

STRONG SPECTRAL GAP FOR GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

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ABSTRACT. Let $\Gamma < G := \mathrm{SO}(d+1, 1)$ for $d \geq 1$ be a Zariski dense, geometrically finite, discrete subgroup with critical exponent strictly greater than $d/2$. We show that $L^2(\Gamma \backslash G)$ admits a strong spectral gap, confirming a conjecture of Mohammadi and Oh. This extends the spherical spectral gap on $L^2(\Gamma \backslash \mathbb{H}^{d+1}) \cong L^2(\Gamma \backslash G / \mathrm{SO}(d+1))$, which follows by the works of Lax–Phillips, Patterson, and Sullivan by different methods. As a consequence, we establish rates of decay of matrix coefficients, and of exponential mixing of the frame flow, that are explicitly determined by the size of the strong spectral gap.

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1. INTRODUCTION

Throughout the paper, $G := \mathrm{SO}(d+1, 1)^\circ$ denotes the connected component of the orthogonal group of signature $(d+1, 1)$ for $d \geq 1$, and $\Gamma < G$ is a discrete subgroup. Recall that G can be realized as the group of orientation-preserving isometries of the $(d+1)$ -dimensional hyperbolic space \mathbb{H}^{d+1} . The discrete subgroup Γ and the corresponding hyperbolic orbifold $\Gamma \backslash \mathbb{H}^{d+1}$ are said to be *geometrically finite* if for any $\epsilon > 0$, the ϵ -neighborhood of the convex core of $\Gamma \backslash \mathbb{H}^{d+1}$ —the closure of the union of all closed geodesics in $\Gamma \backslash \mathbb{H}^{d+1}$ —is of finite volume. The *critical exponent* $\delta_\Gamma \in [0, d]$ is a fundamental parameter associated to Γ which is defined as the abscissa of convergence of the Poincaré series $s \mapsto \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$ for any reference point $o \in \mathbb{H}^{d+1}$. When Γ is geometrically finite, the critical exponent coincides with the Hausdorff dimension of the limit set of Γ in $\partial_\infty(\mathbb{H}^{d+1})$ —the set of limit points of any Γ orbit in $\overline{\mathbb{H}^{d+1}}$. Our primary object of study is the right regular representation of G on the space $L^2(\Gamma \backslash G)$ of \mathbb{C} -valued L^2 functions with respect to a fixed G -invariant measure.

Recall that an irreducible unitary representation of G is called *spherical* if it admits a non-zero fixed vector by a maximal compact subgroup of G . In 1982, Lax–Phillips showed in [LP82] that, when $\Gamma \backslash \mathbb{H}^{d+1}$ is geometrically finite, the intersection of the L^2 spectrum of the Laplace–Beltrami operator with $[0, d^2/4)$ consists of a (possibly empty) finite set of eigenvalues, each occurring with finite multiplicity (so-called *small eigenvalues* of the Laplace–Beltrami operator). Equivalently, since a spherical vector in a spherical complementary series representation is an eigenfunction of the Laplace–Beltrami operator with eigenvalue in $[0, d^2/4)$, at most finitely many *spherical* complementary series can occur in $L^2(\Gamma \backslash G)$ and with finite multiplicities. Moreover, the works of Patterson [Pat76] in the case $d = 1$, and Sullivan [Sul79] in general show that there exists a bottom eigenvalue below the continuous spectrum if and only if $\delta_\Gamma > d/2$ (cf. works of Elstrodt [Els73a, Els73b, Els74]); thereby yielding the existence of a spectral gap in this case, often called the Lax–Phillips spectral gap.

In [MO15, Definition 1.1], Mohammadi–Oh formulated a more general *strong spectral gap* which includes the *non-spherical* complementary series as well as the spherical ones (cf. [KS09, Kel11] for similar types of questions for compact groups). It is natural to extend their definition to the following definition to account for all the quasi-complementary series, which exhaust the non-tempered elements of the unitary dual of G ; see Theorem 2.6. To formulate this definition, denote by \widehat{M}_{s-d} the set of isomorphism classes of self-dual irreducible representations of $M \cong \mathrm{SO}(d)$. Then, the quasi-complementary series are parametrized by pairs (σ, s) for $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$, where $\mathcal{I}_\sigma \subset (d/2, d]$ is an explicit interval determined by σ ; see Subsection 2.5 for precise definitions. The corresponding series will be denoted $\mathcal{U}(\sigma, s)$.

Definition 1.1 (Strong Spectral Gap). We say that $L^2(\Gamma \backslash G)$ admits a *strong spectral gap* if

- (1) for all non-trivial $\sigma \in \widehat{M}_{s-d}$, the quasi-complementary series $\mathcal{U}(\sigma, \delta_\Gamma)$ is not contained in $L^2(\Gamma \backslash G)$; and
- (2) there exists $\eta > 0$ such that for all $\sigma \in \widehat{M}_{s-d}$ and $s \in (\delta_\Gamma - \eta, \delta_\Gamma)$, the quasi-complementary series $\mathcal{U}(\sigma, s)$ is not weakly contained in $L^2(\Gamma \backslash G)$.

Remark 1.2. Note that for all $\sigma \in \widehat{M}_{s-d}$ and $s > \delta_\Gamma$, $\mathcal{U}(\sigma, s)$ is not weakly contained in $L^2(\Gamma \backslash G)$; see Proposition 6.5. Therefore, provided Definition 1.1(2) holds, for any non-trivial $\sigma \in \widehat{M}_{s-d}$, a weak containment of $\mathcal{U}(\sigma, \delta_\Gamma)$ in $L^2(\Gamma \backslash G)$ is automatically a strong containment, which justifies the seemingly stronger requirement of Definition 1.1(1).

In fact, from the Lax–Phillips spectral gap combined with the observation that the non-spherical quasi-complementary series do not exist for parameters $s > d - 1$, it follows that if $\Gamma < G$ is a Zariski-dense geometrically finite discrete subgroup with critical exponent $\delta_\Gamma > \max\{d/2, d - 1\}$, then $L^2(\Gamma \backslash G)$ admits a strong spectral gap (cf. [MO15, Theorem 3.27]). In [MO15, Conjecture 1.2], Mohammadi and Oh conjectured that the optimal condition $\delta_\Gamma > d/2$ suffices for $L^2(\Gamma \backslash G)$ to admit a strong spectral gap for the complementary series, in arbitrary dimension $d + 1 \geq 2$. The following is the main result of this article, which, in particular, confirms their conjecture.

Theorem 1.3. *If $\Gamma < G$ is a Zariski-dense, geometrically finite, discrete subgroup with $\delta_\Gamma > d/2$, then $L^2(\Gamma \backslash G)$ admits a strong spectral gap.*

Remark 1.4. We make the following observations:

- (1) Restricting to the spherical complementary series, we obtain a different proof of the Lax–Phillips spectral gap; cf. Subsection 1.2. We also obtain a different proof of the aforementioned fact, due to Patterson and Sullivan, that the bottom eigenvalue of the L^2 spectrum of the Laplace–Beltrami operator is simple and given by $\delta_\Gamma(d - \delta_\Gamma)$; cf. Proposition 6.8.
- (2) The size of the strong spectral gap η in Definition 1.1(2) provided by Theorem 1.3 remains uniform over any family of subgroups $\Gamma < G$ over which the rate of exponential mixing of the frame flows is uniform. In particular, such uniformity is known to hold for certain families of congruence subgroups of a fixed Zariski-dense convex cocompact subgroup Γ of an arithmetic group of G by work of Oh–Winter for $d = 1$ [OW16], and by the third author for general $d \geq 1$ [Sar22]. These results will be extended to the case Γ is geometrically finite in forthcoming work of the third author.

Having established Theorem 1.3 on the existence of a strong spectral gap, we make the following definition.

Definition 1.5. We define the *strong spectral gap parameter* of Γ , denoted by κ_Γ , as the supremum over all $\eta \in (0, \delta_\Gamma - d/2]$ for which Definition 1.1(2) holds.

Theorem 1.3 yields the following two results on rates of decay of matrix coefficients, and of exponential mixing of the frame flow. This extends the results of Edwards–Oh [EO21], who proved exponential mixing for the geodesic flow without relying on the existence of a strong spectral gap.

Denote by m^{Haar} , m^{BMS} , m^{BR} , and m^{BR^*} , compatibly chosen G -invariant, Bowen–Margulis–Sullivan, and Burger–Roblin measures; see Section 5 and references therein. We also denote by $\mathcal{S}_K^m(\Gamma \backslash G)$ the L^2 -Sobolev space of order $m > 0$ defined using derivatives along a fixed maximal compact subgroup $K < G$; see Subsection 2.3 for a precise definition.

Theorem 1.6. *Let $\Gamma < G$ be a Zariski-dense, geometrically finite, discrete subgroup with $\delta_\Gamma > d/2$. Let $\kappa_0 := \min\{\kappa_\Gamma, 1\}$, where κ_Γ is the strong spectral gap parameter of Γ . There exists $m > d(d + 1)/2$ such that*

for all $\epsilon > 0$, and $\phi, \psi \in \mathcal{S}_K^m(\Gamma \backslash G)$, and $t > 0$, we have

$$e^{(d-\delta_\Gamma)t} \int_{\Gamma \backslash G} (\phi \circ a_t) \cdot \psi \, dm^{\text{Haar}} = \int_{\Gamma \backslash G} \phi \, dm^{\text{BR}} \int_{\Gamma \backslash G} \psi \, dm^{\text{BR}^*} + O_\epsilon(e^{-(\kappa_0-\epsilon)t} \|\phi\|_{\mathcal{S}_K^m(\Gamma \backslash G)} \|\psi\|_{\mathcal{S}_K^m(\Gamma \backslash G)}).$$

Remark 1.7. Theorem 1.6 strengthens the results of [SW21, LPS25] when $\delta_\Gamma > d/2$ in the following three aspects which are important for applications [MO15, KO21]:

- the exponential rate κ_0 of the error term is explicitly determined by the strong spectral gap parameter κ_Γ ;
- the implicit constant does not depend on the supports of the test functions, in particular allowing non-compactly supported functions;
- regularity control on the test functions is weakened to only involve the Sobolev norm \mathcal{S}_K^m .

We note however that the results of [SW21, LPS25] are used as input in our proof of Theorem 1.3. In particular, Theorem 1.6 does not give a different proof of these results.

The following theorem follows from Theorem 1.6 using the same proof as in [MO15, OW16, KO21] which provide effective versions of Roblin's transverse intersection argument [Rob03]. Indeed, the explicit formula for the exponential rate is obtained from a generalization of [OW16, Theorem 5.8] and its proof.

Theorem 1.8. *Let $\Gamma < G$ be a Zariski-dense, geometrically finite, discrete subgroup with $\delta_\Gamma > d/2$. Let $\kappa_1 := \frac{\kappa_0}{2(d+3+\kappa_0)}$, where κ_0 is the constant from Theorem 1.6. There exists $m > d(d+1)/2$ such that for all $\phi, \psi \in C^m(\Gamma \backslash G)$ and $t > 0$, we have*

$$\int_{\Gamma \backslash G} (\phi \circ a_t) \cdot \psi \, dm^{\text{BMS}} = \int_{\Gamma \backslash G} \phi \, dm^{\text{BMS}} \int_{\Gamma \backslash G} \psi \, dm^{\text{BMS}} + O(e^{-\kappa_1 t} \|\phi\|_{C^m(\Gamma \backslash G)} \|\psi\|_{C^m(\Gamma \backslash G)}).$$

Theorem 1.3 also gives the following immediate corollary. Recalling that the Casimir operator acts on the quasi-complementary series $\mathcal{U}(\sigma, s)$ with the scalar $s(d-s)$ and that its action on K -invariant functions coincides with that of the Laplace–Beltrami operator, we have the following:

Corollary 1.9. *The bottom eigenvalue $\delta_\Gamma(d-\delta_\Gamma)$ of the Laplace–Beltrami operator on $L^2(\Gamma \backslash \mathbb{H}^{d+1})$ is isolated from the rest of the spectrum of the Casimir operator on $L^2(\Gamma \backslash G)$.*

Remark 1.10. Note that the eigenvalue $\delta_\Gamma(d-\delta_\Gamma)$ of the Casimir operator may not be simple when d is odd due to the presence of (tempered) discrete series representations for which the Casimir eigenvalue could be the same as that of a (non-tempered) quasi-complementary series representation, and these could also appear in $L^2(\Gamma \backslash G)$. We note, however, that since the Casimir eigenvalues corresponding to discrete series form a discrete set there is no danger of these accumulating near $\delta_\Gamma(d-\delta_\Gamma)$.

1.1. Organization of the paper and outline of the proof. Our strategy is based on the study of the Laplace transform of scaled matrix coefficients of test functions $\phi, \psi \in L^2(\Gamma \backslash G)$ defined as follows:

$$F(z) = \int_0^{+\infty} e^{-(z+\delta_\Gamma-d)t} \langle \phi \circ a_t, \psi \rangle_{L^2(\Gamma \backslash G)} \, dt \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > 0. \quad (1.1)$$

After recalling the necessary representation-theoretic background in Section 2, we recall in Section 3 the work of Edwards and Oh [EO21] on asymptotic expansions of matrix coefficients of complementary series representations, which we use to prove Proposition 3.8 providing precise information on holomorphic continuations of Laplace transforms of such matrix coefficients.

The main terms in the asymptotic expansions of matrix coefficients are expressed in terms of Harish-Chandra functions; Definition 3.2. In Section 4, we recall explicit formulas for the Harish-Chandra functions from [EKM99], which allow us to produce prescribed K -finite vectors for which the main terms in the Edwards–Oh expansions are non-vanishing; Corollary 4.3. We use such non-vanishing statements to provide obstructions of holomorphic extensions of the Laplace transform of matrix coefficients to regions of the real line parametrizing the quasi-complementary series that weakly occur in $L^2(\Gamma \backslash G)$.

In Section 5, we recall the works of Winter and the third author [SW21], and Li, Pan, and the third author [LPS25] on exponential mixing of the frame flow with respect to the Bowen–Margulis–Sullivan measure. Through Roblin's transverse intersection argument, this result provides a power-saving error term in the asymptotic formula for scaled matrix coefficients; Theorem 5.1. This result immediately implies a

meromorphic continuation of the Laplace transform to a strip to the left of the imaginary axis, with at most one simple pole at the origin; Corollary 5.3.

In Section 6, we prove Theorem 1.3 by comparing the Laplace transform $F(z)$ computed in two different ways as above; namely using representation theory in Proposition 3.8 and dynamics in Corollary 5.3. To describe the idea, recall that $L^2(\Gamma \backslash G)$ admits a direct integral decomposition over the unitary dual of G against certain Borel measures, which we refer to as *spectral measures*; cf. Section 2. As a consequence, the main terms in Edwards–Oh’s asymptotic expansions are given by suitable integrals against such measures.

The key idea, due to Avila–Gouëzel [AG13, §3], is to interpret the representation-theoretic formula provided by Proposition 3.8 as a *Stieltjes transform* (recalled in Proposition 6.3) of a certain absolutely continuous measure with respect to the spectral measures. A notable feature of our setting is that the Radon–Nikodym derivatives of the measures in question are complex-valued in general, whereas in the case of $\mathrm{SL}_2(\mathbb{R})$ -representations studied in [AG13], such densities are real-valued and non-negative. One then combines the holomorphic continuation provided by dynamics in Corollary 5.3, with an extension of the Stieltjes inversion formula to complex-valued measures in Corollary 6.4, to get vanishing of the spectral measures on an open interval, with right endpoint the bottom of the spectrum of $L^2(\Gamma \backslash G)$. This argument, carried out in Proposition 6.6, will verify Definition 1.1(2).

Simplicity of the bottom of the spectrum stated in Definition 1.1(1) follows by comparing residues of the Laplace transforms together with the observation that the main term on the dynamical side is a bilinear form of rank 1 (being a product of integrals against Burger–Roblin measures). This argument is carried out in Proposition 6.8.

1.2. Relation to prior works. The overall strategy of combining exponential mixing with representation theory to obtain a spectral gap was employed before in the work of Avila, Gouëzel, and Yoccoz in the setting of $\mathrm{SL}_2(\mathbb{R})$ representations [AGY06, Appendix B], where the argument is attributed to Anantharaman, Bufetov, and Forni. Here, the idea is to consider *spherical vectors*, i.e., those invariant by a maximal compact subgroup, in a given spherical complementary series where it is known that matrix coefficients are non-negative, and in fact are bounded *below* up to uniform constants by $e^{-\lambda t}$, for a suitable $\lambda \geq 0$ depending on the complementary series in question. Together with upper bounds on matrix coefficients coming from exponential mixing, this quickly gives the desired spectral gap.

The above argument in fact readily extends to higher dimensions to give a different proof of Lax–Phillips’s *spherical* spectral gap using exponential mixing of the *geodesic flow*. To our knowledge, there are no analogs of such positivity and lower bounds for matrix coefficients for the non-spherical quasi-complementary series (i.e., those that do not admit non-zero spherical vectors). In our setting, the work of Edwards–Oh, together with the non-vanishing results in Section 4, substitute for the lack of such lower bounds. The lack of positivity introduces further subtleties, which are treated somewhat indirectly using complex analysis in the form of Stieltjes transforms as discussed above.

Finally, we recall the work of Guillarmou and Dyatlov [DG16, DG18] on meromorphic continuations of Laplace transforms of matrix coefficients of sections of vector bundles to the entire complex plane in the setting of Axiom A flows, which includes geodesic flows on convex cocompact manifolds. At least for convex cocompact subgroups,¹ these results imply that, for any given $\sigma \in \widehat{M}_{s-d}$, there are at most finitely many parameters $s \in (d/2, \delta_\Gamma]$ for which the quasi-complementary series $\mathcal{U}(\sigma, s)$ weakly occur in $L^2(\Gamma \backslash G)$. However, due to the fact that \widehat{M}_{s-d} is infinite, such results do not imply the existence of a strong spectral gap in the sense of Definition 1.1.

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¹A similar result was obtained for scalar-valued functions on convex cocompact quotients of negatively curved symmetric spaces by different methods in [Kha23b], which additionally yield meromorphic continuation for geometrically finite groups to an explicit strip to the left of the imaginary axis determined by δ_Γ and the ranks of the cusps of Γ .

2. REPRESENTATION THEORETIC PRELIMINARIES

As stated in the introduction, $G := \mathrm{SO}(d+1, 1)^\circ$ for $d \geq 1$ and $\Gamma < G$ is a discrete subgroup with critical exponent $\delta_\Gamma \in [0, d]$. We assume henceforth that $\Gamma < G$ is a Zariski-dense geometrically finite discrete subgroup with $\delta_\Gamma > d/2$. Correspondingly, $\Gamma \backslash \mathbb{H}^{d+1}$ is a $(d+1)$ -dimensional geometrically finite hyperbolic orbifold. In this section, we collect the necessary background on Lie algebras, Lie groups, and unitary representation theory.

