MIXING, RESONANCES, AND SPECTRAL GAPS ON GEOMETRICALLY FINITE LOCALLY SYMMETRIC SPACES

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Abstract. We prove that the geodesic flow on any geometrically finite locally symmetric space of negative curvature is super-polynomially mixing with respect to the Bowen-Margulis-Sullivan measure. When the critical exponent δ is close enough to the dimension D of the boundary at infinity, we show that the flow is in fact exponentially mixing. The latter result in particular holds when $\delta > 2D/3$ in the real hyperbolic case and $\delta > 5D/6$ in the general case.

The method is dynamical in nature and is based on constructing anisotropic Banach spaces on which the generator of the flow admits an essential spectral gap of size depending only on the critical exponent and the ranks of the cusps of the manifold (if any). Our analysis also yields that the Laplace transform of the correlation function of smooth observables extends meromorphically to the entire complex plane in the convex cocompact case and to a strip of explicit size beyond the imaginary axis in the case the manifold admits cusps. Along the way, we construct a Margulis function to control recurrence to compact sets when the manifold has cusps.

1. Introduction

Let X be the unit tangent bundle of a quotient of a real, complex, quaternionic, or a Cayley hyperbolic space by a discrete, geometrically finite, non-elementary group of isometries Γ. Denote by g_t the geodesic flow on X and by $\mathrm{m}^{\mathrm{BMS}}$ the Bowen-Margulis-Sullivan probability measure of maximal entropy for g_t. Let δ_Γ be the critical exponent of Γ and D be the dimension of the boundary at infinity of the associated symmetric space. We refer the reader to Section 2 for definitions. The following is the main result of this article in its simplest form.

Theorem A. *The geodesic flow is super-polynomially mixing with respect to* $\mathrm{m}^{\mathrm{BMS}}$. *More precisely, for all* $f, g \in C_c^\infty(X)$, *and* $p, t \geq 0$,

$$\int_X f \circ g_t \cdot g \; d\mathrm{m}^{\mathrm{BMS}} = \int_X f \; d\mathrm{m}^{\mathrm{BMS}} \int_X g \; d\mathrm{m}^{\mathrm{BMS}} + O_{f,p}\left(\|g\|_{C^1} t^{-p} \right).$$

If we further assume that $\delta_\Gamma > 2D/3$ *in the real hyperbolic case or that* $\delta_\Gamma > 5D/6$ *in the other cases, then there exist* $\sigma_0 = \sigma_0(X) > 0$ *and* $k \in \mathbb{N}$ *such that*

$$\int_X f \circ g_t \cdot g \; d\mathrm{m}^{\mathrm{BMS}} = \int_X f \; d\mathrm{m}^{\mathrm{BMS}} \int_X g \; d\mathrm{m}^{\mathrm{BMS}} + \|f\|_{C^2} \|g\|_{C^k} O\left(e^{-\sigma_0 t} \right).$$

The implied constants also depend on the injectivity radius of the support of g.

In fact, we show that the correlation function admits a finite resonance expansions. We state this result in the exponential mixing case.

Theorem B. *Suppose that either* $\delta_\Gamma > 2D/3$ *in the real hyperbolic case or* $\delta_\Gamma > 5D/6$ *otherwise. Then, there exists* $\sigma > 0$, *depending only on* δ_Γ *and the ranks of the cusps of* X *(if any) such that the following holds. There exist* $k \in \mathbb{N}$, *finitely many complex numbers* $\lambda_1, \dots, \lambda_N$ *in* $\{-\sigma < \mathrm{Re}(z) < 0\} \cup \{0\}$, *and finitely many bilinear forms* $\Pi_i : C_c^2(X) \times C_c^k(X) \to \mathbb{C}$ *such that for all* $(f, g) \in C_c^2 \times C_c^k$ *and* $t \geq 0$,

$$\int_X f \circ g_t \cdot g \; d\mathrm{m}^{\mathrm{BMS}} = \sum_{i=1}^N \Pi_i(f, g) e^{\lambda_i t} + O_{f,g}(e^{-\sigma t}).$$

Moreover, for $\lambda_i = 0$, *we have* $\Pi_i(f, g) = \int f \; d\mathrm{m}^{\mathrm{BMS}} \int g \; d\mathrm{m}^{\mathrm{BMS}}$.

Remark 1.1. The above results also hold for functions with unbounded support and controlled growth in the cusp; cf. Section 8. The dependence on f in the rapidly mixing case is through its C^1 norm as well as a number of its derivatives in the flow direction depending on the rate of polynomial decay p.

With a little more work, our method can in fact show that exponential mixing along with Theorem B hold whenever $\delta_\Gamma > D/2$ in the real hyperbolic case and when $\delta_\Gamma > 2D/3$ in the other cases. We expect these results to hold without such restrictions on the critical exponent.

The "eigenvalues" λ_i in Theorem B are commonly referred to as *Pollicott-Ruelle resonances*. The phenomenon of having a strip with finitely many resonances in different contexts is commonly referred to as having an *essential spectral gap*.

It is worth noting that Theorem B implies that the the size of the essential spectral gap σ does not change if we replace Γ with a finite index subgroup. The interested reader is referred to [MN20, MN21] for recent developments yielding uniform resonance free regions for the Laplacian operator in random covers of convex cocompact hyperbolic surfaces.

Among the motivations for Theorem B is the Jakobson-Naud conjecture concerning the closely related problem of the size of the essential spectral gap of the hyperbolic Laplacian operator for convex cocompact hyperbolic surfaces [JN12]. It asserts that the size of such essential spectral gap is exactly half the critical exponent. We defer the study of essential spectral gaps for Laplacians, as well as Ruelle/Selberg zeta functions, to future work.

Our analysis also yields the following result. Let δ_Γ denote the critical exponent of Γ and define

$$\sigma(\Gamma) := \begin{cases} \infty, & \text{if } \Gamma \text{ is convex cocompact,} \\ \min\left\{\delta_\Gamma, 2\delta_\Gamma - k_{\max}, k_{\min}\right\}, & \text{otherwise,} \end{cases}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively; cf. Section 2 for definitions.

Given two bounded functions f and g on X, the associated correlation function is defined by

$$\rho_{f,g}(t) := \int_X f \circ g_t \cdot g \; dm^{\mathrm{BMS}}, \qquad t \in \mathbb{R}.$$

Its (one-sided) Laplace transform is defined for any $z \in \mathbb{C}$ with positive real part $\mathrm{Re}(z)$ as follows:

$$\hat{\rho}_{f,g}(z) := \int_0^\infty e^{-zt} \rho_{f,g}(t) \; dt.$$

The following result is valid without restrictions on the critical exponent.

Theorem C. *Let $k \in \mathbb{N}$. For all $f, g \in C_c^{k+1}(X)$, $\hat{\rho}_{f,g}$ is analytic in the half plane $\mathrm{Re}(z) > 0$ and admits a meromorphic continuation to the half plane:*

$$\mathrm{Re}(z) > -\min\left\{k, \sigma(\Gamma)/2\right\},$$

with the only possible pole on the imaginary axis being the origin. In particular, when Γ is convex cocompact and $f, g \in C_c^\infty(X)$, $\hat{\rho}_{f,g}$ admits a meromorphic extension to the entire complex plane.

Theorem C is deduced from an analogous result on the meromorphic continuation of the family of resolvent operators $z \mapsto R(z)$,

$$R(z) := \int_0^\infty e^{-zt} \mathcal{L}_t \; dt : C_c^\infty(X) \to C^\infty(X),$$

defined initially for $\mathrm{Re}(z) > 0$, where $\mathcal{L}_t$ is the transfer operator given by $f \mapsto f \circ g_t$. Analogous results regarding resolvents were obtained for Anosov flows in [GLP13] and Axiom A flows in [DG16, DG18] leading to a resolution of a conjecture of Smale on the meromorphic continuation of the Ruelle zeta function; cf. [Sma67]. We refer the reader to [GLP13] for a discussion the history of the latter problem and to [GBW22] for related results for finite volume negatively curved manifolds with hyperbolic cusps.

1.1. **Prior results.** In the case Γ is convex cocompact, Theorem A is a special case of the results of [Sto11] which extend the arguments of Dolgopyat [Dol98] to Axiom A flows under certain assumptions on the regularity of the foliations and the holonomy maps. The special case of convex cocompact hyperbolic surfaces was treated in earlier work of Naud [Nau05]. The extension to frame flows on convex cocompact hyperbolic manifolds was treated in [SW20].

In the case of real hyperbolic manifolds with δ_Γ strictly greater than half the dimension of the boundary at infinity, Theorem A and B follow from the work of [EO21], with much more precise and explicit estimates on the size of the essential spectral gap. The methods of [EO21] are unitary representation theoretic, building on results of [LP82], for which the restriction on the critical exponent is necessary. Earlier instances of these results under more stringent assumptions on the size of δ_Γ were obtained in [MO15], albeit the latter results are stronger in that they in fact hold for the frame flow rather than the geodesic flow.

The case of real hyperbolic geometrically finite manifolds with cusps and arbitrary critical exponent was only resolved very recently in [LP20] where a symbolic coding of the geodesic flow was constructed. This approach relies on extensions of Dolgopyat's method to suspension flows over shifts with infinitely many symbols; cf. [AM16, AGY06]. However, it seems the approach does not yield information on the size of the essential spectral gap or the meromorphic continuation of $\hat{\rho}_{f,g}$.

Finally, we refer the reader to [DG16] and the references therein for a discussion of the history of the microlocal approach to the problem of spectral gaps via anisotropic Sobolev spaces.

1.2. **Organization of the article.** After recalling some basic facts in Section 2, we prove a key doubling result, Proposition 3.1, in Section 3 for the conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the strong unstable foliation.

In Section 4, we construct a Margulis function which shows, roughly speaking, that generic orbits with respect to $\mathrm{m}^{\mathrm{BMS}}$ are biased to return to the thick part of the manifold. In Section 5, we prove a statement on average exapnsion of vectors in linear representation which is essential for our construction of the Margulis function. The main difficulty in the latter result in comparison with the classical setting lies in controlling the *shape* of sublevel sets of certain polynomials with respect to conditional measures of $\mathrm{m}^{\mathrm{BMS}}$ along the unstable foliation.

In Section 6, we define anisotropic Banach spaces arising as completions of spaces of smooth functions with respect to a dynamically relevant norm and study the norm of the transfer operator as well as the resolvent in their actions on these spaces in Section 7. The proof of Theorem C is completed in Section 7. The approach of these two sections follows closely the ideas of [GL06, GL08, AG13], originating in [BKL02].

The key technical estimate towards establishing Theorems A and B is proven in Section 8, where the proofs of these latter results are completed. This result is a Dolgopyat-type estimate on the norm of resolvents with large imaginary parts. The main idea, going back to Dolgopyat, is to exploit the non-joint integrability of the stable and unstable foliations via a certain oscillatory integral estimate; cf. [Dol98, Liv04, GLP13, BDL18].

A major difficulty in implementing such philosophy lies in[1] estimating certain oscillatory integrals against the (possibly) *fractal* Patterson-Sullivan measures. We introduce a dynamical method which replaces these fractal measures with smooth ones and is based on a refinement of the idea of transverse intersections used in Roblin's thesis in the proof of his mixing theorems [Rob03]. We hope this method can provide a fruitful alternative route to symbolic coding in establishing rates of mixing of hyperbolic flows in greater generality beyond the case of SRB measures.

[1]For low dimensional real hyperbolic manifolds, we refer the reader to [BD17] for a breakthrough in this direction in the case of convex cocompact surfaces and its extension to Schottky 3-manifolds in [LNP21].

Finally, in Sections 9 and 10, we prove auxiliary technical results needed for the main estimate in Section 8. For the reader's convenience, an index of notation for Section 8 is provided at the end of the article.

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2. Preliminaries

We recall here some background and definitions regarding geometrically finite manifolds.

2.1. **Geometrically Finite Manifolds.** The standard reference for the material in this section is [Bow93]. Suppose G is a connected simple Lie group of real rank one. Then, G can be identified with the group of orientation preserving isometries of a real, complex, quaternionic or Cayley hyperbolic space, denoted $\mathbb{H}^d_{\mathfrak{K}}$, of dimension $d \geq 2$, where $\mathfrak{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In the case $\mathfrak{K} = \mathbb{O}$, $d = 2$.

Fix a basepoint $o \in \mathbb{H}^d_{\mathfrak{K}}$. Then, G acts transitively on $\mathbb{H}^d_{\mathfrak{K}}$ and the stabilizer K of o is a maximal compact subgroup of G. We shall identify $\mathbb{H}^d_{\mathfrak{K}}$ with $K\backslash G$. Denote by $A = \{g_t : t \in \mathbb{R}\}$ a one parameter subgroup of G inducing the geodesic flow on the unit tangent bundle of $\mathbb{H}^d_{\mathfrak{K}}$. Let $M < K$ denote the centralizer of A inside K so that the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}$ may be identified with $M\backslash G$. In Hopf coordinates, we can identify $\mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}$ with $\mathbb{R} \times (\partial\mathbb{H}^d_{\mathfrak{K}} \times \partial\mathbb{H}^d_{\mathfrak{K}} - \Delta)$, where $\partial\mathbb{H}^d_{\mathfrak{K}}$ denotes the boundary at infinity and Δ denotes the diagonal.

Let $\Gamma < G$ be an infinite discrete subgroup of G. The limit set of Γ, denoted Λ_Γ, is the set of limit points of the orbit $\Gamma \cdot o$ on $\partial\mathbb{H}^d_{\mathfrak{K}}$. Note that the discreteness of Γ implies that such limit points exist and that they all belong to the boundary. Moreover, Λ_Γ is the smallest closed Γ invariant set in $\partial\mathbb{H}^d_{\mathfrak{K}}$ and as such Γ acts minimally on Λ. In particular, this definition is independent of the choice of o. We often use Λ to denote Λ_Γ when Γ is understood from context. We say Γ is **non-elementary** if Λ_Γ is infinite.

The **hull** of Λ_Γ, denoted $\mathrm{Hull}(\Lambda_\Gamma)$, is the smallest convex subset of $\mathbb{H}^d_{\mathfrak{K}}$ containing all the geodesics joining points in Λ_Γ. The convex core of the manifold $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$ is the smallest convex subset containing the image of $\mathrm{Hull}(\Lambda_\Gamma)$. We say $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$ is **geometrically finite** if the unit neighborhood of the convex core has finite volume, cf. [Bow93]. The non-wandering set for the geodesic flow is the closure of the set of vectors in the unit tangent bundle whose orbit accumulates on itself. In Hopf coordinates, this set, denoted Ω, coincides with the projection of $\mathbb{R} \times (\Lambda_\Gamma \times \Lambda_\Gamma - \Delta)$ mod Γ.

A useful equivalent definition of geometric finiteness is that the limit set of Γ consists entirely of radial and bounded parabolic limit points; cf. [Bow93]. This characterization of geometric finiteness will be of importance to us and so we recall here the definitions of these objects.

A point $\xi \in \Lambda$ is said to be a **radial point** if any geodesic ray terminating at ξ returns infinitely often to a bounded subset of $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$. The set of radial limit points is denoted by Λ_r.

Denote by N^+ the expanding horospherical subgroup of G associated to g_t, $t \geq 0$. A point $p \in \Lambda$ is said to be a **parabolic point** if the stabilizer of p in Γ, denoted by Γ_p, is conjugate in G to an unbounded subgroup of MN^+. A parabolic limit point p is said to be **bounded** if $(\Lambda - \{p\}/\Gamma_p)$ is compact. An equivalent charachterization is that $p \in \Lambda$ is parabolic if and only if any geodesic ray terminating at p eventually leaves every compact subset of $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$. The set of parabolic limit points will be denoted by Λ_p.

Given $g \in G$, we denote by g^+ the coset of P^-g in the quotient $P^-\backslash G$, where $P^- = N^-AM$ is the stable parabolic group associated to $\{g_t : t \geq 0\}$. Similarly, g^- denotes the coset P^+g in $P^+\backslash G$. Since M is contained in $P^\pm$, such a definition makes sense for vectors in the unit tangent bundle $M\backslash G$. Geometrically, for $v \in M\backslash G$, v^+ (resp. v^-) is the forward (resp. backward) endpoint of the geodesic determined by v on the boundary of $\mathbb{H}^d_{\mathfrak{K}}$. Given $x \in G/\Gamma$, we say $x^{\pm}$ belongs to Λ if the same holds for any representative of x in G; this notion being well-defined since Λ is Γ invariant.

Notation. Throughout the remainder of the article, we fix a discrete non-elementary geometrically finite group Γ of isometries of some (irreducible) rank one symmetric space $\mathbb{H}^d_{\mathfrak{K}}$ and denote by X the quotient G/Γ, where G is the isometry group of $\mathbb{H}^d_{\mathfrak{K}}$.

2.2. **Standard horoballs.** Since parabolic points are fixed points of elements of Γ, Λ contains only countably many such points. Moreover, Γ contains at most finitely many conjugacy classes of parabolic subgroups. This translates to the fact that Λ_p consists of finitely many Γ orbits.

Let $\{p_1, \dots, p_s\} \subset \partial\mathbb{H}^d_{\mathfrak{K}}$ be a maximal set of nonequivalent parabolic fixed points under the action of Γ. As a consequence of geometric finiteness of Γ, one can find a finite disjoint collection of **open** horoballs $H_1, \dots, H_s \subset \mathbb{H}^d_{\mathfrak{K}}$ with the following properties (cf. [Bow93]):

(a) H_i is centered on p_i, for $i = 1, \dots, s$.
(b) $\overline{H_i}\Gamma \cap \overline{H_j}\Gamma = \emptyset$ for all $i \neq j$.
(c) For all $i \in \{1, \dots, s\}$ and $\gamma_1, \gamma_2 \in \Gamma$

$$\overline{H_i}\gamma_1 \cap \overline{H_i}\gamma_2 \neq \emptyset \Longrightarrow \overline{H_i}\gamma_1 = \overline{H_i}\gamma_2, \gamma_1^{-1}\gamma_2 \in \Gamma_{p_i}.$$

(d) $\mathrm{Hull}(\Lambda_\Gamma) \setminus (\bigcup_{i=1}^s H_i\Gamma)$ is compact mod Γ.

2.3. **Conformal Densities and the BMS Measure.** The **critical exponent**, denoted δ_Γ, is defined to be the infimum over all real number $s \geq 0$ such that the Poincaré series

$$P_\Gamma(s, o) := \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma\cdot o)} \tag{2.1}$$

converges. We shall simply write δ for δ_Γ when Γ is understood from context. The Busemann function is defined as follows: given $x, y \in \mathbb{H}^d_{\mathfrak{K}}$ and $\xi \in \partial\mathbb{H}^d_{\mathfrak{K}}$, let $\gamma : [0, \infty) \to \mathbb{H}^d_{\mathfrak{K}}$ denote a geodesic ray terminating at ξ and define

$$\beta_\xi(x, y) = \lim_{t\to\infty} \mathrm{dist}(x, \gamma(t)) - \mathrm{dist}(y, \gamma(t)).$$

A Γ-invariant conformal density of dimension s is a collection of Radon measures $\left\{\nu_x : x \in \mathbb{H}^d_{\mathfrak{K}}\right\}$ on the boundary satisfying

$$\frac{d\nu_{\gamma x}}{d\nu_x}(\xi) = e^{-s\beta_\xi(x,\gamma x)}, \qquad \forall \xi \in \partial\mathbb{H}^d_{\mathfrak{K}}.$$

Given a pair of conformal densities $\{\mu_x\}$ and $\{\nu_x\}$ of dimensions s_1 and s_2 respectively, we can form a Γ invariant measure on $\mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}$, denoted by $m^{\mu,\nu}$ as follows: for $x = (\xi_1, \xi_2, t) \in \mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}$

$$dm^{\mu,\nu}(\xi_1, \xi_2, t) = e^{s_1\beta_{\xi_1}(o,x) + s_2\beta_{\xi_2}(o,x)}\, d\mu_o(\xi_1)\, d\nu_o(\xi_2)\, dt.$$

Moreover, the measure $m^{\mu,\nu}$ is invariant by the geodesic flow.

When Γ is geometrically finite and $\mathfrak{K} = \mathbb{R}$, Patterson [Pat76] and Sullivan [Sul79] showed the existence of a unique (up to scaling) Γ-invariant conformal density of dimension δ_Γ, denoted $\left\{\mu_x^{\mathrm{PS}} : x \in \mathbb{H}^d_{\mathbb{R}}\right\}$. When Γ is geometrically finite, the measure $m^{\mu^{\mathrm{PS}},\mu^{\mathrm{PS}}}$ descends to a finite measure of full support on Ω and is the unique measure of maximal entropy for the geodesic flow. This measure is called the Bowen-Margulis-Sullivan (BMS for short) measure and is denoted $\mathrm{m}^{\mathrm{BMS}}$.

Since the fibers of the projection from G/Γ to $\mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}/\Gamma$ are compact and parametrized by the group M, we can lift such a measure to one G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, by taking locally the product with the Haar probability measure on M. Since M commutes with the geodesic flow, this lift is invariant under the group A. We refer the reader to [Rob03] and [PPS15] and references therein for details of the construction in much greater generality than that of $\mathbb{H}^d_{\mathbb{R}}$.

2.4. **Stable and unstable foliations and leafwise measures.** The fibers of the projection $G \to \mathrm{T}^1\mathbb{H}^d_{\mathfrak{K}}$ are given by the compact group M, which is the centralizer of A inside the maximal compact group K. In particular, we may lift $\mathrm{m}^{\mathrm{BMS}}$ to a measure on G/Γ, also denoted $\mathrm{m}^{\mathrm{BMS}}$, and given locally by the product of $\mathrm{m}^{\mathrm{BMS}}$ with the Haar probability measure on M. The leafwise measures of $\mathrm{m}^{\mathrm{BMS}}$ on N^+ orbits are given as follows:

$$d\mu_x^u(n) = e^{\delta_\Gamma \beta_{(nx)^+}(o,nx)} d\mu_o^{\mathrm{PS}}((nx)^+). \tag{2.2}$$

They satisfy the following equivariance property under the geodesic flow:

$$\mu_{g_t x}^u = e^{\delta t} \mathrm{Ad}(g_t)_* \mu_x^u. \tag{2.3}$$

Moreover, it follows readily from the definitions that for all $n \in N^+$,

$$(n)_* \mu_{nx}^u = \mu_x^u, \tag{2.4}$$

where $(n)_*\mu_{nz}^u$ is the pushforward of μ_{nz}^u under the map $u \mapsto un$ from N^+ to itself. Finally, since M normalizes N^+ and leaves $\mathrm{m}^{\mathrm{BMS}}$ invariant, this implies that these conditionals are $\mathrm{Ad}(M)$-invariant: for all $m \in M$,

$$\mu_{mx}^u = \mathrm{Ad}(m)_* \mu_x^u. \tag{2.5}$$

2.5. **Carnot-Caratheodory metrics.** We recall the definition of Carnot-Caratheodory metric on N^+, denoted d_{N^+}. These metrics are right invariant under translation by N^+, and satisfy the following convenient scaling property under conjugation by g_t. For all $r > 0$, if N_r^+ denotes the ball of radius r around identity in that metric and $t \in \mathbb{R}$, then

$$\mathrm{Ad}(g_t)(N_r^+) = N_{e^t r}^+. \tag{2.6}$$

To define the metric, we need some notation which we use throughout the article. For $x \in \mathfrak{K}$, denote by $\bar{x}$ its $\mathfrak{K}$-conjugate and by $|x| := \sqrt{\bar{x}x}$ its modulus. Recall that such norms are multiplicative in the sense that $\|uv\| = \|u\|\,\|v\|$. We let $\mathrm{Im}\mathfrak{K}$ denote those $x \in \mathfrak{K}$ such that $\bar{x} = -x$. For example, $\mathrm{Im}\mathfrak{K}$ is the pure imaginary numbers and the subspace spanned by the quaternions i, j and k in the cases $\mathfrak{K} = \mathbb{C}$ and $\mathfrak{K} = \mathbb{H}$ respectively. For $u \in \mathfrak{K}$, we write $\mathrm{Re}(u) = (u + \bar{u})/2$ and $\mathrm{Im}(u) = (u - \bar{u})/2$.

The Lie algebra $\mathfrak{n}^+$ of N^+ splits under $\mathrm{Ad}(g_t)$ into eigenspaces as $\mathfrak{n}^+_\alpha \oplus \mathfrak{n}^+_{2\alpha}$, where $\mathfrak{n}^+_{2\alpha} = 0$ if and only if $\mathfrak{K} = \mathbb{R}$. Moreover, we have the identification $\mathfrak{n}^+_\alpha \cong \mathfrak{K}^{d-1}$ and $\mathfrak{n}^+_{2\alpha} \cong \mathrm{Im}(\mathfrak{K})$ as real vector spaces; cf. [Mos73, Section 19]. With this notation, we can define the metric as follows: given $(u, s) \in \mathfrak{n}^+_\alpha \oplus \mathfrak{n}^+_{2\alpha}$, the distance of $n := \exp(u, s)$ to identity is given by:

$$d_{N^+}(n, \mathrm{Id}) := \left(\|u\|^4 + \|s\|^2 \right)^{1/4}. \tag{2.7}$$

Given $n_1, n_2 \in N^+$, we set $d_{N^+}(n_1, n_2) = d_{N^+}(n_1 n_2^{-1}, \mathrm{Id})$.

