delta\gamma(w+jT_0)} \right)$$

of the vectors w_ℓ. This completes the proof of Theorem 9.16.

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