2.1. Lie theory of G . Let us recall the structure of G and the main dynamical system associated to Γ for our purposes.

We always use Fraktur letters to denote Lie algebras corresponding to Lie groups, e.g., $\mathfrak{g} := \mathrm{Lie}(G)$. Fix a Cartan involution on \mathfrak{g} . Via the Killing form on \mathfrak{g} , it induces a left G -invariant and right K -invariant Riemannian metric on G . It also gives rise to an Iwasawa decomposition $G = KAN \cong K \times A \times N$ where

$$K < G, \quad A < G, \quad N < G,$$

is a maximal compact subgroup, a maximal diagonalizable subgroup, and a maximal horospherical subgroup, respectively. Let $M := Z_K(A) < K$, i.e., the centralizer of A in K . We denote the Haar measure on G induced by the Riemannian metric on G by μ_G or dg and for other induced unimodular topological groups similarly. Denote by $T^1(\mathbb{H}^{d+1})$ and $F(\mathbb{H}^{d+1})$ the unit tangent bundle and the oriented orthonormal frame bundle of \mathbb{H}^{d+1} , respectively. We may make the following identifications as Riemannian manifolds equipped with left G -actions:

$$\mathbb{H}^{d+1} \cong G/K, \quad T^1(\mathbb{H}^{d+1}) \cong G/M, \quad F(\mathbb{H}^{d+1}) \cong G.$$

Similarly, we may also make the following identifications as Riemannian orbifolds:

$$\Gamma \backslash \mathbb{H}^{d+1} \cong \Gamma \backslash G/K, \quad T^1(\Gamma \backslash \mathbb{H}^{d+1}) \cong \Gamma \backslash G/M, \quad F(\Gamma \backslash \mathbb{H}^{d+1}) \cong \Gamma \backslash G.$$

Recall that G is of rank one meaning that A is a one-parameter subgroup. We parametrize $A := \{a_t\}_{t \in \mathbb{R}}$ such that its right translation action on G and $\Gamma \backslash G$ (resp. G/M and $\Gamma \backslash G/M$) is the unit speed frame flow (resp. geodesic flow). We may further assume that the parametrization is such that N is the *contracting* horospherical subgroup; and let $\bar{N} < G$ be the *expanding* horospherical subgroup. That is, we have

$$N = \left\{ g \in G : \lim_{t \rightarrow -\infty} a_t g a_{-t} = e \right\}, \quad \bar{N} = \left\{ g \in G : \lim_{t \rightarrow +\infty} a_t g a_{-t} = e \right\}.$$

Moreover, N and \bar{N} (resp. \mathfrak{n} and $\bar{\mathfrak{n}}$) are d -dimensional abelian Lie groups (resp. Lie algebras) isomorphic to \mathbb{R}^d . The above parametrization is equivalent to the following adjoint action of a_t :

$$\mathrm{Ad}(a_t)v = e^t v, \quad \mathrm{Ad}(a_t)\bar{v} = e^{-t}\bar{v}, \quad \text{for all } v \in \mathfrak{n}, \bar{v} \in \bar{\mathfrak{n}}, \text{ and } t \in \mathbb{R}.$$

As a consequence of the above discussion, we have unique maps

$$\kappa : G \rightarrow K, \quad H : G \rightarrow \mathbb{R}, \quad n : G \rightarrow N,$$

such that

$$g = \kappa(g)a_{H(g)}n_g.$$

Note that H is then parameterized such that $H(a_t) = t$ for all $t \in \mathbb{R}$.

2.2. Unitary representation theory: generalities. Let us fix some notations and conventions which apply for any unimodular locally compact Hausdorff topological group, which we denote by G only in this subsection. We refer the reader to the books [Dix77, Wal92, War72b] for details. Throughout the paper, all representations are *strongly continuous* and over \mathbb{C} without any specification. As such, a unitary representation of G is a map

$$\pi : G \rightarrow \mathrm{U}(\mathcal{H})$$

into the unitary group $\mathrm{U}(\mathcal{H})$ of a Hilbert space \mathcal{H} equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and a corresponding norm $\| \cdot \|_{\mathcal{H}}$, such that for all $v \in \mathcal{H}$, the map $G \rightarrow \mathcal{H}$ given by $g \mapsto \pi(g)v$ is continuous. For brevity, we often specify it as a unitary representation (π, \mathcal{H}) of G . We also use $\| \cdot \|_{\mathcal{H}}$ for the operator norm of bounded operators on \mathcal{H} .

For any left (resp. right) G -action on a locally compact Hausdorff topological space X endowed with a G -invariant Borel measure μ_X , such as G itself, we denote by $L^2(X)$ the Hilbert space equipped with the

standard inner product $\langle \cdot, \cdot \rangle_{L^2(X)}$ defined by $\langle \phi, \psi \rangle_{L^2(X)} := \int_X \phi \cdot \bar{\psi} d\mu_X$ for all $\phi, \psi \in L^2(X)$. We also denote by $(\lambda_G, L^2(X))$ (resp. $(\rho_G, L^2(X))$) the left (resp. right) regular representation of G which is defined by

$$\lambda_G(g)(\phi)(x) = \phi(g^{-1}x) \quad (\text{resp. } \rho_G(g)(\phi)(x) = \phi(xg))$$

for all $x \in X$, $\phi \in L^2(X)$, and $g \in G$.

A representation (ρ, \mathcal{V}) of G is (strongly) contained in another representation (π, \mathcal{H}) of G if (ρ, \mathcal{V}) is isomorphic to a subrepresentation of (π, \mathcal{H}) . A unitary representation (ρ, \mathcal{V}) of G is *weakly* contained in another unitary representation (π, \mathcal{H}) of G if any diagonal matrix coefficient of (ρ, \mathcal{V}) is a uniform limit on compact subsets of G of a sequence of linear combinations of diagonal matrix coefficients of (π, \mathcal{H}) . Recall that a representation of G is said to be irreducible if it does not strictly contain any non-trivial G -invariant subspace. The *unitary dual* of G , denoted by \widehat{G} , is the set of all isomorphism classes of *irreducible* unitary representations equipped with the hull-kernel/Fell topology. We denote by $\mathbf{1} \in \widehat{G}$ the equivalence class of trivial irreducible representations.

The *dual* of a representation (π, \mathcal{H}) of G is a new representation (π^*, \mathcal{H}^*) of G defined by

$$\pi^*(g)(f)(v) = f(\pi(g^{-1})v)$$

for all $v \in \mathcal{H}$, $f \in \mathcal{H}^*$, and $g \in G$. A representation, (π, \mathcal{H}) , is *self-dual* if it is isomorphic to (π^*, \mathcal{H}^*) . We denote $\widehat{G}_{\text{s-d}} \subset \widehat{G}$ for the subset of self-dual elements.

A unitary representation (π, \mathcal{H}) of G is said to be *tempered* if its matrix coefficients are in $L^{2+\epsilon}(G)$ for all $\epsilon > 0$. We have the decomposition into a disjoint union $\widehat{G} = \widehat{G}_{\text{temp}} \sqcup \widehat{G}_{\text{q-comp}}$, where $\widehat{G}_{\text{temp}}$ is the subset consisting of tempered elements and $\widehat{G}_{\text{q-comp}}$ is the subset consisting of the non-tempered elements.

Let (π, \mathcal{H}) be a unitary representation of G . We cover some decompositions (see [Wal92, Chapter 14, §14.10, Theorem 14.10.5]). There exists an associated Borel spectral measure $m^{\mathcal{H}}$ on \widehat{G} , which can be further decomposed into Borel spectral measures $m_{\bullet}^{\mathcal{H}}$ on \widehat{G}_{\bullet} for $\bullet \in \{\text{temp}, \text{q-comp}\}$ such that we have the following orthogonal direct integral decomposition:

$$\begin{aligned} (\pi, \mathcal{H}) &= \int_{\widehat{G}}^{\oplus} (\pi_{\xi}, \mathcal{H}_{\xi}) dm^{\mathcal{H}}(\xi) \\ &= (\pi_{\text{temp}}, \mathcal{H}_{\text{temp}}) \oplus (\pi_{\text{q-comp}}, \mathcal{H}_{\text{q-comp}}) \\ &= \int_{\widehat{G}_{\text{temp}}}^{\oplus} (\pi_{\xi}, \mathcal{H}_{\xi}) dm_{\text{temp}}^{\mathcal{H}}(\xi) \oplus \int_{\widehat{G}_{\text{q-comp}}}^{\oplus} (\pi_{\xi}, \mathcal{H}_{\xi}) dm_{\text{q-comp}}^{\mathcal{H}}(\xi) \end{aligned}$$

where $(\pi_{\text{temp}}, \mathcal{H}_{\text{temp}})$ is the tempered subrepresentation and $(\pi_{\text{q-comp}}, \mathcal{H}_{\text{q-comp}})$ is its orthogonal complement, and $(\pi_{\xi}, \mathcal{H}_{\xi})$ is isotypic of type $\xi = (\pi_{\xi}, \widehat{\mathcal{H}}_{\xi}) \in \widehat{G}$, i.e., abusing notation we have the isomorphism $(\pi_{\xi}, \mathcal{H}_{\xi}) \cong (\pi_{\xi}, \widehat{\mathcal{H}}_{\xi}) \widehat{\otimes} (\mathbf{1}, \mathcal{M}_{\xi})$ for some (possibly infinite dimensional) multiplicity Hilbert space \mathcal{M}_{ξ} . Recall that atoms of Borel measures must be singletons. Moreover, we have the following well-known properties (see [Wal92, Chapter 14, §14.10, Lemmas 14.10.6 and 14.10.7]).

Lemma 2.1. *Let (π, \mathcal{H}) be a unitary representation of G . The following holds.*

- (1) *An element $\xi \in \widehat{G}$ is strongly contained in (π, \mathcal{H}) if and only if ξ is an atom of $m^{\mathcal{H}}$.*
- (2) *An element $\xi \in \widehat{G}$ is weakly contained in (π, \mathcal{H}) if and only if $\xi \in \text{supp}(m^{\mathcal{H}})$.*

Lemma 2.2. *Let (π, \mathcal{H}) be a unitary representation of G . Suppose that \mathcal{H} is separable. Then, $m^{\mathcal{H}}$ has at most a countable number of atoms.*

Now, let $L < G$ be a compact subgroup. Since we may view (π, \mathcal{H}) as a unitary representation of L , i.e., $(\pi|_L, \mathcal{H})$, we also have the decomposition

$$(\pi|_L, \mathcal{H}) = \widehat{\bigoplus}_{\tau \in \widehat{L}} (\tau, \mathcal{H}_{\tau}).$$

where $(\tau, \mathcal{H}_{\tau})$ is isotypic of type $\tau \in \widehat{L}$ in the sense as above—we often call them L -types. We also say that $\tau \in \widehat{L}$ is contained in (π, \mathcal{H}) whenever \mathcal{H}_{τ} is non-trivial and write $\tau \subset \pi$. The classification of $\tau \in \widehat{L}$

which are contained in (π, \mathcal{H}) are called *branching laws/rules*. Let $\tau \in \widehat{L}$. Recall that there is an associated character $\chi_\tau \in L^2(L)$ defined by $\chi_\tau(l) = \text{tr}(\tau(l))$ and an operator $\mathbf{P}_\tau^{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$\mathbf{P}_\tau^{\mathcal{H}} v = \dim(\tau) \int_L \overline{\chi_\tau(l)} \cdot \pi(l)v \, dl \quad \text{for all } v \in \mathcal{H}, \quad (2.1)$$

where dl is the Haar probability measure on L . Then it is in fact an orthogonal projection operator onto $\mathbf{P}_\tau^{\mathcal{H}}(\mathcal{H}) = \mathcal{H}_\tau$. For a maximal compact subgroup $K < G$, the unitary representation (π, \mathcal{H}) is *spherical* if it contains a non-trivial K -invariant vector.

2.3. Sobolev norms. Given a unitary representation (π, \mathcal{H}) of K , a fixed basis $\{X_j\}_{j=1}^{\dim(\mathfrak{k})}$ of \mathfrak{k} , and $m \in \mathbb{Z}_{\geq 0}$, the Sobolev norm $\|\cdot\|_{\mathcal{S}_K^m(\mathcal{H})}$ is defined on smooth vectors $v \in \mathcal{H}^\infty \subset \mathcal{H}$ as

$$\|v\|_{\mathcal{S}_K^m(\mathcal{H})}^2 = \|v\|_{\mathcal{H}}^2 + \sum_U \|d\pi(U)(v)\|_{\mathcal{H}}^2,$$

where the sum is over all monomials U of degree m in $X_1, \dots, X_{\dim(\mathfrak{k})}$. Let $\mathcal{S}_K^m(\mathcal{H})$ denote the completion of $\{v \in \mathcal{H}^\infty : \|v\|_{\mathcal{S}_K^m(\mathcal{H})} < +\infty\} \subset \mathcal{H}$ with respect to the Sobolev norm. While the Sobolev norm depends on the choice of basis of \mathfrak{k} , different choices give rise to equivalent norms. We note that when \mathcal{H} is finite dimensional, $\mathcal{S}_K^m(\mathcal{H}) = \mathcal{H}$.

Given a unitary representation (π, \mathcal{H}) of G we define the Sobolev norm on \mathcal{H} by viewing $(\pi|_K, \mathcal{H})$ as a unitary representation of K . It is well known that K -finite vectors are smooth and hence have finite Sobolev norm. Moreover, we note that for vectors of a fixed K -type the Sobolev norm is equivalent to the L^2 norm. Explicitly, we have the following.

Lemma 2.3. *Let (π, \mathcal{H}) be a unitary representation of G on a separable Hilbert space \mathcal{H} . For any $\tau \in \widehat{K}$ there is a constant $C(\tau) > 0$ such that for any $\phi \in \mathcal{H}_\tau$, we have*

$$\|\phi\|_{\mathcal{S}_K^1(\mathcal{H})} \leq C(\tau) \|\phi\|_{\mathcal{H}}.$$

Proof. Let V_τ denote the finite dimensional Hilbert space realizing $\tau \in \widehat{K}$ and note that as K -representations, we have $(\tau, \mathcal{H}_\tau) \cong (\tau, V_\tau) \otimes (\mathbf{1}, \mathcal{M}_\tau)$, where K acts trivially on a (possibly infinite dimensional) separable multiplicity space \mathcal{M}_τ . That is,

$$\pi(k)(u \otimes v) = (\tau(k)u) \otimes v \quad \text{for all } u \otimes v \in V_\tau \otimes \mathcal{M}_\tau \text{ and } k \in K.$$

Let $\{v_j\}_{j=1}^n$ for some $n \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ be a countable orthonormal Hilbert basis for \mathcal{M}_τ . Let $\phi \in \mathcal{H}_\tau \cong V_\tau \otimes \mathcal{M}_\tau$. Note that it has a decomposition $\phi = \sum_{j=1}^n \phi_j \otimes v_j$, with $\phi_j \in V_\tau$ and $\|\phi\|_{\mathcal{H}}^2 = \sum_{j=1}^n \|\phi_j\|_{V_\tau}^2 < +\infty$. Letting $X \in \mathfrak{k}$, we have

$$d\pi(X)\phi = \sum_{j=1}^n (d\tau(X)\phi_j) \otimes v_j,$$

so that by orthonormality of the basis $\{v_j\}_{j=1}^n$, we have

$$\begin{aligned} \|d\pi(X)\phi\|_{\mathcal{H}}^2 &= \left\langle \sum_{i=1}^n (d\tau(X)\phi_i) \otimes v_i, \sum_{j=1}^n (d\tau(X)\phi_j) \otimes v_j \right\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle d\tau(X)\phi_i, d\tau(X)\phi_j \rangle_{V_\tau} \cdot \langle v_i, v_j \rangle_{\mathcal{M}_\tau} \\ &= \sum_{j=1}^n \|d\tau(X)\phi_j\|_{V_\tau}^2. \end{aligned}$$

Since V_τ is finite dimensional, for any $v \in V_\tau$ we can bound $\|d\tau(X)v\|_{V_\tau} \leq \|d\tau(X)\|_{V_\tau} \|v\|_{V_\tau}$ where $\|d\tau(X)\|_{V_\tau}$ is the operator norm. Hence,

$$\|d\pi(X)\phi\|_{\mathcal{H}}^2 \leq \|d\tau(X)\|_{V_\tau}^2 \sum_{j=1}^n \|\phi_j\|_{V_\tau}^2 = \|d\tau(X)\|_{V_\tau}^2 \|\phi\|_{\mathcal{H}}^2.$$

Finally, for the fixed basis $\{X_j\}_{j=1}^{\dim(\mathfrak{k})}$ of \mathfrak{k} , taking $C(\tau) = 1 + \max_{1 \leq j \leq \dim(\mathfrak{k})} \|d\tau(X_j)\|_{V_\tau}$ gives the desired bound for $\|\phi\|_{\mathcal{S}_K^1(\mathcal{H})}$ and finishes the proof. \blacksquare

2.4. Unitary representation theory of $\mathrm{SO}(n)$. We recall some fundamental facts regarding irreducible representations of $\mathrm{SO}(n)$ for any $n \geq 1$, which then apply to both $K \cong \mathrm{SO}(d+1)$ and its subgroup $M \cong \mathrm{SO}(d)$ (see for instance [Thi73]).

Firstly, irreducible representations of $\mathrm{SO}(n)$ are unitarizable by compactness of $\mathrm{SO}(n)$, and consequently form the unitary dual $\widehat{\mathrm{SO}(n)}$. By the Peter–Weyl theorem, they are finite-dimensional and contained in $(\lambda_{\mathrm{SO}(n)}, L^2(\mathrm{SO}(n)))$; in fact

$$(\lambda_{\mathrm{SO}(n)}, L^2(\mathrm{SO}(n))) = \bigoplus_{\tau \in \widehat{\mathrm{SO}(n)}} (\tau, V_\tau)^{\oplus \dim(\tau)}.$$

2.4.1. Case $2 \mid n$. Let us write $n = 2m$ for $m \in \mathbb{Z}_{>0}$. The unitary dual $\widehat{\mathrm{SO}(2m)}$ can be canonically identified with

$$\{(\tau_1, \dots, \tau_m) \in \mathbb{Z}^m : \tau_1 \geq \tau_2 \geq \dots \geq \tau_{m-1} \geq |\tau_m|\}$$

as topological spaces. As such, we write $\tau = (\tau_1, \dots, \tau_m) \in \widehat{\mathrm{SO}(2m)}$ to specify an irreducible representation where the constraints on the integers are understood. Note that, if $4 \mid n$ or equivalently $2 \mid m$, then τ is self-dual; otherwise, $\tau^* = (\tau_1, \dots, \tau_{m-1}, -\tau_m)$. For the moment, let us view $\mathrm{SO}(2m-1) \subset \mathrm{SO}(2m)$ as a subgroup and let $\sigma = (\sigma_1, \dots, \sigma_{m-1}) \in \widehat{\mathrm{SO}(2m-1)}$ (see Case $2 \nmid n$ below). Then, the associated branching law says that σ is contained in τ if and only if they satisfy the interlacing property

$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \dots \geq \tau_{m-1} \geq \sigma_{m-1} \geq |\tau_m|;$$

and in which case the multiplicity is 1.