2.6. **Local stable holonomy.** In this Section, we recall the definition of (stable) holonomy maps which are essential for our arguments. We give a simplified discussion of this topic which is sufficient in our homogeneous setting homogeneous. Let $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. Since the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism near identity, we can choose the norm on the Lie algebra so that the following holds. We can find maps $p^- : N_1^+ \to P^- = N^- AM$ and $u^+ : N_2^+ \to N^+$ so that

$$nu^- = p^-(n) u^+(n), \qquad \forall n \in N_2^+. \tag{2.8}$$

Then, it follows by (2.2) that for all $n \in N_2^+$, we have

$$d\mu_y^u(u^+(n)) = e^{\delta \beta_{(nx)^+}(u^+(n)y, nx)} d\mu_x^u(n).$$

Moreover, by further scaling the metrics if necessary, we can ensure that these maps are diffeomorphisms onto their images. In particular, writing $\Phi(nx) = u^+(n)y$, we obtain the following change of variables formula: for all $f \in C(N_2^+)$,

$$\int f(n)\, d\mu_x^u(n) = \int f((u^+)^{-1}(n)) e^{-\delta\beta_{\Phi^{-1}(ny)}(ny,\Phi^{-1}(ny))}\, d\mu_y^u(n). \tag{2.9}$$

Remark 2.1. To avoid cluttering the notation with auxiliary constants, we shall assume that the N^- component of $p^-(n)$ belongs to N_2^- for all $n \in N_2^+$ whenever u^- belongs to N_1^-.

2.7. **Notational convention.** Throughout the article, given two quantities A and B, we use the Vingogradov notation $A \ll B$ to mean that there exists a constant $C \geq 1$, possibly depending on Γ and the dimension of G, such that $|A| \leq CB$. In particular, this dependence on Γ is suppressed in all of our implicit constants, except when we wish to emphasize it. The dependence on Γ may include for instance the diameter of the complement of our choice of cusp neighborhoods inside Ω and the volume of the unit neighborhood of Ω. We write $A \ll_{x,y} B$ to indicate that the implicit constant depends parameters x and y. We also write $A = O_x(B)$ to mean $A \ll_x B$.

3. Doubling Properties of Leafwise Measures

The goal of this section is to prove the following useful consequence of the global measure formula on the doubling properties of the leafwise measures. The result is immediate in the case Γ is convex cocompact. In particular, the content of the following result is the uniformity, even in the case Ω is not compact. The argument is based on the topological transitivity of the flow.

Define the following exponents:

$$\begin{aligned} \Delta &:= \min\{\delta, 2\delta - k_{\max}, k_{\min}\}, \\ \Delta_+ &:= \max\{\delta, 2\delta - k_{\min}, k_{\max}\}. \end{aligned} \tag{3.1}$$

where $k_{\max}$ and $k_{\min}$ denote the maximal and minimal ranks of parabolic fixed points of Γ respectively. If Γ has no parabolic points, we set $k_{\max} = k_{\min} = \delta$, so that $\Delta = \Delta_+ = \delta$.

Proposition 3.1 (Global Doubling and Decay)**.** *For every $0 < \sigma \leq 5$, $x \in N_2^-\Omega$ and $0 < r \leq 1$, we have*

$$\mu_x^u(N_{\sigma r}^+) \ll \begin{cases} \sigma^{\Delta} \cdot \mu_x^u(N_r^+) & \forall 0 < \sigma \leq 1, 0 < r \leq 1, \\ \sigma^{\Delta_+} \cdot \mu_x^u(N_r^+) & \forall \sigma > 1, 0 < r \leq 5/\sigma. \end{cases}$$

Remark 3.2. The above proposition has very different flavor when applied with $\sigma < 1$, compared with $\sigma > 1$. In the former case, we obtain a global rate of decay of the measure of balls on the boundary, centered in the limit set. In the latter case, we obtain the so-called Federer property for our leafwise measures.

Remark 3.3. The restriction that $r \leq 5/\sigma$ in the case $\sigma > 1$ allows for a uniform implied constant. The proof shows that in fact, when $\sigma > 1$, the statement holds for any $0 < r \leq 1$, but with an implied constant depending on σ.

3.1. **Global Measure Formula.** Our basic tool in proving Proposition 3.1 is the extension of Sullivan's shadow lemma known as the global measure formula, which we recall in this section.

Given a parabolic fixed point $p \in \Lambda$, with stabilizer $\Gamma_p \subset \Gamma$, we define **the rank of** p to be twice the critical exponent of the Poincaré series $P_{\Gamma_p}(s, o)$ associated with Γ_p; cf. (2.1). This rank is always an integer. In the case of real hyperbolic spaces, it agrees with the dimension of the unipotent radical N_p of the Zariski closure of Γ_p inside the parabolic subgroup of G stabilizing p. For general hyperbolic spaces, such a unipotent radical is a nilpotent group of step size at most 2 and fits in an exact sequence

$$0 \to \mathbb{R}^l \to N_p \to \mathbb{R}^k \to 0,$$

extending the abelian group $\mathbb{R}^k$ by the center $\mathbb{R}^l$. In this case, the rank is equal to $2l + k$.

Given $\xi \in \partial\mathbb{H}^d_{\mathfrak{K}}$, we let $[o\xi)$ denote the geodesic ray. For $t \in \mathbb{R}_+$, denote by $\xi(t)$ the point at distance t from o on $[o\xi)$. For $x \in \mathbb{H}^d_{\mathfrak{K}}$, define the $\mathcal{O}(x)$ to be the **shadow** of unit ball $B(x, 1)$ in $\mathbb{H}^d_{\mathfrak{K}}$ on the boundary as viewed from o. More precisely,

$$\mathcal{O}(x) := \left\{ \xi \in \partial\mathbb{H}^d_{\mathfrak{K}} : [o\xi) \cap B(x,1) \neq \emptyset \right\}.$$

Shadows form a convenient, dynamically defined, collection of neighborhoods of points on the boundary.

The following generalization of Sullivan's shadow lemma gives precise estimates on the measures of shadows with respect to Patterson-Sullivan measures.

Theorem 3.4 (Theorem 3.2, [Sch04])**.** *There exists $C = C(\Gamma, o) \geq 1$ such that for every $\xi \in \Lambda$ and all $t > 0$,*

$$C^{-1} \leq \frac{\mu_o^{\mathrm{PS}}(\mathcal{O}(\xi(t)))}{e^{-\delta t} e^{d(t)(k(\xi(t)) - \delta)}} \leq C,$$

where

$$d(t) = \mathrm{dist}(\xi(t), \Gamma \cdot o),$$

and $k(\xi(t))$ denotes the rank of a parabolic fixed point p if $\xi(t)$ is contained in a standard horoball centered at p and otherwise $k(\xi(t)) = \delta$.

A version of Theorem 3.4 was obtained earlier for real hyperbolic spaces in [SV95] and for complex and quaternionic hyperbolic spaces in [New03].

3.2. **Proof of Proposition 3.1.** Assume that $\sigma \leq 1$, the proof in the case $\sigma > 1$ being identical.

Fix a non-negative C^∞ bump function ψ supported inside N_1^+ and having value identically 1 on $N_{1/2}^+$. Given $\varepsilon > 0$, let $\psi_\varepsilon(n) = \psi(\mathrm{Ad}(g_{-\log\varepsilon})(n))$. Note that the condition that $\psi_\varepsilon(\mathrm{Id}) = \psi(\mathrm{Id}) = 1$ implies that for $x \in X$ with $x^+ \in \Lambda$,

$$\mu_x^u(\psi_\varepsilon) > 0, \qquad \forall \varepsilon > 0. \tag{3.2}$$

Note further that for any $r > 0$, we have that $\chi_{N_r^+} \leq \psi_r \leq \chi_{N_{2r}^+}$.

First, we establish a uniform bound over $x \in \Omega$. Consider the following function $f_\sigma : \Omega \to (0, \infty)$:

$$f_\sigma(x) = \sup_{0 < r \leq 1} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)}.$$

We claim that it suffices to prove that

$$f_\sigma(x) \ll \sigma^\Delta, \tag{3.3}$$

uniformly over all $x \in \Omega$ and $0 < \sigma \leq 1$. Indeed, fix some $0 < r \leq 1$ and $0 < \sigma \leq 1$. By enlarging our implicit constant if necessary, we may assume that $\sigma \leq 1/4$. From the above properties of ψ, we see that

$$\mu_x^u(N_{\sigma r}^+) \leq \mu_x^u(\psi_{(4\sigma)(r/2)}) \ll \sigma^\Delta \mu_x^u(\psi_{r/2}) \leq \sigma^\Delta \mu_x^u(N_r^+).$$

Hence, it remains to prove (3.3). By [Rob03, Lemme 1.16], for each given $r > 0$, the map $x \mapsto \mu_x^u(\psi_{\sigma r})/\mu_x^u(\psi_r)$ is a continuous function on Ω. Indeed, the weak-$*$ continuity of the map $x \mapsto \mu_x^u$ is the reason we work with bump functions instead of indicator functions directly. Moreover, continuity of these functions implies that f_σ is lower semi-continuous.

The crucial observation regarding f_σ is as follows. In view of (2.3), we have for $t \geq 0$,

$$f_\sigma(g_t x) = \sup_{0 < r \leq e^{-t}} \frac{\mu_x^u(\psi_{\sigma r})}{\mu_x^u(\psi_r)} \leq f_\sigma(x).$$

Hence, for all $B \in \mathbb{R}$, the sub-level sets $\Omega_{<B} := \{f_\sigma < B\}$ are invariant by g_t for all $t \geq 0$. On the other hand, the restriction of the (forward) geodesic flow to Ω is topologically transitive. In particular, any invariant subset of Ω with non-empty interior must be dense in Ω. Hence, in view of the lower semi-continuity of f_σ, to prove (3.3), it suffices to show that f_σ satisfies (3.3) for all x in some open subset of Ω.

Recall we fixed a basepoint $o \in \mathbb{H}^d_{\mathfrak{K}}$ belonging to the hull of the limit set. Let $x_o \in G$ denote a lift of o whose projection to G/Γ belongs to Ω. Let E denote the unit neighborhood of x_o. We show that $E \cap \Omega \subset \left\{ f_\sigma \ll \sigma^\Delta \right\}$. Without loss of generality, we may further assume that $\sigma < 1/2$, by enlarging the implicit constant if necessary.

First, note that the definition of the conditional measures μ_x^u immediately gives

$$\mu_x^u|_{N_4^+} \asymp \mu_o^{\mathrm{PS}}|_{(N_4^+ \cdot x)^+}, \qquad \forall x \in E.$$

It follows that

$$\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+) \ll \mu_x^u(\psi_r) \ll \mu_o^{\mathrm{PS}}((N_{2r}^+ \cdot x)^+),$$

for all $0 \leq r \leq 2$ and $x \in E$. Hence, it will suffice to show

$$\frac{\mu_o^{\mathrm{PS}}((N_{\sigma r}^+ \cdot x)^+)}{\mu_o^{\mathrm{PS}}((N_r^+ \cdot x)^+)} \ll \sigma^\Delta,$$

for all $0 < \sigma < 1$.

To this end, there is a constant $C_1 \geq 1$ such that the following holds; cf. [Cor90, Theorem 2.2]. For all $x \in E$, if $\xi = x^+$, then, the shadow $S_r = \{(nx)^+ : n \in N_r^+\}$ satisfies

$$\mathcal{O}(\xi(|\log r| + C_1)) \subseteq S_r \subseteq \mathcal{O}(\xi(|\log r| - C_1)), \qquad \forall 0 < r \leq 2. \tag{3.4}$$

Here, and throughout the rest of the proof, if $s \leq 0$, we use the convention

$$\mathcal{O}(\xi(s)) = \mathcal{O}(\xi(0)) = \partial \mathbb{H}^d_{\mathfrak{K}}.$$

Fix some arbitrary $x \in E$ and let $\xi = x^+$. To simplify notation, set for any $t, r > 0$,

$$\begin{aligned} t_\sigma &:= \max\left\{|\log \sigma r| - C_1, 0\right\}, & t_r &:= |\log r| + C_1, \\ d(t) &:= \mathrm{dist}(\xi(t), \Gamma \cdot o), & k(t) &:= k(\xi(t)), \end{aligned}$$

where $k(\xi(t))$ is as in the notation of Theorem 3.4.

By further enlarging the implicit constant, we may assume for the rest of the argument that

$$-\log \sigma > 2C_1.$$

This insures that $t_\sigma \geq t_r$ and avoids some trivialities.

Let $0 < r \leq 1$ be arbitrary. We define constants $\sigma_0 := \sigma \leq \sigma_1 \leq \sigma_2 \leq \sigma_3 := 1$ as follows. If $k(t_\sigma) = \delta$ (i.e. $\xi(t_\sigma)$ is in the complement of the cusp neighborhoods), we set $\sigma_1 = \sigma$. Otherwise, we define σ_1 by the property that $\xi(|\log \sigma_1 r|)$ is the first point along the geodesic segment joining $\xi(t_\sigma)$ and $\xi(t_r)$ (travelling from the former point to the latter) meets the boundary of the horoball containing $\xi(t_\sigma)$. Similarly, if $k(t_r) = \delta$, we set $\sigma_2 = 1$. Otherwise, we define σ_2 by the property that $\xi(|\log \sigma_2 r|)$ is the first point along the same segment, now travelling from $\xi(t_r)$ towards $\xi(t_\sigma)$, which intersects the boudary of the horoball containing $\xi(t_r)$. Define

$$t_{\sigma_0} := t_\sigma, \qquad t_{\sigma_3} := t_r, \qquad t_{\sigma_i} := |\log \sigma_i r| \quad \text{for } i = 1, 2.$$

In this notation, we first observe that $k(t_{\sigma_1}) = k(t_{\sigma_2}) = \delta$. In particular, Theorem 3.4 yields

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \ll \left(\frac{\sigma_1}{\sigma_2}\right)^\delta.$$

Note further that the projection map $\mathbb{H}^d_{\mathfrak{K}} \to \mathbb{H}^d_{\mathfrak{K}}/\Gamma$ restricts to an (isometric) embedding on cusp horoballs. Combined with convexity of horoballs and the fact that geodesics in $\mathbb{H}^d_{\mathfrak{K}}$ are unique distance minimizers, this implies that, for $i = 0, 2$, the distance between the projections of $\xi(t_{\sigma_i})$ and $\xi(t_{\sigma_{i+1}})$ to $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$ is equal to $|t_{\sigma_i} - t_{\sigma_{i+1}}|$. In particular, there is a constant $C_2 \geq 1$, depending only on the diameter of the complement of the cusp neighborhoods in the quotient $\mathbb{H}^d_{\mathfrak{K}}$ and on the constant C_1, such that, for $i = 0, 2$, we have

$$-C_2 - \log(\sigma_i/\sigma_{i+1}) \leq d(t_{\sigma_i}) \leq -\log(\sigma_i/\sigma_{i+1}) + C_2.$$

Hence, it follows using Theorem 3.4 and the above discussion that

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \ll \left(\frac{\sigma_0}{\sigma_1}\right)^{\delta} e^{d(t_{\sigma_0})(k(t_{\sigma_0})-\delta)} \ll \left(\frac{\sigma_0}{\sigma_1}\right)^{2\delta - k(t_{\sigma_0})}.$$

Similarly, we obtain

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_3 r})} \ll \left(\frac{\sigma_2}{\sigma_3}\right)^{\delta} e^{-d(t_{\sigma_3})(k(t_{\sigma_3})-\delta)} \ll \left(\frac{\sigma_2}{\sigma_3}\right)^{k(t_{\sigma_3})}.$$

Therefore, using the following trivial identity

$$\frac{\mu_o^{\mathrm{PS}}(S_{\sigma r})}{\mu_o^{\mathrm{PS}}(S_r)} = \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_0 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_1 r})}{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})} \frac{\mu_o^{\mathrm{PS}}(S_{\sigma_2 r})}{\mu_o^{\mathrm{PS}}(S_r)},$$

we see that $f(x) \ll \sigma^{\Delta}$, where Δ is as in the statement of the proposition. As $x \in E$ was arbitrary, we find that $E \subset \left\{ f_\sigma \ll \sigma^{\Delta} \right\}$, thus concluding the proof in the case $\sigma \leq 1$. Note that in the case $\sigma > 1$, the constants σ_i satisfy $\sigma_i/\sigma_{i+1} \geq 1$, so that combining the 3 estimates requires taking the maximum over the exponents, yielding the bound with Δ_+ in place of Δ in this case.

Now, let $r \in (0,1]$ and suppose $x = u^- y$ for some $y \in \Omega$ and $u^- \in N_2^-$. By [Cor90, Theorem 2.2], the analog of (3.4) holds, but with shadows from the viewpoint of x and y, in place of the fixed basepoint o. Recalling the map $n \mapsto u^+(n)$ in (2.8), one checks that this implies that this map is Lipschitz on N_1^+ with respect to the Carnot metric, with Lipschitz constant $\asymp C_1$. Moreover, the Jacobian of the change of variables associated to this map with respect to the measures μ_x^u and μ_y^u is bounded on N_1^+, independently of y and u^-; cf. (2.9) for a formula for this Jacobian. Hence, the estimates for $x \in N_2^- \Omega$ follow from their counterparts for points in Ω.

4. Margulis Functions In Infinite Volume

We construct Margulis functions on Ω which allow us to obtain quantitative recurrence estimates to compact sets. Our construction is similar to the one in [BQ11] in the case of lattices in rank 1 groups. We use geometric finiteness of Γ to establish the analogous properties more generally. The idea of Margulis functions originated in [EMM98].

Throughout this section, we assume Γ is non-elementary, geometrically finite group containing parabolic elements. The following is the main result of this section.

Theorem 4.1. *Let $\Delta > 0$ denote the constant in (3.1). For every $0 < \beta < \Delta/2$, there exists a proper function $V_\beta : N_1^- \Omega \to \mathbb{R}_+$ such that the following holds. There is a constant $c \geq 1$ such that for all $x \in N_1^- \Omega$ and $t \geq 0$,*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_t n x) \, d\mu_x^u(n) \leq c e^{-\beta t} V_\beta(x) + c.$$

Our key tool in establishing Theorem 4.1 is Proposition 4.2, which is a statement regarding average expansion of vectors in linear represearntations of G. The fractal nature of the conditional measures μ_x^u poses serious difficulties in establishing this latter result.

4.1. **Construction of Margulis functions.** Let $p_1, \dots, p_d \in \Lambda$ be a maximal set of inequivalent parabolic fixed points and for each i, let Γ_i denote the stabilizer of p_i in Γ. Let $P_i < G$ denote the parabolic subgroup of G fixing p_i. Denote by U_i the unipotent radical of P_i and by A_i a maximal $\mathbb{R}$-split torus inside P_i. Then, each U_i is a maximal connected unipotent subgroup of G admitting a closed (but not necessarily compact) orbit from identity in G/Γ. As all maximal unipotent subgroups of G are conjugate, we fix elements $h_i \in G$ so that $h_i U_i h_i^{-1} = N^+$. Note further that G admits an Iwasawa decomposition of the form $G = KA_iU_i$ for each i, where K is our fixed maximal compact subgroup.

Denote by W the the adjoint representation of G on its Lie algebra. The specific choice of representation is not essential for the construction, but is convenient for making some parameters more explicit. We endow that W is endowed with a norm that is invariant by K.

Let $0 \neq v_0 \in W$ denote a vector that is fixed by N^+. In particular, v_0 is a highest weight vector for the diagonal group A (with respect to the ordering determined by declaring the roots in N^+ to be positive). Let $v_i = h_i v_0 / \|h_i v_0\|$. Note that each of the vectors v_i is fixed by U_i and is a weight vector for A_i. In particular, there is an additive character $\chi_i : A_i \to \mathbb{R}$ such that

$$a \cdot v_i = e^{\chi_i(a)} v_i, \qquad \forall a \in A_i. \tag{4.1}$$

We denote by A_i^+ the subsemigroup of A_i which expands U_i (i.e. the positive Weyl chamber determined by U_i). We let $\alpha_i : A_i \to \mathbb{R}$ denote the simple root of A_i in $\mathrm{Lie}(U_i)$. Then,

$$\chi_i = \chi_{\mathfrak{K}} \alpha_i, \qquad \chi_{\mathfrak{K}} = \begin{cases} 1, & \text{if } \mathfrak{K} = \mathbb{R}, \\ 2 & \text{if } \mathfrak{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}. \end{cases} \tag{4.2}$$

Given $\beta > 0$, we define a function $V_\beta : G/\Gamma \to \mathbb{R}_+$ as follows:

$$V_\beta(g\Gamma) := \max_{w \in \bigcup_{i=1}^d g\Gamma \cdot v_i} \|w\|^{-\beta/\chi_{\mathfrak{K}}}. \tag{4.3}$$

The fact that $V_\beta(g\Gamma)$ is indeed a maximum will follow from Lemma 4.6.

4.2. **Linear expansion.** The following result is our key tool in establishing the contraction estimate on V_β in Theorem 4.1. A similar result was obtained in [MO20, Lemma 5.6] in the case of representations of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 4.2. *For every $0 \le \beta < \Delta/2$, there exists $C = C(\beta) \ge 1$ so that for all $t > 0$, $x \in N_1^- \Omega$, and all non-zero vectors v in the orbit $G \cdot v_0 \subset W$, we have*

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathfrak{K}}} \, d\mu_x^u(n) \le C e^{-\beta t} \|v\|^{-\beta/\chi_{\mathfrak{K}}}.$$

We postpone the proof of Proposition 4.2 to Section 5. Let $\pi_+ : W \to W^+$ denote the projection onto the highest weight space of g_t. The difficulty in the proof of Proposition 4.2 beyond the case $G = \mathrm{SL}_2(\mathbb{R})$ lies in controlling the *shape* of the subset of N^+ on which $\|\pi_+(n \cdot v)\|$ is small, so that we may apply the decay results from Proposition 3.1, that are valid only for balls of the form N_ε^+. We deal with this problem by using a convexity trick. A suitable analog of the above result holds for any non-trivial linear representation of G.

The following proposition establishes several geometric properties of the functions V_β which are useful in proving, and applying, Theorem 4.1. summarizes the main geometric properties of the functions V_β. This result is proved in Section 4.4.

Proposition 4.3. *Suppose V_β is as in (4.3). Then,*

(1) For every x in the unit neighborhood of Ω, we have that

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathfrak{K}}/\beta}(x),$$

where $\mathrm{inj}(x)$ denotes the injectivity radius at x. In particular, V_β is proper on Ω.

(2) For all $g \in G$ and all $x \in X$,

$$\|g\|^{-\beta} V_\beta(x) \le V_\beta(gx) \le \left\|g^{-1}\right\|^\beta V_\beta(x).$$

(3) There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \le \varepsilon_0$.

4.3. **Proof of Theorem 4.1.** In this section, we use Proposition 4.3 to translate the linear expansion estimates in Proposition 4.2 into a contraction estimate for the functions V_β.

Let $t_0 > 0$ be be given and define

$$\omega_0 := \sup_{n \in N_1^+} \max\left\{\|g_{t_0} n\|^{1/\chi_{\mathfrak{K}}}, \left\|(g_{t_0} n)^{-1}\right\|^{1/\chi_{\mathfrak{K}}}\right\},$$

where $\|\cdot\|$ denotes the operator norm of the action of G on W. Then, for all $n \in N_1^+$ and all $x \in X$, we have

$$\omega_0^{-1} V_1(x) \le V_1(g_{t_0} n x) \le \omega_0 V_1(x), \tag{4.4}$$

where $V_1 = V_\beta$ for $\beta = 1$.

Let ε_0 be as in Proposition 4.3(3). Suppose $x \in X$ is such that $V_1(x) \le \omega_0/\varepsilon_0$. Then, by (4.4), for any $\beta > 0$, we have that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x)\, d\mu_x^u(n) \le B_0 := (\omega_0^2 \varepsilon_0^{-1})^\beta. \tag{4.5}$$

Now, suppose $x \in N_1^- \Omega$ is such that $V_1(x) \ge \omega_0/\varepsilon_0$ and write $x = g\Gamma$ for some $g \in G$. Then, by Proposition 4.3(3), there exists a unique vector $v_\star \in \bigcup_i g\Gamma \cdot v_i$ satisfying $V_1(x) = \|v_\star\|^{-1/\chi_{\mathfrak{K}}}$. Moreover, by (4.4), we have that $V_1(g_{t_0} n x) \ge 1/\varepsilon_0$ for all $n \in N_1^+$. And, by definition of ω_0, for all $n \in N_1^+$, $\|g_{t_0} n v_\star\|^{1/\chi_{\mathfrak{K}}} \le \varepsilon_0$. Thus, applying Proposition 4.3(3) once more, we see that $g_{t_0} n v_\star$ is the unique vector in $\bigcup_i g_{t_0} n g\Gamma \cdot v_i$ satisfying

$$V_\beta(g_{t_0} n x) = \|g_{t_0} n v_\star\|^{-1/\chi_{\mathfrak{K}}}, \qquad \forall n \in N_1^+.$$

Moreover, since the vectors v_i all belong to the G-orbit of v_0, it follows that $v_\star$ also belongs to $G \cdot v_0$. Thus, we may apply Proposition 4.2 as follows. Fix some $\beta > 0$ and let $C = C(\beta) \ge 1$ be the constant in the conclusion of the proposition. Then,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x) d\mu_x^u = \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_{t_0} n v_\star\|^{-\beta/\chi_{\mathfrak{K}}} d\mu_x^u \le C e^{-\beta t_0} \|v_\star\|^{-\beta/\chi_{\mathfrak{K}}} = C e^{-\beta t_0} V_\beta(x).$$

Combining this estimate with (4.5), we obtain for any fixed t_0,

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{t_0} n x)\, d\mu_x^u(n) \le C e^{-\beta t_0} V_\beta(x) + B_0, \tag{4.6}$$

for all $x \in \Omega$. We claim that there is a constant $c_1 = c_1(\beta) > 0$ such that, if t_0 is large enough, depending on β, then

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} V_\beta(g_{k t_0} n x)\, d\mu_x^u(n) \le c_1^k e^{-\beta k t_0} V_\beta(x) + 2B_0, \tag{4.7}$$

for all $k \in \mathbb{N}$. By Proposition 4.3, this claim completes the proof since $V_\beta(g_t y) \ll V_\beta(g_{\lfloor t/t_0 \rfloor t_0} y)$, for all $t \ge 0$ and $y \in X$, with an implied constant depending only on t_0 and β.