2.4.2. Case $2 \nmid n$. Let us write $n = 2m + 1$ for $m \in \mathbb{Z}_{\geq 0}$. The unitary dual $\widehat{\mathrm{SO}(2m+1)}$ can be canonically identified with

$$\{(\tau_1, \dots, \tau_m) \in \mathbb{Z}^m : \tau_1 \geq \tau_2 \geq \dots \geq \tau_m \geq 0\}$$

as topological spaces; for $m = 0$, we use the convention that the latter is a singleton containing the empty tuple. Again, we write $\tau = (\tau_1, \dots, \tau_m) \in \widehat{\mathrm{SO}(2m+1)}$ to specify an irreducible representation. Note that τ is always self-dual. Again for the moment, let us view $\mathrm{SO}(2m) \subset \mathrm{SO}(2m+1)$ as a subgroup and let $\sigma = (\sigma_1, \dots, \sigma_m) \in \widehat{\mathrm{SO}(2m)}$ (see Case $2 \mid n$ above). Then, the associated branching law says that σ is contained in τ if and only if they satisfy the interlacing property

$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \dots \geq \tau_{m-1} \geq \sigma_{m-1} \geq \tau_m \geq |\sigma_m|;$$

and in which case the multiplicity is 1.

2.5. Unitary representation theory of G . The necessary background covered in this subsection is contained in [Kna01] with further details in references therein. Working in the *compact picture* (in the terminology of [Kna01, Chapter VII, §1]), for all $s \in \mathbb{C}$, define the representation $(U^s, L^2(K))$ of G by

$$U^s(g)(\phi)(k) = e^{-sH(g^{-1}k)} \phi(\kappa(g^{-1}k)) \quad \text{for all } k \in K, \phi \in L^2(K), \text{ and } g \in G.$$

Though the latter is not obvious in the compact picture, we have the following facts:

- $(U^s|_K, L^2(K)) = (\lambda_K, L^2(K))$ is the left regular representation of K ;
- $(U^s, L^2(K))$ is a *unitary* representation if and only if $s \in d/2 + i\mathbb{R}$.

Recall that $P := MAN \cong M \times A \times N$. For any $s \in \mathbb{C}$, let $\chi_s : A \rightarrow \mathbb{C}$ be the character defined by $\chi_s(a_t) = e^{st}$ for all $t \in \mathbb{R}$. For any $\sigma \in \widehat{M}$ and $s \in \mathbb{C}$, viewing $\sigma \otimes \chi_s \otimes \mathbf{1}$ as a representation of P , we may consider the induced representation $\mathrm{Ind}_P^G(\sigma \otimes \chi_s \otimes \mathbf{1})$; this is called parabolic induction (see [Kna01, Chapter VII, §1]). If $s \in i\mathbb{R}$, they are unitary representations called the *principal series*, and they are irreducible if $s \neq 0$ (see [KS71, Theorem 5]). More generally, the isomorphism classes of the so-called irreducible admissible representations of G (strictly containing the unitary dual \widehat{G}) arise as unique quotients of $\mathrm{Ind}_P^G(\sigma \otimes \chi_s \otimes \mathbf{1})$ with $\sigma \in \widehat{M}$ and $s \in \mathbb{C}$; these are called the Langlands quotients. Moreover, they are tempered only if $s \in i\mathbb{R}$.

We refer the reader to [Kna01, Chapter XIV, §17, Theorem 14.92], [Kna01, Chapter VIII, §14, Theorem 8.53], and [Lan89] for more details. For all $\sigma \in \widehat{M}$, define

$$L^2(K : \sigma) := \left\{ \phi \in L^2(K) : \int_M \chi_\sigma(m) \rho_K(m) \phi dm = \phi \right\}.$$

It turns out that for all $\sigma \in \widehat{M}$ and $s \in \mathbb{C}$, we may identify

$$(U^s, L^2(K : \sigma)) \cong \text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})^{\oplus \dim(\sigma)}. \quad (2.2)$$

We now introduce an intertwining operator which plays a central role in the classification of \widehat{G} [KS67, KS71, Kna01]. Recall that $\widehat{M}_{s-d} \subset \widehat{M}$ denotes the subset of self-dual elements. Let $\sigma \in \widehat{M}_{s-d}$. For all $s > d/2$, define a bounded operator $\mathcal{A}(\sigma, s) : L^2(K : \sigma) \rightarrow L^2(K : \sigma)$ by

$$\mathcal{A}(\sigma, s) = \sigma(w_0) \gamma_\sigma(s)^{-1} \int_{\overline{N}} e^{-sH(\overline{n})} \rho_G(w_0 \kappa(\overline{n})^{-1}) d\overline{n}$$

where $w_0 \in N_K(A)/M$ is the unique nontrivial Weyl element, $\sigma(w_0)$ is a unitary operator according to the unique extension (up to sign) of σ from M to $N_K(A)$, and $\gamma_\sigma : \mathbb{C} \rightarrow \mathbb{C}$ is a certain meromorphic normalizing function. It enjoys the intertwining property

$$\mathcal{A}(\sigma, s) U^s(g) = U^{-s}(g) \mathcal{A}(\sigma, s) \quad \text{for all } g \in G. \quad (2.3)$$

Abusing notation, we also denote by $\mathcal{A}(\sigma, s)$ the induced operator on $\text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})$ via any isomorphic subrepresentation of $L^2(K : \sigma)$.

Let $\sigma \in \widehat{M}$. Writing $\sigma = (\sigma_1, \dots, \sigma_{\lfloor d/2 \rfloor})$, define $1 \leq \ell(\sigma) \leq \lfloor d/2 \rfloor$ to be the largest index with a positive integer coordinate and put $\ell(\sigma) = 0$ if σ is the trivial irreducible representation $\mathbf{1} \in \widehat{M}$. Define the corresponding half-closed interval

$$\mathcal{I}_\sigma := (d/2, d - \ell(\sigma)] \subset (d/2, d].$$

The following is a theorem of Knapp–Stein [KS71, Theorem 6, and Propositions 49 and 50].

Theorem 2.4. *Let $\sigma \in \widehat{M}$ and $s \in \mathbb{R}$. Then, $(U^s, L^2(K : \sigma))$ is unitarizable with a semidefinite inner product if and only if σ is self-dual and $s \in \mathcal{I}_\sigma$. Additionally, such an inner product is positive definite if and only if $s \in \text{int } \mathcal{I}_\sigma$. The semidefinite inner product $\langle \cdot, \cdot \rangle_{(\sigma, s)}$ on $L^2(K : \sigma)$ can be defined by*

$$\langle u, v \rangle_{(\sigma, s)} = \langle \mathcal{A}(\sigma, s) u, v \rangle_{L^2(K)} \quad \text{for all } u, v \in L^2(K : \sigma). \quad (2.4)$$

Remark 2.5. Indeed, unitarity with respect to the (possibly semidefinite) inner product follows from the intertwining property of $\mathcal{A}(\sigma, s)$.

Let $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathbb{R}$. Denote by

$$\mathcal{Z}(\sigma, s) := \{v \in L^2(K : \sigma) : \|v\|_{(\sigma, s)} = 0\} \subset L^2(K : \sigma)$$

the subspace of null vectors in $L^2(K : \sigma)$ for the seminorm $\|\cdot\|_{(\sigma, s)}$ corresponding to the semidefinite inner product $\langle \cdot, \cdot \rangle_{(\sigma, s)}$, which is nontrivial if and only if $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$. Then, observe that $(U^s, \ker \mathcal{A}(\sigma, s)) = (U^s, \mathcal{Z}(\sigma, s))$ and is a subrepresentation of $(U^s, L^2(K : \sigma))$.

We have the following theorem according to the classification results of Hirai [Hir62]. We also refer the reader to [KZ77, Pg 152] for a deduction of this result from the general results of Langlands [Lan89] and Knapp–Stein [KS71].

Theorem 2.6. *The non-tempered elements of \widehat{G} are exhausted by the quasi-complementary series which is itself the union of the two families in the following list:*

(1) *the complementary series consisting of*

$$(\pi_{\sigma, s}, \mathcal{U}(\sigma, s)) := \text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1}) \quad \text{for } \sigma \in \widehat{M}_{s-d} \text{ and } s \in \text{int } \mathcal{I}_\sigma;$$

(2) *the ends of complementary series consisting of*

$$(\pi_{\sigma, s}, \mathcal{U}(\sigma, s)) := \text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1}) / \ker \mathcal{A}(\sigma, s) \quad \text{for } \sigma \in \widehat{M}_{s-d} \text{ and } s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma;$$

and in both cases equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}(\sigma, s)}$ induced by the (possibly semidefinite) inner product $\langle \cdot, \cdot \rangle_{(\sigma, s)}$ on $L^2(K : \sigma)$.

Remark 2.7. Indeed, since $\ker \mathcal{A}(\sigma, s) = \mathcal{Z}(\sigma, s)$, the inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}(\sigma, s)}$ is positive-definite for the ends of complementary series $\mathcal{U}(\sigma, s)$ for $\sigma \in \widehat{M}_{s-d}$ and $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$.

When $\sigma = \mathbf{1} \in \widehat{M}_{s-d}$, then there exists a K -invariant vector and so these are called the spherical quasi-complementary series. The remaining ones are called the non-spherical quasi-complementary series. In particular, as G is non-compact, the trivial irreducible representation is not tempered and arises as the end of spherical complementary series $\mathbf{1} = (\pi_{\mathbf{1}, d}, \mathcal{U}(\mathbf{1}, d)) \in \widehat{G}$. Note that for $\sigma \in \widehat{M}_{s-d}$ and $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$, the intertwining operator further descends to an operator on $\mathcal{U}(\sigma, s)$ which, again by abuse of notation, we also denote by $\mathcal{A}(\sigma, s)$.

In light of Eq. (2.2), we deduce the following elementary but crucial corollary of Theorem 2.6.

Corollary 2.8. *We have the following as unitary representations of G :*

- (1) $(\pi_{\sigma, s}, \mathcal{U}(\sigma, s))$ is contained in $(U^s, L^2(K : \sigma))$ for all $\sigma \in \widehat{M}_{s-d}$ and $s \in \text{int } \mathcal{I}_\sigma$;
- (2) $(\pi_{\sigma, s}, \mathcal{U}(\sigma, s))$ is contained in $(U^s, L^2(K : \sigma) / \ker \mathcal{A}(\sigma, s))$ for all $\sigma \in \widehat{M}_{s-d}$ and $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$.

Remark 2.9. We warn the reader that for the ends of complementary series $(\pi_{\sigma, s}, \mathcal{U}(\sigma, s))$ for $\sigma \in \widehat{M}_{s-d}$ and $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$, we do not have the stronger form: $(\pi_{\sigma, s}, \mathcal{U}(\sigma, s))$ is contained in $(U^s, L^2(K : \sigma))$. Nevertheless, the weaker form suffices for our purposes; see the lifting argument in the proof of Corollary 3.6.

As a consequence of Frobenius reciprocity (cf. [War72a, Chapter 5, §5.5.1]), for any $\sigma \in \widehat{M}$ and $s \in \mathbb{C}$, the multiplicity of a K -type $\tau \in \widehat{K}$ occurring in $\text{Ind}_P^G(\sigma \otimes \chi_s \otimes \mathbf{1})$ is exactly the multiplicity of σ occurring in τ . Therefore, τ is contained in $\text{Ind}_P^G(\sigma \otimes \chi_s \otimes \mathbf{1})$ if and only if σ is contained in τ , and in which case both multiplicities are 1. Now, for $\sigma \in \widehat{M}_{s-d}$, we apply Theorem 2.6 to conclude the following. For all $s \in \text{int } \mathcal{I}_\sigma$, the previous characterization applies for the K -types in the complementary series $\mathcal{U}(\sigma, s)$. For $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$, further care is needed: a K -type $\tau \in \widehat{K}$ is contained in the end of complementary series $\mathcal{U}(\sigma, s)$ if and only if σ is contained in τ and τ is not contained in $\ker \mathcal{A}(\sigma, s)$, and in which case the multiplicity is 1. This is not a simple criteria; however, we have a result of Vogan quoted in Theorem 2.12 which will be useful. For this, we need to introduce the concept of minimal K -types as defined in loc. cit.; for convenience, we write it explicitly for our setting where $K \cong \text{SO}(d+1)$ (cf. [Thi73, §4]).

Definition 2.10 (Minimal K -type). A K -type $\tau = (\tau_1, \dots, \tau_{\lceil d/2 \rceil}) \in \widehat{K}$ of a unitary representation (π, \mathcal{H}) of G is called a *minimal K -type* of (π, \mathcal{H}) if

$$\lambda_\tau := \sum_{j=1}^{\lceil d/2 \rceil} \left(\tau_j + \frac{d+1-2j}{2} \right)^2 \quad (2.5)$$

is minimized among all the K -types contained in (π, \mathcal{H}) .

Remark 2.11. In light of the formula for the (intrinsic) Casimir eigenvalue [Kna96, Chapter 5, §4, Proposition 5.28], the above definition is equivalent to the condition that the (intrinsic) Casimir eigenvalue is minimized among all the K -types contained in (π, \mathcal{H}) .

The following is the result of Vogan from [Kna01, Chapter XV, §3, Theorem 15.10].

Theorem 2.12. *Let $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$. All minimal K -types of $\text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})$ are contained in $\mathcal{U}(\sigma, s)$.*

Recall the decomposition $\widehat{G} = \widehat{G}_{\text{temp}} \sqcup \widehat{G}_{\text{q-comp}}$. We further have the identification

$$\widehat{G}_{\text{q-comp}} \cong \{(\sigma, s) : \sigma \in \widehat{M}_{s-d}, s \in \mathcal{I}_\sigma\}$$

as topological spaces, where \widehat{G} is equipped with the Fell topology and the right hand side is equipped with the standard topology.

Let (π, \mathcal{H}) be a unitary representation of G . Using the above decomposition, together with the direct integral decomposition from Subsection 2.2, we obtain a corresponding decomposition of the measure $m_{\text{q-comp}}$ into measures $\{m_\sigma\}_{\sigma \in \widehat{M}_{s-d}}$ such that

$$\text{supp}(m_\sigma) \subset \mathcal{I}_\sigma \quad \text{for all } \sigma \in \widehat{M}_{s-d},$$

and

$$(\pi, \mathcal{H}) = \int_{\widehat{G}}^{\oplus} (\pi_{\xi}, \mathcal{H}_{\xi}) dm(\xi) = \int_{\widehat{G}_{\text{temp}}}^{\oplus} (\pi_{\xi}, \mathcal{H}_{\xi}) dm_{\text{temp}}(\xi) \oplus \widehat{\bigoplus}_{\sigma \in \widehat{M}_{s-d}} \int_{\mathcal{I}_{\sigma}}^{\oplus} (\pi_{\sigma,s}, \mathcal{H}(\sigma, s)) dm_{\sigma}(s) \quad (2.6)$$

where each $(\pi_{\xi}, \mathcal{H}_{\xi})$ is isotypic of type $\xi \in \widehat{G}$.

By definition of the direct integral, we obtain the following explicit decomposition for any vector $\phi \in \mathcal{H}$: there exists a measurable section $\{\phi_{\xi}\}_{\xi \in \widehat{G}} = \{\phi_{\xi}\}_{\xi \in \widehat{G}_{\text{temp}}} \sqcup \{\phi_{\sigma,s}\}_{\sigma \in \widehat{M}_{s-d}, s \in \mathcal{I}_{\sigma}}$ with $\phi_{\xi} \in \mathcal{H}_{\xi}$ and $\phi_{\sigma,s} \in \mathcal{H}(\sigma, s)$, such that

$$\phi = \int_{\widehat{G}}^{\oplus} \phi_{\xi} dm(\xi) = \int_{\widehat{G}_{\text{temp}}}^{\oplus} \phi_{\xi} dm_{\text{temp}}(\xi) + \sum_{\sigma \in \widehat{M}_{s-d}} \int_{\mathcal{I}_{\sigma}}^{\oplus} \phi_{\sigma,s} dm_{\sigma}(s). \quad (2.7)$$

Finally, we record the following useful lemma on density of smooth compactly supported functions of a fixed K -type. Recall the projection operators which we obtain for any $L \in \{K, M\}$ and denote by P_{τ} for any $\tau \in \widehat{K}$ or P_{σ} for any $\sigma \in \widehat{M}$.

Lemma 2.13. *Let $\tau \in \widehat{K}$ be a K -type. The subspace $C_c^{\infty}(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)_{\tau}$ is dense in $L^2(\Gamma \backslash G)_{\tau}$.*

Proof. Let $\tau \in \widehat{K}$ be a K -type, $\tilde{\phi} \in L^2(\Gamma \backslash G)_{\tau} \subset L^2(\Gamma \backslash G)$, and $\epsilon > 0$. Since $C_c^{\infty}(\Gamma \backslash G) \subset L^2(\Gamma \backslash G)$ is dense, there exists $\phi \in C_c^{\infty}(\Gamma \backslash G)$ such that $\|\phi - \tilde{\phi}\|_{L^2(\Gamma \backslash G)} < \epsilon$. We now apply P_{τ} . Firstly, since $\tilde{\phi} \in L^2(\Gamma \backslash G)_{\tau}$, we have $P_{\tau}\tilde{\phi} = \tilde{\phi}$. Secondly, it follows from the definition of P_{τ} in Eq. (2.1) that $P_{\tau}\phi$ remains as a compactly supported smooth function. Therefore, applying P_{τ} gives $P_{\tau}\phi \in C_c^{\infty}(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)_{\tau}$ with

$$\|P_{\tau}\phi - \tilde{\phi}\|_{L^2(\Gamma \backslash G)} \leq \|\phi - \tilde{\phi}\|_{L^2(\Gamma \backslash G)} < \epsilon$$

as desired. ■

3. ASYMPTOTIC EXPANSION OF MATRIX COEFFICIENTS IN TERMS OF THE HARISH-CHANDRA C-FUNCTION AND LAPLACE TRANSFORMS

In this section, we recount a purely representation theoretic asymptotic expansion of matrix coefficients of Edwards–Oh [EO21]. We then use it to prove Proposition 3.8, which is a general result on analytic continuations of Laplace transforms of matrix coefficients. Roughly speaking, combined with the results of the next section, Proposition 3.8 characterizes weak containment of quasi-complementary series within a given representation of G in terms of failure of holomorphic extension of the Laplace transform to certain subsets of the negative real line corresponding to the set of parameters of the quasi-complementary series in question.

3.1. Intertwining operators, Eisenstein integrals, and the Harish-Chandra C-function. We define constants related to the intertwining operator introduced previously, and define other important operators.

Let $\sigma \in \widehat{M}_{s-d}$ and $s > d/2$. From the intertwining property in Eq. (2.3), we know that $\mathcal{A}(\sigma, s)$ intertwines with λ_K . Consequently, by Schur's lemma, $\mathcal{A}(\sigma, s)$ acts as a scalar on each K -type of $L^2(K : \sigma)$. By the characterization of the K -types which occur in $\text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})$, we may write

$$\mathcal{A}(\sigma, s) = \sum_{\tau \in \widehat{K} : \tau \supset \sigma} a(\sigma, s, \tau) P_{\tau}$$

as operators on $L^2(K : \sigma)$, for some set of scalars $\{a(\sigma, s, \tau)\}_{\tau \in \widehat{K} : \tau \supset \sigma} \subset \mathbb{C}$. Positive semi-definiteness of the inner product $\langle \cdot, \cdot \rangle_{(\sigma, s)}$ implies that for all $\tau \in \widehat{K}$ containing σ , the scalars are real and non-negative. We also deduce from the intertwining property that $(\lambda_K, \ker \mathcal{A}(\sigma, s))$ is a unitary representation of K . Note that using $\ker \mathcal{A}(\sigma, s) = \mathcal{Z}(\sigma, s)$, for all K -types $\tau \in \widehat{K}$ containing σ , we have

$$a(\sigma, s, \tau) = 0 \iff \tau \subset (\lambda_K, \ker \mathcal{A}(\sigma, s)).$$

Let $\sigma \in \widehat{M}$. For each K -type $\tau \in \widehat{K}$ of $L^2(K : \sigma)$, we use any orthonormal basis $\{v_j\}_{j=1}^{\dim(\sigma)\dim(\tau)} \subset L^2(K : \sigma)_{\tau}$ to define a corresponding special vector $\omega_{\tau} \in L^2(K : \sigma)_{\tau}$ defined by

$$\omega_{\tau} = \sum_{j=1}^{\dim(\sigma)\dim(\tau)} \overline{v_j(e)} v_j.$$

Then, for each vector $v \in L^2(K : \sigma)_\tau$, we have the identity

$$v(k) = \langle v, \lambda_K(k) \omega_\tau \rangle_{L^2(K)} \quad \text{for all } k \in K.$$

For any pair of K -types $\tau_1, \tau_2 \in \widehat{K}$, define an operator $\mathbb{T}_{\tau_1}^{\tau_2} : L^2(K : \sigma)_{\tau_1} \rightarrow L^2(K : \sigma)_{\tau_2}$ by

$$\mathbb{T}_{\tau_1}^{\tau_2} v = \int_M v(m) \lambda_K(m) \omega_{\tau_2} dm \quad \text{for all } v \in L^2(K : \sigma)_{\tau_1}.$$

These are called *Eisenstein integrals*. Note that $\mathbb{T}_{\tau_1}^{\tau_2}$ intertwines with λ_M . In particular, for $\tau_1 = \tau_2 = \tau$, we have by Schur's lemma that \mathbb{T}_τ^τ preserves M -types. Then, for any $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$, the operator \mathbb{T}_τ^τ preserves $\ker \mathcal{A}(\sigma, s)$ and hence descends to an operator on $\mathcal{U}(\sigma, s)_\tau$. We also have the following lemma.

Lemma 3.1 ([EO21, Corollary 3.9]). *Let $\sigma \in \widehat{M}$. Let $\tau \in \widehat{K}$ be a K -type of $L^2(K : \sigma)$. Then, we have*

$$\mathbb{T}_\tau^\tau|_{L^2(K:\sigma)_\tau} = \frac{\dim(\tau)}{\dim(\sigma)} \mathbb{P}_{\sigma^*}|_{L^2(K:\sigma)_\tau}.$$

When $s \in \mathcal{I}_\sigma$, the same holds on $\mathcal{U}(\sigma, s)$.

Definition 3.2 (Harish-Chandra C-function). Let $\sigma \in \widehat{M}$. For all $s > d/2$, define a bounded operator $C_+(s) : L^2(K : \sigma) \rightarrow L^2(K : \sigma)$, by

$$C_+(s) = \int_{\overline{N}} e^{-sH(\overline{n})} U^s(\kappa(\overline{n})^{-1}) d\overline{n}.$$

The bounded operator-valued function given by $s \mapsto C_+(s)$ is called the *Harish-Chandra C-function*.

Indeed, $C_+(s)$ is a well-defined bounded operator for all $s > d/2$ since U^s is unitary and

$$\int_{\overline{N}} e^{-sH(\overline{n})} d\overline{n} < +\infty$$

by [HC58, Corollary of Lemma 45] (see [HC58, pp. 303] for an explicit formula in terms of the Γ function). The Harish-Chandra C-function $C_+(s)$ intertwines with $U^s|_M = \lambda_M$ (see the proof of [EO21, Lemma 4.2]). When $s \in \mathcal{I}_\sigma$, it descends as above to a bounded operator on $\mathcal{U}(\sigma, s)$. By Schur's lemma we may record a slightly refined version of [EO21, Lemma 4.2].

Lemma 3.3. *Let $\sigma \in \widehat{M}$ and $s > d/2$. The Harish-Chandra C-function $C_+(s)$ acts as a scalar on each irreducible M -submodule of $L^2(K : \sigma)$. In particular, they preserve the M - and K -types of $L^2(K : \sigma)$. Consequently, when $s \in \mathcal{I}_\sigma$, the same holds on $\mathcal{U}(\sigma, s)$.*

3.2. Asymptotic expansion of matrix coefficients following Edwards–Oh. For $s \in (d/2, d]$, we set

$$\eta_s := \min \{2s - d, 1\} > 0. \quad (3.1)$$

We recall the following expansion of matrix coefficients due to Edwards and Oh.

Theorem 3.4 ([EO21, Theorem 4.3]). *Let I be a compact subset of $(d/2, d]$. Let $\sigma \in \widehat{M}$, $s \in \mathcal{I}_\sigma \cap I$, and $\tau_1, \tau_2 \in \widehat{K}$ be K -types. For all $u \in L^2(K : \sigma)_{\tau_1}$, $v \in L^2(K : \sigma)_{\tau_2}$, and $t \geq 0$, we have*

$$\langle \pi_{\sigma, s}(a_t)u, v \rangle_{L^2(K)} = e^{-(d-s)t} \langle \mathbb{T}_{\tau_1}^{\tau_2} C_+(s)u, v \rangle_{L^2(K)} + O_I \left(e^{-(d-s+\eta_s)t} \|\mathbb{T}_{\tau_1}^{\tau_2}\|_{L^2(K)} \|u\|_{L^2(K)} \|v\|_{\mathcal{S}_K^1(L^2(K))} \right).$$

Remark 3.5. Although the original statement and proof of [EO21, Theorem 4.3] is for $\mathcal{U}(\sigma, s)_{\tau_1}$ and $\mathcal{U}(\sigma, s)_{\tau_2}$ for $\sigma \in \widehat{M}_{s-d}$ and $s \in \text{int } \mathcal{I}_\sigma$, the proof for $L^2(K : \sigma)_{\tau_1}$ and $L^2(K : \sigma)_{\tau_2}$ goes through verbatim.

Moreover, the uniformity of the implicit constant in loc. cit. is stated for compact subsets I of the open interval $(d/2, d)$; however, their proof shows that such uniformity continues to hold for compact subsets of the half-closed interval $(d/2, d]$.

Specializing Theorem 3.4 to the case $\tau_1 = \tau_2 = \tau$, we have the following corollary.

Corollary 3.6. *Let I be a compact subset of $(d/2, d]$. Let $\sigma \in \widehat{M}_{s-d}$, and $s \in \mathcal{I}_\sigma \cap I$. Let $(\pi, \mathcal{H}) = (\pi_{\sigma, s}, \mathcal{U}(\sigma, s)) \widehat{\otimes} (\mathbf{1}, \mathcal{M}(\sigma, s))$ be a unitary representation of G , with $\mathcal{M}(\sigma, s)$ a (possibly infinite dimensional) separable multiplicity space. Then, for any K -type $\tau \in \widehat{K}$ containing σ , for all $\phi, \psi \in \mathcal{H}_\tau$ and $t \geq 0$, we have*

$$\langle \pi(a_t)\phi, \psi \rangle_{\mathcal{H}} = e^{-(d-s)t} \langle \mathbb{T}_\tau^\tau C_+(s)\phi, \psi \rangle_{\mathcal{H}} + O_{I, \tau} \left(e^{-(d-s+\eta_s)t} \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \right). \quad (3.2)$$

Proof. Let I , σ , s , and τ be as in the lemma. We first obtain the desired result for the unitary representation $(\pi_{\sigma,s}, \mathcal{U}(\sigma, s))$ of G , i.e., assuming $\mathcal{M}(\sigma, s)$ is a 1-dimensional multiplicity space. First, we treat the complementary series, i.e., the case $s \in \text{int } \mathcal{I}_\sigma$. This case is simpler since we may fix an embedding $(\pi_{\sigma,s}, \mathcal{U}(\sigma, s)) \subset (U^s, L^2(K : \sigma)) \subset (U^s, L^2(K))$ by Corollary 2.8. From Eq. (2.4), for any $u \in \mathcal{U}(\sigma, s)_\tau$ and $v \in \mathcal{U}(\sigma, s)$, we have

$$\langle u, v \rangle_{\mathcal{U}(\sigma, s)} = a(\sigma, s, \tau) \langle u, v \rangle_{L^2(K)}.$$

Letting $u, v \in \mathcal{U}(\sigma, s)_\tau$, we can use Theorem 3.4 and then the above identity to obtain

$$\begin{aligned} & \langle \pi_{\sigma,s}(a_t)u, v \rangle_{\mathcal{U}(\sigma, s)} \\ &= a(\sigma, s, \tau) \langle \pi_{\sigma,s}(a_t)u, v \rangle_{L^2(K)} \\ &= e^{-(d-s)t} a(\sigma, s, \tau) \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(s)u, v \rangle_{L^2(K)} + O_I \left(e^{-(d-s+\eta_s)t} a(\sigma, s, \tau) \|\mathbb{T}_\tau^\tau\|_{L^2(K)} \|u\|_{L^2(K)} \|v\|_{\mathcal{S}_K^1(L^2(K))} \right) \\ &= e^{-(d-s)t} \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(s)u, v \rangle_{\mathcal{U}(\sigma, s)} + O_I \left(e^{-(d-s+\eta_s)t} a(\sigma, s, \tau) \|\mathbb{T}_\tau^\tau\|_{L^2(K)} \|u\|_{L^2(K)} \|v\|_{\mathcal{S}_K^1(L^2(K))} \right). \end{aligned}$$

By Lemma 2.3, there exists a constant $C(\tau) > 0$ such that $\|v\|_{\mathcal{S}_K^1(L^2(K))} \leq C(\tau) \|v\|_{L^2(K)}$. Since $\|\mathbb{T}_\tau^\tau\|_{L^2(K)}$ is $O_\tau(1)$ and $a(\sigma, s, \tau) \|v\|_{L^2(K)}^2 = \|v\|_{\mathcal{U}(\sigma, s)}^2$, we have the bound

$$a(\sigma, s, \tau) \|\mathbb{T}_\tau^\tau\|_{L^2(K)} \|u\|_{L^2(K)} \|v\|_{\mathcal{S}_K^1(L^2(K))} \ll_\tau \|u\|_{\mathcal{U}(\sigma, s)} \|v\|_{\mathcal{U}(\sigma, s)}.$$

Thus, we get

$$\langle \pi_{\sigma,s}(a_t)u, v \rangle_{\mathcal{U}(\sigma, s)} = e^{-(d-s)t} \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(s)u, v \rangle_{\mathcal{U}(\sigma, s)} + O_{I,\tau} \left(e^{-(d-s+\eta_s)t} \|u\|_{\mathcal{U}(\sigma, s)} \|v\|_{\mathcal{U}(\sigma, s)} \right). \quad (3.3)$$

Now, we treat the ends of complementary series, i.e., the case $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$. In this case, although we do not have the same embedding as in the first case, we have $(\pi_{\sigma,s}, \mathcal{U}(\sigma, s)) \subset (U^s, L^2(K : \sigma) / \ker \mathcal{A}(\sigma, s))$ from Corollary 2.8. For any given $u, v \in \mathcal{U}(\sigma, s)_\tau$, we may take *arbitrary* lifts $\tilde{u}, \tilde{v} \in L^2(K : \sigma)_\tau \subset L^2(K)$ and carry out the same argument as above with $u, v, \langle \cdot, \cdot \rangle_{\mathcal{U}(\sigma, s)}$, and $\|\cdot\|_{\mathcal{U}(\sigma, s)}$, replaced with $\tilde{u}, \tilde{v}, \langle \cdot, \cdot \rangle_{(\sigma, s)}$, and $\|\cdot\|_{(\sigma, s)}$, respectively, to obtain the analog of Eq. (3.3) for \tilde{u} and \tilde{v} . Finally, observe that Eq. (3.3) then holds for u and v since the inner products and norms descend to $\mathcal{U}(\sigma, s)$ as we have $(U^s, \ker \mathcal{A}(\sigma, s)) = (U^s, \mathcal{Z}(\sigma, s))$ and is a subrepresentation of $(U^s, L^2(K : \sigma))$.

Now, consider the more general unitary representation $(\pi, \mathcal{H}) = (\pi_{\sigma,s}, \mathcal{U}(\sigma, s)) \widehat{\otimes} (\mathbf{1}, \mathcal{M}(\sigma, s))$ of G , with $\mathcal{M}(\sigma, s)$ a separable multiplicity space. Let $\{v_j\}_{j=1}^n$ for some $n \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ be a countable orthonormal Hilbert basis for $\mathcal{M}(\sigma, s)$. Let $\phi, \psi \in \mathcal{H}_\tau$. They have decompositions $\phi = \sum_{i=1}^n \phi_i \otimes v_i$ and $\psi = \sum_{j=1}^n \psi_j \otimes v_j$, with $\phi_i, \psi_j \in \mathcal{U}(\sigma, s)_\tau$ and $\|\phi\|_{\mathcal{H}}^2 = \sum_{i=1}^n \|\phi_i\|_{\mathcal{U}(\sigma, s)}^2 < +\infty$ and $\|\psi\|_{\mathcal{H}}^2 = \sum_{j=1}^n \|\psi_j\|_{\mathcal{U}(\sigma, s)}^2 < +\infty$. Since $\{v_j\}_{j=1}^n$ is orthonormal and G acts trivially on $\mathcal{M}(\sigma, s)$ we have

$$\begin{aligned} \langle \pi(a_t)\phi, \psi \rangle_{\mathcal{H}} &= \left\langle \sum_{i=1}^n \pi_{\sigma,s}(a_t)\phi_i \otimes v_i, \sum_{j=1}^n \psi_j \otimes v_j \right\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle \pi_{\sigma,s}(a_t)\phi_i, \psi_j \rangle_{\mathcal{U}(\sigma, s)} \cdot \langle v_i, v_j \rangle_{\mathcal{M}} \\ &= \sum_{j=1}^n \langle \pi_{\sigma,s}(a_t)\phi_j, \psi_j \rangle_{\mathcal{U}(\sigma, s)}. \end{aligned}$$

The corollary now follows by applying Eq. (3.3) to each summand. ■

The following proposition is essentially known in the literature, though it is difficult to pinpoint a reference dealing with the full quasi-complementary series. We provide a proof for the sake of completeness.

Proposition 3.7. *Let $s_* \in [d/2, d)$. Let (π, \mathcal{H}) be a unitary representation of G which does not weakly contain any quasi-complementary series $\mathcal{U}(\sigma, s)$ with $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma \cap (s_*, d]$. Then, for all $\epsilon > 0$ and K -finite vectors $u, v \in \mathcal{H}$, we have the following bound on matrix coefficients:*

$$\langle \pi(a_t)u, v \rangle_{\mathcal{H}} \ll_G (1+t) e^{-(d-s_*)t} (\dim \langle \pi(K)u \rangle \dim \langle \pi(K)v \rangle)^{1/2} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \quad \text{for all } t > 0.$$

Proof. In the proof, we will encounter the spherical functions $\varphi_s : G \rightarrow \mathbb{R}$ for any $s \in \mathbb{R}$ defined by

$$\varphi_s(g) = \int_K e^{-sH(g^{-1}k)} dk \quad \text{for all } g \in G.$$

They coincide with the matrix coefficient of $(U^s, L^2(K : \mathbf{1}))$ corresponding to $\mathbf{1}_K \in L^2(K : \mathbf{1})$ (or corresponding to any K -invariant unit vector in $\mathcal{U}(\mathbf{1}, s)$; see [HC58, Lemma 5]). The particular spherical function $\Xi := \varphi_{d/2}$ is called the *Harish-Chandra function*. We refer the reader to [Kna01, Chapter VII, §8] and [GV88, Chapter 4, §4.6] for further details on spherical functions. In particular, we shall use their bounds which are originally due to Harish-Chandra [HC58, Theorem 3].

Let s_* and (π, \mathcal{H}) be as in the proposition. By definition, we have the orthogonal decomposition $\mathcal{H} = \mathcal{H}_{\text{temp}} \oplus \mathcal{H}_{\text{q-comp}}$ into the tempered part $\mathcal{H}_{\text{temp}}$ and the non-tempered part $\mathcal{H}_{\text{q-comp}}$. We treat each part separately.