The proof of (4.7) is by now a standard argument, with the key ingredient in carrying it out being the doubling estimate Proposition 3.1. We proceed by induction. Let $k \in \mathbb{N}$ be arbitrary and assume that (4.7) holds for such k. Let $\left\{n_i \in \mathrm{Ad}(g_{k t_0})(N_1^+) : i \in I\right\}$ denote a finite collection of points in the support of $\mu^u_{g_{kt_0}x}$ such that $N_1^+ n_i$ covers the part of the support inside $\mathrm{Ad}(g_{kt_0}(N_1^+))$. We can find such a cover with uniformly bounded multiplicity, depending only on N^+. That is

$$\sum_{i\in I} \chi_{N_1^+ n_i}(n) \ll \chi_{\cup_i N_1^+ n_i}(n), \qquad \forall n \in N^+.$$

Let $x_i = n_i g_{kt_0} x$. By (4.6), and a change of variable, cf. (2.3) and (2.4), we obtain

$$e^{\delta k t_0} \int_{N_1^+} V_\beta(g_{(k+1)t_0} nx)\, d\mu^u_x \leq \sum_{i\in I} \int_{N_1^+} V_\beta(g_{t_0} n x_i)\, d\mu^u_{x_i} \leq \sum_{i\in I} \mu^u_{x_i}(N_1^+) \left(Ce^{-\beta t_0} V_\beta(x_i) + B_0 \right).$$

It follows using Proposition 4.3 that $\mu^u_y(N_1^+) V_\beta(y) \ll \int_{N_1^+} V_\beta(ny)\, d\mu^u_y(n)$ for all $y \in X$. Hence,

$$\int_{N_1^+} V_\beta(g_{(k+1)t_0} nx)\, d\mu^u_x(n) \ll e^{-\delta k t_0} \sum_{i\in I} \int_{N_1^+} \left(Ce^{-\beta t_0} V_\beta(n x_i) + B_0 \right)\, d\mu^u_{x_i}(n).$$

Note that since g_t expands N^+ by at least e^t, we have

$$\mathcal{A}_k := \mathrm{Ad}(g_{-kt_0}) \left(\bigcup_i N_1^+ n_i \right) \subseteq N_2^+.$$

Using the bounded multiplicity property of the cover, we see that, for any non-negative function φ, we have

$$\sum_{i\in I} \int_{N_1^+} \varphi(n x_i)\, d\mu^u_{x_i} = \int_{N^+} \varphi(n g_{kt_0} x) \sum_{i\in I} \chi_{N_1^+ n_i}(n)\, d\mu^u_{g_{kt_0}x} \ll \int_{\bigcup_i N_1^+ n_i} \varphi(n g_{kt_0} x)\, d\mu^u_{g_{kt_0}x}.$$

Changing variables back so the integrals take place against μ^u_x, we obtain

$$\begin{aligned} e^{-\delta k t_0} \sum_{i\in I} \int_{N_1^+} \left(Ce^{-\beta t_0} V_\beta(n x_i) + B_0 \right) d\mu^u_{x_i} &\ll \int_{\mathcal{A}_k} \left(Ce^{-\beta t_0} V_\beta(g_{kt_0} nx) + B_0 \right)\, d\mu^u_x \\ &\leq Ce^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} nx)\, d\mu^u_x + B_0 \mu^u_x(N_2^+). \end{aligned}$$

To apply the induction hypothesis, we again pick a cover of N_2^+ by balls of the form $N_1^+ n$, for a collection of points $n \in N_2^+$ in the support of μ^u_x. We can arrange for such a collection to have a uniformly bounded cardinality and multiplicity. By essentially repeating the above argument, and using our induction hypothesis for k, in addition to the doubling property in Proposition 3.1, we obtain

$$Ce^{-\beta t_0} \int_{N_2^+} V_\beta(g_{kt_0} nx)\, d\mu^u_x + B_0 \mu^u_x(N_2^+) \ll (C c_1^k e^{-\beta(k+1)t_0} V_\beta(x) + 2B_0 C e^{-\beta t_0} + B_0) \mu^u_x(N_1^+),$$

where we also used Proposition 4.3 to ensure that $V_\beta(nx) \ll V_\beta(x)$, for all $n \in N_3^+$. Taking c_1 to be larger than the product of C with all the uniform implied constants accumulated thus far in the argument, we obtain

$$\frac{1}{\mu^u_x(N_1^+)} \int_{N_1^+} V_\beta(g_{(k+1)t_0} nx)\, d\mu^u_x(n) \leq c_1^{k+1} e^{-\beta(k+1)t_0} V_\beta(x) + 2c_1 e^{-\beta t_0} B_0 + B_0.$$

Taking t_0 large enough so that $2c_1 e^{-\beta t_0} \leq 1$ completes the proof.

4.4. **Geometric properties of Margulis functions and proof of Proposition 4.3.** In this section, we give a geometric interpretation of the functions V_β which allows us to prove Proposition 4.3. Item (2) follows directly from the definitions, so we focus on the remaining properties.

The data in the definition of V_β allows us to give a linear description of cusp neighborhoods as follows. Given $g \in G$ and i, write $g = kau$ for some $k \in K$, $a \in A_i$ and $u \in U_i$. Geometrically, the size of the A component in the Iwasawa decomposition $G = KA_iU_i$ corresponds to the value of the Busemann cocycle $|\beta_{p_i}(Kg, o)|$, where Kg is the image of g in $K\backslash G$; cf. [BQ16, Remark 6.5] and the references therein for the precise statement. This has the following consequence. We can find $0 < \varepsilon_i < 1$ such that

$$\left\| \mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)} \right\| < \varepsilon_i \Longleftrightarrow Kg \in H_{p_i}, \tag{4.8}$$

where H_{p_i} is the standard horoball based at p_i in $\mathbb{H}^d_{\mathfrak{K}} \cong K\backslash G$.

The functions $V_\beta(x)$ roughly measure how far into the cusp x is. More precisely, we have the following lemma.

Lemma 4.4. *The restriction of V_β to any bounded neighborhood of Ω is a proper map.*

Proof. In view of Property (2) of Proposition 4.3, it suffices to prove that V_β is proper on Ω. Now, suppose that for some sequence $g_n \in G$, we have $g_n\Gamma$ tends to infinity in Ω. Then, since Γ is geometrically finite, this implies that the injectivity radius at $g_n\Gamma$ tends to 0. Hence, after passing to a subsequence, we can find $\gamma_n \in \Gamma$ such that $g_n\gamma_n$ belongs to a single horoball among the horoballs constituting our fixed standard cusp neighborhood; cf. Section 2.2. By modifying γ_n on the right by a fixed element in Γ if necessary, we can assume that $Kg_n\gamma_n$ converges to one of the parabolic points p_i (say p_1) on the boundary of $\mathbb{H}^d_{\mathfrak{K}} \cong K\backslash G$.

Moreover, geometric finiteness implies that $(\Lambda_\Gamma \setminus \{p_1\})/\Gamma_1$ is compact. Thus, by multiplying $g_n\gamma_n$ by an element of Γ_1 on the right if necessary, we may assume that $(g_n\gamma_n)^-$ belongs to a fixed compact subset of the boundary, which is disjoint from $\{p_1\}$.

Thus, for all large n, we can write $g_n\gamma_n = k_na_nu_n$, for $k_n \in K$, $a_n \in A_i$ and $u_n \in U_i$, such that the eigenvalues of $\mathrm{Ad}(a_n)$ are bounded above; cf. (4.8). Moreover, as $(g_n\gamma_n)^-$ belongs to a compact set that is disjoint from $\{p_1\}$ and $(g_n\gamma_n)^+ \to p_1$, the set $\{u_n\}$ is bounded. To show that $V_\beta(g_n\Gamma) \to \infty$, since U_i fixes v_i and K is a compact group, it remains to show that a_n contracts v_i to 0. Since $g_n\gamma_n$ is unbounded in G while k_n and u_n remain bounded, this shows that the sequence a_n is unbounded. Upper boundedness of the eigenvalues of $\mathrm{Ad}(a_n)$ thus implies the claim. $\square$

Remark 4.5. The above lemma is false without restricting to Ω in the case Γ has infinite covolume since the injectivity radius is not bounded above on G/Γ. Note also that this lemma is false in the case Γ is not geometrically finite, since the complement of cusp neighborhoods inside Ω is compact if and only if Γ is geometrically finite.

The next crucial property of the functions V_β is the following linear manifestation of the existence of cusp neighborhoods consisting of disjoint horoballs. This lemma implies Proposition 4.3(3).

Lemma 4.6. *There exists a constant $\varepsilon_0 > 0$ such that for all $x = g\Gamma \in X$, there exists at most one vector $v \in \bigcup_i g\Gamma \cdot v_i$ satisfying $\|v\| \leq \varepsilon_0$.*

Remark 4.7. The constant ε_0 roughly depends on the distance from a fixed basepoint to the cusp neighborhoods.

Proof of Lemma 4.6. Let $g \in G$ and i be given. Write $g = kau$, for some $k \in K$, $a \in A_i$ and $u \in U_i$. Since U_i fixes v_i and the norm on W is K-invariant, we have $\|g \cdot v_i\| = \|a \cdot v_i\| = e^{\chi_i(a)}$; cf. (4.1). Moreover, since W is the adjoint representation, we have

$$\left\| \mathrm{Ad}(a)|_{\mathrm{Lie}(U_i)} \right\| \asymp e^{\chi_i(a)},$$

and the implied constant, denoted C, depends only on the norm on the Lie algebra.

Let $0 < \varepsilon_i < 1$ be the constants in (4.8) and define $\varepsilon_0 := \min_i \varepsilon_i / C$. Let $x = g\Gamma \in G/\Gamma$. Suppose that there are elements $\gamma_1, \gamma_2 \in \Gamma$ and vectors v_{i_1}, v_{i_2} in our finite fixed collection of vectors v_i such that $\left\| g\gamma_j \cdot v_{i_j} \right\| < \varepsilon_0$ for $j = 1, 2$. Then, the above discussion, combined with the choice of ε_i in (4.8), imply that $Kg\gamma_j$ belongs to the standard horoball H_j in $\mathbb{H}^d_{\mathfrak{K}}$ based at p_{i_j}. However, this implies that the two standard horoballs $H_1\gamma_1^{-1}$ and $H_2\gamma_2^{-1}$ intersect non-trivially. By choice of these standard horoballs, this implies that the two horoballs $H_j\gamma_j^{-1}$ are the same and that the two parabolic points p_{i_j} are equivalent under Γ. In particular, the two vectors v_{i_1}, v_{i_2} are in fact the same vector, call it v_{i_0}. It also follows that $\gamma_1^{-1}\gamma_2$ sends H_1 to itself and fixes the parabolic point it is based at. Thus, $\gamma_1^{-1}\gamma_2$ fixes v_{i_0} by definition. But, then, we get that

$$g\gamma_2 \cdot v_{i_0} = g\gamma_1(\gamma_1^{-1}\gamma_2) \cdot v_{i_0} = g\gamma_1 \cdot v_{i_0}.$$

This proves uniqueness of the vector in $\bigcup_i g\Gamma \cdot v_i$ with length less than ε_0, if it exists, and concludes the proof.

□

Finally, we verify Proposition 4.3 (1) relating the injectivity radius to V_β.

Lemma 4.8. *For all x in the unit neighborhood of Ω, we have*

$$\mathrm{inj}(x)^{-1} \ll_\Gamma V_\beta^{\chi_{\mathfrak{K}}/\beta}(x), \qquad e^{\mathrm{dist}(x,o)} \ll_\Gamma V_\beta^{1/\beta}(x),$$

where $\chi_{\mathfrak{K}}$ is given in (4.2) and $o \in \Omega$ is our fixed basepoint.

Proof. Let $x \in \Omega$ and set $\tilde{x}_0 = Kx$. Let $x_0 \in K\backslash G \cong \mathbb{H}^d_{\mathfrak{K}}$ denote a lift of $\tilde{x}_0$. Then, x_0 belongs to the hull of the limit set of Γ; cf. Section 2.

Since $\mathrm{inj}(\cdot)^{-1}$ and V_β are uniformly bounded above and below on the complement of the cusp neighborhoods inside Ω, it suffices to prove the lemma under the assumption that x_0 belongs to some standard horoball H based at a parabolic fixed point p. We may also assume that the lift x_0 is chosen so that p is one of our fixed finite set of inequivalent parabolic points $\{p_i\}$.

Geometric finiteness of Γ implies that there is a compact subset $\mathcal{K}_p$ of $\partial\mathbb{H}^d_{\mathfrak{K}} \setminus \{p\}$, depending only on the stabilizer Γ_p in Γ, with the following property. Every point in the hull of the limit set is equivalent, under Γ_p, to a point on the set of geodesics joining p to points in $\mathcal{K}_p$. Thus, after adjusting x_0 by an element of Γ_p if necessary, we may assume that x_0 belongs to this set. In particular, we can find $g \in G$ so that $x_0 = Kg$ and g can be written as kau in the Iwasawa decomposition associated to p, for some $k \in K, a \in A_p$, and $u \in U_p$[2] with the property that $\mathrm{Ad}(a)$ is contracting on U_p and u is of uniformly bounded size.

Note that it suffices to prove the statement assuming the injectivity radius of x is smaller than $1/3$, while the distance of x_0 to the boundary of the cusp horoball H_p is at least 1. Now, let $\gamma \in \Gamma$ be a non-trivial element such that $x_0\gamma$ is at distance at most $1/2$ from x_0. Then, this implies that both x_0 and $x_0\gamma$ belong to H_p. In particular, the standard horoballs H_p and $H_p\gamma$ intersect non-trivially, and hence must be the same. It follows that γ belongs to Γ_p.

Let M_p denote the centralizer of A_p inside K. Since Γ_p is a subgroup of M_pU_p, we can find v in the Lie algebra of M_pU_p so that $\gamma = \exp(v)$. In view of the discreteness of Γ, we have that $\|v\| \gg 1$. Since the exponential map is close to an isometry near the origin, we see that

$$\mathrm{dist}(g\gamma g^{-1}, \mathrm{Id}) \asymp \|\mathrm{Ad}(au)(v)\| \geq e^{\chi_{\mathfrak{K}}\alpha(a)} \|\mathrm{Ad}(u)(v)\|,$$

where $\chi_{\mathfrak{K}}$ is given in (4.2) and we used K-invariance of the norm. Here, α is the simple root of A_p in the Lie algebra of U_p and $e^{\chi_{\mathfrak{K}}\alpha(a)}$ is the smallest eigenvalue of $\mathrm{Ad}(a)$ on the Lie algebra of the parabolic group stabilizing p. Note that since x_0 belongs to H_p, $\alpha(a)$ is strictly negative.

[2]The groups A_p and U_p were defined at the beginning of the section.

Recalling that u belongs to a uniformly bounded neighborhood of identity in G and that $\|v\| \gg 1$, it follows that $\mathrm{dist}(g\gamma g^{-1}, \mathrm{Id}) \gg e^{\chi_{\mathfrak{K}}\alpha(a)}$. Since γ was arbitrary, this shows that the injectivity radius at x satisfies the same lower bound.

Finally, let $v_p \in \{v_i\}$ denote the vector fixed by U_p. Using the above Iwasawa decomposition, we see that

$$V_\beta^{1/\beta}(x) \geq \|av_p\|^{-1/\chi_{\mathfrak{K}}} = e^{-\chi_p(a)/\chi_{\mathfrak{K}}}, \tag{4.9}$$

where χ_p is the character on A_p determined by v_p, cf. (4.1). This concludes the proof of the first estimate in view of (4.2) and the fact that $\chi_p = \chi_{\mathfrak{K}}\alpha$.

The proof of the second estimate is very similar. We again note that it suffices to establish the estimate in the case x_0 belongs to a horoball H based at a parabolic point p. Let y be an arbitrary point on the boundary of H. The above argument then shows that $|\mathrm{dist}(x_0, o) - |\beta_p(x_0, y)|| \ll 1$, since the Busemann function $|\beta_p(x_0, y)|$ provides the distance between x_0 and the boundary of H. By [BQ16, Remark 6.5], we have $|-\alpha(a) - |\beta_p(x_0, y)|| \ll 1$, where $a \in A_p$ is as above. The second estimate then follows from (4.9). $\square$

5. Shadow Lemmas, Convexity, and Linear Expansion

The goal of this section is to prove Proposition 4.2 estimating the average rate of expansion of vectors with respect to leafwise measures. This completes the proof of Theorem 4.1.

5.1. **Proof of Proposition 4.2.** We may assume without loss of generality that $\|v\| = 1$. Let W^+ denote the highest weight subspace of W for $A_+ = \{g_t : t > 0\}$. Denote by π_+ the projection from W onto W^+. In our choice of representation W, the eigenvalue of A_+ in W^+ is $e^{\chi_{\mathfrak{K}}t}$, , where $\chi_{\mathfrak{K}}$ is given in (4.2). It follows that

$$\frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|g_t n \cdot v\|^{-\beta/\chi_{\mathfrak{K}}} \, d\mu_x^u(n) \leq e^{-\beta t} \frac{1}{\mu_x^u(N_1^+)} \int_{N_1^+} \|\pi_+(n \cdot v)\|^{-\beta/\chi_{\mathfrak{K}}} \, d\mu_x^u(n).$$

Hence, it suffices to show that, for a suitable choice of β, the integral on the right side is uniformly bounded, independently of v and x (but possibly depending on β).

For simplicity, set $\beta_{\mathfrak{K}} = \beta/\chi_{\mathfrak{K}}$. A simple application of Fubini's Theorem yields

$$\int_{N_1^+} \|\pi_+(n \cdot v)\|^{-\beta_{\mathfrak{K}}} \, d\mu_x^u(n) = \int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n \cdot v)\|^{\beta_{\mathfrak{K}}} \leq t^{-1}\Big) \, dt.$$

For $v \in W$, we define a polynomial map on N^+ by $n \mapsto p_v(n) := \|\pi_+(n \cdot v)\|^2$ and set

$$S(v, \varepsilon) := \left\{n \in N^+ : p_v(n) \leq \varepsilon\right\}.$$

To apply Proposition 3.1, we wish to efficiently estimate the radius of a ball in N^+ containing the sublevel sets $S\big(v, t^{-2/\beta_{\mathfrak{K}}}\big) \cap N_1^+$. We have the following claim.

Claim 5.1. There exists a constant $C_0 > 0$, such that, for all $\varepsilon > 0$, the diameter of $S(v, \varepsilon) \cap N_1^+$ is at most $C_0 \varepsilon^{1/4\chi_{\mathfrak{K}}}$.

Let us show how to conclude the proof assuming this claim. By estimating the integral over $[0, 1]$ trivially, we obtain

$$\int_0^\infty \mu_x^u\Big(n \in N_1^+ : \|\pi_+(n \cdot v)\|^{\beta_{\mathfrak{K}}} \leq t^{-1}\Big) \, dt \leq \mu_x^u(N_1^+) + \int_1^\infty \mu_x^u\Big(S\big(v, t^{-2/\beta_{\mathfrak{K}}}\big) \cap N_1^+\Big) \, dt. \tag{5.1}$$

Claim 5.1 implies that if $\mu_x^u\left(S(v, \varepsilon) \cap N_1^+\right) > 0$ for some $\varepsilon > 0$, then $S(v, \varepsilon) \cap N_1^+$ is contained in a ball of radius $2C_0\varepsilon^{1/4\chi_{\mathfrak{K}}}$, centered at a point in the support of the measure $\mu_x^u|_{N_1^+}$. Recalling that $\beta_{\mathfrak{K}} = \beta/\chi_{\mathfrak{K}}$, we thus obtain

$$\int_1^\infty \mu_x^u\Big(S\big(v, t^{-2/\beta_{\mathfrak{K}}}\big) \cap N_1^+\Big)\, dt \le \int_1^\infty \sup_{n\in\operatorname{supp}(\mu_x^u)\cap N_1^+} \mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big)\, dt, \tag{5.2}$$

where for $n \in N^+$ and $r > 0$, $B_{N^+}(n, r)$ denotes the ball of radius r centered at n.

To estimate the integral on the right side of (5.2), we use the doubling results in Proposition 3.1. Note that if $n \in \operatorname{supp}(\mu_x^u)$, then nx belongs to the limit set Λ_Γ. Since $x \in N_1^-\Omega$ by assumption, this implies that nx belongs to $N_2^-\Omega$ for all $n \in N_1^+$ in the support of μ_x^u; cf. Remark 2.1. Hence, changing variables using (2.4) and applying Proposition 3.1, we obtain for all $n \in \operatorname{supp}(\mu_x^u) \cap N_1^+$,

$$\mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big) = \mu_{nx}^u\Big(B_{N^+}\big(\mathrm{Id}, 2C_0 t^{-1/2\beta}\big)\Big) \ll t^{-\Delta/2\beta} \mu_{nx}^u(N_1^+).$$

Moreover, for $n \in N_1^+$, we have, again by Proposition 3.1, that

$$\mu_{nx}^u(N_1^+) \le \mu_x^u(N_2^+) \ll \mu_x^u(N_1^+).$$

Put together, this gives

$$\int_1^\infty \sup_{n\in\operatorname{supp}(\mu_x^u)\cap N_1^+} \mu_x^u\Big(B_{N^+}\big(n, 2C_0 t^{-1/2\beta}\big)\Big)\, dt \ll \mu_x^u(N_1^+) \int_1^\infty t^{-\Delta/2\beta}\, dt.$$

The integral on the right side above converges whenever $\beta < \Delta/2$, which concludes the proof.

5.2. **Prelimiary facts.** We begin by recalling the Bruhat decomposition of G. Denote by P^- the subgroup MAN^- of G.

Proposition 5.2 (Theorem 5.15, [BT65]). *Let $w \in G$ denote a non-trivial Weyl "element" satisfying $wg_tw^{-1} = g_{-t}$. Then,*

$$G = P^-N^+ \bigsqcup P^-w. \tag{5.3}$$

We shall need the following result. It is a special case of the general results in [Yan20] which does not require any tools from invariant theory since we work with vectors in the orbit of a highest weight vector. This result is yet another reflection in linear representations of G of the fact that G has real rank 1.

Proposition 5.3. *Let V be a normed finite dimensional representation of G, and $v_0 \in V$ be any highest weight vector for g_t $(t > 0)$ with weight $e^{\lambda t}$ for some $\lambda \ge 0$. Let v be any vector in the orbit $G \cdot v_0$ and define*

$$G(v, V^{<\lambda}(g_t)) = \left\{ g \in G : \lim_{t\to\infty} \frac{\log \|g_t g v\|}{t} < \lambda \right\}.$$

Then, there exists $g_v \in G$ such that

$$G(v, V^{<\lambda}(g_t)) \subseteq P^- g_v.$$

Proof. Let $h \in G$ be such that $v = hv_0$ and let $g \in G(v, V^{<\lambda}(g_t))$. By the Bruhat decomposition, either $gh = pn$ for some $p \in P^-$ and $n \in N^+$, or $gh = pw$ for some $p \in P^-$ and w being the long Weyl "element". Suppose we are in the first case, and note that N^+ fixes v_0 since it is a highest weight vector for g_t. Moreover, $\mathrm{Ad}(g_t)(p)$ converges to some element in G as t tends to ∞. Since $g_tgv = e^{\lambda t}\mathrm{Ad}(g_t)(p)v_0$, we see that $\log \|g_tgv\|/t \to \lambda$ as t tends to ∞, thus contradicting the assumption that g belongs to $G(v, V^{<\lambda}(g_t))$. Hence, gh must belong to P^-w. This implies the conclusion by taking $g_v := wh^{-1}$.

∎

The following immediate corollary is the form we use this result in our arguments.

Corollary 5.4. *Let the notation be as in Proposition 5.3. Then, $N^+ \cap G(v, W^{0-}(g_t))$ contains at most one point.*

Proof. Recall the Bruhat decomposition of G in Proposition 5.2. Let $g_v \in G$ be as in Proposition 5.3 and suppose that $n_0 \in P^- g_v \cap N^+$. Let $p_0 \in P^-$ be such that $n_0 = p_0 g_v$.

First, assume $g_v = p_v n_v$ for some $p_v \in P^-$ and $n_v \in N^+$. Then, $n_0 = p_0 p_v n_v$. Then, $n_0 n_v^{-1} \in P^- \cap N^+ = \{\mathrm{Id}\}$. In particular, $n_0 = n_v$, and the claim follows in this case.

Now assume that $g_v = p_v w$ for some $p_v \in P^-$, so that $n_0 = p_0 p_v w \in P^- w \cap N^+$. This is a contradiction, since the latter intersection is empty as follows from the Bruhat decomposition. ◻

5.3. **Convexity and Proof of Claim 5.1.** Let $B_1 \subset \mathrm{Lie}(N^+)$ denote a compact convex set whose image under the exponential map contains N_1^+ and denote by B_2 a compact set containing B_1 in its interior.

Define $\mathfrak{n}_1^+$ to be the unit sphere in the Lie algebra $\mathfrak{n}^+$ of N^+ in the following sense:

$$\mathfrak{n}_1^+ := \left\{ u \in \mathfrak{n}^+ : d_{N^+}(\exp(u), \mathrm{Id}) = 1 \right\},$$

where d_{N^+} is the Carnot-Caratheodory metric on N^+; cf. Section 2.5. Given $u, b \in \mathfrak{n}^+$, define a line $\ell_{u,b} : \mathbb{R} \to \mathfrak{n}^+$ as follows:

$$\ell_{u,b}(t) := tu + b,$$

and denote by $\mathcal{L}$ the space of all such lines $\ell_{u,b}$ such that $u \in \mathfrak{n}_1^+$. We endow $\mathcal{L}$ with the topology inherited from its natural identification with its $\mathfrak{n}_1^+ \times \mathfrak{n}^+$. Then, the subset $\mathcal{L}(B_1)$ of all such lines such that b belongs to the compact set B_1 is compact in $\mathcal{L}$.

Recall that a vector $v \in W$ is said to be unstable if the closure of the orbit $G \cdot v$ contains 0. Highest weight vectors are examples of unstable vectors. Let $\mathcal{N}$ denote the null cone of G in W, i.e., the closed cone consisting of all unstable vectors. Let $\mathcal{N}_1 \subset \mathcal{N}$ denote the compact set of unit norm unstable vectors. Note that, for any $v \in \mathcal{N}$, the restriction of p_v to any $\ell \in \mathcal{L}$ is a polynomial in t of degree at most that of p_v. We note further that the function

$$\rho(v, \ell) := \sup \{ p_v(\ell(t)) : \ell(t) \in B_2 \}$$

is continuous and non-negative on the compact space $\mathcal{N}_1 \times \mathcal{L}(B_1)$. We claim that

$$\rho_\star := \inf \{ \rho(v, \ell) : (v, \ell) \in \mathcal{N}_1 \times \mathcal{L}(B_1) \}$$

is strictly positive. Indeed, by continuity and compactness, it suffices to show that ρ is non-vanishing. Suppose not and let (v, ℓ) be such that $\rho(v, \ell) = 0$. Since B_1 is contained in the interior of B_2, the intersection

$$I(\ell) := \{ t \in \mathbb{R} : \ell(t) \in B_2 \}$$

is an interval (by convexity of B_2) with non-empty interior. Since $p_v(\ell(\cdot))$ is a polynomial vanishing on a set of non-empty interior, this implies it vanishes identically. On the other hand, Corollary 5.4 shows that p_v has at most 1 zero in all of $\mathfrak{n}^+$, a contradiction.

Positivity of $\rho_\star$ has the following consequence. Our choice of the representation W implies that the degree of the polynomial p_v is at most $4\chi_{\mathfrak{K}}$, where $\chi_{\mathfrak{K}}$ is given in (4.2). This can be shown by direct calculation in this case.[3] By the so-called (C, α)-good property (cf. [Kle10, Proposition 3.2]), we have for all $\varepsilon > 0$

$$| \{ t \in I(\ell) : p_v(\ell(t)) \leq \varepsilon \} | \leq C_d \, (\varepsilon / \rho_\star)^{1/4\chi_{\mathfrak{K}}} \, |I(\ell)|,$$

where $C_d > 0$ is a constant depending only on the degree of p_v, and $| \cdot |$ denotes the Lebesgue measure on $\mathbb{R}$.

[3]In general, such a degree can be calculated from the largest eigenvalue of g_t in W; for instance by restricting the representation to suitable subalgebras of the Lie algebra of G that are isomoprhic to $\mathfrak{sl}_2(\mathbb{R})$ and using the explicit description of $\mathfrak{sl}_2(\mathbb{R})$ representations.

To use this estimate, we first note that the length of the intervals $I(\ell)$ is uniformly bounded over $\mathcal{L}(B_1)$. Indeed, suppose for some $u = (u_\alpha, u_{2\alpha}), b \in \mathfrak{n}^+$ and $\ell = \ell_{u,b} \in \mathcal{L}(B_1)$, $I(\ell)$ has endpoints $t_1 < t_2$ so that the points $\ell(t_i)$ belong to the boundary of B_2. Recall that the Lie algebra $\mathfrak{n}^+$ of N^+ decomposes into g_t eigenspaces as $\mathfrak{n}^+_\alpha \oplus \mathfrak{n}^+_{2\alpha}$, where $\mathfrak{n}^+_{2\alpha} = 0$ if and only if $\mathfrak{K} = \mathbb{R}$. Set $x_1 = \ell(t_1)$ and $x_2 = \ell(t_2)$. Since N^+ is a nilpotent group of step at most 2, the Campbell-Baker-Hausdorff formula implies that $\exp(x_2)\exp(-x_1) = \exp(Z)$, where $Z \in \mathfrak{n}^+$ is given by

$$Z = x_2 - x_1 + \frac{1}{2}[x_2, -x_1] = (t_2 - t_1)u + \frac{1}{2}(t_2 - t_1)[b, u].$$

Note that since $\mathfrak{n}^+_{2\alpha}$ is the center of $\mathfrak{n}^+$, $[b, u] = [b, u_\alpha]$ belongs to $\mathfrak{n}^+_{2\alpha}$. Hence, we have by (2.7) that

$$d_{N^+}(\exp(x_1), \exp(x_2)) = \left((t_2 - t_1)^4 \|u_\alpha\|^4 + (t_2^2 - t_1^2)^2 \left\| u_{2\alpha} + \frac{1}{2}[b, u] \right\|^2 \right)^{1/4}.$$

Since $\exp(u)$ is at distance 1 from identity, at least one of $\|u_\alpha\|$ and $\|u_{2\alpha}\|$ is bounded below by 10^{-1}. Moreover, we can find a constant $\theta \in (0, 10^{-2})$ so that for all $b \in B_1$ and all $y_\alpha \in \mathfrak{n}^+_\alpha$ with $\|y_\alpha\| \leq \theta$ such that $\|[b, y_\alpha]\| \leq 10^{-2}$. Together this implies that

$$\min\left\{ t_2 - t_1, (t_2^2 - t_1^2)^{1/2} \right\} \ll \operatorname{diam}(B_1),$$

where $\operatorname{diam}(B_1)$ denotes the diameter of B_1. This proves that $|I(\ell)| = t_2 - t_1 \ll 1$, where the implicit constant depends only on the choice of B_1. We have thus shown that

$$|\{t \in I(\ell) : p_v(\ell(t)) \leq \varepsilon\}| \ll \varepsilon^{1/4\chi_{\mathfrak{K}}}. \tag{5.4}$$

We now use our assumption that v belongs to the G orbit of a highest weight vector v_0. Since v_0 is a highest weight vector, it is fixed by N^+. Hence, the Bruhat decomposition, cf. (5.3) with the roles of P^- and P^+ reversed, implies that the orbit $G \cdot v_0$ can be written as

$$G \cdot v_0 = P^+ \cdot v_0 \bigsqcup P^+ w \cdot v_0,$$

where w is the long Weyl "element". Recall that $P^+ = N^+MA$, where M is the centralizer of $A = \{g_t\}$ in the maximal compact group K. In particular, M preserves eigenspaces of A and normalizes N^+. Recall further that the norm on W is chosen to be K-invariant.

First, we consider the case $v \in P^+ w \cdot v_0$ and has unit norm. For $v' \in W$, we write $[v']$ for its image in the projective space $\mathbb{P}(W)$. Then, since $w \cdot v_0$ is a joint weight vector of A, we see that the image of $P^+ w \cdot v_0$ in $\mathbb{P}(W)$ has the form $N^+M \cdot [w \cdot v_0]$. Setting $v_1 := w \cdot v_0$, we see that

$$S(nm \cdot v_1, \varepsilon) = S(mv_1, \varepsilon) \cdot n^{-1} = \mathrm{Ad}(m^{-1})(S(v_1, \varepsilon)) \cdot n^{-1}, \tag{5.5}$$

where we implicitly used the fact that M commutes with the projection π_+ and preserves the norm on W. Since the metric on N^+ is right invariant under translations by N^+ and is invariant under $\mathrm{Ad}(M)$, the above identity implies that it suffices to estimate the diameter of $S(v_1, \varepsilon) \cap N_1^+$ in the case $v \in P^+ w \cdot v_0$. Similarly, in the case $v \in P^+ \cdot v_0$, it suffices to estimate the diameter of $S(v_0, \varepsilon) \cap N_1^+$.

Let $\tilde{S}(v, \varepsilon) = \log S(v, \varepsilon)$ denote the pre-image of $S(v, \varepsilon)$ in the Lie algebra $\mathfrak{n}^+$ of N^+ under the exponential map. By Corollary 5.4, for any non-zero $v \in \mathcal{N}$, either $S(v, \varepsilon)$ is empty for all small enough ε, or there is a unique global minimizer of $p_v(\cdot)$ on N^+, at which p_v vanishes. In either case, for any given $v \in \mathcal{N} \setminus \{0\}$, the set $\tilde{S}(v, \varepsilon)$ is convex for all small enough $\varepsilon > 0$, depending on v. Let $s_0 > 0$ be such that $\tilde{S}(v, \varepsilon)$ is convex for $v \in \{v_0, v_1\}$ and for all $0 \leq \varepsilon \leq s_0$.

Fix some $v \in \{v_0, v_1\}$ and $\varepsilon \in [0, s_0]$. Suppose that $x_1 \neq x_2 \in \tilde{S}(v, \varepsilon) \cap B_1$. Let r denote the distance $d_{N^+}(x_1, x_2)$. Let $u' = x_2 - x_1$, $u = u'/r$ and $b = x_1$. Set $\ell = \ell_{u,b}$ and note that $\ell_{u,b}(0) = x_1$ and $\ell_{u,b}(r) = x_2$. Since B_1 is convex, the set $\tilde{S}(v, \varepsilon) \cap B_1$ is also convex. Hence, the entire interval

$(0,r)$ belongs to the set on the left side of (5.4) and, hence, that $r \ll \varepsilon^{1/4\chi_{\mathfrak{K}}}$. Since x_1 and x_2 were arbitrary, this shows that the diameter of $\tilde{S}(v,\varepsilon) \cap B_1$ is $O(\varepsilon^{1/4\chi_{\mathfrak{K}}})$ as desired.

6. Anisotropic Banach Spaces and Transfer Operators

In this section, we define the Banach spaces on which the transfer operator and resolvent associated to the geodesic flow have good spectral properties.

The transfer operator, denoted $\mathcal{L}_t$, acts on continuous functions as follows: for a continuous function f, let

$$\mathcal{L}_t f := f \circ g_t. \tag{6.1}$$

For $z \in \mathbb{C}$, the resolvent $R(z) : C_c(X) \to C(X)$ is defined formally as follows:

$$R(z)f := \int_0^\infty e^{-zt} \mathcal{L}_t f \; dt.$$

If Γ is not convex cocompact, we fix a choice of $\beta > 0$ so that Theorem 4.1 holds and set $V = V_\beta$. If Γ is convex cocompact, we take $V = V_\beta \equiv 1$ and we may take β as large as we like in this case. Note that the conclusion of Theorem 4.1 holds trivially with this choice of V. In particular, we shall use its conclusion throughout the argument regardless of whether Γ admits cusps.

Denote by $C_c^{k+1}(X)^M$ the subspace of $C_c^{k+1}(X)$ consisting of M-invariant functions, where M is the centralizer of the geodesic flow inside the maximal compact group K. In particular, $C_c^{k+1}(X)^M$ is naturally identified with the space of C_c^{k+1} functions on the unit tangent bundle of $\mathbb{H}^d_{\mathfrak{K}}/\Gamma$; cf. Section 2. The following is the main result of this section.

Theorem 6.1 (Essential Spectral Gap). *Let $k \in \mathbb{N}$ be given. Then, there exists a seminorm $\|\cdot\|_k$ on $C_c^{k+1}(X)^M$, non-vanishing on functions whose support meets Ω, and such that for every $z \in \mathbb{C}$, with $\mathrm{Re}(z) > 0$, the resolvent $R(z)$ extends to a bounded operator on the completion of $C_c^{k+1}(X)^M$ with respect to $\|\cdot\|_k$ and having spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ is bounded above by $1/(\mathrm{Re}(z) + \sigma_0)$, where*

$$\sigma_0 := \min\{k, \beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0 = k$.

By the completion of a topological vector space V with respect to a seminorm $\|\cdot\|$, we mean the Banach space obtained by completing the quotient topological vector space V/W with respect to the induced norm, where W is the kernel of $\|\cdot\|$.

The proof of Theorem 6.1 occupies Sections 6 and 7.

6.1. Anisotropic Banach Spaces.

We construct a Banach space of functions on X containing C^∞ functions satisfying Theorem 6.1.

Given $r \in \mathbb{N}$, let $\mathcal{V}_r^-$ denote the space of all C^r vector fields on N^+ pointing in the direction of the Lie algebra $\mathfrak{n}^-$ of N^- and having norm at most 1. More precisely, $\mathcal{V}_r^-$ consists of all C^r maps $v : N^+ \to \mathfrak{n}^-$, with C^r norm at most 1. Similarly, we denote by $\mathcal{V}_r^0$ the set of C^r vector fields $v : N^+ \to \mathfrak{a} := \mathrm{Lie}(A)$, with C^r norm at most 1. Note that if $\omega \in \mathfrak{a}$ is the vector generating the flow g_t, i.e. $g_t = \exp(t\omega)$, then each $v \in \mathcal{V}_r^0$ is of the form $v(n) = \phi(n)\omega$, for some $\phi \in C^r(N^+)$ such that $\|\phi\|_{C^r(N^+)} \leq 1$. Define

$$\mathcal{V}_r = \mathcal{V}_r^- \cup \mathcal{V}_r^0.$$

For $v \in \mathcal{V}$, denote by L_v the differential operator on $C^1(X)$ given by differentiation with respect to the vector field generated by v. Hence, for $\varphi \in C^1(G/\Gamma)$,

$$L_v \varphi(x) = \lim_{s \to 0} \frac{\varphi(\exp(sv)x) - \varphi(x)}{s}.$$

For each $k \in \mathbb{N}$, we define a norm on $C^k(N^+)$ functions as follows. Letting $\mathcal{V}^+$ be the unit ball in the Lie algebra of N^+, $0 \leq \ell \leq k$, and $\phi \in C^k(N^+)$, we define $c_\ell(\phi)$ to be the supremum of

$|L_{v_1}\cdots L_{v_\ell}(\phi)|$ over N^+ and all tuples $(v_1,\dots,v_\ell)\in(\mathcal{V}^+)^\ell$. We define $\|\phi\|_{C^k}$ to be $\sum_{\ell=0}^k 2^{-\ell}c_\ell(\phi)$. One then checks that for all $\phi_1,\phi_2\in C^k(N^+)$, we have

$$\|\phi_1\phi_2\|_{C^k}\leq\|\phi_1\|_{C^k}\|\phi_2\|_{C^k}. \tag{6.2}$$

Following [GL06, GL08], we define a norm on $C_c^{k+1}(X)$ as follows. Given $f\in C_c^{k+1}(X)$, k,ℓ non-negative integers, $\gamma=(\gamma_1,\dots,\gamma_\ell)\in\mathcal{V}_{k+\ell}^\ell$ (i.e. ℓ tuple of $C^{k+\ell}$ vector fields) and $x\in X$, define

$$e_{k,\ell,\gamma}(f;x):=\frac{1}{V(x)}\sup\frac{1}{\mu_x^u\left(N_1^+\right)}\left|\int_{N_1^+}\phi(n)L_{\gamma_1}\cdots L_{\gamma_\ell}(f)(g_snx)\,d\mu_x^u(n)\right|, \tag{6.3}$$

where the supremum is taken over all $s\in[0,1]$ and all functions $\phi\in C^{k+\ell}(N_1^+)$ which are compactly supported in the interior of N_1^+ and having $\|\phi\|_{C^{k+\ell}(N_1^+)}\leq 1$.

For $\gamma\in\mathcal{V}_{k+\ell+1}^\ell$, we define $e'_{k,\ell,\gamma}(f;x)$ analogously to $e_{k,\ell,\gamma}(f;x)$, but where we take $s=0$ and take the supremum over $\phi\in C^{k+\ell+1}(N_{1/10}^+)$ instead[4] of $C^{k+\ell}(N_1^+)$. Given $r>0$, set

$$\Omega_r^-:=N_r^-\Omega. \tag{6.4}$$

We define

$$e_{k,\ell,\gamma}(f):=\sup_{x\in\Omega_1^-}e_{k,\ell,\gamma}(f;x),\qquad e_{k,\ell}(f)=\sup_{\gamma\in\mathcal{V}_{k+\ell}^\ell}e_{k,\ell,\gamma}(f). \tag{6.5}$$

Finally, we define $\|f\|_k$ and $\|f\|'_k$ by

$$\|f\|_k:=\max_{0\leq\ell\leq k}e_{k,\ell}(f),\qquad \|f\|'_k:=\max_{0\leq\ell\leq k-1}\sup_{\gamma\in\mathcal{V}_{k+\ell+1}^\ell,x\in\Omega_{1/2}^-}e'_{k,\ell,\gamma}(f;x). \tag{6.6}$$

Note that the (semi-)norm $\|f\|'_k$ is weaker than $\|f\|_k$ since we are using more regular test functions and vector fields, and we are testing fewer derivatives of f.

Remark 6.2. Since the suprema in the definition of $\|\cdot\|_k$ are restricted to points on Ω_1^-, $\|\cdot\|_k$ defines a seminorm on $C_c^{k+1}(X)^M$. Moreover, since Ω_1^- is invariant by g_t for all $t\geq 0$, the kernel of this seminorm, denoted W_k, is invariant by $\mathcal{L}_t$. The seminorm $\|\cdot\|_k$ induces a norm on the quotient $C_c^{k+1}(X)^M/W_k$, which we continue to denote $\|\cdot\|_k$.

Definition 6.3. We denote by $\mathcal{B}_k$ the Banach space given by the completion of the quotient $C_c^{k+1}(X)^M/W_k$ with respect to the norm $\|\cdot\|_k$, where $C_c^{k+1}(X)^M$ denotes the subspace consisting of M-invariant functions.

Note that since $\|\cdot\|'_k$ is dominated by $\|\cdot\|_k$, $\|\cdot\|'_k$ descends to a (semi-)norm on $C_c^{k+1}(X)^M/W_k$ and extends to a (semi-)norm on $\mathcal{B}_k$, again denoted $\|\cdot\|'_k$.

The following is a reformulation of Theorem 6.1 in the above setup.

Theorem 6.4. *For all $z\in\mathbb{C}$, with $\mathrm{Re}(z)>0$, and for all $k\in\mathbb{N}$, the operator $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$. Moreover, the essential spectral radius of $R(z)$ acting on $\mathcal{B}_k$ is bounded above by $1/(\mathrm{Re}(z)+\sigma_0)$, where*

$$\sigma_0:=\min\{k,\beta\}.$$

In particular, if Γ is convex cocompact, we can take $\sigma_0=k$.

[4]The restriction on the supports allows us to handle non-smooth conditional measures; cf. proof of Prop. 6.6.

6.2. **Hennion's Theorem and Compact Embedding.** Our key tool in estimating the essential spectral radius is the following refinement of Hennion's Theorem, based on Nussbaum's formula.

Theorem 6.5 (cf. [Hen93] and Lemma 2.2 in [BGK07])**.** *Suppose that $\mathcal{B}$ is a Banach space with norm $\|\cdot\|$ and that $\|\cdot\|'$ is a seminorm on $\mathcal{B}$ so that the unit ball in $(\mathcal{B}, \|\cdot\|)$ is relatively compact in $\|\cdot\|'$. Suppose R is a bounded operator on $\mathcal{B}$ such that for some $n \in \mathbb{N}$, there exist constants $r > 0$ and $C > 0$ satisfying*

$$\|R^n v\| \leq r^n \|v\|_{\mathcal{B}} + C \|v\|', \tag{6.7}$$

for all $v \in \mathcal{B}$. Then, the essential spectral radius of R is at most r.

In this Section, we show, roughly speaking, that the unit ball in $\mathcal{B}_k$ is relatively compact in the weak norm $\|\cdot\|'_k$; Proposition 6.6.

Proposition 6.6. *Let $K \subseteq X$ be such that*

$$\sup \{V(x) : x \in K\} < \infty.$$

Then, every sequence $f_n \in C_c^{k+1}(X)^M$, such that f_n is supported in K and has $\|f_n\|_k \leq 1$ for all n, admits a Cauchy subsequence in $\|\cdot\|'_k$.

6.3. **Proof of Proposition 6.6.** We adapt the arguments in [GL06,GL08] with the main difference being that we bypass the step involving integration by parts over N^+ since our conditionals μ_x^u need not be smooth in general. The idea is to show that since all directions in the tangent space of X are accounted for in the definition of $\|\cdot\|_k$ (differentiation along the weak stable directions and integration in the unstable directions), one can estimate $\|\cdot\|'_k$ using finitely many coefficients $e_k(f; x_i)$. More precisely, we first show that there exists $C \geq 1$ so that for all sufficiently small $\varepsilon > 0$, there exists a finite set $\Xi \subset \Omega$ so that for all $f \in C_c^{k+1}(X)^M$, which is supported in K,

$$\|f\|'_k \leq C\varepsilon \|f\|_k + C \sup \int_{N_1^+} \phi L_{v_1} \cdots L_{v_\ell} f \, d\mu^u_{x_i}, \tag{6.8}$$

where the supremum is over all $0 \leq \ell \leq k-1$, all $(v_1, \ldots, v_\ell) \in \mathcal{V}^\ell_{k+\ell+1}$, all functions $\phi \in C^{k+\ell+1}(N_2^+)$ with $\|\phi\|_{C^{k+\ell+1}} \leq 1$ and all $x_i \in \Xi$.

First, we show how (6.8) completes the proof. Let $f_n \in C_c^{k+1}(K)$ be as in the statement. Let $\varepsilon > 0$ be small enough so that (6.8) holds. Since $C^{k+\ell+1}(N_2^+)$ is compactly included inside $C^{k+\ell}(N_2^+)$, we can find a finite collection $\{\phi_j : j\} \subset C^{k+\ell}(N_2^+)$ which is ε dense in the unit ball of $C^{k+\ell+1}(N_2^+)$. Similarly, we can find a finite collection of vector fields $\{(v_1^m, \ldots, v_\ell^m) : m\} \subset \mathcal{V}^\ell_{k+\ell}$ which is ε dense in $\mathcal{V}^\ell_{k+\ell+1}$ in the $C^{k+\ell+1}$ topology. Then, we can find a subsequence, also denoted f_n, so that the finitely many quantities

$$\left\{ \int_{N_1^+} \phi_j L_{v_1^m} \cdots L_{v_\ell^m} f_n \, d\mu^u_{x_i} : i, j, m \right\}$$

converge. Together with (6.8), this implies that

$$\|f_{n_1} - f_{n_2}\|'_k \ll \varepsilon,$$

for all large enough n_1, n_2, where we used the fact that $\|f_n\|_k \leq 1$ for all n. As ε was arbitrary, one can extract a Cauchy subequence by a standard diagonal argument. Thus, it remains to prove (6.8).

Fix some $f \in C_c^{k+1}(X)^M$ which is supported inside K. Let an arbitrary tuple $\gamma = (v_1, \ldots, v_\ell) \in \mathcal{V}^\ell_{k+\ell+1}$ be given and set

$$\psi = L_{v_1} \cdots L_{v_\ell} f.$$

Let $\phi \in C^{k+\ell+1}(N^+_{1/10})$ and write $Q = N^+_{1/10}$. To estimate $e'_{k,\ell,\gamma}(f; z)$ using the right side of (6.8), we need to estimate integrals of the form

$$\frac{1}{V(z)} \frac{1}{\mu^u_z\left(N^+_1\right)} \int_{N^+_1} \phi(n)\psi(nz)\, d\mu^u_z(n), \tag{6.9}$$

for all $z \in \Omega^-_{1/2}$.

Denote by $\rho : X \to [0, 1]$ a smooth function which is identically one on the 1-neighborhood Ω^1 of Ω and vanishes outside its 2-neighborhood. Note that if f is supported outside of Ω^1, then the integral in (6.9) vanishes for all z and the estimate follows. The same reasoning implies that

$$\|\rho f\|_k = \|f\|_k\,, \qquad \|\rho f\|'_k = \|f\|'_k\,.$$

Hence, we may assume that f is supported inside the intersection of K with Ω^1. In particular, for the remainder of the argument, we may replace K with (the closure of) its intersection with Ω^1.

This discussion has the important consequence that we may assume that K is a compact set in light of Proposition 4.3. Let K_1 denote the 1-neighborhood of K and fix some $z \in K_1 \cap \Omega^-_{1/2}$. By shrinking ε, we may assume it is smaller than the injectivity radius of K_1. Hence, we can find a finite cover $B_1, \ldots, B_M$ of $K_1 \cap \Omega^-_{1/2}$ with flow boxes of radius ε and with centers $\Xi := \{x_i\} \subset \Omega^-_{1/2}$.