We first treat the tempered part. Let $u, v \in \mathcal{H}_{\text{temp}}$ be K -finite vectors. Then, by a result of Cowling–Haagerup–Howe [CHH88, Corollary] and bounds on the Harish-Chandra function, we have

$$|\langle \pi(a_t)u, v \rangle_{\mathcal{H}}| \leq \Xi(a_t) (\dim \langle \pi(K)u \rangle \dim \langle \pi(K)v \rangle)^{1/2} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \quad (3.4)$$

$$\ll_G (1+t)e^{-(d/2)t} (\dim \langle \pi(K)u \rangle \dim \langle \pi(K)v \rangle)^{1/2} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \quad \text{for all } t > 0. \quad (3.5)$$

Now we treat the non-tempered part. By the classification result quoted in Theorem 2.6, only the quasi-complementary series can be weakly contained in $\mathcal{H}_{\text{q-comp}}$. We first establish Eq. (3.7) for all quasi-complementary series $\mathcal{U}(\sigma, s)$ with parameters $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_{\sigma} \setminus (s_*, d]$. As the ends of complementary series can be handled using a lifting argument as in the proof of Corollary 3.6, we restrict our attention to the complementary series. To this end, let $\sigma \in \widehat{M}_{s-d}$ and $s \in \text{int } \mathcal{I}_{\sigma} \setminus (s_*, d]$, and let $u, v \in \mathcal{U}(\sigma, s)$ be K -finite vectors. By [Kna01, Chapter VII, §8, Proposition 7.14], we have

$$|\langle \pi(a_t)u, v \rangle_{L^2(K)}| \ll_{u,v} \varphi_s(a_t).$$

Define the subset

$$\mathcal{T} := \{\tau \in \widehat{K} : \tau \supset \sigma, P_{\tau}(u) \neq 0, P_{\tau}(v) \neq 0\} \subset \widehat{K}.$$

By K -finiteness of u and v , the subset \mathcal{T} is finite. Converting to the unitary inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}(\sigma, s)}$ on $\mathcal{U}(\sigma, s)$ and again using bounds on spherical functions, we have

$$|\langle \pi_{\sigma, s}(a_t)u, v \rangle_{\mathcal{U}(\sigma, s)}| \ll_{u,v} \#\mathcal{T} \cdot \max_{\tau \in \mathcal{T}} a(\sigma, s, \tau) \cdot \varphi_s(a_t) \quad (3.6)$$

$$\ll_{u,v,s} (1+t)e^{-(d-s)t} \quad \text{for all } t > 0. \quad (3.7)$$

Combining Eq. (3.7) with the fact from [Hel00, Chapter I, §5, Theorem 5.8] that the Haar measure on G is given by

$$dg = \mu_M(M)^{-1} \sinh(t)^d dk dt dk$$

and using the asymptotics $\sinh(t)^d \asymp e^{dt}$ for $t \gg 1$, we conclude that $\mathcal{U}(\sigma, s)$ is strongly L^p for $p := \frac{d}{d-s_*}$ (in the terminology of Shalom [Sha00]) for any $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_{\sigma} \setminus (s_*, d]$, i.e., the matrix coefficient $G \rightarrow \mathbb{C}$ given by $g \mapsto \langle \pi(g)u, v \rangle_{\mathcal{H}}$ is in $L^{p+\epsilon}(G)$ for all $\epsilon > 0$. Since this holds true for all elements of \widehat{G} weakly contained in $\mathcal{H}_{\text{q-comp}}$, the claimed bound of the proposition on matrix coefficients of $\mathcal{H}_{\text{q-comp}}$ follows by the equivalence of condition 2 and condition 4 in [Sha00, Theorem 2.1] and the fact that $d/p = d - s_*$. Combined with the bounds on matrix coefficients of $\mathcal{H}_{\text{temp}}$ in Eq. (3.5), this completes the proof of the proposition. ■

We will now use the gathered tools to prove the following main proposition of this section.

Proposition 3.8. *Let $\delta_{\mathcal{H}} \in (d/2, d]$ and let (π, \mathcal{H}) be a unitary representation of G which does not weakly contain any quasi-complementary series $\mathcal{U}(\sigma, s)$ with parameters $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_{\sigma} \cap (\delta_{\mathcal{H}}, d]$.*

Let $\tau \in \widehat{K}$ be a K -type. For all $\phi, \psi \in \mathcal{H}_{\tau}$, define the Laplace transform of the real and imaginary parts of the scaled matrix coefficient $F_{\spadesuit} : \{z \in \mathbb{C} : \Re(z) > 0\} \rightarrow \mathbb{C}$ by

$$F_{\spadesuit}(z) = \int_0^{+\infty} e^{-(z+\delta_{\mathcal{H}}-d)t} \spadesuit(\langle \pi(a_t)\phi, \psi \rangle_{\mathcal{H}}) dt \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > 0,$$

for $\spadesuit \in \{\Re, \Im\}$. Let $\eta_0 = \min\{\delta_{\mathcal{H}} - d/2, 1\}$. Then, both F_{\Re} and F_{\Im} admit holomorphic extensions to $\{z \in \mathbb{C} : \Re(z) > -\eta_0\} \setminus (-\eta_0, 0]$. Moreover, on the half-plane $\{z \in \mathbb{C} : \Re(z) > -\eta\}$ for any fixed $\eta < \eta_0$, we may write

$$F_{\spadesuit} = A_{\spadesuit} + B_{\spadesuit}$$

where A_{\spadesuit} is a holomorphic function and B_{\spadesuit} is a meromorphic function defined by

$$B_{\spadesuit}(z) = \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{[\delta_{\mathcal{H}} - \eta, \delta_{\mathcal{H}}]} \frac{\spadesuit(\langle \mathbb{T}_{\tau}^{\tau} \mathbf{C}_+(s) \phi_{\sigma,s}, \psi_{\sigma,s} \rangle_{\mathcal{H}(\sigma,s)})}{z + \delta_{\mathcal{H}} - s} dm_{\sigma}(s) \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > -\eta,$$

for $\spadesuit \in \{\Re, \Im\}$. Moreover, A_{\Re} and A_{\Im} are uniformly bounded on the half-plane $\{z \in \mathbb{C} : \Re(z) \geq -\eta_1\}$ for any $\eta_1 < \eta$.

Proof. Assume the hypotheses of the proposition. We give the proof for F_{\Re} as the proof for F_{\Im} is identical. Recall the direct integral decomposition and the spectral measures $\{m_{\sigma}\}_{\sigma \in \widehat{M}_{s-d}}$ given in Eq. (2.6). For any $\eta \in (0, \delta_{\mathcal{H}} - d/2)$, define $I_{\eta} := [\delta_{\mathcal{H}} - \eta, \delta_{\mathcal{H}}]$ and the Hilbert subspace

$$\mathcal{H}[\eta] = \bigoplus_{\sigma \in \widehat{M}_{s-d}} \int_{I_{\eta}}^{\oplus} \mathcal{H}(\sigma, s) dm_{\sigma}(s) \quad \subset \mathcal{H}.$$

Then, $\mathcal{H} = \mathcal{H}[\eta] \oplus \mathcal{H}[\eta]^{\perp}$ is an orthogonal decomposition into subrepresentations, and for any $\varphi \in \mathcal{H}$, denote by $\varphi_{\eta} \in \mathcal{H}[\eta]$ and $\varphi_{\eta}^{\perp} \in \mathcal{H}[\eta]^{\perp}$ its projections to $\mathcal{H}[\eta]$ and $\mathcal{H}[\eta]^{\perp}$, respectively. By orthogonality of these spaces, we have

$$\langle \pi(a_t) \phi, \psi \rangle_{\mathcal{H}} = \langle \pi(a_t) \phi_{\eta}, \psi_{\eta} \rangle_{\mathcal{H}} + \langle \pi(a_t) \phi_{\eta}^{\perp}, \psi_{\eta}^{\perp} \rangle_{\mathcal{H}} \quad \text{for all } t \in \mathbb{R},$$

and taking real parts, we have

$$\Re(\langle \pi(a_t) \phi, \psi \rangle_{\mathcal{H}}) = \Re(\langle \pi(a_t) \phi_{\eta}, \psi_{\eta} \rangle_{\mathcal{H}}) + \Re(\langle \pi(a_t) \phi_{\eta}^{\perp}, \psi_{\eta}^{\perp} \rangle_{\mathcal{H}}) \quad \text{for all } t \in \mathbb{R}.$$

Using the above decomposition we get a corresponding decomposition $F_{\Re}(z) = F_{\Re, \eta}(z) + F_{\Re, \eta}^{\perp}(z)$ where

$$F_{\Re, \eta}(z) = \int_0^{+\infty} e^{-(z + \delta_{\mathcal{H}} - d)t} \Re(\langle \pi(a_t) \phi_{\eta}, \psi_{\eta} \rangle_{\mathcal{H}}) dt$$

and similarly for $F_{\Re, \eta}^{\perp}(z)$. By Proposition 3.7 we have the bound

$$|\Re(\langle \pi(a_t) \phi_{\eta}^{\perp}, \psi_{\eta}^{\perp} \rangle_{\mathcal{H}})| \leq |\langle \pi(a_t) \phi_{\eta}^{\perp}, \psi_{\eta}^{\perp} \rangle_{\mathcal{H}}| \ll_{\tau} (|t| + 1) e^{-(d - \delta_{\mathcal{H}} + \eta)t} \quad \text{for all } t \in \mathbb{R}.$$

Thus, for $\Re(z) > -\eta$, the integral defining $F_{\Re, \eta}^{\perp}(z)$ converges absolutely:

$$|F_{\Re, \eta}^{\perp}(z)| \leq \int_0^{+\infty} |e^{-(z + \delta_{\mathcal{H}} - d)t} \Re(\langle \pi(a_t) \phi_{\eta}^{\perp}, \psi_{\eta}^{\perp} \rangle_{\mathcal{H}})| dt \ll \int_0^{+\infty} (t + 1) e^{-(\Re(z) + \eta)t} dt < +\infty,$$

and hence defines a holomorphic function that is uniformly bounded on $\{z \in \mathbb{C} : \Re(z) \geq -\eta_1\}$, for any $\eta_1 < \eta$.

Next, we treat $F_{\Re, \eta}(z)$. Note that for any $\varphi \in \mathcal{H}_{\tau}$ of the fixed K -type τ , we have that for all $\sigma \in \widehat{M}_{s-d}$ and m_{σ} -almost all $s \in I_{\eta}$, the component $\varphi_{\sigma, s}$ is also of K -type τ . Since $\mathcal{U}(\sigma, s)_{\tau}$ is non-trivial only if $\sigma \subset \tau$, we get

$$\varphi_{\eta} = \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_{\eta}}^{\oplus} \varphi_{\sigma, s} dm_{\sigma}(s) \quad \in \mathcal{H}[\eta]_{\tau}.$$

Using such a decomposition for ϕ and ψ , we have

$$\langle \pi(a_t) \phi_{\eta}, \psi_{\eta} \rangle_{\mathcal{H}} = \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_{\eta}} \langle \pi_{\sigma, s}(a_t) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)} dm_{\sigma}(s).$$

For the compact interval $I_{\eta} = [\delta_{\mathcal{H}} - \eta, \delta_{\mathcal{H}}] \subset (d/2, d]$ and each of the finitely many $\sigma \in \widehat{M}_{s-d}$ with $\sigma \subset \tau$, we use Corollary 3.6 to estimate for all $s \in I_{\eta}$ and $t \geq 0$ that

$$\langle \pi_{\sigma, s}(a_t) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)} = e^{-(d-s)t} \langle \mathbb{T}_{\tau}^{\tau} \mathbf{C}_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)} + O_{I_{\eta}, \tau} \left(e^{-(d-s+\eta_s)t} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \right).$$

Hence, taking real parts and integrating, we get that

$$\begin{aligned} F_{\Re, \eta}(z) &= \int_0^{+\infty} \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta} e^{-(z+\delta_{\mathcal{H}}-d)t} \Re(\langle \pi_{\sigma, s}(a_t) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)}) dm_\sigma(s) dt \\ &= \int_0^{+\infty} \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta} \left(e^{-(z+\delta_{\mathcal{H}}-s)t} \Re(\langle \mathbb{T}_\tau^T C_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)}) \right. \\ &\quad \left. + O_{I_\eta, \tau} \left(e^{-(z+\delta_{\mathcal{H}}-s+\eta_s)t} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \right) \right) dm_\sigma(s) dt. \end{aligned}$$

Write

$$F_{\Re, \eta}(z) = B_{\Re}(z) + E(z)$$

where $B_{\Re}(z)$ is the main term and $E(z)$ is the error term. Assume $\Re(z) > 0$ so that the integrals converge absolutely and we may interchange the order of integration. Then, we get

$$B_{\Re}(z) = \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta} \frac{\Re(\langle \mathbb{T}_\tau^T C_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{H}(\sigma, s)})}{z + \delta_{\mathcal{H}} - s} dm_\sigma(s).$$

Note that $B_{\Re}(z)$ is holomorphic on $\mathbb{C} \setminus [-\eta, 0]$. Moreover, the integral defining $E(z)$ is absolutely convergent on $\{z \in \mathbb{C} : \Re(z) > -\eta\}$ as we now show. Decomposing $I_\eta = (I_\eta \setminus ((d+1)/2, d]) \sqcup (I_\eta \setminus (d/2, (d+1)/2])$, and using the definition of η_s in Eq. (3.1), we have

$$\begin{aligned} |E(z)| &\ll_{I_\eta, \tau} \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \int_0^{+\infty} e^{-(\Re(z)+\delta_{\mathcal{H}}-s+\eta_s)t} dt dm_\sigma(s) \\ &= \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta \setminus ((d+1)/2, d]} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \int_0^{+\infty} e^{-(\Re(z)+\delta_{\mathcal{H}}+s-d)t} dt dm_\sigma(s) \\ &\quad + \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta \setminus (d/2, (d+1)/2]} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \int_0^{+\infty} e^{-(\Re(z)+\delta_{\mathcal{H}}-s+1)t} dt dm_\sigma(s) \\ &\leq \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta \setminus ((d+1)/2, d]} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \int_0^{+\infty} e^{-(\Re(z)+2\delta_{\mathcal{H}}-d-\eta)t} dt dm_\sigma(s) \\ &\quad + \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta \setminus (d/2, (d+1)/2]} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \int_0^{+\infty} e^{-(\Re(z)+1)t} dt dm_\sigma(s) \\ &\leq \left(\frac{1}{\Re(z) + 2\delta_{\mathcal{H}} - d - \eta} + \frac{1}{\Re(z) + 1} \right) \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \int_{I_\eta} \|\phi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} \|\psi_{\sigma, s}\|_{\mathcal{H}(\sigma, s)} dm_\sigma(s) \\ &\leq \left(\frac{1}{\Re(z) + \delta_{\mathcal{H}} - d/2} + \frac{1}{\Re(z) + 1} \right) \|\phi_\eta\|_{\mathcal{H}[\eta]} \|\psi_\eta\|_{\mathcal{H}[\eta]}. \end{aligned}$$

It follows that $E(z)$ is bounded and holomorphic on $\{z \in \mathbb{C} : \Re(z) > -\min\{\eta, 1\}\}$. The result now follows with $A_{\Re}(z) = F_{\Re, \eta}^\perp(z) + E(z)$. ■

4. NON-VANISHING OF MAIN TERMS OF MATRIX COEFFICIENTS

In the work of Edwards–Oh [EO21], a key point is the *vanishing* of the main terms for non-trivial $\sigma \in \widehat{M}_{s-d}$ and M -invariant vectors $u \in \mathcal{U}(\sigma, s)_{\tau_1}$ in the asymptotic expansion of matrix coefficients, Theorem 3.4—for their purpose of eventually extracting exponential mixing of the geodesic flow with respect to the Bowen–Margulis–Sullivan measure. On the other hand, for our purpose of obtaining a strong spectral gap, we require the *non-vanishing* of the main terms for non-trivial $\sigma \in \widehat{M}_{s-d}$ for suitable choices of test vectors. This has the consequence that presence of a given quasi-complementary series can be detected using asymptotics of matrix coefficients. The goal of this section is to prove such a non-vanishing result.

Let $s > d/2$. Recall from Lemma 3.3 that the Harish-Chandra $C_+(s)$ act as a scalar on each irreducible M -submodule of $L^2(K : \sigma)$. Denote by $C_+(\tau : \sigma; s)$ the scalar that $C_+(s)$ acts by on an M -type $\sigma \in \widehat{M}$ contained in a K -type $\tau \in \widehat{K}$ (which occurs with multiplicity 1; see Subsection 2.4). Precisely these scalars were computed by Eguchi–Koizumi–Mamiuda in [EKM99]. We quote their theorem below.

Theorem 4.1 ([EKM99, Theorem 8.2]). *Let $d \geq 2$ and $s > d/2$. Let $\sigma = (\sigma_1, \dots, \sigma_{\lfloor d/2 \rfloor}) \in \widehat{M}$ be an M -type and $\tau = (\tau_1, \dots, \tau_{\lfloor d/2 \rfloor}) \in \widehat{K}$ be a K -type such that σ is contained in τ . We have the following:*

(1) if $2 \mid d$, then

$$C_+(\tau : \sigma; s) = \frac{(d-1)!}{(d/2-1)!} \cdot \frac{\prod_{j=1}^{d/2} \Gamma(s-d/2+j-\sigma_j) \prod_{j=1}^{d/2} \Gamma(s+d/2-j+\sigma_j)}{\prod_{j=1}^{d/2} \Gamma(s-d/2+j-\tau_j) \prod_{j=1}^{d/2} \Gamma(s+d/2-j+1+\tau_j)};$$

(2) if $2 \nmid d$, then

$$C_+(\tau : \sigma; s) = \left(\frac{d-1}{2}\right)! 2^{-2s+d} \Gamma(2s) \cdot \frac{\prod_{j=1}^{(d-1)/2} \Gamma(s-d/2+j-\sigma_j) \prod_{j=1}^{(d-1)/2} \Gamma(s+d/2-j+\sigma_j)}{\prod_{j=1}^{(d+1)/2} \Gamma(s-d/2+j-\tau_j) \prod_{j=1}^{(d+1)/2} \Gamma(s+d/2-j+1+\tau_j)}.$$

We use Theorem 4.1 to prove the following proposition. Recall the constant λ_τ associated to K -types $\tau \in \widehat{K}$ from Eq. (2.5).

Proposition 4.2. *Let $\sigma \in \widehat{M}$ be an M -type. There exists a K -type $\tau \in \widehat{K}$ such that*

- (1) τ contains both σ and its dual σ^* ;
- (2) τ minimizes λ_τ among the K -types satisfying Property (1);
- (3) $C_+(\tau : \sigma^*; s) \neq 0$ for all $s > d/2$.

Proof. Let $\sigma \in \widehat{M}$ be an M -type. Proceeding by cases, we take an explicit K -type $\tau \in \widehat{K}$ and verify the claimed properties.

Case 1: $2 \mid d$. Let us write $\sigma = (\sigma_1, \dots, \sigma_{d/2})$ where

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{d/2-1} \geq |\sigma_{d/2}|.$$

Let us also write $\sigma^* = (\sigma_1^*, \dots, \sigma_{d/2}^*)$. In the current case, recall that

- if $4 \mid d$, then $\sigma^* = \sigma = (\sigma_1, \dots, \sigma_{d/2})$;
- otherwise, $\sigma^* = (\sigma_1, \dots, \sigma_{d/2-1}, -\sigma_{d/2})$.

Take $\tau = (\tau_1, \dots, \tau_{d/2-1}, \tau_{d/2}) := (\sigma_1, \dots, \sigma_{d/2-1}, |\sigma_{d/2}|)$. Then τ contains both σ and σ^* since they satisfy the interlacing property

$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \dots \geq \tau_{d/2-1} \geq \sigma_{d/2-1} \geq \tau_{d/2} \geq |\pm \sigma_{d/2}|.$$

This proves Property (1). Inspecting Eq. (2.5) term by term verifies Property (2).