Step 1: We first handle the case where z belongs to the same unstable manifold as one of the x_i's. Note that we may assume that Q intersects the support of μ^u_z non-trivially, since otherwise the integral in question is 0. Let $u \in Q$ be one point in this intersection and let $x = uz$. Thus, by (2.4), we get

$$\int_{N^+_1} \phi(n)\psi(nz)\, d\mu^u_z(n) = \int_Q \phi(n)\psi(nz)\, d\mu^u_z(n) = \int_{Qu^{-1}} \phi(nu)\psi(nx)\, d\mu^u_x(n).$$

Let $\phi_u(n) := \phi(nu)$. Then, ϕ_u is supported inside Qu^{-1}. Moreover, since $u \in Q$, $Q_u := Qu^{-1}$ is a ball of radius 1/10 containing the identity element. Hence, $Qu^{-1} \subset N^+_1$ and, thus,

$$\int_{Q_u} \phi(nu)\psi(nx)\, d\mu^u_x(n) = \int_{N^+_1} \phi_u(n)\psi(nx)\, d\mu^u_x(n).$$

Fix some $\varepsilon > 0$. We may assume that $\varepsilon < 1/10$. Note that x belongs to the 1-neighborhood of K. Then, $x = u_2^{-1} x_i$ for some i and some $u_2 \in N^+_\varepsilon$, by our assumption in this step that z belongs to the unstable manifold of one of the x_i's. By repeating the above argument with z, u, x, Q and ϕ replaced with x, u_2, x_i, Q_u and ϕ_u respectively, we obtain

$$\int_{N^+_1} \phi_u(n)\psi(nx)\, d\mu^u_x(n) = \int_{Q_u u_2^{-1}} \phi_u(nu_2)\psi(nx_i)\, d\mu^u_{x_i}(n).$$

Note that Q_u is contained in the ball of radius 1/5 centered around identity. Since $u_2 \in N^+_\varepsilon$ and $\varepsilon < 1/10$, we see that $Q_u u_2^{-1} \subset N^+_1$. It follows that

$$\int_{N^+_1} \phi_u(n)\psi(nx_i)\, d\mu^u_{x_i}(n) = \int_{N^+_1} \phi_{u_2 u}(n)\psi(nx_i)\, d\mu^u_{x_i}(n),$$

where $\phi_{u_2 u}(n) = \phi_u(nu_2) = \phi(nu_2 u)$. The function $\phi_{u_2 u}$ satisfies $\|\phi_{u_2 u}\|_{C^{k+\ell+1}} = \|\phi\|_{C^{k+\ell+1}} \leq 1$. Finally, let $\varphi_1, \varphi_2 : N^+ \to [0, 1]$ be non-negative bump C^0 functions where $\varphi_1 \equiv 1$ on N^+_1 and while φ_2 is equal to 1 at identity and its support is contained inside N^+_1. Since $y \mapsto \mu^u_y(\varphi_i)$ is continuous for $i = 1, 2$, by [Rob03, Lemme 1.16], and is non-zero on Ω^-_1, we can find, by compactness of K_1, a constant $C \geq 1$, depending only on K (and the choice of φ_1, φ_2), such that

$$1/C \leq \mu^u_y\left(N^+_1\right) \leq C, \qquad \forall y \in K_1 \cap \Omega^-_1. \tag{6.10}$$

Hence, recalling that $\psi = L_{v_1} \cdots L_{v_\ell} f$ and that $V(z) \gg 1$, we conclude that the integral in (6.9) is bounded by the second term in (6.8).

Step 2: We reduce to the case where z is contained in the unstable manifolds one of the x_i's. Let i be such that $z \in B_i$. Set $z_1 = z$ and let $z_0 \in (N_\varepsilon^+ \cdot x_i)$ be the unique point in the intersection of $N_\varepsilon^+ \cdot x_i$ with the local weak stable leaf of z_1 inside B_i. Let $p_1^- \in P^- := MAN^-$ be an element of the ε neighborhood of identity P_ε^- in P^- such that $z_1 = p_1^- z_0$.

We will estimate the integral in (6.9) using integrals at z_0. The idea is to perform weak stable holonomy between the local strong unstable leaves of z_0 and z_1. To this end, we need some notation. Let $Y \in \mathfrak{p}^-$ be such that $p_1^- = \exp(Y)$ and set

$$p_t^- = \exp(tY), \qquad z_t = p_t^- z_0,$$

for $t \in [0,1]$. Let us also consider the following maps $u_t^+ : N_1^+ \to N^+$ and $\tilde{p}_t^- : N_1^+ \to P^-$ defined by the following commutation relations

$$np_t^- = \tilde{p}_t^-(n) u_t^+(n), \qquad \forall n \in N_1^+.$$

Recall we are given a test function $\phi \in C^{k+\ell+1}(N_{1/10}^+)$. We can rewrite the integral we wish to estimate as follows:

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u(n) = \int_{N_1^+} \phi(n)\psi(np_1^- z_0)\, d\mu_{z_1}^u(n) = \int \phi(n)\psi(\tilde{p}_1^-(n)u_1^+(n)z_0)\, d\mu_{z_1}^u(n).$$

Let $U_t^+ \subset N^+$ denote the image of u_t^+. Note that if ε is small enough, $U_t^+ \subseteq N_2^+$ for all $t \in [0,1]$. We may further assume that ε is small enough so that the map u_t^+ is invertible on U_t^+ for all $t \in [0,1]$ and write $\phi_t := \phi \circ (u_t^+)^{-1}$. For simplicity, set

$$p_t^-(n) := \tilde{p}_t^-((u_t^+)^{-1}(n)).$$

Write $m_t(n) \in M$ and $b_t^-(n) \in AN^-$ for the components of $p_t^-(n)$ along M and AN^- respectively so that

$$p_t^-(n) = m_t(n) b_t^-(n).$$

We denote by J_t the Radon-Nikodym derivative of the pushforward of $\mu_{z_1}^u$ by u_t^+ with respect to $\mu_{z_t}^u$; cf. (2.9) for an explicit formula. Thus, changing variables using $n \mapsto u_1^+(n)$, and using the M-invariance of f, we obtain

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu_{z_1}^u = \int \phi_1(n)\psi(p_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u = \int \phi_1(n)\tilde{\psi}_1(b_1^-(n)nz_0)J_1(n)\, d\mu_{z_0}^u,$$

where $\tilde{\psi}_t$ is given by

$$\tilde{\psi}_t := L_{\tilde{v}_1^t} \cdots L_{\tilde{v}_\ell^t} f, \qquad \tilde{v}_i(n) := \mathrm{Ad}(m_t((u_t^+)^{-1}(n)))(v_i((u_t^+)^{-1}(n))).$$

Here, we recall that $\mathrm{Ad}(M)$ commutes with A and normalizes N^- so that $\tilde{v}_i^t$ is a vector field with the same target as v_i.

Let $\mathfrak{b}^-$ denote the Lie algebra of AN^- and denote by $\tilde{w}_t' : U_t^+ \times [0,1] \to \mathfrak{b}^-$ the vector field tangent to the paths defined by b_t^-. More explicitly, $\tilde{w}_t'$ is given by the projection of tY to $\mathfrak{b}^-$. Denote $\tilde{w}_t(n) := \mathrm{Ad}(m_t(n))(\tilde{w}_t'(n))$. Then, using the M-invariance of f as above once more, we can write

$$\psi(b_1^-(n)nz_0) - \psi(nz_0)) = \int_0^1 \frac{\partial}{\partial t}\tilde{\psi}_t(b_t^-(n)nz_0)\, dt = \int_0^1 L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)\, dt.$$

To simplify notation, let us set $w_t = \tilde{w}_t \circ u_t^+$, and

$$F_t := L_{\tilde{v}_1^t \circ u_t^+} \cdots L_{\tilde{v}_\ell^t \circ u_t^+} f.$$

Using a reverse change of variables, we obtain for every $t \in [0,1]$ that

$$\int \phi_1(n) L_{\tilde{w}_t}(\tilde{\psi}_t)(p_t^-(n)nz_0)J_1(n)\, d\mu^u_{z_0} = \int (\phi_1 J_1)\circ u_t^+(n) L_{w_t}(F_t)(\tilde{p}_t^-(n)u_t^+(n)z_0)J_t^{-1}(n)\, d\mu^u_{z_t}$$
$$= \int (\phi_1 J_1)\circ u_t^+(n)\cdot L_{w_t}(F_t)(nz_t)\cdot J_t^{-1}(n)\, d\mu^u_{z_t}(n),$$

where we used the identities $\tilde{p}_t^-(n)u_t^+(n) = np_t^-$ and $z_t = p_t^- z_0$. Let us write

$$\Phi_t(n) := (\phi_1 J_1)\circ u_t^+(n)\cdot J_t^{-1}(n),$$

which we view as a test function[5]. Hence, the last integral above amounts to integrating $\ell+1$ weak stable derivatives of f against a $C^{k+\ell}$ function. Moreover, since ϕ is supported in $N^+_{1/10}$, we may assume that ε is small enough so that Φ_t is supported in N_1^+ for all $t\in[0,1]$, and meets the requirements on the test functions in the definition of $\|f\|_k$. Since $z = z_1$ belongs to $\Omega^-_{1/2}$ by assumption, we may further shrink ε if necessary so that the points z_t all[6] belong to Ω_1^-. Thus, decomposing w_t into its A and N^- components, and noting that $\|w_t\| \ll \varepsilon$, we obtain the estimate

$$\int \Phi_t(n)\cdot L_{w_t}(F_t)(nz_t)\, d\mu^u_{z_t}(n) \ll \varepsilon \|f\|_k V(z_t)\mu^u_{z_t}(N_1^+). \tag{6.11}$$

To complete the argument, note that the integral we wish to estimate satisfies

$$\int_{N_1^+} \phi(n)\psi(nz_1)\, d\mu^u_{z_1} = \int (\phi_1 J_1)(n)\psi(nz_0)\, d\mu^u_{z_0} + \int_0^1 \int \Phi_t(n)\cdot L_{w_t}(F_t)(nz_t)\, d\mu^u_{z_t}(n)\, dt. \tag{6.12}$$

Moreover, recall that z_0 belongs to the same unstable manifold as some $x_i \in \Xi$. Additionally, since ϕ is supported in $N^+_{1/10}$, by taking ε small enough, we may assume that ϕ_1 is supported inside $N^+_{1/5}$. Hence, arguing similarly to Step 1, viewing $\phi_1 J_1$ as a test function, we can estimate the first term on the right side above using the right side of (6.8).

The second term in (6.12) is also bounded by the right side of (6.8), in view of (6.11). Here we are using that $y \mapsto \mu^u_y(N_1^+)$ and $y\mapsto V(y)$ are uniformly bounded as y varies in the compact set K_1; cf. (6.10). This completes the proof of (6.8) in all cases, since ϕ and z were arbitrary.

7. The Essential Spectral Radius of Resolvents

In this section, we study the operator norm of the transfer operators $\mathcal{L}_t$ and the resolvents $R(z)$ on the Banach spaces constructed in the previous section. These estimates constitute the proof of Theorem 6.1. With these results in hand, we deduce Theorem C at the end of the section.

7.1. **Strong continuity of transfer operators.**

Recall that a collection of measurable subsets $\{B_i\}$ of a space Y are said to have **intersection multiplicity** bounded by a constant $C\geq 1$ if for all i, the number of sets B_j in the collection that intersect B_i non-trivially is at most C. In this case, one has

$$\sum_i \chi_{B_i}(y) \leq C\chi_{\cup_i B_i}(y), \qquad \forall y \in Y.$$

The following lemma implies that the operators $\mathcal{L}_t$ are uniformly bounded on $\mathcal{B}_k$ for $t\geq 0$.

Lemma 7.1. *For every $k,\ell\in\mathbb{N}\cup\{0\}$, $\gamma\in\mathcal{V}^\ell_{k+\ell}$, $t\geq 0$, and $x\in\Omega_1^-$,*

$$e_{k,\ell,\gamma}(\mathcal{L}_t f;x) \ll_\beta e^{-\varepsilon(\gamma)t} e_{k,\ell,\gamma}(f)(e^{-\beta t} + 1/V(x)),$$

where $\varepsilon(\gamma)\geq 0$ is the number of stable derivatives determined by γ. In partitcular, $\varepsilon(\gamma)=0$ if only if $\ell = 0$ or all components of γ point in the flow direction.

[5]The Jacobians are smooth maps as they are given in terms of Busemann functions; cf. (2.9).

[6]This type of estimate is the reason we use stable thickenings Ω_r^- of Ω in the definition of the norm instead of Ω.

Proof. Fix some $x \in \Omega$ and $\gamma = (v_1, \dots, v_\ell) \in \mathcal{V}^\ell_{k+\ell}$. Since the Lie algebra of N^- has the orthogonal decomposition $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, where α is the simple positive root in $\mathfrak{g}$ with respect to g_t, we have that g_t contracts the norm of each stable vector $v \in \mathcal{V}^-_{k+\ell}$ by at least e^{-t}. It follows that for all $v \in \mathcal{V}^-_{k+\ell}$ and $w \in \mathcal{V}^0_{k+\ell}$,

$$L_v(\mathcal{L}_t f)(x) = \|v_t\| \, L_{\bar{v}_t}(f)(g_t x), \qquad L_w(\mathcal{L}_t f)(x) = L_w(f)(g_t x), \tag{7.1}$$

for all $f \in C^{k+1}(X)^M$, where $v_t = \mathrm{Ad}(g_t)(v)$ and $\bar{v}_t = v_t / \|v_t\|$. Moreover, we have

$$\|v_t\| \le e^{-t} \|v\| = e^{-t} \|v\| .$$

Let ϕ be a test function and $\psi \in C(X)^M$. Using (2.3) to change variables, we get

$$\int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu^u_x(n) = e^{-\delta t} \int_{\mathrm{Ad}(g_t)(N_1^+)} \phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu^u_{g_t x}(n).$$

Let $\{\rho_i : i \in I\}$ be a partition of unity of $\mathrm{Ad}(g_t)(N_1^+)$ so that each ρ_i is non-negative, C^∞, and supported inside some ball of radius 1 centered inside $\mathrm{Ad}(g_t)(N_1^+)$. Such a partition of unity can be chosen so that the supports of ρ_i have a uniformly bounded multiplicity[7], depending only on N^+. Denote by $I(\Lambda)$ the subset of indices $i \in I$ such that there is $n_i \in N^+$ in the support of the measure $\mu^u_{g_t x}$ with the property that the support of ρ_i is contained in $N_1^+ \cdot n_i$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu^u_{g_t x}$ is the 0 measure. Then, we obtain

$$\int_{\mathrm{Ad}(g_t)(N_1^+)} \phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu^u_{g_t x}(n) = \sum_{i \in I(\Lambda)} \int_{N_1^+ \cdot n_i} \rho_i(n) \phi(g_{-t} n g_t)\psi(n g_t x) \, d\mu^u_{g_t x}(n).$$

Setting $x_i = n_i g_t x$ and changing variables using (2.4), we obtain

$$\int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu^u_x(n) = e^{-\delta t} \sum_{i \in I(\Lambda)} \int_{N_1^+} \rho_i(n n_i) \phi(g_{-t} n n_i g_t)\psi(n x_i) \, d\mu^u_{x_i}(n). \tag{7.2}$$

The bounded multiplicity of the partition of unity implies that the balls $N_1^+ \cdot n_i$ have intersection multiplicity bounded by a constant C_0, depending only on N^+. Enlarging C_0 if necessary, we may also choose ρ_i so that $\|\rho_i\|_{C^{k+\ell}} \le C_0$. In particular, C_0 is independent of t and x.

For each i, let $\bar{\phi}_i(n) = \rho_i(n n_i)\phi(g_{-t} n n_i g_t)$. Since ρ_i is chosen to be supported inside $N_1^+ n_i$, then $\bar{\phi}_i$ is supported inside N_1^+. Moreover, since ρ_i is C^∞, $\bar{\phi}_i$ is of the same differentiability class as ϕ. Since conjugation by g_{-t} contracts N^+, we see that $\|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} \le \|\phi\|_{C^{k+\ell}} \le 1$ (note that the supremum norm of $\phi \circ \mathrm{Ad}(g_{-t})$ does not decrease, and hence we do not gain from this contraction). Hence, since $\|\rho_i\|_{C^{k+\ell}} \le C_0$, (6.2) implies that $\left\|\bar{\phi}_i\right\|_{C^{k+\ell}} \le C_0$.

First, let us suppose that $t \ge 1$. Then, using Remark 2.1, since $x \in N_1^- \Omega$, one checks that x_i belongs to $N_1^- \Omega$ as well for all i. Applying (7.2) with $\psi = L_{v_1} \cdots L_{v_\ell} f$, we obtain

$$\begin{aligned} \int_{N_1^+} \phi(n)\psi(g_t n x) \, d\mu^u_x &= e^{-\delta t} \sum_{i \in I(\Lambda)} \int_{N_1^+} \bar{\phi}_i(n)\psi(n x_i) \, d\mu^u_{x_i} \\ &\le C_0 e_{k,\ell,\gamma}(f) \|\phi \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} e^{-\delta t} \sum_{i \in I(\Lambda)} \mu^u_{x_i}(N_1^+) V(x_i). \end{aligned} \tag{7.3}$$

[7]Note that the analog of the classical Besicovitch covering theorem fails to hold for N^+ with the Carnot-Caratheodory metric when N^+ is not abelian; cf. [KR95, pg. 17]. Instead, such a partition of unity can be constructed using the Vitali covering lemma with the aid of the right invariance of the Haar measure. To obtain a uniform bound on the multiplicity here and throughout, it is important that such an argument is applied to balls with uniformly comparable radii.

By the log Lipschitz property of V provided by Proposition 4.3, and by enlarging C_0 if necessary, we have $V(x_i) \leq C_0 V(nx_i)$ for all $n \in N_1^+$. It follows that

$$\sum_{i \in I(\Lambda)} \mu^u_{x_i}(N_1^+) V(x_i) \leq C_0 \sum_{i \in I(\Lambda)} \int_{N_1^+} V(nx_i) \, d\mu^u_{x_i}(n).$$

Recall that the balls $N_1^+ \cdot n_i$ have intersection multiplicity at most C_0. Moreover, since the support of ρ_i is contained inside $\mathrm{Ad}(g_t)(N_1^+)$, the balls $N_1^+ n_i$ are all contained in $N_2^+ \mathrm{Ad}(g_t)(N_1^+)$. Hence, applying the equivariance properties (2.3) and (2.4) once more yields

$$\sum_{i \in I(\Lambda)} \int_{N_1^+} V(nx_i) \, d\mu^u_{x_i}(n) \leq C_0 \int_{N_2^+ \mathrm{Ad}(g_t)(N_1^+)} V(ng_t x) \, d\mu^u_{g_t x}(n) \leq C_0 e^{\delta t} \int_{N_3^+} V(g_t nx) \, d\mu^u_x(n).$$

Here, we used the positivity of V and that $\mathrm{Ad}(g_{-t})(N_2^+) N_1^+ \subseteq N_3^+$. Combined with (7.2) and the contraction estimate on V, Theorem 4.1, it follows that

$$\int_{N_1^+} \phi(n) \psi(g_t nx) \, d\mu^u_x \leq C_0^3 (c e^{-\beta t} V(x) + c) \mu^u_x(N_3^+) e_{k,0}(f),$$

for a constant $c \geq 1$ depending on β. By Proposition 3.1, we have $\mu^u_x(N_3^+) \leq C_1 \mu^u_x(N_1^+)$, for a uniform constant $C_1 \geq 1$, which is independent of x. This estimate concludes the proof in view of (7.1).

Now, let $s \in [0,1]$ and $t \geq 0$. If $t + s \geq 1$, then the above argument applied with $t + s$ in place of t implies that

$$\left| \int_{N_1^+} \phi(n) \psi(g_{t+s} nx) \, d\mu^u_x \right| \ll_\beta e^{-\varepsilon(\gamma) t} e_{k,\ell,\gamma}(f) (e^{-\beta t} V(x) + 1) \mu^u_x(N_1^+),$$

as desired. Otherwise, if $t + s < 1$, then by definition of $e_{k,\ell,\gamma}$, we have that

$$\left| \int_{N_1^+} \phi(n) \psi(g_{t+s} nx) \, d\mu^u_x \right| \leq e_{k,\ell,\gamma}(f) V(x) \mu^u_x(N_1^+).$$

Since t is at most 1 in this case and $V(x) \gg 1$ on Ω_1^-, the conclusion of the lemma follows in this case as well.

□

As a corollary, we deduce the following strong continuity statement which implies that the infinitesimal generator of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. When restricted to $C_c^{k+1}(X)^M$, this generator is nothing but the differentiation operator in the flow direction. This strong continuity is also important in applying the results of [But16a] to deduce exponential mixing from our spectral bounds on the resolvent in Section 8.

Corollary 7.2. *The semigroup $\{\mathcal{L}_t : t \geq 0\}$ is strongly continuous; i.e. for all $f \in \mathcal{B}_k$,*

$$\lim_{t \downarrow 0} \|\mathcal{L}_t f - f\|_k = 0.$$

Proof. For all $f \in C_c^{k+1}(X)^M$, one easily checks that since $V(\cdot) \gg 1$ on any bounded neighborhood of Ω, then

$$\|\mathcal{L}_t f - f\|_k \ll \sup_{0 \leq s \leq 1} \|\mathcal{L}_{t+s} f - \mathcal{L}_s f\|_{C^k(X)}.$$

Moreover, since f belongs to C^{k+1}, the right side above tends to 0 as $t \to 0^+$ by the mean value theorem. Now, let f be a general element of $\mathcal{B}_k$ and let $f_n \in C_c^{k+1}$ be a sequence tending to f in $\|\cdot\|_k$. Then, by the triangle inequality, we have

$$\|\mathcal{L}_t f - f\|_k \leq \|\mathcal{L}_t f - \mathcal{L}_t f_n\|_k + \|\mathcal{L}_t f_n - f_n\|_k + \|f_n - f\|_k.$$

We note that the first term satisfies the bound

$$\|\mathcal{L}_t f - \mathcal{L}_t f_n\|_k \ll \|f - f_n\|_k ,$$

uniformly in $t \geq 0$, by Lemma 7.1. The conclusion of the corollary thus follows by the previous estimate for elements of $C_c^{k+1}(X)^M$. □

7.2. **Towards a Lasota-Yorke inequality for the resolvent.** Recall that for all $n \in \mathbb{N}$,

$$R(z)^n = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t \, dt, \tag{7.4}$$

as follows by induction on n. The following corollary is immediate from Lemma 7.1 and the fact that

$$\left| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \, dt \right| \leq \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-\mathrm{Re}(z)t} \, dt = 1/\mathrm{Re}(z)^n, \tag{7.5}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Corollary 7.3. *For all $n, k, \ell \in \mathbb{N} \cup \{0\}$, $f \in C_c^{k+1}(X)^M$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we have*

$$e_{k,\ell}(R(z)^n f; x) \ll_\beta e_{k,\ell}(f) \left(\frac{1}{(\mathrm{Re}(z) + \beta)^n} + \frac{V(x)^{-1}}{\mathrm{Re}(z)^n} \right) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n.$$

In particular, $R(z)$ extends to a bounded operator on $\mathcal{B}_k$ with spectral radius at most $1/\mathrm{Re}(z)$.

Note that Lemma 7.1 does not provide contraction in the part of the norm that accounts for the flow direction. In particular, the estimate in this lemma is not sufficient to control the essential spectral radius of the resolvent. The following lemma provides the first step towards a Lasota-Yorke inequality for resolvents for the coefficients $e_{k,\ell}$ when $\ell < k$. The idea, based on regularization of test functions, is due to [GL06]. The doubling estimates on conditional measures in Proposition 3.1 are crucial for carrying out the argument.

Lemma 7.4. *For all $t \geq 2$ and $0 \leq \ell < k$, we have*

$$e_{k,\ell}(\mathcal{L}_t f) \ll_{k,\beta} e^{-kt} e_{k,\ell}(f) + e'_{k,\ell}(f).$$

Proof. Fix some $0 \leq \ell < k$. Let $x \in \Omega_1^-$ and $\phi \in C^{k+\ell}(N_1^+)$. Let $(v_i)_i \in \mathcal{V}_{k+\ell}^\ell$ and set $F = L_{v_1} \cdots L_{v_\ell} f$. We wish to estimate the following:

$$\sup_{0 \leq s \leq 1} \int_{N_1^+} \phi(n) F(g_{t+s} n x) \, d\mu_x^u.$$

To simplify notation, we prove the desired estimate for $s = 0$, the general case being essentially identical.

Let $\varepsilon > 0$ to be determined and choose ψ_ε to be a C^∞ bump function supported inside N_ε^+ and satisfying $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$. Define the following regularization of ϕ

$$\mathcal{M}_\varepsilon(\phi)(n) = \frac{\int_{N^+} \phi(un) \psi_\varepsilon(u) \, du}{\int_{N^+} \psi_\varepsilon(u) \, du},$$

where du denotes the right-invariant Haar measure on N^+. Recall the definition of the coefficients c_r above (6.2). Let $0 \leq m < k + \ell$ and $(w_j) \in (\mathcal{V}^+)^m$. Then,

$$\begin{aligned} |L_{w_1} \cdots L_{w_m}(\phi - \mathcal{M}_\varepsilon(\phi))(n)| &\leq \frac{\int |L_{w_1} \cdots L_{w_m}(\phi)(n) - L_{w_1} \cdots L_{w_m}(\phi)(un)| \psi_\varepsilon(u) \, du}{\int \psi_\varepsilon(u) \, du} \\ &\ll c_{m+1}(\phi) \frac{\int \mathrm{dist}(n, un) \psi_\varepsilon(u) \, du}{\int \psi_\varepsilon(u) \, du}. \end{aligned}$$

Now, note that if $\psi_\varepsilon(u) \neq 0$, then $\mathrm{dist}(u, \mathrm{Id}) \leq \varepsilon$. Hence, right invariance of the metric on N^+ implies that $c_m(\phi - \mathcal{M}_\varepsilon(\phi)) \ll \varepsilon c_{m+1}(\phi)$.

Moreover, we have that $c_m(\mathcal{M}_\varepsilon(\phi)) \leq c_m(\phi)$ for all $0 \leq m \leq k+\ell$. It follows that $c_{k+\ell}(\phi - \mathcal{M}_\varepsilon(\phi)) \leq 2c_{k+\ell}(\phi)$. Finally, given $(w_i) \in (\mathcal{V}^+)^{k+\ell+1}$, integration by parts implies

$$L_{w_1} \cdots L_{w_{k+\ell+1}}(\mathcal{M}_\varepsilon(\phi))(n) = \frac{\int_{N^+} L_{w_2} \cdots L_{w_{k+\ell+1}}(\phi)(un) \cdot L_{w_1}(\psi_\varepsilon)(u)\, du}{\int_{N^+} \psi_\varepsilon(u)\, du}.$$

In particular, since $\|\psi_\varepsilon\|_{C^1} \ll \varepsilon^{-1}$, we get $c_{k+\ell+1}(\mathcal{M}_\varepsilon(\phi)) \ll \varepsilon^{-1} c_{k+\ell}(\phi)$. Since g_t expands N^+ by at least e^t, this discussion shows that for any $t \geq 0$, if $\|\phi\|_{C^{k+\ell}} \leq 1$, then

$$\begin{aligned} \|(\phi - \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell}} &\ll \varepsilon \sum_{m=0}^{k+\ell-1} \frac{e^{-mt}}{2^m} + \frac{e^{-(k+\ell)t}}{2^{k+\ell}}, \\ \|\mathcal{M}_\varepsilon(\phi) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} &\ll \sum_{m=0}^{k+\ell} \frac{e^{-mt}}{2^m} + \frac{\varepsilon^{-1} e^{-(k+\ell+1)t}}{2^{k+\ell+1}}. \end{aligned} \tag{7.6}$$

Set $\mathcal{A}_t = \mathrm{Ad}(g_t)(N_1^+)$. Then, taking $\varepsilon = e^{-kt}$, we obtain

$$\begin{aligned} \int_{N_1^+} \phi(n) F(g_t n x)\, d\mu_x^u &= \int \phi(n) F(g_t n x)\, d\mu_x^u \\ &= \int (\phi - \mathcal{M}_\varepsilon(\phi))(n) F(g_t n x)\, d\mu_x^u + \int \mathcal{M}_\varepsilon(\phi)(n) F(g_t n x)\, d\mu_x^u. \end{aligned} \tag{7.7}$$

To estimate the second term, we recall that the test functions for the weak norm were required to be supported inside $N_{1/10}^+$. On the other hand, the support of $\mathcal{M}_\varepsilon(\phi)$ may be larger, but still inside $N_{1+\varepsilon}^+$. To remedy this issue, we pick a partition of unity $\{\rho_i : i \in I\}$ of N_2^+, so that each ρ_i is smooth, non-negative, and supported inside some ball of radius $1/20$. We also require that $\|\rho_i\|_{C^{k+\ell+1}} \ll_k 1$. We can find such a partition of unity with cardinality and multiplicity, depending only on N^+ (through its dimension and metric).

Similarly to Lemma 7.1, we denote by $I(\Lambda) \subseteq I$, the subset of those indices i such that there is some $n_i \in N^+$ in the support of of μ_x^u so that the support of ρ_i is contained inside $N_{1/10}^+$. In particular, for $i \in I \setminus I(\Lambda)$, $\rho_i \mu_x^u$ is the 0 measure.

Now, observe that the functions $n \mapsto \rho_i(nn_i)\mathcal{M}_\varepsilon(\phi)(nn_i)$ are supported inside $N_{1/10}^+$. Thus, writing $x_i = n_i g_1 x$, using a change of variable, and arguing as in the proof of Lemma 7.1, cf. (7.3), we obtain

$$\begin{aligned} \int \mathcal{M}_\varepsilon(\phi)(n) F(g_t n x)\, d\mu_x^u &= e^{-\delta} \sum_{i \in I(\Lambda)} \int (\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-1})(n) F(g_{t-1} n g_1 x)\, d\mu_{g_1 x}^u \\ &\ll e'_{k,\ell}(f) \cdot \sum_{i \in I(\Lambda)} \|(\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})\|_{C^{k+\ell+1}} \cdot V(x_i) \mu_{x_i}^u(N_1^+). \end{aligned}$$

The point of replacing x with $g_1 x$ is that since x belongs to $N_1^- \Omega$, $g_1 x$ belongs to $N_{1/2}^- \Omega$, which satisfies the requirement on the basepoints in the definition of the weak norm.

Note that the bounded multiplicity property of the partition of unity, together with the doubing property in Proposition 3.1, imply that

$$\sum_{i \in I} \mu_{x_i}^u(N_1^+) \ll \mu_x^u(N_3^+) \ll \mu_x^u(N_1^+).$$

Moreover, combining the Leibniz estimate (6.2) with (7.6), we see that the $C^{k+\ell+1}$ norm of $(\rho_i \mathcal{M}_\varepsilon(\phi)) \circ \mathrm{Ad}(g_{-t})$ is $O_k(1)$. Hence, by properties of the height function V in Proposition 4.3, it follows that

$$\int \mathcal{M}_\varepsilon(\phi)(n)F(g_t nx)\, d\mu_x^u \ll_k e'_{k,\ell}(f)V(x)\mu_x^u(N_1^+).$$

Using a completely analogous argument to handle the issues of the support of the test function, we can estimate the first term in (7.7) as follows:

$$\frac{1}{V(x)\mu_x^u(N_1^+)}\int_{N_1^+}(\phi-\mathcal{M}_\varepsilon(\phi))(n)F(g_t nx)\, d\mu_x^u \ll_k e^{-kt}e_{k,\ell}(f).$$

Since $(v_i)\in\mathcal{V}_{k+\ell}^\ell$, $x\in\Omega_1^-$ and $\phi\in C^{k+\ell}(N_1^+)$ were all arbitrary, this completes the proof. □

It remains to estimate the coefficients $e_{k,k}$. First, the following estimate in the case all the derivatives point in the stable direction follows immediately from Lemma 7.1.

Lemma 7.5. *For all $\gamma=(v_i)\in(\mathcal{V}_{2k}^-)^k$, we have*

$$e_{k,k,\gamma}(R(z)^n f)\ll_\beta \frac{1}{(\mathrm{Re}(z)+k)^n}e_{k,k}(f).$$

Proof. Indeed, Lemma 7.1 shows that

$$e_{k,k,\gamma}(\mathcal{L}_t f)\ll e^{-kt}e_{k,k}(f).$$

Moreover, induction and integration by parts give $|\int_0^\infty t^{n-1}e^{-(z+k)t}/(n-1)!dt|\le 1/(\mathrm{Re}(z)+k)^n$. This completes the proof. □

To give improved estimates on the the coefficient $e_{k,k,\gamma}$ in the case some of the components of γ point in the flow direction, the idea (cf. [AG13, Lem. 8.4] and [GLP13, Lem 4.5]) is to take advantage of the fact that the resolvent is defined by integration in the flow direction, which provides additional smoothing. This is leveraged through integration by parts to estimate the coefficient $e_{k,k}$ by $e_{k,k-1}$.

To see how such estimate can be turned into a gain on the norm of the resolvents, following [AG13], we define the following equivalent norms to $\|\cdot\|_k$. First, let us define the following coefficients:

$$e_{k,\ell,s}:=\begin{cases} e_{k,\ell} & 0\le\ell<k,\\ \sup_{\gamma\in(\mathcal{V}_{2k}^-)^k} e_{k,k,\gamma} & \ell=k,\end{cases}, \qquad e_{k,k,\omega}:=\sup_{\gamma\in\mathcal{V}_{2k}^k\setminus(\mathcal{V}_{2k}^-)^k} e_{k,k,\gamma}.$$

Given $B\ge 1$, define

$$\|f\|_{k,B,s}:=\sum_{\ell=0}^k\frac{e_{k,\ell,s}(f)}{B^\ell}, \qquad \|f\|_{k,B,\omega}:=\frac{e_{k,k,\omega}(f)}{B^k}.$$

Finally, we set

$$\|f\|_{k,B}:=\|f\|_{k,B,s}+\|f\|_{k,B,\omega}. \tag{7.8}$$

Lemma 7.6. *Let $n,k\in\mathbb{N}$ and $z\in\mathbb{C}$ with $\mathrm{Re}(z)>0$ be given. Then, if B is large enough, depending on n,k,β and z, we obtain for all $f\in C_c^{k+1}(X)^M$ that*

$$\|R(z)^n f\|_{k,B,\omega}\le\frac{1}{(\mathrm{Re}(z)+k+1)^n}\|f\|_{k,B}.$$

Proof. Fix an integer $n\ge 0$. We wish to estimate integrals of the form

$$\int_{N_1^+}\phi(u)L_{v_1}\cdots L_{v_k}\left(\int_0^\infty\frac{t^n e^{-zt}}{n!}\mathcal{L}_{t+s}f\, dt\right)(ux)\, d\mu_x^u(u)$$
$$=\int_{N_1^+}\phi(u)\int_0^\infty\frac{t^n e^{-zt}}{n!}L_{v_1}\cdots L_{v_k}(\mathcal{L}_{t+s}f)(ux)\, dt\, d\mu_x^u(u),$$

with $0 \le s \le 1$ and at least one of the v_i pointing in the flow direction.

First, let us consider the case v_k points in the flow direction. Then, $v_k(u) = \psi_k(u)\omega$, where ω is the vector field generating the geodesic flow, for some function ψ_k in the unit ball of $C^{2k}(N^+)$. Hence, for a fixed $u \in N_1^+$, integration by parts in t, along with the fact that f is bounded, yields

$$\begin{aligned}&\int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} L_{v_2} \cdots L_{v_k}(\mathcal{L}_{t+s} f)(ux)\, dt \\ &= \psi_k(u) z \int_0^\infty \frac{t^n e^{-zt}}{n!} L_{v_1} \cdots L_{v_{k-1}}(\mathcal{L}_{t+s} f)(ux)\, dt - \psi_k(u) \int_0^\infty \frac{t^{n-1} e^{-zt}}{(n-1)!} L_{v_1} \cdots L_{v_{k-1}}(\mathcal{L}_{t+s} f)(ux)\, dt \\ &= \psi_k(u) z L_{v_1} \cdots L_{v_{k-1}}(\mathcal{L}_s R(z)^{n+1} f)(ux) - \psi_k(u) L_{v_1} \cdots L_{v_{k-1}}(\mathcal{L}_s R^n(z) f)(ux).\end{aligned}$$

Recall by Lemma 7.1 that $e_{k,\ell}(R(z)^n f) \ll_\beta e_{k,\ell}(f)/\mathrm{Re}(z)^n$ for all $n \in \mathbb{N}$; cf. Corollary 7.3. It follows that

$$e_{k,k,\gamma}(R(z)^{n+1} f) \le e_{k,k-1}(R(z)^n f) + |z| e_{k,k-1}(R(z)^{n+1} f) \ll_\beta \left(\frac{\mathrm{Re}(z) + |z|}{\mathrm{Re}(z)^{n+1}} \right) e_{k,k-1}(f).$$

In the case v_k points in the stable direction instead, we note that $L_v L_w = L_w L_v + L_{[v,w]}$ for any two vector fields v and w, where $[v,w]$ is their Lie bracket. In particular, we can write $L_{v_1} \cdots L_{v_k}$ as a sum of at most k terms involving $k-1$ derivatives in addition to one term of the form $L_{w_1} \cdots L_{w_k}$, where w_k points in the flow direction. Each of the terms with one fewer derivative can be bounded by $e_{k,k-1}(R(z)^{n+1} f) \ll_\beta e_{k,k-1}(f)/\mathrm{Re}(z)^{n+1}$, while the term with k derivatives is controlled as in the previous case. Hence, taking the supremum over $\gamma \in \mathcal{V}_{2k}^k \setminus (\mathcal{V}_{2k}^-)^k$ and choosing B to be large enough, we obtain the conclusion. □

7.3. **Decomposition of the transfer operator according to recurrence of orbits.** In order to make use of the compact embedding result in Proposition 6.6, we need to localize our functions to a fixed compact set. This is done with the help of the Margulis function V. In this section, we introduce some notation and prove certain preliminary estimates for that purpose.

Recall the notation in Theorem 4.1. Let $T_0 \ge 1$ be a constant large enough so that $e^{\beta T_0} > 1$. We will enlarge T_0 over the course of the argument to absorb various auxiliary uniform constants. Define V_0 by

$$V_0 = e^{3\beta T_0}. \tag{7.9}$$

Let $\rho_{V_0} \in C_c^\infty(X)$ be a non-negative M-invariant function satisfying $\rho_{V_0} \equiv 1$ on the unit neighborhood of $\{x \in X : V(x) \le V_0\}$ and $\rho_{V_0} \equiv 0$ on $\{V > 2V_0\}$. Moreover, we require that $\rho_{V_0} \le 1$. Note that since T_0 is at least 1, we can choose ρ_{V_0} so that its C^{2k} norm is independent of T_0.

Let $\psi_1 = \rho_{V_0}$ and $\psi_2 = 1 - \psi_1$. Then, we can write

$$\mathcal{L}_{T_0} f = \tilde{\mathcal{L}}_1 f + \tilde{\mathcal{L}}_2 f,$$

where $\tilde{\mathcal{L}}_i f = \mathcal{L}_{T_0}(\psi_i f)$, for $i \in \{1,2\}$. It follows that for all $j \in \mathbb{N}$, we have

$$\mathcal{L}_{jT_0} f = \sum_{\varpi \in \{1,2\}^j} \tilde{\mathcal{L}}_{\varpi_1} \cdots \tilde{\mathcal{L}}_{\varpi_j} f = \sum_{\varpi \in \{1,2\}^j} \mathcal{L}_{jT_0}(\psi_\varpi f), \qquad \psi_\varpi = \prod_{i=1}^j \psi_{\varpi_i} \circ g_{-(j-i)T_0}. \tag{7.10}$$

Note that if $\varpi_i = 1$ for some $1 \le i \le j$, then, by Proposition 4.3, we have

$$\sup_{x \in \mathrm{supp}(\psi_\varpi)} V(x) \le e^{\beta I_\varpi T_0} V_0, \qquad I_\varpi = j - \max\{1 \le i \le j : \varpi_i = 1\}. \tag{7.11}$$

For simplicity, let us write

$$f_\varpi := \psi_\varpi f.$$

The following lemma estimates the effect of multiplying by a fixed smooth function such as ψ_ϖ.

Lemma 7.7. *Let $\psi \in C^{2k}(X)$ be given. Then, if $B \geq 1$ is large enough, depending on k and $\|\psi\|_{C^{2k}}$, we have*
$$\|\psi f\|_{k,B,s} \leq \|f\|_{k,B,s}\,.$$

Proof. Given $0 \leq \ell \leq k$ and $0 \leq s \leq 1$, we wish to estimate integrals of the form
$$\int_{N_1^+} \phi(n) L_{v_1} \cdots L_{v_\ell}(\psi f)(g_s n x)\, d\mu_x^u(n).$$
The term $L_{v_1} \cdots L_{v_\ell}(\psi f)$ can be written as a sum of ℓ terms, each consisting of a product of i derivatives of ψ by $\ell - i$ derivatives of f, for $0 \leq i \leq \ell$. Viewing the product of ϕ by i derivatives of ψ as a $C^{k+\ell-i}$ test function, and using (6.2) to bound the $C^{k+\ell-i}$ norm of such a product, we obtain a bound of the form
$$e_{k,\ell,s}(\psi f) \leq \|\psi\|_{C^{2k}} \sum_{i=0}^{\ell} e_{k,i,s}(f).$$
Hence, given $B \geq 1$, we obtain
$$\|f\|_{k,B,s} = \sum_{\ell=0}^{k} \frac{1}{B^\ell} e_{k,\ell}(\psi f) \leq \|\psi\|_{C^{2k}} \sum_{\ell=0}^{k} \frac{1}{B^\ell} \sum_{i=0}^{\ell} e_{k,i,s}(f) \leq \|\psi\|_{C^{2k}} \sum_{\ell=0}^{k} \frac{k-\ell}{B} \frac{e_{k,\ell,s}(f)}{B^\ell}.$$
Thus, the conclusion follows as soon as B is large enough, depending only on k and $\|\psi\|_{C^{2k}}$. $\square$

The above lemma allows us to estimate the norms of the operators $\tilde{\mathcal{L}}_i$, for $i = 1, 2$ as follows.

Lemma 7.8. *If $B \geq 1$ is large enough, depending on k and $\|\rho_{V_0}\|_{C^{2k}}$ we obtain*
$$\left\|\tilde{\mathcal{L}}_1 f\right\|_{k,B,s} \ll_\beta \|f\|_{k,B,s}\,, \qquad \left\|\tilde{\mathcal{L}}_2 f\right\|_{k,B,s} \ll_\beta e^{-\beta T_0} \|f\|_{k,B,s}\,.$$

Proof. The first inequality follows by Lemmas 7.1 and 7.7, since $\|\psi_i\|_{C^k} \ll 1$ for $i = 1, 2$. The second inequality follows similarly since
$$\psi_2(g_{T_0} n x) \neq 0 \Longrightarrow V(g_{T_0} n x) \geq V_0, \qquad \forall n \in N_1^+.$$
By Proposition 4.3, this in turn implies that, whenever $\psi_2(g_{T_0} n x) \neq 0$ for some $n \in N_1^+$, we have that $V(x) \gg e^{\beta T_0}$, by choice of V_0. $\square$

7.4. **Proof of Theorems 6.1 and 6.4.** Theorem 6.1 follows at once from 6.4. Theorem 6.4 will follow upon verifying the hypotheses of Theorem 6.5. The boundedness assertion follows by Corollary 7.3. It remains to estimate the essential spectral radius of the resolvent $R(z)$.

Write $z = a + ib \in \mathbb{C}$. Fix some parameter $0 < \theta < 1$ and define
$$\sigma := \min\{k, \beta\theta\}\,.$$
Let $0 < \epsilon < \sigma/5$ be given. We show that for a suitable choice of r and B, the following Lasota-Yorke inequality holds:
$$\left\|R(z)^{r+1} f\right\|_{k,B} \leq \frac{\|f\|_{k,B}}{(a + \sigma - 3\epsilon)^{r+1}} + C'_{k,r,z,\beta} \|\Psi_r f\|'_k\,, \tag{7.12}$$
where $C'_{k,r,z,\beta} \geq 1$ is a constant depending on k, r and z, while Ψ_r is a compactly supported smooth function on X, and whose support depends on r.

First, we show how (7.12) implies the result. Note that, since the norms $\|\cdot\|_k$ and $\|\cdot\|_{k,B}$ are equivalent, the Lasota-Yorke inequality (7.12) holds with $\|\cdot\|_k$ in place of $\|\cdot\|_{k,B}$ (with a different constant $C'_{k,r,z,\beta}$). Hennion's Theorem, Theorem 6.5, applied with the strong norm $\|\cdot\|_k$ and the weak semi-norm $\|\Psi_r \bullet\|'_k$, implies that the essential spectral radius ρ_{ess} of $R(z)$ is at most $1/(a + \sigma - 3\epsilon)$. Note that the compact embedding requirement follows by Proposition 6.6. Since $\epsilon > 0$ was arbitrary, this shows that $\rho_{ess}(R(z)) \leq 1/(a+\sigma)$. Finally, as $0<\theta<1$ was arbitrary, we obtain that

$$\rho_{ess}(R(z)) \leq \frac{1}{\mathrm{Re}(z)+\sigma_0},$$

completing the proof.

To show (7.12), let an integer $r \geq 0$ be given and $J_r \in \mathbb{N}$ to be determined. Using (7.10) and a change of variable, we obtain

$$\begin{aligned} R(z)^{r+1} f &= \int_0^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \\ &= \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt + \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt + \sum_{j=1}^{J_r} \int_{jT_0}^{(j+1)T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt. \end{aligned}$$

First, by Lemma 7.6, if B is large enough, depending on r,k and z, we obtain

$$\left\| R(z)^{r+1}(z) f \right\|_{k,B,\omega} \leq \frac{1}{(a+k+1)^{r+1}} \left\| f \right\|_{k,B}.$$

It remains to estimate $\left\| R(z)^{r+1} f \right\|_{k,B,s}$. Note that $\int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq T_0^{r+1}/r!$. Hence, taking r large enough, depending on k, a, β and T_0, and using Lemma 7.1, we obtain for any $B \geq 1$,

$$\left\| \int_0^{T_0} \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \right\|_{k,B,s} \ll_\beta \left\| f \right\|_{k,B} \int_0^{T_0} \frac{t^r e^{-at}}{r!} dt \leq \frac{1}{(a+k+1)^{r+1}} \left\| f \right\|_{k,B}.$$

Similarly, taking J_r to be large enough, depending on k, a, β, and r, we obtain for any $B \geq 1$,

$$\left\| \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-zt}}{r!} \mathcal{L}_t f \; dt \right\|_{k,B,s} \ll_\beta \left\| f \right\|_{k,B} \int_{(J_r+1)T_0}^\infty \frac{t^r e^{-at}}{r!} \; dt \leq \frac{1}{(a+k+1)^{r+1}} \left\| f \right\|_{k,B}.$$

To estimate the remaining term in $R(z)^{r+1} f$, let $1 \leq j \leq J_r$ and $\varpi = (\varpi_i)_i \in \{1,2\}^j$ be given. Let θ_ϖ denote the number of indices i such that $\varpi_i = 2$. Then, taking B large enough, depending on k and $C^{2k}(\psi_\varpi)$, it follows from Lemma 7.1 and induction on Lemma 7.8 that

$$\left\| \mathcal{L}_{t+jT_0}(\psi_\varpi f) \right\|_{k,B,s} \leq C_0 \left\| \mathcal{L}_{jT_0}(\psi_\varpi f) \right\|_{k,B,s} \leq C_0^{j+1} e^{-\beta \theta_\varpi j T_0} \left\| f \right\|_{k,B,s},$$

where we take $C_0 \geq 1$ to be larger than the implied uniform constant in Lemma 7.8 and the implied constant in Lemma 7.1. Suppose $\theta_\varpi \geq \theta$. Then, by taking T_0 to be large enough, we obtain

$$\left\| \mathcal{L}_{t+jT_0}(\psi_\varpi f) \right\|_{k,B,s} \leq e^{-(\beta\theta-\epsilon) j T_0} \left\| f \right\|_{k,B,s}.$$

On the other hand, if $\theta_\varpi < \theta$, we apply Lemma 7.4 to obtain for all $0 \leq \ell < k$,

$$e_{k,\ell}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_{k,\beta} e^{-(t+jT_0)k} e_{k,\ell}(\psi_\varpi f) + e'_{k,\ell}(\psi_\varpi f),$$

where we may assume that T_0 is at least 2 so that the same holds for $t+jT_0$, thus verifying the hypothesis of the lemma. Moreover, we note that (7.11), implies that ψ_ϖ is supported inside a sublevel set of V, depending only on θ and J_r. Let Ψ_r denote a smooth bump function on X which is identically 1 on the union of the (finitely many) supports of ψ_ϖ as ϖ ranges over tuples in $\{1,2\}^j$ with $\theta_\varpi < \theta$ and for $1 \leq j \leq J_r$. Note that for any such ϖ, arguing as in the proof of Lemma 7.7, we obtain

$$e'_{k,\ell}(\psi_\varpi f) = e'_{k,\ell}(\psi_\varpi \Psi_r f) \ll_k \left\| \Psi_r f \right\|'_k.$$

For the coefficient $e_{k,k}$, Lemma 7.5 shows that for any $\gamma \in (\mathcal{V}_{2k}^-)^k$, we have

$$e_{k,k,\gamma}(\mathcal{L}_{t+jT_0}(\psi_\varpi f)) \ll_\beta e^{-(t+jT_0)k} e_{k,k}(\psi_\varpi f).$$

Combining these estimates, and using Lemma 7.7, we obtain

$$\begin{aligned}\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s} &\leq C_0 e^{-(\sigma-\epsilon)jT_0}\|\psi_\varpi f\|_{k,B,s} + C_{k,r,z,\beta}\|\Psi_r f\|'_k \\ &\leq e^{-(\sigma-2\epsilon)jT_0}\|\psi_\varpi f\|_{k,B,s} + C_{k,r,z,\beta}\|\Psi_r f\|'_k,\end{aligned}$$

where we enlarge the constant C_0 as necessary to subsume the implied constants and the constant $C_{k,r,z,\beta} \geq 1$ is large enough, depending on B, so the above inequality holds. The inequality on the second line follows by taking T_0 large enough depending on C_0 and ϵ.

Putting the above estimates together, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{J_r}\int_{jT_0}^{(j+1)T_0}\frac{t^r e^{-zt}}{r!}\mathcal{L}_t f\,dt\right\|_{k,B,s} &\leq \sum_{j=1}^{J_r} e^{-ajT_0}\sum_{\varpi\in\{1,2\}^j}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}\|\mathcal{L}_{t+jT_0}(\psi_\varpi f)\|_{k,B,s}\,dt \\ &\leq \|f\|_{k,B,s}\sum_{j=1}^{J_r} e^{-(a+\sigma-2\epsilon)jT_0}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}dt \\ &+ C_{k,r,z,\beta}\|\Psi_r f\|'_k\sum_{j=1}^{J_r}2^j e^{-ajT_0}\int_0^{T_0}\frac{(t+jT_0)^r e^{-at}}{r!}\,dt \\ &\leq e^{(\sigma-2\epsilon)T_0}\|f\|_{k,B,s}\int_1^{J_r}\frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!}\,dt + C'_{k,r,z,\beta}\|\Psi_r f\|'_k,\end{aligned}$$

where we take $C'_{k,r,z,\beta} \geq 1$ to be a constant large enough so that the last inequality holds.