Let $s > d/2$. The formula provided by Theorem 4.1 for σ^* and the choice of τ gives

$$C_+(\tau : \sigma^*; s) = \frac{(d-1)!}{(d/2-1)!} \cdot \frac{\Gamma(s-\sigma_{d/2}^*)N_2}{\Gamma(s-|\sigma_{d/2}|)D_2}$$

where all but the last factor in the first iterated products in the numerator and denominator canceled out, and $N_2 = \prod_{j=1}^{d/2} \Gamma(s+d/2-j+\sigma_j^*)$ and $D_2 = \prod_{j=1}^{d/2} \Gamma(s+d/2-j+1+\tau_j)$ denote the second iterated products in the numerator and denominator. To prove Property (3), we need to verify that zeros in the numerator $\Gamma(s-\sigma_{d/2}^*)N_2$ and poles in the denominator $\Gamma(s-|\sigma_{d/2}|)D_2$ are canceled out. The first condition holds simply because the Γ function does not have zeros. Now we verify the second condition. Recall that the poles of Γ are all simple and occur exactly at the non-positive integers. The factor D_2 does not have poles since

$$s+d/2-j+1+\tau_j \geq s > 0 \quad \text{for all } 1 \leq j \leq d/2. \quad (4.1)$$

So we are left to examine the first factor $\Gamma(s-|\sigma_{d/2}|)$ in the denominator. Fortunately, this must cancel with one of the factors $\Gamma(s-\sigma_{d/2})$ or $\Gamma(s+\sigma_{d/2})$ in the numerator using the observation that

$$\Gamma(s-\sigma_{d/2}^*)\Gamma(s+\sigma_{d/2}^*) = \Gamma(s-\sigma_{d/2})\Gamma(s+\sigma_{d/2}).$$

Therefore, $C_+(\tau : \sigma^*; s) \neq 0$.

Case 2: $2 \nmid d$. Let us write $\sigma = (\sigma_1, \dots, \sigma_{(d-1)/2})$ where

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{(d-1)/2} \geq 0.$$

In the current case, $\sigma^* = \sigma$. Take $\tau = (\tau_1, \dots, \tau_{(d-1)/2}, \tau_{(d+1)/2}) := (\sigma_1, \dots, \sigma_{(d-1)/2}, 0)$. Then τ contains σ (and hence also σ^*) since they satisfy the interlacing property

$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \dots \geq \tau_{(d-1)/2} \geq \sigma_{(d-1)/2} \geq |\tau_{(d+1)/2}|.$$

This proves Property (1). Inspecting Eq. (2.5) term by term verifies Property (2).

Let $s > d/2$. As in Case 1, the formula provided by Theorem 4.1 for σ^* and the choice of τ gives

$$C_+(\tau : \sigma^*; s) = \left(\frac{d-1}{2}\right)! 2^{-2s+d} \cdot \frac{\Gamma(2s)N_2}{\Gamma(s+1/2)D_2}$$

and to prove Property (3), we only need to verify that the poles in the denominator $\Gamma(s+1/2)D_2$ are canceled out. Again, D_2 does not have poles by the same inequality Eq. (4.1) for all $1 \leq j \leq (d+1)/2$. Similarly, $\Gamma(s+1/2)$ does not have poles since $s+1/2 > 0$. Therefore, $C_+(\tau : \sigma^*; s) \neq 0$. ■

The following corollary shows that the main terms in the asymptotic expansion provided by Theorem 3.4 are non-vanishing for a suitable choice of test vectors.

Corollary 4.3. *Let $\sigma \in \widehat{M}_{s-d}$ be a self-dual M -type. Let $\tau \in \widehat{K}$ be the K -type provided by Proposition 4.2. Then, for all $s \in \mathcal{I}_\sigma$, the subspace $\mathcal{U}(\sigma, s)_\tau \cap \mathcal{U}(\sigma, s)_\sigma \subset \mathcal{U}(\sigma, s)$ is non-trivial and the operator $\mathbb{T}_\tau^\tau C_+(s)$ acts on it by a non-zero scalar.*

Proof. Let σ and τ be as in the corollary. By Proposition 4.2(1), τ contains σ , whence τ is contained in $\text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})$ (see Subsection 2.5). By Proposition 4.2(2), τ is a minimal K -type of $\text{Ind}_P^G(\sigma \otimes \chi_{s-d/2} \otimes \mathbf{1})$ and so by Theorem 2.12, we further deduce that the subspace $\mathcal{U}(\sigma, s)_\tau$ is non-trivial. Again since τ contains σ , we conclude that $\mathcal{U}(\sigma, s)_\tau \cap \mathcal{U}(\sigma, s)_\sigma$ is non-trivial. Recall that Lemma 3.1 gives $\mathbb{T}_\tau^\tau|_{\mathcal{U}(\sigma, s)_\tau} = \frac{\dim(\tau)}{\dim(\sigma)} \mathbb{P}_{\sigma^*}|_{\mathcal{U}(\sigma, s)_\tau}$, and that $\mathbb{P}_{\sigma^*} = \mathbb{P}_\sigma$ acts as identity on $\mathcal{U}(\sigma, s)_\sigma$. Moreover, we have by Proposition 4.2(3) that $C_+(s)$ acts by the scalar $C_+(\tau : \sigma; s) \neq 0$ on $\mathcal{U}(\sigma, s)_\tau \cap \mathcal{U}(\sigma, s)_\sigma$. Putting everything together, we have

$$[\mathbb{T}_\tau^\tau C_+(s)]|_{\mathcal{U}(\sigma, s)_\tau \cap \mathcal{U}(\sigma, s)_\sigma} = \frac{\dim(\tau)}{\dim(\sigma)} C_+(\tau : \sigma; s) \cdot \text{Id}_{\mathcal{U}(\sigma, s)_\tau \cap \mathcal{U}(\sigma, s)_\sigma} \neq 0,$$

as desired. ■

5. DECAY OF MATRIX COEFFICIENTS WITH POWER SAVING ERROR TERMS AND LAPLACE TRANSFORMS

In this section, we introduce the key dynamical input: the decay of matrix coefficients with power saving error terms which is a consequence of exponential mixing of the frame flow. We use this to obtain meromorphic extensions of Laplace transforms of the scaled matrix coefficients to a strip to the left of the imaginary axis with at most one simple pole at the origin.

We quote the following theorem from [SW21, Theorem 1.2] and [LPS25, Theorem 1.2] in a simplified form.

Theorem 5.1 ([SW21, LPS25]). *There exist non-zero Borel measures m^{BR} and $m^{\text{BR}*}$, and $\eta > 0$ such that for all $\phi, \psi \in C_c^\infty(\Gamma \backslash G)$ and $t > 0$, we have*

$$\int_{\Gamma \backslash G} (\phi \circ a_t) \psi dm^{\text{Haar}} = e^{-(d-\delta_\Gamma)t} \int_{\Gamma \backslash G} \phi dm^{\text{BR}} \int_{\Gamma \backslash G} \psi dm^{\text{BR}*} + O_{\phi, \psi}(e^{-(d-\delta_\Gamma+\eta)t}),$$

where the implicit constant depends on Sobolev norms of ϕ and ψ , as well as their supports.

As alluded to previously, the above theorem is derived from [SW21, Theorem 1.1] and [LPS25, Theorem 1.1] regarding exponential mixing of the frame flow with respect to the Bowen–Margulis–Sullivan (BMS) measure m^{BMS} which is the Borel probability measure of maximal entropy. In the case $\delta_\Gamma > \max\{d/2, d-1\}$, this result was obtained previously by Mohammadi–Oh in [MO15]. Exponential mixing of the geodesic flow is due to Li–Pan [LP23]; see also [Kha23a, Kha23b]. In the convex cocompact setting, exponential mixing of the frame flow is due to [SW21]. The aforementioned derivation uses Roblin’s transverse intersection argument [Rob03]; see also [OS13, MO15, KO21, ELO22] for more expositions of the argument, the latter being the most general. We also refer to loc. cit. for explicit formulas and further details for the BMS measure m^{BMS}

and the Burger–Roblin (BR) measures m^{BR} and m^{BR^*} which appear in the above theorem—in this article, knowing their existence and the form of the main term in the above theorem suffices.

Remark 5.2. [SW21, Theorem 1.2] and [LPS25, Theorem 1.2] are stated, a priori, for torsion-free Γ . It also holds for non-torsion-free Γ by passing to a torsion-free subgroup of finite index $\tilde{\Gamma} \leq \Gamma$ by Selberg’s lemma; the measures m^{Haar} , m^{BMS} , m^{BR} , and m^{BR^*} on $\tilde{\Gamma} \backslash G$ are simply the lifts of those on $\Gamma \backslash G$ up to normalization by the constant factor $[\Gamma : \tilde{\Gamma}]$.

Corollary 5.3. *Let $\eta > 0$ be as in Theorem 5.1. For all $\phi, \psi \in C_c^\infty(\Gamma \backslash G)$, define the Laplace transform of the real and imaginary parts of the scaled matrix coefficient $F_\spadesuit : \{z \in \mathbb{C} : \Re(z) > d - \delta_\Gamma\} \rightarrow \mathbb{C}$ by*

$$F_\spadesuit(z) = \int_0^{+\infty} e^{-(z+\delta_\Gamma-d)t} \spadesuit(\langle \phi \circ a_t, \psi \rangle_{L^2(\Gamma \backslash G)}) dt \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > d - \delta_\Gamma, \quad (5.1)$$

for $\spadesuit \in \{\Re, \Im\}$. Then, F_\spadesuit admits a meromorphic extension to the half-plane $\{z \in \mathbb{C} : \Re(z) > -\eta\}$ with at most a simple pole at $z = 0$ of residue $\spadesuit\left(\int_{\Gamma \backslash G} \phi dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} dm^{\text{BR}^*}\right)$, for $\spadesuit \in \{\Re, \Im\}$.

Proof. Let $\eta > 0$ be a parameter satisfying Theorem 5.1 and let $\phi, \psi \in C_c^\infty(\Gamma \backslash G)$. We give the proof for F_\Re as the proof for F_\Im is identical. Denote $M_{\phi, \psi} := \Re\left(\int_{\Gamma \backslash G} \phi dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} dm^{\text{BR}^*}\right)$. Then, by Theorem 5.1, the function

$$F_\Re(z) - \int_0^{+\infty} e^{-zt} M_{\phi, \psi} dt = F_\Re(z) - \frac{M_{\phi, \psi}}{z},$$

defined initially on $\{z \in \mathbb{C} : \Re(z) > d - \delta_\Gamma\}$, extends holomorphically to the half-plane $\{z \in \mathbb{C} : \Re(z) > -\eta\}$. The corollary now follows since $M_{\phi, \psi}/z$ extends meromorphically to \mathbb{C} with at most a simple pole at $z = 0$ of residue $M_{\phi, \psi}$. \blacksquare

6. COMPARISON OF LAPLACE TRANSFORMS AND THE PROOFS OF THEOREMS 1.3 AND 1.6

In this section, we prove Theorem 1.3. Property (2) of Definition 1.1 is established in Proposition 6.6, while Property (1) is proved in Proposition 6.8. An outline of the proof strategy is given in Subsection 1.1.

6.1. Preliminaries on Stieltjes transforms. Recall that for a signed measure ν , infinite values $+\infty$ and $-\infty$ are permitted but not both. Consequently, in the Jordan decomposition $\nu = \nu^+ - \nu^-$ into the positive and negative parts ν^\pm of ν (which are positive measures), at least one of ν^+ or ν^- must be finite. Moreover, if ν is finite, then both ν^+ and ν^- must be finite, and hence its total variation $|\nu| = \nu^+ + \nu^-$ is also finite. We refer the reader to [Fol99, Chapter 3] for further details. We also record following elementary lemma.

Lemma 6.1. *Let $I \subset \mathbb{R}$ be an open interval and $E \subset I$ be a dense subset. Let ν be a signed Borel measure on I such that $\nu([a, b]) = 0$ for all closed intervals $[a, b] \subset I$ with $a, b \in E$. Then, ν is the 0 measure.*

Proof. Let I , E , and ν be as in the lemma. We have the Jordan decomposition $\nu = \nu^+ - \nu^-$ where the positive and negative parts ν^\pm are positive Borel measures on I . It follows by hypothesis that $\nu^+([a, b]) = \nu^-([a, b])$ for all intervals $[a, b] \subset I$ with $a, b \in E$. Since the set of closed intervals whose endpoints are in the dense subset $E \subset I$ generate the Borel σ -algebra on I , it follows that $\nu^+ = \nu^-$. Hence, the lemma follows. \blacksquare

Remark 6.2. We cannot repeat the above proof for closed intervals $I \subset \mathbb{R}$ since the dense subset $E \subset I$ may not contain either endpoints of I .

The following is the Stieltjes inversion formula corresponding to a Stieltjes transform for *signed* measures. It is proved in [Wal48, Chapter XIII, §65] for *positive* measures; however, its elementary proof can be adapted for signed measures. We include it here for completeness, especially because the proposition plays a crucial role in the proof of our main theorem.

Proposition 6.3 (Stieltjes Inversion Formula). *Let ν be a finite signed Borel measure on \mathbb{R} . Let $F : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ be the **Stieltjes transform** of ν defined by*

$$F(z) = \int_{\mathbb{R}} \frac{1}{z - t} d\nu(t) \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

Then F is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and for any closed interval $[a, b] \subset \mathbb{R}$, we have

$$\frac{\nu([a, b]) + \nu((a, b))}{2} = -\frac{1}{\pi} \lim_{y \searrow 0} \int_a^b \Im F(x + iy) dx.$$

In particular, if $\nu(\{a\}) = \nu(\{b\}) = 0$ and F extends continuously to $(\mathbb{C} \setminus \mathbb{R}) \cup [a, b]$, and hence holomorphically to $(\mathbb{C} \setminus \mathbb{R}) \cup (a, b)$, then $\nu([a, b]) = 0$.

Proof. Let ν , F , and $[a, b]$ be as in the proposition. Define a finite signed Borel measure $\tilde{\nu}$ on $[a, b] \times \mathbb{R}$ by $d\tilde{\nu}(x, t) = dx d\nu(t)$. For any compact subset $\mathcal{K} \subset \mathbb{R}_{>0}$ and $y \in \mathcal{K}$, we have the uniform bound

$$\int_{[a, b] \times \mathbb{R}} \left| \frac{1}{x + iy - t} \right| d|\tilde{\nu}|(x, t) \leq \frac{1}{\inf \mathcal{K}} \cdot |\tilde{\nu}|([a, b] \times \mathbb{R}) < +\infty.$$

By similar uniform bounds, we may apply Fubini's theorem for integrals of F over arbitrary line segments in $\mathbb{C} \setminus \mathbb{R}$ which are parallel to the real or imaginary axis. Hence, holomorphicity of $z \mapsto \frac{1}{z-t}$ on $\mathbb{C} \setminus \mathbb{R}$ together with Fubini's theorem imply vanishing of contour integrals of F over rectangular contours in $\mathbb{C} \setminus \mathbb{R}$ (with sides parallel to the real and imaginary axes). It follows that F is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ by Morera's theorem (see [SS03, Chapter 2, §5, Theorem 5.1] and [SS03, Chapter 2, §7, Problem 3]).

By the above bound, for any $y > 0$, we may integrate the imaginary part of F over $[a, b]$ and apply Fubini's theorem again to obtain

$$\begin{aligned} \int_a^b \Im F(x + iy) dx &= \int_a^b \int_{\mathbb{R}} \frac{-y}{(x-t)^2 + y^2} d\nu(t) dx = \int_{\mathbb{R}} \int_a^b \frac{-1/y}{(x-t)^2/y^2 + 1} dx d\nu(t) \\ &= \int_{\mathbb{R}} \arctan((t-b)/y) d\nu(t) - \int_{\mathbb{R}} \arctan((t-a)/y) d\nu(t). \end{aligned}$$

Now, we introduce the step function $h = -\chi_{(-\infty, 0)} + \chi_{(0, +\infty)} : \mathbb{R} \rightarrow \mathbb{R}$ and observe the pointwise limit

$$\lim_{y \searrow 0} \arctan(t/y) = \frac{\pi}{2} h(t) \quad \text{for all } t \in \mathbb{R}.$$

Recall that $|\arctan| \leq \frac{\pi}{2}$ and the total variation $|\nu|$ is finite since ν is finite. Thus, we may apply the Lebesgue dominated convergence theorem to obtain the limit

$$\begin{aligned} \lim_{y \searrow 0} \int_a^b \Im F(x + iy) dx &= \lim_{y \searrow 0} \int_{\mathbb{R}} \arctan((t-b)/y) d\nu(t) - \lim_{y \searrow 0} \int_{\mathbb{R}} \arctan((t-a)/y) d\nu(t) \\ &= \frac{\pi}{2} \int_{\mathbb{R}} h(t-b) d\nu(t) - \frac{\pi}{2} \int_{\mathbb{R}} h(t-a) d\nu(t) \\ &= -\frac{\pi}{2} (\nu((-\infty, b)) - \nu((b, +\infty)) - \nu((-\infty, a)) + \nu((a, +\infty))) \\ &= -\pi \frac{\nu([a, b]) + \nu((a, b))}{2}. \end{aligned}$$

Let us deduce the last claim of the proposition. Suppose F extends continuously to $(\mathbb{C} \setminus \mathbb{R}) \cup [a, b]$. Observe that for any rectangular contour (with sides parallel to the real and imaginary axes) that intersects (a, b) , we may decompose it into 3 rectangular contours where only one of them intersects (a, b) and is arbitrarily thin in the imaginary direction. In light of this observation, we again deduce using Morera's theorem (see loc. cit. in [SS03]) that F is holomorphic on $(\mathbb{C} \setminus \mathbb{R}) \cup (a, b)$. Now, since ν is a real-valued measure, we have the property that $\overline{F(z)} = F(\bar{z})$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ and hence $\Im F(z) = 0$ for all $z \in [a, b]$ by continuity. Thus, by uniform continuity of F on $[a, b] + i[-1, 1]$, we have

$$\lim_{y \searrow 0} \int_a^b \Im F(x + iy) dx = \int_a^b \lim_{y \searrow 0} \Im F(x + iy) dx = 0.$$

Suppose further that $\nu(\{a\}) = \nu(\{b\}) = 0$. Then, it follows from the proven identities that $\nu([a, b]) = \frac{\nu([a, b]) + \nu((a, b))}{2} = 0$. \blacksquare

Corollary 6.4. Let $\nu = \nu_{\Re} + i\nu_{\Im}$ be a complex Borel measure on \mathbb{R} where ν_{\Re} and ν_{\Im} are signed measures, and consider the functions F_{\Re} and F_{\Im} defined by

$$F_{\Re}(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\nu_{\Re}(t), \quad F_{\Im}(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\nu_{\Im}(t), \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}.$$

Suppose there exists a closed interval $[a, b] \subset \mathbb{R}$ such that the following holds:

- (1) there exists a dense subset $E \subset [a, b]$ such that $\nu(\{x\}) = 0$ for all $x \in E$;
- (2) $F_{\mathfrak{R}}$ and $F_{\mathfrak{I}}$ extend continuously to $(\mathbb{C} \setminus \mathbb{R}) \cup [a, b]$.