Next, we note that

$$\int_1^{J_r}\frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!}\,dt \leq \int_0^{\infty}\frac{t^r e^{-(a+\sigma-2\epsilon)t}}{r!}\,dt = \frac{1}{(a+\sigma-2\epsilon)^{r+1}}.$$

Thus, taking r to be large enough depending on a and T_0, and combining the estimates on $\left\|R(z)^{r+1}f\right\|_{k,B,\omega}$ and $\left\|R(z)^{r+1}f\right\|_{k,B,s}$, we obtain (7.12) as desired.

7.5. **Proof of Theorem C.** Recall the notation in the statement of the theorem. We note that switching the order of integration in the definition of the Laplace transform shows that

$$\hat{\rho}_{f,g}(z) = \int R(z)(f)g\,d\mathrm{m}^{\mathrm{BMS}}, \qquad \mathrm{Re}(z) > 0.$$

In particular, the poles of $\hat{\rho}_{f,g}$ are contained in those of the resolvent $R(z)$.

On the other hand, Corollary 7.2 implies that the infinitesimal generator $\mathfrak{X}$ of the semigroup $\mathcal{L}_t$ is well-defined as a closed operator on $\mathcal{B}_k$ with dense domain. Moreover, $R(z)$ coincides with the resolvent operator $(\mathfrak{X}-z\mathrm{Id})^{-1}$ associated to $\mathfrak{X}$, whenever z belongs to the resolvent set (complement of the spectrum) of $\mathfrak{X}$.

We further note that the spectra of $\mathfrak{X}$ and $R(z)$ are related by the formula $\sigma(\mathfrak{X}) = z - 1/\sigma(R(z))$. In particular, by Theorem 6.4, in the half plane $\mathrm{Re}(z) > -\sigma_0$, the poles of $R(z)$ coincide with the eigenvalues of $\mathfrak{X}$. In view of this relationship between the spectra, the fact that the imaginary axis does not contain any poles for the resolvent, apart from 0, follows from the mixing property of the geodesic flow with respect to $\mathrm{m}^{\mathrm{BMS}}$. The latter property follows from [Bab02]. We refer the reader to the proof of [BDL18, Corollary 5.4] for a deduction of this assertion[8].

Finally, we note that in the case Γ has cusps, β was an arbitrary constant in $(0, \Delta/2)$, so that we may take σ_0 in the conclusion of Theorem 6.4 to be the minimum of k and $\Delta/2$ in this case. This completes the proof of Theorem C.

[8]The analog of [BDL18, Lemma 2.11] needed in the proof of the quoted result is furnished in Lemma 8.3 below.

8. Spectral gap for resolvents with large imaginary parts

In this Section, we complete the proof of Theorems A and B. The estimates in Sections 6 and 7 allow us to show that there is a half plane $\{\mathrm{Re}(z) > -\eta\}$, for a suitable $\eta > 0$, containing at most countably many isolated eigenvalues for the generator of the geodesic flow. To show exponential mixing, it is important to rule out the accumulation of such eigenvalues on the imaginary axis as their imaginary part tends to ∞.

Remark 8.1. Throughout the rest of this section, if X has cusps, we require the Margulis function $V = V_\beta$ in the definition of all the norms we use to have

$$\beta = \Delta/4 \tag{8.1}$$

in the notation of Theorem 4.1. In particular, the contraction estimate in Theorem 4.1 holds with V^p in place of V for all $1 \le p \le 2$. Recall that the constant Δ is given in (3.1).

Similarly to (7.8), we define for $B > 0$ a similar norm to those defined in (6.6) as follows:

$$\|f\|_{1,B} := e_{k,0}(f) + \frac{e_{1,1}(f)}{B}. \tag{8.2}$$

The following result is one of the main technical contributions of this article.

Theorem 8.2. *There exist constants $b_\star \ge 1$, $k \in \mathbb{N}$, $\varrho \ge 0$, and $\varkappa, a_\star, \sigma_\star > 0$, depending only on the critical exponent δ_Γ and the ranks of the cusps of Γ (if any), such that the following holds. For all $z = a_\star + ib \in \mathbb{C}$ with $|b| \ge b_\star$ and for $m = \lceil \log |b| \rceil$, we have that*

$$e_{k,0}(R(z)^m f) \le C_\Gamma \left(\frac{e_{k,0}(f)}{(a_\star + |b|^{-\varrho})^m} + \frac{\|f\|_{1,B}}{(a_\star + \sigma_\star)^m} \right),$$

where $C_\Gamma \ge 1$ is a constant depending only on the fundamental group Γ and $B = |b|^{1+\varkappa}$.

If we assume in addition that

$$\begin{cases} \delta_\Gamma > 2D/3, & \mathfrak{K} = \mathbb{R}, \\ \delta_\Gamma > 5D/6, & \mathfrak{K} = \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O}, \end{cases} \tag{8.3}$$

then have that

$$e_{k,0}(R(z)^m f) \le C_\Gamma \frac{\|f\|_{1,B}}{(a_\star + \sigma_\star)^m}.$$

8.1. **Proof of Theorems A and B.** We show here the deduction of the exponential mixing assertion from Theorem 8.2 in the case $\varrho = 0$ using the results in [But16a]. The deduction of the rapid mixing assertion is very similar and so it is omitted.

The link between the norms we introduced and decay of correlations is furnished in the following lemma.

Lemma 8.3. *For all $f \in C_c^2(X)^M$ and $\varphi \in C_c^k(X)^M$, we have that*

$$\int f \cdot \varphi \, d\mathrm{m}^{\mathrm{BMS}} \ll \|\varphi\|_{C^k} \, e_{k,0}(f),$$

where the implied constant depends on the injectivity radius of the support of φ.

Proof. Using a partition of unity, we may assume φ is supported inside a flow box. The implied constant then depends on the number of elements of the partition of unity needed to cover the support of φ. Inside each such flow box, the measure $\mathrm{m}^{\mathrm{BMS}}$ admits a local product structure of the conditional measures μ_x^u with a suitable measure on the transversal to the strong unstable foliation. Thus, the lemma follows by definition of the norm by viewing the restriction of φ to each local unstable leaf as a test function. ∎

In particular, this lemma implies that decay of correlations (for mean 0 functions) would follow at once if we verify that $e_{k,0}(\mathcal{L}_t f)$ decays in t with a suitable rate. It is shown in [But16a][9] that such decay follows from suitable spectral bounds on the resolvent. We list here the results that verify the hypotheses of [But16a] and refer the reader to [BDL18, Section 9] where such application of Butterley's result is carried out in detail in a similar setting.

We take $e'_{k,0}$ (defined above (6.5)) to be the weak norm $\|\cdot\|_{\mathcal{A}}$ in the notation of [But16a], while we take the following as the strong norm:

$$\|f\|_{\mathcal{B}} := e_{k,0}(f) + e_{1,1}(f).$$

The following corollary verifies [But16a, Assumption 3A].

Corollary 8.4. *Let the notation be as in Theorem 8.2 and assume that (8.3) holds. Then, there exist constants $c_\star, \lambda_\star > 0$, depending only on the critical exponent δ_Γ and the ranks of the cusps of Γ (if any), such that the following holds. For all $z = a_\star + ib \in \mathbb{C}$ and for $m = \lceil c_\star \log |b| \rceil$, we have the following bound on the operator norm of $R(z)$:*

$$\|R(z)^m\|_{\mathcal{B}} \leq \frac{1}{(a+\lambda_\star)^m},$$

whenever $|b| \geq b_\Gamma$, where $b_\Gamma \geq 1$ is a constant depending on Γ.

Proof. First, we verify the corollary for the norm $\|\cdot\|_{1,B}$ in (8.2), with $B = |b|^{1+\varkappa}$. Let $e_{1,1,b}$ be the scaled seminorm $e_{1,1}/|b|^{1+\varkappa}$. Note that the arguments of Lemmas 7.5 and 7.6 imply that for $z = a_\star + ib$ with $|b| \geq a_\star$, we have

$$e_{1,1,b}(R(z)^m f) \leq C_\Gamma \frac{\|f\|_{1,B}\,(a_\star + |z|)}{a_\star^m b^{1+\varkappa}} \leq \frac{3C_\Gamma \|f\|_{1,B}}{a_\star^m |b|^{\varkappa}},$$

for some constant $C_\Gamma \geq 1$ depending only on Γ, where we used the fact that $a_\star + |z| \leq 3|b|$.

Recall that $m = \lceil \log |b| \rceil$. Hence, in view of the inequality $\log(1+x) \leq x$ for $x \geq 0$, we see that $a_\star^m |b|^{\varkappa}$ is at least $(a_\star + \sigma_0)^m$, for some $\sigma_0 > 0$ depending on $a_\star$ and $\varkappa$. Hence, we obtain

$$e_{1,1,b}(R(z)^m f) \leq \frac{3C_\Gamma \|f\|_{1,B}}{(a_\star + \sigma_0)^m},$$

This estimate, combined with the estimate in Theorem 8.2 implies that whenever $|b| \geq b_\star$,

$$\|R(z)^m\|_{1,B} \ll_\Gamma (a_\star + \sigma_1)^{-m},$$

where $\sigma_1 > 0$ is the minimum of $\sigma_\star$ and σ_0. In particular, if $|b|$ is large enough, depending on Γ, we can absorb the implied constant in the estimate above to obtain

$$\|R(z)^m\|_{1,B} \leq (a_\star + \sigma_1/2)^{-m}.$$

Set $\sigma_2 = \sigma_1/2$. Let $p \in \mathbb{N}$ be a large integer to be chosen. To obtain the claimed estimate for the norm $\|\cdot\|_{\mathcal{B}}$, note that since $\|\cdot\|_{1,B} \leq \|\cdot\|_{\mathcal{B}} \leq |b|^{1+\varkappa} \|\cdot\|_{1,B}$, iterating the above estimate yields

$$\left\|R(z)^{2pm} f\right\|_{\mathcal{B}} \leq \frac{B \left\|R(z)^{pm} f\right\|_{1,B}}{(a_\star + \sigma_2)^{pm}} \leq \frac{B \|f\|_{\mathcal{B}}}{(a_\star + \sigma_2)^{2pm}}.$$

Since $m = \lceil \log |b| \rceil$, choosing p large enough, depending only on $a_\star$ and σ_2, we can ensure that $B/(a_\star + \sigma_2)^{pm} \leq 1/a_\star^{pm}$. In particular, taking $\lambda_\star$ to be the positive root of the quadratic polynomial $x \mapsto x^2 + 2a_\star x - a_\star \sigma_2$, we obtain the desired estimate with $c_\star = 4p$. $\square$

Remark 8.5. In the rapid mixing case, to verify [But16a, Assumption 3B], one uses the identity $R(z+w) = R(z)(\mathbf{id} - wR(z))^{-1}$ for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $w \in \mathbb{C}$ with $|w| > 1/a$ to estimate the norm of the resolvents to the left of the imaginary axis.

[9] See also the erratum [But16b].

Assumption 2 of [But16a] is verified in Theorem 6.4. The strong continuity of the semigroup $\mathcal{L}_t$ is given in Corollary 7.2. Finally, the following lemma verifies the weak Lipschitz property in [But16a, Assumption 1], completing the proof of Theorems A and B.

Lemma 8.6. *For all $t \geq 0$,*

$$e_{k,0}(\mathcal{L}_t f - f) \ll t e_{1,1}(f) \leq t \left\| f \right\|_{\mathcal{B}}.$$

Proof. Let $x \in N_1^- \Omega$, $t \geq 0$ and $s \in [0,1]$. Then, given any test function ϕ, we have that

$$\int_{N_1^+} \phi(n)(f(g_{t+s}nx) - f(g_s nx))\, d\mu_x^u = \int_0^t \int_{N_1^+} \phi(n) L_\omega f(g_{s+r}nx)\, d\mu_x^u dr,$$

where L_ω denotes the derivative with respect to the vector field generating the geodesic flow. Hence, Lemma 7.1 implies that

$$\left| \int_{N_1^+} \phi(n)(f(g_{t+s}nx) - f(g_s nx))\, d\mu_x^u \right| \leq V(x)\mu_x^u(N_1^+) \int_0^t e_{1,1}(\mathcal{L}_r f)\, dr \ll tV(x)\mu_x^u(N_1^+) e_{1,1}(f).$$

This completes the proof since x and ϕ are abitrary. $\square$

8.2. **Proof of Theorem 8.2.** The remainder of this section is dedicated to the proof of Theorem 8.2. Let $a \in (0,2]$ to be determined (cf. (8.53)). We assume that $z = a + ib$ with $b > 0$, the other case being identical. For the convenience of the reader, an index of notation for this section is provided at the end of the article.

Time partition. Let $p : \mathbb{R} \to [0,1]$ be a smooth bump function supported in $(-1,1)$ with the property that

$$\sum_{j \in \mathbb{Z}} p(t-j) = 1, \qquad \forall t \in \mathbb{R}. \tag{8.4}$$

Let $m \in \mathbb{N}$ and $T_0 > 0$ be parameters to be specified later. Changing variables, we obtain

$$\begin{aligned} R(z)^m &= \int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!} \mathcal{L}_t\, dt \\ &= \int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!} p(t/T_0) \mathcal{L}_t R(z)^m\, dt + \sum_{j=0}^\infty \frac{((j+2)T_0)^{m-1} e^{-zjT_0}}{(m-1)!} \int_{\mathbb{R}} p_j(t) e^{-zt} \mathcal{L}_{t+jT_0}\, dt, \end{aligned} \tag{8.5}$$

where we define p_j as follows:

$$p_j(t) := \left(\frac{jT_0 + t}{(j+2)T_0} \right)^{m-1} p\left(\frac{t - T_0}{T_0} \right). \tag{8.6}$$

Note that p_j is supported in the interval $(0, 2T_0)$ for all $j \geq 0$.

We will estimate the contribution of each term in the sum over j in (8.5) individually. We will restrict our attention to small values of j, compared to b. For this purpose, let $\eta > 0$ be a small parameter to be determined. Then, similarly to (8.10), we have

$$\sum_{j : jT_0 > \eta m} \frac{((j+2)T_0)^{m-1} e^{-ajT_0}}{(m-1)!} \int_{\mathbb{R}} p_j(t) e^{-at} e_{k,0}(\mathcal{L}_{t+jT_0} f)\, dt \ll e_{k,0}(f) \int_{\eta m}^\infty \frac{t^{m-1}e^{-at}}{(m-1)!}\, dt \tag{8.7}$$

The following lemma estimates the tail of the resolvent integral.

Lemma 8.7. *Suppose that $a\eta > 1$. Then, there exists $\theta \in (0,1)$, such that*

$$\int_{\eta m}^\infty \frac{t^{m-1}e^{-at}}{(m-1)!}\, dt \ll_{a,\eta} \left(\frac{\theta}{a} \right)^m.$$

Alternatively, if $a\eta < 1$, then there exists $\xi \in (0,1)$ such that if m is large enough, we have

$$\int_{\eta m}^{\infty} \frac{t^{m-1}e^{-at}}{(m-1)!}\,dt \ll \frac{1}{(a+\xi^m)^m}.$$

Proof. Integration by parts and induction on m yield

$$\int_{\eta m}^{\infty} \frac{t^{m-1}e^{-at}}{(m-1)!}\,dt = \frac{e^{-a\eta m}}{a^m}\sum_{k=0}^{m-1}\frac{(a\eta m)^k}{k!} = \frac{e^{-a\eta m}(a\eta m)^m}{a^m m!}\sum_{k=0}^{m-1}\frac{(m)\cdots(k+1)}{(a\eta m)^{m-k}}.$$

Note that the k^{th} term of the latter sum is at most $(a\eta)^{-m+k}$. Moreover, from Stirling's formula, we have that $m! \gg m^{m+1/2}e^{-m}$. Hence, when $a\eta > 1$, we get

$$\int_{\eta m}^{\infty} \frac{t^{m-1}e^{-at}}{(m-1)!}\,dt \ll \frac{e^{(1-a\eta)m}(a\eta)^m}{a^m}.$$

Taking $\theta = a\eta e^{1-a\eta}$ and noting that xe^{1-x} is strictly less than 1 for all $x \geq 0$ with $x \neq 1$, concludes the proof of the first claim.

For the second estimate, let $M = a\eta m$. Estimating the tail of the power series of e^M from below by its first term, we get

$$\int_{\eta m}^{\infty} \frac{t^{m-1}e^{-at}}{(m-1)!}\,dt = \frac{1-e^{-M}\sum_{k=m}^{\infty}\frac{M^k}{k!}}{a^m} \leq \frac{1-(e^{-M}M^m/m!)}{a^m}.$$

Using Stirling's approximation, we see that $(e^{-M}M^m/m!) \gg \theta^m m^{-1/2}$, for $\theta = e^{(1-a\eta)}a\eta$. Thus, Bernoulli's inequality yields

$$\int_{\eta m}^{\infty} \frac{t^{m-1}e^{-at}}{(m-1)!}\,dt \ll \frac{1-m^{-1/2}\theta^m}{a^m} \leq \left(\frac{1-m^{-3/2}\theta^m}{a}\right)^m,$$

Finally, we note that since $\theta < 1$, when m is large enough, $(1-m^{-3/2}\theta^m)/a$ is at most $1/(a + a(\theta/2)^m)$. Thus, the estimate follows with $\xi = \theta/4$ for all m large enough. $\square$

In view of this lemma and (8.7), in what follows, we restrict to the case

$$jT_0 \leq \eta m. \tag{8.8}$$

Let $J_0 \in \mathbb{N}$ be a parameter to be specified later. By the triangle inequality for the seminorm $e_{k,0}$ and Lemma 7.1, we have

$$\begin{aligned} &e_{k,0}\left(\sum_{j=0}^{J_0}\frac{((j+2)T_0)^{m-1}e^{-zjT_0}}{(m-1)!}\int_{\mathbb{R}} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0}f\,dt\right) \\ &\leq \int_0^{(J_0+2)T_0}\frac{t^{m-1}e^{-at}}{(m-1)!}e_{k,0}(\mathcal{L}_t f)dt \ll \frac{((J_0+2)T_0)^m e_{k,0}(f)}{(m-1)!}. \end{aligned}$$

We will choose

$$m = \lceil \log b \rceil. \tag{8.9}$$

Hence, since $a \leq 2$ by assumption, when b is large enough[10], we get

$$e_{k,0}\left(\sum_{j=0}^{J_0}\frac{((j+2)T_0)^{m-1}e^{-zjT_0}}{(m-1)!}\int_{\mathbb{R}} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0}f\,dt\right) \ll \frac{e_{k,0}(f)}{(a+1)^m}. \tag{8.10}$$

[10]Over the course of the proof, b will be assumed large depending on all the parameters we choose in the argument.

A similar argument also shows that

$$e_{k,0}\left(\int_0^\infty \frac{t^{m-1}e^{-zt}}{(m-1)!}p(t/T_0)\mathcal{L}_t f\ dt\right) \ll \frac{e_{k,0}(f)}{(a+1)^m},$$

where we used the fact that $p(t/T_0)$ is supported in $(-T_0,T_0)$. Thus, we may assume for the remainder of the section that

$$j > J_0. \tag{8.11}$$

Let $0<\epsilon\ll 1$ be a small parameter to be chosen later. The advantage of taking J_0 large is that it allows us to give a reasonable estimate on the sum of the errors of each term in (8.5). Indeed, taking J_0 large enough so that $2/J_0\leq\epsilon$, in view of (7.5), we have that

$$\sum_{j=J_0+1}^{\infty}\frac{((j+2)T_0)^{m-1}e^{-ajT_0}}{(m-1)!} \leq e^{2aT_0}\left(1+\frac{2}{J_0}\right)^m\int_0^\infty \frac{t^{m-1}e^{-at}}{(m-1)!}\ dt = e^{2aT_0}\left(\frac{1+\epsilon}{a}\right)^m. \tag{8.12}$$

We will take J_0 large enough (independently of b) so that the loss of a factor of $1+\epsilon$ does not exceed the gains we make over the course of the proof.

Contribution of points in the cusp. Let $x\in N_1^-\Omega$ be arbitrary. Then, Lemma 7.1 implies that

$$e_{k,0}\left(\int_{\mathbb{R}} p_j(t)e^{-zt}\mathcal{L}_{t+jT_0}f\ dt;x\right)\leq\int_{\mathbb{R}}p_j(t)e^{-at}e_{k,0}\left(\mathcal{L}_{t+jT_0}f;x\right)\ dt\ll T_0e^{-(a+\beta\alpha)jT_0}e_{k,0}(f),$$

provided $V(x)>e^{\beta\alpha jT_0}$. In light of (8.12), summing the above errors over j, we obtain an error term of the form

$$T_0e^{2aT_0}e_{k,0}(f)\left(\frac{1+\epsilon}{a+\beta\alpha}\right)^m \leq e_{k,0}(f)\left(\frac{1+2\epsilon}{a+\beta\alpha}\right)^m\leq\frac{e_{k,0}(f)}{(a+\beta\alpha-\epsilon)^m}, \tag{8.13}$$

where the first inequality can be ensured to hold by taking b large enough in view of (8.9) and the second inequality holds whenever ϵ is small enough.

Thus, we may assume for the remainder of the section that

$$V(x)\leq e^{\beta\alpha jT_0}. \tag{8.14}$$

Fix some suitable test function ϕ for $e_{k,0}$. In particular, ϕ has $C^1(N^+)$ norm at most 1. The integrals we wish to estimate take the form

$$\int_{N_1^+}\phi(n)\int_{\mathbb{R}}p_j(t)e^{-zt}\mathcal{L}_{t+jT_0}(f)(g_snx)\ dtd\mu_x^u(n),$$

for all $s\in[0,1]$. We again only provide the estimate in the case $s=0$ to simplify notation, the general case being essentially identical.

Recall that p_j is supported in the interval $(0,2T_0)$. In particular, the extra t in $\mathcal{L}_{t+jT_0}$ could be rather large, which will ruin certain trivial estimates later. To remedy this, recall the partition of unity of $\mathbb{R}$ given in (8.4) and set

$$p_{j,w}(t):=p_j(t+w)p(t),\qquad\forall w\in\mathbb{Z}. \tag{8.15}$$

Using a change of variable, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}e^{-zt}\int_{N_1^+}p_j(t)\phi(n)f(g_{t+jT_0}nx)\ d\mu_x^u(n)dt\\ &\qquad=\sum_{w\in\mathbb{Z}}e^{-zw}\int_{\mathbb{R}}e^{-zt}\int_{N_1^+}p_{j,w}(t)\phi(n)f(g_{t+w+jT_0}nx)\ d\mu_x^u(n)dt.\end{aligned} \tag{8.16}$$

Note the above sum is supported on

$$0\leq w\ll T_0, \tag{8.17}$$

and the support of each integral in t is now $(-1, 1)$. For the remainder of the section, we fix some $w \in \mathbb{Z}$ in that support.

To simplify notation, we set

$$g_j^w := g_{w+jT_0}. \tag{8.18}$$

Partitions of unity and flow boxes. Let us define

$$K_j := \left\{ y \in X : V(y) \le e^{(2\beta\alpha j + 3\beta)T_0} \right\}, \qquad \iota_j := \min\left\{1, \mathrm{inj}(K_j)\right\}. \tag{8.19}$$

We let $\mathcal{P}_j$ denote a partition of unity of the unit neighborhood of K_j so that each $\rho \in \mathcal{P}_j$ is M-invariant and supported inside a flow box B_ρ of radius $\iota_j/10$. With the aid of the Vitali covering lemma, we can arrange for the collection $\{B_\rho\}$ to have a uniformly bounded multiplicity, depending only on the dimension of G. We can choose such a partition of unity so that for all $\rho \in \mathcal{P}_j$,

$$\|\rho\|_{C^k} \ll_k \iota_j^{-k}. \tag{8.20}$$

We also need the following subcollection of $\mathcal{P}_j$:

$$\mathcal{P}_j^0 := \left\{ \rho \in \mathcal{P}_j : B_\rho \cap N_{1/2}^- \Omega \neq \emptyset \right\}. \tag{8.21}$$

We shall need an estimate on the cardinality of $\mathcal{P}_j^0$. To this end, note that the cardinality of the collection $\mathcal{P}_j^0$ is controlled in terms of the injectivity radius ι_j in (8.19). Indeed, since Γ is geometrically finite, the unit neighborhood of Ω has finite volume. Moreover, the flow boxes B_ρ with $\rho \in \mathcal{P}^0$ are all contained in such a unit neighborhood and have uniformly bounded multiplicity; cf. (8.21). Finally, each B_ρ has radius at least ι_j for all $\rho \in \mathcal{P}_j$. Thus, we have that

$$\#\mathcal{P}_j^0 \ll_\Gamma \iota_j^{-(2D+1)}, \tag{8.22}$$

where D is the dimension of N^+. Note that the dimension of X is $2D + 1 + \dim(M)$, however the bound above involves $2D + 1$ only since each flow box is M-invariant.