Then $\nu|_{(a,b)}$ is the zero measure.

Proof. Since $F_{\mathfrak{R}}$ and $F_{\mathfrak{I}}$ extend continuously to $[a, b]$, then, by Proposition 6.3, we conclude that $\nu([\alpha, \beta]) = 0$ for any $\alpha, \beta \in E$ with $\alpha < \beta$. Hence, by Lemma 6.1 we get that both $\nu_{\mathfrak{R}}$ and $\nu_{\mathfrak{I}}$ restricted to (a, b) are the zero measures. The corollary follows since $\nu = \nu_{\mathfrak{R}} + i\nu_{\mathfrak{I}}$. \blacksquare

6.2. Proof of Theorem 1.3. We now prove the main theorem. We need the following fact in order to be able apply Proposition 3.8.

Proposition 6.5. *For all $v \in \widehat{M}_{s-d}$ and $s > \delta_{\Gamma}$, the quasi-complementary series $\mathcal{U}(v, s)$ is not weakly contained in $L^2(\Gamma \backslash G)$.*

Proof. By qualitative mixing of the frame flow together with Roblin's transverse intersection argument, cf. [Win15, Theorem 1.4], we have for all $\phi, \psi \in C_c(\Gamma \backslash G)$, the estimate $\langle \phi \circ a_t, \psi \rangle_{L^2(\Gamma \backslash G)} = O_{\phi, \psi}(e^{-(d-\delta_{\Gamma})t})$ as $t \rightarrow +\infty$.² Hence, the matrix coefficients $g \mapsto \langle \rho_G(g)\phi, \psi \rangle_{L^2(\Gamma \backslash G)}$ belong to $L^{p+\epsilon}(G)$ for all $\epsilon > 0$, and for $p := \frac{d}{d-\delta_{\Gamma}}$. This follows by a similar argument to the proof of Proposition 3.7 using the formula for the Haar measure on G . Since $C_c(\Gamma \backslash G)$ is dense in $L^2(\Gamma \backslash G)$, this means that the representation $(\rho_G, L^2(\Gamma \backslash G))$ is strongly L^p by definition (in the terminology of [Sha00]). Hence, [Sha00, Theorem 2.1] implies that every representation that is weakly contained in $L^2(\Gamma \backslash G)$ is also strongly L^p . On the other hand, given $v \in \widehat{M}_{s-d}$ and $s > \delta_{\Gamma}$, the quasi-complementary series $\mathcal{U}(v, s)$ is not strongly L^p , and hence cannot be weakly contained in $L^2(\Gamma \backslash G)$. Indeed, by Corollary 4.3, there exists a K -type $\tau \in \widehat{K}$ such that for all $s \in \mathcal{I}_v$, we have that the operator $T_{\tau}^r C_+(s)$ is non-zero on $\mathcal{U}(v, s)_{\tau}$. Combined with Corollary 3.6, this shows that there exist K -finite vectors in $\mathcal{U}(v, s)$ for which the matrix coefficients are not in $L^{p+\epsilon}(G)$ for any sufficiently small $\epsilon > 0$. Hence, [Sha00, Theorem 2.1] implies that the representation $\mathcal{U}(v, s)$ cannot be strongly L^p . \blacksquare

In what follows, we denote by $\eta_{\text{rep}} > 0$ and $\eta_{\text{mix}} > 0$ any constants for which Proposition 3.8 and Corollary 5.3 hold, respectively.

Proposition 6.6. *Fix $\eta := \min\{\eta_{\text{rep}}, \eta_{\text{mix}}\} \in (0, \delta_{\Gamma} - d/2)$. For all $v \in \widehat{M}_{s-d}$ and $s \in (\delta_{\Gamma} - \eta, \delta_{\Gamma})$, the quasi-complementary series $\mathcal{U}(v, s)$ is not weakly contained in $L^2(\Gamma \backslash G)$.*

Proof. Fix the constant $\eta := \min\{\eta_{\text{rep}}, \eta_{\text{mix}}\} \in (0, \delta_{\Gamma} - d/2)$ as in the proposition. Let us denote $\mathcal{L} := L^2(\Gamma \backslash G)$.

For the purposes of our proof, let us define the following objects for all K -types $\tau \in \widehat{K}$ and M -types $\sigma \in \widehat{M}_{s-d}$. First define the Hilbert subspaces

$$\begin{aligned} \mathcal{L}_{\tau, \sigma} &:= \int_{\mathcal{I}_{\sigma} \cap (\delta_{\Gamma} - \eta, \delta_{\Gamma})}^{\oplus} \mathcal{L}(\sigma, s)_{\tau} dm_{\sigma}(s) \quad \subset \mathcal{L}_{\tau} \subset \mathcal{L}, \\ \mathcal{J}_{\tau, \sigma} &:= \int_{\mathcal{I}_{\sigma} \cap (\delta_{\Gamma} - \eta, \delta_{\Gamma})}^{\oplus} \mathcal{L}(\sigma, s)_{\tau} \cap \mathcal{L}(\sigma, s)_{\sigma} dm_{\sigma}(s) \quad \subset \mathcal{L}_{\tau, \sigma} \subset \mathcal{L}_{\tau} \subset \mathcal{L}. \end{aligned}$$

Note that the Hilbert subspaces corresponding to different elements of \widehat{M}_{s-d} are mutually orthogonal, i.e.,

$$\sigma, \sigma' \in \widehat{M}_{s-d} \text{ such that } \sigma \neq \sigma' \implies \mathcal{L}_{\tau, \sigma} \perp \mathcal{L}_{\tau, \sigma'}. \quad (6.1)$$

Define the bounded operator $\mathbf{Q}_{\tau, \sigma} : \mathcal{L}_{\tau, \sigma} \rightarrow \mathcal{L}_{\tau, \sigma}$ by

$$\mathbf{Q}_{\tau, \sigma} := \int_{\mathcal{I}_{\sigma} \cap (\delta_{\Gamma} - \eta, \delta_{\Gamma})}^{\oplus} T_{\tau}^r|_{\mathcal{L}(\sigma, s)_{\tau}} C_+(s)|_{\mathcal{L}(\sigma, s)_{\tau}} dm_{\sigma}(s).$$

We then extend it to an operator $\mathbf{Q}_{\tau, \sigma} : \mathcal{L} \rightarrow \mathcal{L}$ by setting it to be the 0 operator on $(\mathcal{L}_{\tau, \sigma})^{\perp}$. Finally, we define the operator $\mathbf{Q}_{\tau} : \mathcal{L} \rightarrow \mathcal{L}$ by

$$\mathbf{Q}_{\tau} := \sum_{\sigma \in \widehat{M}_{s-d} : \sigma \subset \tau} \mathbf{Q}_{\tau, \sigma}. \quad (6.2)$$

²This is the qualitative form of Theorem 5.1 with no error rate.

Now we begin the proof in earnest. Suppose for the sake of contradiction that $\mathcal{U}(v, s_*)$ is weakly contained in \mathcal{L} for some $v \in \widehat{M}_{s-d}$ and $s_* \in (\delta_\Gamma - \eta, \delta_\Gamma)$. Let $\tau \in \widehat{K}$ be the corresponding K -type provided by Proposition 4.2. We first claim that

$$\text{there exist functions } \phi, \psi \in C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau \text{ such that } \langle \mathbf{Q}_\tau \phi, \psi \rangle_{\mathcal{L}} \neq 0. \quad (6.3)$$

Indeed, by our assumption, $s_* \in \text{supp}(m_v)$ and hence $m_v((\delta_\Gamma - \eta, \delta_\Gamma)) > 0$. By Corollary 4.3, for all $s \in \mathcal{I}_v$, the Hilbert subspace $\mathcal{L}(v, s)_\tau \cap \mathcal{L}(v, s)_v \subset \mathcal{L}(v, s)$ is non-trivial and the operator $\mathbf{T}_\tau^T \mathbf{C}_+(s)$ acts on it by a non-zero scalar. It follows from the definition of direct integrals that:

- $\mathcal{J}_{\tau, v} \subset \mathcal{L}_{\tau, v}$ are *non-trivial* Hilbert subspaces of $\mathcal{L}_\tau \subset \mathcal{L}$;
- $\mathbf{Q}_{\tau, v}$ acts as a *non-zero* bounded operator on $\mathcal{L}_{\tau, v}$, \mathcal{L}_τ , and \mathcal{L} ;
- \mathbf{Q}_τ acts as a *non-zero* bounded operator on \mathcal{L}_τ and \mathcal{L} , due to the orthogonality relations in Eq. (6.1).

By the above facts, there exist $\tilde{\phi}, \tilde{\psi} \in \mathcal{L}_\tau$ such that

$$\langle \mathbf{Q}_\tau \tilde{\phi}, \tilde{\psi} \rangle_{\mathcal{L}} \neq 0. \quad (6.4)$$

Since $C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$ is a dense subspace of \mathcal{L}_τ by Lemma 2.13, Claim (6.3) follows.

Next, we can rewrite Eq. (6.3) more explicitly as follows:

$$\langle \mathbf{Q}_\tau \phi, \psi \rangle_{\mathcal{L}} = \sum_{\sigma \in \widehat{M}_{s-d}: \sigma \subset \tau} \int_{(\delta_\Gamma - \eta, \delta_\Gamma)} \mathbf{1}_{\mathcal{I}_\sigma}(s) \langle \mathbf{T}_\tau^T \mathbf{C}_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{L}(\sigma, s)} dm_\sigma(s) \neq 0.$$

This motivates introducing the complex measure ν_τ on \mathbb{R} defined by

$$d\nu_\tau(s) = \sum_{\sigma \in \widehat{M}_{s-d}: \sigma \subset \tau} \mathbf{1}_{\mathcal{I}_\sigma}(s) \langle \mathbf{T}_\tau^T \mathbf{C}_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{L}(\sigma, s)} dm_\sigma(s).$$

Then, we may further rewrite Eq. (6.3) as

$$\nu_\tau((\delta_\Gamma - \eta, \delta_\Gamma)) \neq 0. \quad (6.5)$$

By Lemma 2.2, ν_τ has at most a countable number of atoms, which we recall are all singletons. As a consequence, we obtain a dense subset

$$E_\tau = \{x \in \mathbb{R} : \{x\} \text{ is not an atom of } \nu_\tau\} \subset \mathbb{R}$$

with the property that $\nu_\tau(\{x\}) = 0$ for all $x \in E_\tau$. Thus, Property (1) of Corollary 6.4 is satisfied.

Now, Proposition 6.5 says that (ρ_G, \mathcal{L}) does not weakly contain any quasi-complementary series $\mathcal{U}(\sigma, s)$ with parameters $\sigma \in \widehat{M}_{s-d}$ and $s > \delta_\Gamma$. Thus, Proposition 3.8 applies to the representation $(\pi, \mathcal{H}) = (\rho_G, \mathcal{L})$ with $\delta_{\mathcal{H}} = \delta_\Gamma$. Let $\spadesuit \in \{\Re, \Im\}$. Applying the proposition, the Laplace transform of the \spadesuit part of the scaled matrix coefficient defined by

$$F_{\spadesuit}(z) = \int_0^{+\infty} e^{-(z + \delta_\Gamma - d)t} \spadesuit(\langle \phi \circ a_t, \psi \rangle_{L^2(\Gamma \backslash G)}) dt \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > 0 \quad (6.6)$$

admits a holomorphic extension to $\{z \in \mathbb{C} : \Re(z) > -\eta_{\text{rep}}\} \setminus (-\eta_{\text{rep}}, 0]$ and can be written as a sum

$$F_{\spadesuit} = A_{\spadesuit} + B_{\spadesuit}$$

where A_{\spadesuit} is a bounded holomorphic function and B_{\spadesuit} is a meromorphic function both on the half-plane $\{z \in \mathbb{C} : \Re(z) > -\eta_{\text{rep}}\}$ and B_{\spadesuit} is defined explicitly by the formula

$$B_{\spadesuit}(z) = \sum_{\sigma \in \widehat{M}_{s-d}: \sigma \subset \tau} \int_{[\delta_\Gamma - \eta, \delta_\Gamma]} \frac{\spadesuit(\langle \mathbf{T}_\tau^T \mathbf{C}_+(s) \phi_{\sigma, s}, \psi_{\sigma, s} \rangle_{\mathcal{L}(\sigma, s)})}{z + \delta_\Gamma - s} dm_\sigma(s) \quad \text{for all } z \in \mathbb{C} \text{ with } \Re(z) > -\eta_{\text{rep}}. \quad (6.7)$$

Note that the integral in the definition of B_{\spadesuit} is over the *closed* interval $[\delta_\Gamma - \eta, \delta_\Gamma]$ as opposed to the integral in the definition of \mathbf{Q}_τ which is over the *open* interval $(\delta_\Gamma - \eta, \delta_\Gamma)$; only the right endpoint δ_Γ is of significance here. In particular, note that $s = \delta_\Gamma$, a priori, may contribute a pole to F_{\spadesuit} ; but in this proposition, we are only concerned with detecting whether the parameters $s \in (\delta_\Gamma - \eta, \delta_\Gamma)$ contribute poles to F_{\spadesuit} . Proposition 6.8 below will treat the endpoint $s = \delta_\Gamma$.

Write $\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]} = (\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]})_{\Re} + i(\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]})_{\Im}$ where $(\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]})_{\Re}$ and $(\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]})_{\Im}$ are signed measures. Observe also that $B_{\spadesuit}(z)$ is the Stieltjes transform of $(\nu_\tau|_{[\delta_\Gamma - \eta, \delta_\Gamma]})_{\spadesuit}$ at $z + \delta_\Gamma$.

Now, Corollary 5.3 also applies for our choice of ϕ and ψ . Let $\spadesuit \in \{\heartsuit, \clubsuit\}$. Applying the corollary, F_\spadesuit admits a meromorphic extension to $\{z \in \mathbb{C} : \Re(z) > -\eta_{\text{mix}}\}$, with exactly one simple pole at $z = 0$ of residue $\spadesuit \left(\int_{\Gamma \backslash G} \phi \, dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} \, dm^{\text{BR}*} \right)$. In particular, B_\spadesuit extends holomorphically to intervals on the real line of the form $[a, b]$, for all $-\eta < a < b < 0$. Thus, Property (2) of Corollary 6.4 is satisfied. We may now apply Corollary 6.4 to conclude that $\nu_\tau|_{(\delta_\Gamma - \eta, \delta_\Gamma)}$ is the zero measure. In particular, this contradicts Eq. (6.5) and completes the proof. \blacksquare

To prepare for the proof of the second proposition, we set some notation. Given a unitary representation (π, \mathcal{H}) of G , and more generally, of M , we write

$$\mathcal{H} = \mathcal{H}^M \oplus \mathcal{H}^0$$

for the orthogonal decomposition of \mathcal{H} into the subspace \mathcal{H}^M of M -invariant vectors and its orthogonal complement \mathcal{H}^0 . In particular, given a K -type $\tau \in \widehat{K}$, we also write

$$\mathcal{H}_\tau = \mathcal{H}_\tau^M \oplus \mathcal{H}_\tau^0.$$

For $v \in \mathcal{H}$, we denote by v^M and v^0 its orthogonal projections onto \mathcal{H}^M and \mathcal{H}^0 respectively, i.e.,

$$v^M = \int_M \pi(m)v \, dm, \quad v^0 = v - v^M. \quad (6.8)$$

We record the following lemma, which is an immediate consequence of Lemma 3.1.

Lemma 6.7. *Let $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$. Let $\tau \in \widehat{K}$ be a K -type containing σ . Then, the operator Γ_τ^τ preserves the orthogonal decomposition $\mathcal{U}(\sigma, s)_\tau = \mathcal{U}(\sigma, s)_\tau^M \oplus \mathcal{U}(\sigma, s)_\tau^0$. Moreover, if σ is non-trivial, then $\Gamma_\tau^\tau|_{\mathcal{U}(\sigma, s)_\tau^M}$ vanishes.*

Proof. Let $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$. The first assertion follows by recalling that Γ_τ^τ preserves M -types. Lemma 3.1 asserts that the operator Γ_τ^τ restricted to $\mathcal{U}(\sigma, s)_\tau$ is a scalar multiple of the orthogonal projection onto $\mathcal{U}(\sigma, s)_\sigma$ as σ is self-dual. The second assertion follows upon noting that $\mathcal{U}(\sigma, s)_\tau^M$ is orthogonal to the subspace $\mathcal{U}(\sigma, s)_\sigma$ since σ is non-trivial. \blacksquare

Recall from Remark 1.2 that having established Property (2) of Definition 1.1 in Proposition 6.6, for any non-trivial $v \in \widehat{M}_{s-d}$, a weak containment of $\mathcal{U}(v, \delta_\Gamma)$ in $L^2(\Gamma \backslash G)$ is automatically a strong containment. We now prove the second proposition, establishing Definition 1.1(1) and completing the proof of Theorem 1.3.

Proposition 6.8. *Let $v \in \widehat{M}_{s-d}$ be such that $\delta_\Gamma \in \mathcal{I}_v$. Then, we have the following.*

- (1) *If $v = \mathbf{1}$ is trivial, then the quasi-complementary series $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$ is contained in $L^2(\Gamma \backslash G)$ exactly with multiplicity 1.*
- (2) *If v is non-trivial, then the quasi-complementary series $\mathcal{U}(v, \delta_\Gamma)$ is not contained in $L^2(\Gamma \backslash G)$.*

Proof. Fix some $v \in \widehat{M}_{s-d}$ such that $\delta_\Gamma \in \mathcal{I}_v$. If $\delta_\Gamma = d$, then $v = \mathbf{1}$ and hence $\mathcal{U}(v, \delta_\Gamma) = \mathcal{U}(\mathbf{1}, d) = \mathbf{1} \in \widehat{G}$, and $\Gamma < G$ is a lattice. Thus, Part (1) trivially follows. We may now assume $\delta_\Gamma < d$. If $v = \mathbf{1}$, let $\tau = \mathbf{1} \in \widehat{K}$ be the trivial K -type. Otherwise, if v is non-trivial, let $\tau \in \widehat{K}$ be the K -type provided by Proposition 4.2.

Let us denote $\mathcal{L} = L^2(\Gamma \backslash G)$. First, having established Proposition 6.6, we may isolate the contribution of $\mathcal{L}(\sigma, \delta_\Gamma)$ for $\sigma \in \widehat{M}_{s-d}$ satisfying $\sigma \subset \tau$ and $\mathcal{U}(\sigma, \delta_\Gamma) \subset \mathcal{L}$ in the following fashion (cf. Remark 1.2). Fix $s_\star := \delta_\Gamma - \eta$ where $\eta := \min\{\eta_{\text{rep}}, \eta_{\text{mix}}\}$ as in Proposition 6.6 and define

$$\mathcal{L}(s_\star) := \bigoplus_{\sigma \in \widehat{M}_{s-d}} \int_{\mathcal{I}_\sigma \cap (d/2, s_\star]}^\oplus \mathcal{L}(\sigma, s) \, dm_\sigma(s).$$

By Proposition 6.6, $(s_\star, \delta_\Gamma) \subset \mathcal{I}_\sigma \setminus \text{supp}(m_\sigma)$ for all $\sigma \in \widehat{M}_{s-d}$ and so using Lemma 2.1, we have the orthogonal decomposition

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{temp}} \oplus \bigoplus_{\sigma \in \widehat{M}_{s-d}} \int_{\mathcal{I}_\sigma}^\oplus \mathcal{L}(\sigma, s) \, dm_\sigma(s) \\ &= \mathcal{L}_{\text{temp}} \oplus \bigoplus_{\sigma \in \widehat{M}_{s-d}} \int_{\mathcal{I}_\sigma \cap (d/2, \delta_\Gamma)}^\oplus \mathcal{L}(\sigma, s) \, dm_\sigma(s) \oplus \bigoplus_{\sigma \in \widehat{M}_{s-d} : m_\sigma(\{\delta_\Gamma\}) > 0} \mathcal{L}(\sigma, \delta_\Gamma) \end{aligned}$$

$$= \mathcal{L}_{\text{temp}} \oplus \mathcal{L}(s_*) \oplus \bigoplus_{\sigma \in \widehat{M}_{s-d}: \mathcal{U}(\sigma, \delta_\Gamma) \subset \mathcal{L}} \mathcal{L}(\sigma, \delta_\Gamma).$$

Let $\phi, \psi \in C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$; in particular, they are K -finite. Let ϕ_* and ψ_* denote the orthogonal projections of ϕ and ψ onto $\mathcal{L}_* := \mathcal{L}_{\text{temp}} \oplus \mathcal{L}(s_*)$. Applying Proposition 3.7 for \mathcal{L}_* , we have

$$|\langle \phi_* \circ a_t, \psi_* \rangle_{\mathcal{L}}| \ll_{\phi_*, \psi_*} (1+t)e^{-(d-s_*)t} \quad \text{for all } t > 0. \quad (6.9)$$

Recall the direct integral notation from Eq. (2.7). Applying Corollary 3.6 for each orthogonal summand $\mathcal{L}(\sigma, \delta_\Gamma)$ above, we have

$$\begin{aligned} \langle \pi_{\sigma, \delta_\Gamma}(a_t) \phi_{\sigma, \delta_\Gamma}, \psi_{\sigma, \delta_\Gamma} \rangle_{\mathcal{L}(\sigma, \delta_\Gamma)} &= e^{-(d-\delta_\Gamma)t} \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(\delta_\Gamma) \phi_{\sigma, \delta_\Gamma}, \psi_{\sigma, \delta_\Gamma} \rangle_{\mathcal{L}(\sigma, \delta_\Gamma)} \\ &\quad + O_\tau \left(e^{-(d-\delta_\Gamma+\eta_{\delta_\Gamma})t} \|\phi_{\sigma, \delta_\Gamma}\|_{\mathcal{L}(\sigma, \delta_\Gamma)} \|\psi_{\sigma, \delta_\Gamma}\|_{\mathcal{L}(\sigma, \delta_\Gamma)} \right) \quad \text{for all } t > 0. \end{aligned}$$

Note that for all $\sigma \in \widehat{M}_{s-d}$, the orthogonal projections $\phi_{\sigma, \delta_\Gamma}$ and $\psi_{\sigma, \delta_\Gamma}$ are non-trivial only if $\sigma \subset \tau$ and $\mathcal{U}(\sigma, \delta_\Gamma) \subset \mathcal{L}$. Thus, we have

$$e^{(d-\delta_\Gamma)t} \langle \phi \circ a_t, \psi \rangle_{\mathcal{L}} = \sum_{\sigma \in \widehat{M}_{s-d}: \sigma \subset \tau, \mathcal{U}(\sigma, \delta_\Gamma) \subset \mathcal{L}} \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(\delta_\Gamma) \phi_{\sigma, \delta_\Gamma}, \psi_{\sigma, \delta_\Gamma} \rangle_{\mathcal{L}(\sigma, \delta_\Gamma)} + O_{\phi, \psi}(e^{-\eta' t}) \quad \text{for all } t > 0 \quad (6.10)$$

where we choose any positive $\eta' < \min\{\eta, \eta_{\delta_\Gamma}\}$. Here, we note that the sum on the right hand side of Eq. (6.10) has finitely many terms since there are finitely many $\sigma \in \widehat{M}_{s-d}$ with $\sigma \subset \tau$.

On the other hand, by Theorem 5.1, we have

$$e^{(d-\delta_\Gamma)t} \langle \phi \circ a_t, \psi \rangle_{\mathcal{L}} = \int_{\Gamma \backslash G} \phi dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} dm^{\text{BR}*} + O_{\phi, \psi}(e^{-\eta_{\text{mix}} t}) \quad \text{for all } t > 0. \quad (6.11)$$

In light of Eqs. (6.10) and (6.11), we respectively define sesquilinear forms Q_{rep} on \mathcal{L}_τ and Q_{mix} on $C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$ as follows:

$$\begin{aligned} Q_{\text{rep}}(\phi, \psi) &:= \sum_{\sigma \in \widehat{M}_{s-d}: \sigma \subset \tau, \mathcal{U}(\sigma, \delta_\Gamma) \subset \mathcal{L}} \langle \mathbb{T}_\tau^\tau \mathbb{C}_+(\delta_\Gamma) \phi_{\sigma, \delta_\Gamma}, \psi_{\sigma, \delta_\Gamma} \rangle_{\mathcal{L}(\sigma, \delta_\Gamma)} \quad \text{for all } \phi, \psi \in \mathcal{L}_\tau, \\ Q_{\text{mix}}(\phi, \psi) &:= \int_{\Gamma \backslash G} \phi dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} dm^{\text{BR}*} \quad \text{for all } \phi, \psi \in C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau. \end{aligned}$$

Then, equating Eqs. (6.10) and (6.11), and taking $t \rightarrow +\infty$, we obtain

$$Q_{\text{rep}}(\phi, \psi) = Q_{\text{mix}}(\phi, \psi) \quad \text{for all } \phi, \psi \in C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau. \quad (6.12)$$

We now prove Part (1). In this case, $\tau = \mathbf{1} \in \widehat{K}$ is the trivial K -type and we have the following:

- $\sigma = \mathbf{1} \in \widehat{M}_{s-d}$ is the only M -type such that $\sigma \subset \tau$;
- \mathbb{T}_τ^τ acts by the identity operator on $\mathcal{L}(\mathbf{1}, \delta_\Gamma)_\tau$ by Lemma 3.1;
- $\mathbb{C}_+(\delta_\Gamma)$ acts by a non-zero scalar $\mathbb{C}_+(\tau : \mathbf{1}; \delta_\Gamma)$ on $\mathcal{L}(\mathbf{1}, \delta_\Gamma)_\tau$; see Section 4.

Hence, we have

$$Q_{\text{rep}}(\phi, \psi) = \mathbb{C}_+(\tau : \mathbf{1}; \delta_\Gamma) \cdot \langle \phi_{\mathbf{1}, \delta_\Gamma}, \psi_{\mathbf{1}, \delta_\Gamma} \rangle_{\mathcal{L}(\mathbf{1}, \delta_\Gamma)} \quad \text{for all } \phi, \psi \in \mathcal{L}_\tau. \quad (6.13)$$

Note that $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$ is contained in \mathcal{L} with multiplicity ≥ 1 . Indeed, otherwise, Q_{rep} would be the zero sesquilinear form, contradicting Eq. (6.12) and the fact that Q_{mix} is certainly not the zero sesquilinear form.

Now, suppose that $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$ is contained in \mathcal{L} with multiplicity at least 2. Then, we can find two orthogonal unit vectors $u, v \in \mathcal{L}(\mathbf{1}, \delta_\Gamma)_\tau$. Since $C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$ is a dense subspace of \mathcal{L}_τ by Lemma 2.13, we can find sequences $\{\phi^{(n)}\}_{n \in \mathbb{N}}$ and $\{\psi^{(n)}\}_{n \in \mathbb{N}}$ in $C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$ such that $\phi^{(n)} \rightarrow u$ and $\psi^{(n)} \rightarrow v$ in \mathcal{L}_τ as $n \rightarrow \infty$. In particular,

$$\langle \phi_{\mathbf{1}, \delta_\Gamma}^{(n)}, \phi_{\mathbf{1}, \delta_\Gamma}^{(n)} \rangle_{\mathcal{L}(\mathbf{1}, \delta_\Gamma)}, \langle \psi_{\mathbf{1}, \delta_\Gamma}^{(n)}, \psi_{\mathbf{1}, \delta_\Gamma}^{(n)} \rangle_{\mathcal{L}(\mathbf{1}, \delta_\Gamma)} \xrightarrow{n \rightarrow \infty} 1, \quad \text{and} \quad \langle \psi_{\mathbf{1}, \delta_\Gamma}^{(n)}, \phi_{\mathbf{1}, \delta_\Gamma}^{(n)} \rangle_{\mathcal{L}(\mathbf{1}, \delta_\Gamma)} \xrightarrow{n \rightarrow \infty} 0. \quad (6.14)$$

For sufficiently large $n \in \mathbb{N}$, the set $\{\phi^{(n)}, \psi^{(n)}\}$ is linearly independent and we may define the 2-dimensional subspace generated by it, $V^{(n)} := \text{span}\{\phi^{(n)}, \psi^{(n)}\} \subset C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau$. Further define $Q_{\text{rep}}^{(n)}$ and $Q_{\text{mix}}^{(n)}$ to be the restrictions of the sesquilinear forms Q_{rep} and Q_{mix} to $V^{(n)}$, respectively. It follows from Eqs. (6.13)

and (6.14) and continuity of the determinant that for sufficiently large $n \in \mathbb{N}$, the sesquilinear form $Q_{\text{rep}}^{(n)}$ is non-degenerate and hence has rank 2.

On the other hand, the sesquilinear form Q_{mix} has rank at most 1 since it is given by a *scalar* product of two linear forms. Consequently, for sufficiently large $n \in \mathbb{N}$, the same holds for the restriction $Q_{\text{mix}}^{(n)}$. This contradicts the fact that $Q_{\text{rep}}^{(n)} = Q_{\text{mix}}^{(n)}$ for all sufficiently large $n \in \mathbb{N}$ by Eq. (6.12) and finishes the proof of Part (1).

We now turn to Part (2). Recall the orthogonal decomposition $\mathcal{L}_\tau = \mathcal{L}_\tau^M \oplus \mathcal{L}_\tau^0$ into a subspace of M -invariant vectors and its orthocomplement in the notation introduced above Eq. (6.8). Since the disintegration of the measures m^{BR} and $m^{\text{BR}*}$ along fibers of the projection $\Gamma \backslash G \rightarrow \Gamma \backslash G/M$ is given by our fixed Haar measure on M , we have

$$Q_{\text{mix}}(\phi, \psi) = 0, \quad \text{for all } \phi, \psi \in C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau^0.$$

Now, $C_c^\infty(\Gamma \backslash G) \cap \mathcal{L}_\tau^0$ is a dense subspace of \mathcal{L}_τ^0 by Lemma 2.13 and continuity of the orthogonal projection $\mathcal{L}_\tau \rightarrow \mathcal{L}_\tau^0$. Moreover, Q_{rep} is continuous on \mathcal{L}_τ due to the fact that both T_τ^r and $C_+(\delta_\Gamma)$ are bounded operators since $\delta_\Gamma > d/2$. Combined with Eq. (6.12), this implies

$$Q_{\text{rep}}(\phi, \psi) = 0 \quad \text{for all } \phi, \psi \in \mathcal{L}_\tau^0.$$

Moreover, $\{\mathcal{L}(\sigma, \delta_\Gamma)_\tau^0\}_{\sigma \in \widehat{M}_{s-d}}$ is a set of mutually orthogonal subspaces. We conclude, in particular, that for all $\sigma \in \widehat{M}_{s-d}$ contained in τ , the operator $T_\tau^r C_+(\delta_\Gamma)$ vanishes on $\mathcal{L}(\sigma, \delta_\Gamma)_\tau^0$. Moreover, Lemma 6.7 also shows that this operator vanishes on $\mathcal{L}(\sigma, \delta_\Gamma)_\tau^M$ whenever σ is non-trivial, and hence on all of $\mathcal{L}(\sigma, \delta_\Gamma)_\tau$ in this case. Specializing further, this holds for $\sigma = v$, contradicting our choice of the K -type τ due to Corollary 4.3 and completing the proof. ■

6.3. Proof of Theorem 1.6. We now prove Theorem 1.6 using the established Theorem 1.3 on strong spectral gap. We require [EO21, Theorem 4.8] of Edwards–Oh which is derived from another theorem of theirs quoted earlier as Theorem 3.4.

Theorem 6.9 ([EO21, Theorem 4.8]). *There exists $m > d(d+1)/2$ such that for all $\sigma \in \widehat{M}_{s-d}$ and $s \in \mathcal{I}_\sigma$, if the quasi-complementary series $\mathcal{U}(\sigma, s)$ contains non-zero M -invariant vectors, then for all $u, v \in \mathcal{S}_K^m(\Gamma \backslash G)$ and $t \geq 0$, we have*

$$e^{(d-\delta_\Gamma)t} \langle \pi_{\sigma, s}(a_t)u, v \rangle_{\mathcal{U}(\sigma, s)} = \sum_{\tau_1, \tau_2 \in \widehat{K}} \langle T_{\tau_1}^{\tau_2} C_+(s) P_{\tau_1} u, P_{\tau_2} v \rangle_{\mathcal{U}(\sigma, s)} + O_s(e^{-\eta_s t} \|u\|_{\mathcal{S}_K^m(\Gamma \backslash G)} \|v\|_{\mathcal{S}_K^m(\Gamma \backslash G)}).$$

Here, the sum

$$\sum_{\tau_1, \tau_2 \in \widehat{K}} \langle T_{\tau_1}^{\tau_2} C_+(s) P_{\tau_1} u, P_{\tau_2} v \rangle_{\mathcal{U}(\sigma, s)}$$

converges absolutely.

Remark 6.10. Though originally not included, [EO21, Theorem 4.8] of Edwards–Oh also holds for the ends of complementary series $\mathcal{U}(\sigma, s)$ for $\sigma \in \widehat{M}_{s-d}$ and $s = d - \ell(\sigma) \in \partial \mathcal{I}_\sigma$ thanks to the lifting argument in the proof of Corollary 3.6.

Proof of Theorem 1.6. Let $\kappa_0 := \min\{\kappa_\Gamma, 1\} > 0$ be as in the theorem, where κ_Γ is the strong spectral gap parameter of Γ defined in Definition 1.5. Let

$$s_1 := \delta_\Gamma - \kappa_0.$$

We use the same decompositions as in the proof of Proposition 6.8 with s_* replaced by the optimal parameter s_1 . In particular, as in the proof of Proposition 6.8, we have the following decomposition:

$$\mathcal{L} = \mathcal{L}_{\text{temp}} \oplus \mathcal{L}(s_1) \oplus \mathcal{U}(\mathbf{1}, \delta_\Gamma).$$

Denote $\mathcal{L}_1 = \mathcal{L}_{\text{temp}} \oplus \mathcal{L}(s_1)$. Let $\phi, \psi \in \mathcal{S}_K^m(\Gamma \backslash G)$ and denote their orthogonal decompositions by

$$\phi = \phi_1 + \phi_{1, \delta_\Gamma} \in \mathcal{L}_1 \oplus \mathcal{U}(\mathbf{1}, \delta_\Gamma), \quad \psi = \psi_1 + \psi_{1, \delta_\Gamma} \in \mathcal{L}_1 \oplus \mathcal{U}(\mathbf{1}, \delta_\Gamma).$$

Exactly as in Eq. (6.9), applying Proposition 3.7 for \mathcal{L}_1 , we have

$$|\langle \phi_1 \circ a_t, \psi_1 \rangle_{\mathcal{L}}| \ll_{\phi_1, \psi_1} (1+t)e^{-(d-s_1)t} \quad \text{for all } t > 0.$$

It remains to treat the orthogonal direct summand $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$. Clearly, as $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$ is spherical, it contains non-zero M -invariant vectors. Therefore, we may apply Theorem 6.9 to $\mathcal{U}(\mathbf{1}, \delta_\Gamma)$ to obtain for all $t > 0$

$$e^{(d-\delta_\Gamma)t} \langle \pi_{\mathbf{1}, \delta_\Gamma}(a_t) \phi_{\mathbf{1}, \delta_\Gamma}, \psi_{\mathbf{1}, \delta_\Gamma} \rangle_{\mathcal{U}(\mathbf{1}, \delta_\Gamma)} = \sum_{\tau_1, \tau_2 \in \widehat{K}} \langle \mathbf{T}_{\tau_1}^{\tau_2} \mathbf{C}_+(s) \mathbf{P}_{\tau_1} \phi_{\mathbf{1}, \delta_\Gamma}, \mathbf{P}_{\tau_2} \psi_{\mathbf{1}, \delta_\Gamma} \rangle_{\mathcal{U}(\mathbf{1}, \delta_\Gamma)} \\ + O(e^{-\eta_{\delta_\Gamma} t} \|\phi_{\mathbf{1}, \delta_\Gamma}\|_{\mathcal{S}_K^m(\Gamma \backslash G)} \|\psi_{\mathbf{1}, \delta_\Gamma}\|_{\mathcal{S}_K^m(\Gamma \backslash G)}),$$

where the sum in the first term converges absolutely. As in the proof of Proposition 6.8, taking $t \rightarrow +\infty$, and applying Theorem 5.1, we see that the sum in the first term is in fact equal to $\int_{\Gamma \backslash G} \phi \, dm^{\text{BR}} \int_{\Gamma \backslash G} \bar{\psi} \, dm^{\text{BR}*}$.

Thus, combining these two estimates, we obtain a decay rate of $\min\{d - s_1, \eta_{\delta_\Gamma}\} - \varepsilon$ for every $\varepsilon > 0$. Note that we have the trivial inequality $d - s_1 \geq \delta_\Gamma - s_1 = \kappa_0$. We also have $\eta_{\delta_\Gamma} = \min\{2\delta_\Gamma - d, 1\}$ by definition. Note further that $s_1 \geq d/2$ by definition of κ_Γ ; cf. Definition 1.5. Together with $\delta_\Gamma > d/2$, we have $2\delta_\Gamma - d > \delta_\Gamma - d/2 \geq \delta_\Gamma - s_1 = \kappa_0$, and hence $\eta_{\delta_\Gamma} \geq \kappa_0$. This shows that $\min\{d - s_1, \eta_{\delta_\Gamma}\} \geq \kappa_0$ and completes the proof. \blacksquare

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