Localizing away from the cusp. We begin by restricting the support of the integral away from the cusp. Define the following smoothed cusp indicator function $\zeta_j : X \to [0, 1]$:

$$\zeta_j(y) := 1 - \sum_{\rho \in \mathcal{P}_j} \rho(y).$$

We also fix a parameter $\gamma \in (0, 1)$ as follows:

$$\gamma = \begin{cases} 1/3, & \mathfrak{K} = \mathbb{R}, \\ 1/6, & \mathfrak{K} = \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O}, \end{cases} \tag{8.23}$$

where we recall that our underlying manifold is a quotient of $\mathbb{H}_{\mathfrak{K}}^d$. To simplify notation, we set

$$g^\gamma := g_{\gamma(w+jT_0)}. \tag{8.24}$$

It will be convenient to take T_0 large enough depending on γ so that

$$\min\left\{(1-\gamma)(w + jT_0), \gamma(w + jT_0)\right\} \ge 2. \tag{8.25}$$

First, by taking

$$\alpha \le 1 - \gamma,$$

we note that the bounded multiplicity property of $\mathcal{P}_j$ and (8.20) imply that

$$\left\|\zeta_j \circ g^{\gamma-1}\right\|_{C^k(N^+)} \ll_k 1.$$

Moreover, by definition, ζ_j is supported outside of the sublevel set K_j in (8.19). Hence, changing variables and repeating the argument in the proof of Lemma 7.1 by picking a partition of unity of N_1^+, with supports contained in N_2^+, and suitable points x_i, we obtain

$$\begin{aligned}\int \phi(n)\zeta_j(g^\gamma nx)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u &= e^{-\delta(w+jT_0)}\sum_i \int_{N_1^+} \phi_i(n)\zeta_j(g^{\gamma-1}nx_i)\mathcal{L}_t f(nx_i)\, d\mu_{x_i}^u \\ &\ll_k e_{k,0}(f)e^{-\delta(w+jT_0)} \sum_{i:\zeta_j(g^{\gamma-1}x_i)\neq 0} V(x_i)\mu_{x_i}^u(N_1^+) \\ &\ll e_{k,0}(f)e^{-\delta(w+jT_0)}\sum_i \int_{N_1^+} \mathbb{1}_{K_j^c}(g^{\gamma-1}nx_i)V(nx_i)\, d\mu_{x_i}^u \\ &\ll e_{k,0}(f)\int_{N_2^+} \mathbb{1}_{K_j^c}(g^\gamma nx)V(g_j^w nx)\, d\mu_x^u,\end{aligned}$$

where we regarded $\phi_i(n)\xi_j(g^{\gamma-1}nx_i)$ as test functions. Thus, the Cauchy-Schwarz inequality yields

$$\left|\int_{N_2^+} \mathbb{1}_{K_j^c}(g^\gamma nx)\mathcal{L}_t V(g_j^w nx)\, d\mu_x^u\right|^2 \leq \mu_x^u\left(n\in N_2^+ : V(g^\gamma nx) > e^{2\beta\alpha jT_0}\right)\times \int_{N_2^+}\mathcal{L}_t V^2(g_j^w nx)\, d\mu_x^u.$$

Recall that we are assuming that V^2 satisfies the Margulis inequality in Theorem 4.1; cf. Remark 8.1. Hence, by Theorem 4.1 and Chebychev's inequality, we obtain

$$\left|\int_{N_1^+}\phi(n)\zeta_j(g^\gamma nx)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u\right| \ll_k e_{k,0}(f)\mu_x^u(N_2^+)V^{3/2}(x)e^{-\beta\alpha jT_0}.$$

Using the bound on $V(x)$ in (8.14) and the doubling estimate in Proposition 3.1, we thus obtain

$$\begin{aligned}&\int_{N_1^+}\phi(n)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u(n)\\ &= \sum_{\rho\in\mathcal{P}_j}\int_{N_1^+}\phi(n)\rho(g^\gamma nx)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u + O\left(e_{k,0}(f)\mu_x^u(N_1^+)V(x)e^{-\beta\alpha jT_0/2}\right).\end{aligned}$$

Recall the sub-partition of unity $\mathcal{P}_j^0$ in (8.21). Since $x\in N_1^-\Omega$, it follows that $g^\gamma nx$ belongs to $N_{1/2}^-\Omega$ for all $n\in N_1^+$ in the support of μ_x^u (i.e. for all $n\in N_1^+$ with $(nx)^+$ in the limit set Λ_Γ); cf. Remark 2.1. Hence, the only non-zero terms in the above sum correspond to those ρ in $\mathcal{P}_j^0$. Hence, we see that

$$\begin{aligned}&\int_{N_1^+}\phi(n)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u(n)\\ &= \sum_{\rho\in\mathcal{P}_j^0}\int_{N_1^+}\phi(n)\rho(g^\gamma nx)\mathcal{L}_t f(g_j^w nx)\, d\mu_x^u + O\left(e_{k,0}(f)\mu_x^u(N_1^+)V(x)e^{-\beta\alpha jT_0/2}\right).\end{aligned} \tag{8.26}$$

Finally, using (8.12) and taking b large enough and ϵ small enough, we see that the sum of the above error terms over j gives an error term of the form

$$O\left(\frac{e_{k,0}(f)\mu_x^u(N_1^+)V(x)}{(a+\beta\alpha/2-\epsilon)^m}\right). \tag{8.27}$$

Pre-localization. It will be convenient to replace the function f with one supported near Ω and away from the cusp. To simplify notation, we set

$$s := (1-\gamma)(w+jT_0). \tag{8.28}$$

We also define

$$F := \sum_{\rho_0 \in \mathcal{P}_j^0} \rho_0 f. \tag{8.29}$$

By a very similar argument to the proof of (8.26), we obtain

$$\begin{aligned} &\int_{N_1^+} \phi(n) \mathcal{L}_t f(g_j^w nx)\, d\mu_x^u(n) \\ &= \sum_{\rho \in \mathcal{P}_j^0} \int_{N_1^+} \phi(n)\rho(g^\gamma nx) \mathcal{L}_t F(g_j^w nx)\, d\mu_x^u + O\left(e_{k,0}(f)\mu_x^u(N_1^+)V(x)e^{-\beta\alpha j T_0/2}\right). \end{aligned} \tag{8.30}$$

The remainder of the section is dedicated to estimating the right side (8.30).

Saturation and post-localization. Our next step is to partition the integral over N_1^+ into pieces according to the flow box they land in under flowing by g^γ. To simplify notation, we write

$$x_j := g^\gamma x.$$

We denote by $N_1^+(j)$ a neighborhood of N_1^+ defined by the property that the intersection

$$B_\rho \cap (\mathrm{Ad}(g^\gamma)(N_1^+(j)) \cdot x_j)$$

consists entirely of full local strong unstable leaves in B_ρ. We note that since $\mathrm{Ad}(g^\gamma)$ expands N^+ and B_ρ has radius < 1, $N_1^+(j)$ is contained inside the N_2^+. Since ϕ is supported inside N_1^+, we have

$$\chi_{N_1^+}(n)\phi(n) = \chi_{N_1^+(j)}(n)\phi(n), \qquad \forall n \in N^+. \tag{8.31}$$

For simplicity, we set

$$\varphi_j(n) := \phi(\mathrm{Ad}(g^\gamma)^{-1} n), \qquad \mathcal{A}_j := \mathrm{Ad}(g^\gamma)(N_1^+(j)).$$

For $\rho \in \mathcal{P}$, we let $\mathcal{W}_{\rho,j}$ denote the collection of connected components of the set

$$\{n \in \mathcal{A}_j : nx_j \in B_\rho\}.$$

In view of (8.31), changing variables using (2.3) yields

$$\begin{aligned} &\sum_{\rho \in \in \mathcal{P}_j^0} \int_{N_1^+} \phi(n)\rho(g^\gamma nx) \mathcal{L}_{s+t}(F)(g_{\gamma(w+jT_0)} nx)\, d\mu_x^u(n) \\ &= e^{-\delta\gamma(w+jT_0)} \sum_{\rho \in \mathcal{P}_j^0, W \in \mathcal{W}_{\rho,j}} \int_{n \in W} \varphi_j(n)\rho(nx_j)\mathcal{L}_{s+t}(F)(nx_j)\, d\mu_{x_j}^u(n). \end{aligned} \tag{8.32}$$

Transversals. We fix a system of transversals $\{T_\rho\}$ to the strong unstable foliation inside the boxes B_ρ. Since B_ρ meets $N_{1/2}^-\Omega$ for all $\rho \in \mathcal{P}_j^0$, we take y_ρ in the intersection $B_\rho \cap N_{1/2}^-\Omega$. In this notation, we can find neighborhoods of identity $P_\rho^- \subset P^- = MAN^-$ and $N_\rho^+ \subset N^+$ such that

$$B_\rho = N_\rho^+ P_\rho^- \cdot y_\rho, \qquad T_\rho = P_\rho^- \cdot y_\rho. \tag{8.33}$$

We also let M_ρ, A_ρ, and N_ρ^- be neighborhoods of identity in M, A and N^- respectively so that $P_\rho^- = M_\rho A_\rho N_\rho^-$.

Centering the integrals. It will be convenient to center all the integrals in (8.32) so that their basepoints belong to the transversals T_ρ of the respective flow box B_ρ; cf. (8.33).

Let $I_{\rho,j}$ denote an index set for $\mathcal{W}_{\rho,j}$. For $W \in \mathcal{W}_{\rho,j}$ with index $\ell \in I_{\rho,j}$, let $n_{\rho,\ell} \in W$, $m_{\rho,\ell} \in M_\rho$, $n^-_{\rho,\ell} \in N^-_\rho$, and $t_{\rho,\ell} \in (-\iota_j, \iota_j)$ be such that

$$x_{\rho,\ell} := m_{\rho,\ell} g_{-t_{\rho,\ell}} n_{\rho,\ell} \cdot x_j = n^-_{\rho,\ell} \cdot y_\rho \in T_\rho. \tag{8.34}$$

Note that since x belongs to $N_1^-\Omega$, we have that

$$x_{\rho,\ell} \in N_1^-\Omega, \tag{8.35}$$

cf. (8.25) and Remark 2.1.

For each such ℓ and W, let us denote $W_\ell = \mathrm{Ad}(m_{\rho,\ell} g_{t_{\rho,\ell}})(W n^{-1}_{\rho,\ell})$ and

$$\widetilde{\phi}_{\rho,\ell}(t,n) := p_{j,w}(t - t_{\rho,\ell}) \cdot e^{z t_{\rho,\ell}} \cdot \phi(\mathrm{Ad}(m_{\rho,\ell} g^\gamma g_{-t_{\rho,\ell}})^{-1}(n n_{\rho,\ell})) \cdot \rho(g_{t_{\rho,\ell}} n x_{\rho,\ell}). \tag{8.36}$$

Note that $\widetilde{\phi}_{\rho,\ell}$ has bounded support in the t direction and (8.20) implies

$$\left\|\widetilde{\phi}_{\rho,\ell}\right\|_{C^0(\mathbb{R}\times N^+)} \leq 1, \qquad \left\|\widetilde{\phi}_{\rho,\ell}(t,\cdot)\right\|_{C^k(N^+)} \ll \iota_j^{-k}, \tag{8.37}$$

for all $t \in \mathbb{R}$. Moreover, recalling (8.6), we see that

$$\left\|\widetilde{\phi}_{\rho,\ell}\right\|_{C^1(\mathbb{R}\times N^+)} \ll \iota_j^{-k} m^k. \tag{8.38}$$

Changing variables using (2.3) and (2.4), we can rewrite the integral in t of the right side of (8.32) as follows:

$$\begin{aligned} e^{-\delta\gamma(w+jT_0)} &\int_{\mathbb{R}} e^{-zt} p_{j,w}(t) \sum_{\rho\in\mathcal{P}_j^0, W\in\mathcal{W}_{\rho,j}} \int_{n\in W} \varphi_j(n)\rho(nx_j)\mathcal{L}_s(F)(g_t n x_j)\, d\mu^u_{x_j}(n)dt \\ &= e^{-\delta\gamma(w+jT_0)} \sum_{\rho\in\mathcal{P}_j^0} \sum_{\ell\in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n\in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathcal{L}_s(F)(g_{t+t_{\rho,\ell}} n x_{\rho,\ell})\, d\mu^u_{x_{\rho,\ell}}(n)dt, \end{aligned} \tag{8.39}$$

where we also used M-invariance of F.

Mass estimates. We record here certain counting estimates which will allow us to sum error terms in later estimates over $\mathcal{P}_j^0$. Note that by definition of $N_1^+(j)$, we have $\bigcup_{\rho\in\mathcal{P}_j, W\in\mathcal{W}_{\rho,j}} W \subseteq \mathcal{A}_j$. Thus, using the log-Lipschitz and contraction properties of V, it follows that

$$\begin{aligned} \sum_{\rho\in\mathcal{P}_j^0, \ell\in I_{\rho,j}} \mu^u_{x_{\rho,\ell}}(W_\ell)V(x_{\rho,\ell}) &\ll \int_{\mathcal{A}_j} V(nx_j)\, d\mu^u_{x_j}(n) \\ &= e^{\delta\gamma(w+jT_0)} \int_{N_1^+(j)} V(g_{\gamma(w+jT_0)} n x)\, d\mu^u_x(n) \ll e^{\delta\gamma(w+jT_0)} \mu^u_x(N_1^+)V(x), \end{aligned} \tag{8.40}$$

where we used the fact that $|t_{\rho,\ell}| < 1$ and the last inequality follows by Proposition 3.1 since $N_1^+(j) \subseteq N_2^+$. We also used the fact that the partition of unity $\mathcal{P}_j^0$ has uniformly bounded multiplicity.

Remark 8.8. We note the exact same argument as above gives

$$\sum_{\rho\in\mathcal{P}_j^0, \ell\in I_{\rho,j}} \mu^u_{x_{\rho,\ell}}(W_\ell)V^2(x_{\rho,\ell}) \ll e^{\delta\gamma(w+jT_0)} \mu^u_x(N_1^+)V^2(x), \tag{8.41}$$

in view of our choice of V at the beginning of the section; cf. Remark 8.1.

Transverse intersections and Lebesgue conditionals. We will view the integrals in the definition of the resolvent as an oscillatory integrals to take advantage of the large phase ib. For this purpose, it is essential for our method to replace the integrals against μ_x^u with ones against a smooth measure so we may use integration by parts.

Denote by dn the Haar measure on N^+ and by $|S|$ the Haar measure of S for any measurable subset $S \subset N^+$. Recall the subcollection $\mathcal{P}_j^0$ of the partition of unity $\mathcal{P}_j$ defined in (8.21). For each $\rho \in \mathcal{P}_j$, we define functions $\mathscr{F}_\rho$ on $B_\rho \subset X$ as follows:

$$\mathscr{F}_\rho(u^+p^-y_\rho) := |N_\rho^+|^{-1} \int_{N_\rho^+} (\rho f)(n^+p^-y_\rho)\, d\mu^u_{p^-y_\rho}(n^+), \qquad \forall u^+ \in N_\rho^+, p^- \in P_\rho^-. \tag{8.42}$$

In particular, $\mathscr{F}_\rho$ depends only on the "transversal coordinate" p^-.

The following simple, but crucial, result allows us to replace μ_x^u with the Haar measure. In fact, the proof allows for exchanging any two conformal densities, once $\mathscr{F}_\rho$ is defined appropriately. The lemma is a simple quantitative refinement of ideas appearing in [Rob03, Sch05].

Proposition 8.9. *Let $0 < r \leq 1$ and $\psi \in C_c^1(N_r^+)$ be given. For all $y \in N_1^-\Omega$, $s \geq 0$, $\rho_0 \in \mathcal{P}_j^0$, and $m \geq 1$, we have*

$$\begin{aligned}
&\int_{N_r^+} \psi(n)(\rho_0 f)(g_s n y)\, d\mu_y^u \\
&= e^{(D-\delta)s} \int_{N_r^+} \psi(n)\mathscr{F}_{\rho_0}(g_s n y)\, dn + O\left(\frac{\|\psi\|_{C^1}\, e^{-s} e_{k,0}(f)V(y)\mu_y^u(N_r^+)}{r^{\Delta_+}\iota_j^{\Delta_+}}\right),
\end{aligned}$$

where ι_j is the radius of the flow box B_{ρ_0} supporting ρ_0 and Δ_+ is given in (3.1).

The proof of Proposition 8.9 is given in Section 9.1. Setting

$$\mathscr{F}_\star := e^{(D-\delta)s} \sum_{\rho_0 \in \mathcal{P}_j^0} \mathcal{L}_s(\mathscr{F}_{\rho_0}), \tag{8.43}$$

we apply Proposition 8.9 to switch to integrating against the Lebesgue measure in (8.39) to obtain:

$$\begin{aligned}
&\int_{W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathcal{L}_s(F_\gamma)(g_{t+t_{\rho,\ell}} n x_{\rho,\ell})\, d\mu^u_{x_{\rho,\ell}} \\
&= e^{(D-\delta)(t+t_{\rho,\ell})} \int_{W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathscr{F}_\star(g_{t+t_{\rho,\ell}} n x_{\rho,\ell}) dn + O\left(\frac{e_{k,0}(f)V(x_{\rho,\ell})\mu^u_{x_{\rho,\ell}}(W_\ell)}{e^{(1-\gamma)(w+jT_0)}\iota_j^{2\Delta_+ + 2D+2}}\right).
\end{aligned}$$

Here, we applied the proposition with $\psi = \tilde{\phi}_{\rho,\ell}$ while noting that the C^1 norm of ψ is estimated in (8.37). The factor of $\iota_j^{-(2D+1)}$ comes from the cardinality of the partition of unity $\mathcal{P}_j^0$; cf. (8.22). We also recall that the radius of W_ℓ is ι_j.

Estimating the sum of the error terms using (8.40), and recalling that the support of the integrals in t is uniformly bounded, we obtain

$$\begin{aligned}
&e^{-\delta\gamma(w+jT_0)} \sum_{\rho \in \mathcal{P}_j^0} \sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n \in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathcal{L}_s(F_\gamma)(g_{t+t_{\rho,\ell}} n x_{\rho,\ell})\, d\mu^u_{x_{\rho,\ell}}(n) dt \\
=&e^{-\delta\gamma(w+jT_0)} \sum_{\rho \in \mathcal{P}_j^0, \ell \in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n \in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathscr{F}_\star(g_{t+t_{\rho,\ell}} n x_{\rho,\ell})\, dndt \\
&+ e_{k,0}(f)V(x)\mu_x^u(N_1^+) \times O\left(e^{-(1-\gamma)(w+jT_0)}\iota_j^{-(2\Delta_+ + 2D+2)}\right).
\end{aligned} \tag{8.44}$$

To sum the above errors over j and w, we first note that (8.19) and Proposition 4.3 imply that

$$\iota_j^{-1} \ll e^{(4\alpha j+6)T_0}, \tag{8.45}$$

where we used the fact that $\chi_{\mathfrak{K}} \leq 2$; cf. (4.2). Thus, as before, using (8.12), taking α and ϵ small enough, the sum of the above error terms over j and w is bounded by

$$e_{k,0}(f)V(x)\mu_x^u(N_1^+) \times O_{T_0}\left(\frac{(1+\epsilon)^m}{(a+(1-\gamma)-4\alpha(2\Delta_+ + 2D+2)-\epsilon)^m}\right).$$

Taking α and ϵ small enough, while taking b large enough to absorb the factors depending on T_0 and remembering (8.23), we obtain an error of the form

$$e_{k,0}(f)V(x)\mu_x^u(N_1^+) \times O\left(\frac{1}{(a+0.6)^m}\right). \tag{8.46}$$

Stable holonomy. Fix some $\rho \in \mathcal{P}_j^0$. Recall the points $y_\rho \in T_\rho$ and $n_{\rho,\ell}^- \in N_\rho^-$ satisfying (8.34). The product map $M \times N^- \times A \times N^+ \to G$ is a diffeomorphism on a ball of radius 1 around identity; cf. Section 2.6. Hence, given $\ell \in I_{\rho,j}$, we can define maps $\tilde{u}_\ell$, $\tilde{\tau}_\ell$, m_ℓ and $\tilde{u}_\ell^-$ from W_ℓ to N^+, $\mathbb{R}$, M and N^- respectively by the following formula

$$g_{t+t_{\rho,\ell}} n n_{\rho,\ell}^- = g_{t+t_{\rho,\ell}} m_\ell(n)\tilde{u}_\ell^-(n) g_{\tilde{\tau}_\ell(n)}\tilde{u}_\ell(n) = m_\ell(n)\tilde{u}_\ell^-(t,n) g_{t+t_{\rho,\ell}+\tilde{\tau}_\ell(n)}\tilde{u}_\ell(n), \tag{8.47}$$

where we set $\tilde{u}_\ell^-(t,n) = \mathrm{Ad}(g_{t+t_{\rho,\ell}})(\tilde{u}_\ell^-(n))$. We define the following change of variable map:

$$\Phi_\ell : \mathbb{R} \times W_\ell \to \mathbb{R} \times N^+, \qquad \Phi_\ell(t,n) = (t+\tilde{\tau}_\ell(n), \tilde{u}_\ell(n)). \tag{8.48}$$

We suppress the dependence on ρ and j to ease notation. Then, Φ_ℓ induces a map between the weak unstable manifolds of $x_{\rho,\ell}$ and y_ρ, also denoted Φ_ℓ, and defined by

$$\Phi_\ell(g_t n x_{\rho,\ell}) = g_{t+\tilde{\tau}_\ell(n)}\tilde{u}_\ell(n) y_\rho.$$

In particular, this induced map coincides with the local strong stable holonomy map inside B_ρ.

Note that we can find a neighborhood $W_\rho \subset N^+$ of identity of radius $\asymp \iota_j$ such that

$$\Phi_\ell(\mathbb{R} \times W_\ell) \subseteq \mathbb{R} \times W_\rho, \tag{8.49}$$

for all $\ell \in I_{\rho,j}$. Moreover, by shrinking the radius ι_j of the flow boxes by an absolute amount (depending only on the metric on G) if necessary, we may assume that all the maps Φ_ℓ in (8.48) are invertible on $\mathbb{R} \times W_\rho$. Hence, we can define the following:

$$\tau_\ell(n) = \tilde{\tau}_\ell(\tilde{u}_\ell^{-1}(n)) + t_{\rho,\ell} \in \mathbb{R}, \qquad u_\ell^-(t,n) = \tilde{u}_\ell^-(t-\tau_\ell(n), \tilde{u}_\ell^{-1}(n)) \in N^-,$$
$$\phi_{\rho,\ell}(t,n) = e^{-a(t-\tau_\ell(n))} \times J\Phi_\ell(n) \times \widetilde{\phi}_{\rho,\ell}(t-\tau_\ell(n), \tilde{u}_\ell^{-1}(n)),$$

and $J\Phi_\ell$ denotes the Jacobian of the change of variable Φ_ℓ with respect to the measure $dndt$.

Changing variables and using M-invariance of $\mathscr{F}_\star$, we obtain

$$\sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} e^{-zt} \int_{n \in W_\ell} \widetilde{\phi}_{\rho,\ell}(t,n)\mathscr{F}_\star(g_{t+t_{\rho,\ell}} n x_{\rho,\ell}) \, dndt$$
$$= \sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} \int_{W_\rho} e^{-ib(t-\tau_\ell(n))}\phi_{\rho,\ell}(t,n)\mathscr{F}_\star(u_\ell^-(t,n) g_t n y_\rho) \, dndt. \tag{8.50}$$

Stable derivatives. Our next step is to remove $\mathscr{F}_\star$ from the sum over ℓ in (8.50). Due to non-joint integrability of the stable and unstable foliations, our estimate involves a derivative of f in the flow direction. In particular, in view of the way we obtain contraction in the norm of flow derivatives in Lemma 7.6, this step is the most "expensive" estimate in our argument.

Recalling the definition of $\mathscr{F}_\star$ in (8.43) and of s in (8.28), we have that

$$|\mathscr{F}_\star(u_\ell^-(t,n) g_t n y_\rho) - \mathscr{F}_\star(g_t n y_\rho)| \leq e^{(D-\delta)s} \sum_{\rho_0 \in \mathcal{P}_j^0} |\mathscr{F}_{\rho_0}(g_s u_\ell^-(t,n) g_t n y_\rho) - \mathscr{F}_{\rho_0}(g_{t+s} n y_\rho)|.$$

The following lemma provides an estimate on the above integral. Its proof is given in Section 9.2.

Lemma 8.10. *For all $s \geq 0$, $u^- \in N^-_{1/10}$, $\rho_0 \in \mathcal{P}^0_j$, and $y \in N^-_{1/2}\Omega$, we have that*

$$e^{(D-\delta)s} \int_{N_1^+} |\mathscr{F}_{\rho_0}(u^- g_s n y) - \mathscr{F}_{\rho_0}(g_s n y)| \, dn \ll_k \operatorname{dist}(u^-, \mathrm{Id}) \iota_j^{-k} \|f\|_1 \mu_y^u(N_1^+) V(y),$$

where $k \in \mathbb{N}$ is the order of regularity of the test functions for the seminorm $e_{k,0}$.

Since y_ρ belongs to $N^-_{1/2}\Omega$ and $u^-_\ell(t,n)$ belongs to a neighborhood of identity in N^- of radius $O(\iota_j)$ (cf. (8.19)), uniformly over (t,n) in the support of our integrals, Lemma 8.10, combined with (8.22), yield

$$\int_{W_\rho} |\mathscr{F}_\star(u^-_\ell(t,n) g_t n y_\rho) - \mathscr{F}_\star(g_t n y_\rho)| \, dn \ll_k e^{-(1-\gamma)(w+jT_0)} \|f\|_1 \mu^u_{y_\rho}(N_1^+) V(y_\rho) \iota_j^{-(2D+1+k)}, \quad (8.51)$$

where we implicitly used the fact that $W_\rho \subset N_1^+$ and $|t| \leq 1$. Indeed, the additional gain is due to the fact that g_s contracts N^- by at least e^{-s}.

To sum the above errors over ℓ and ρ, we wish to use (8.40). We first note that Proposition 3.1 and the fact W_ρ has diameter $\asymp \iota_j$ imply that

$$\mu^u_{y_\rho}(N_1^+) \ll \iota_j^{-\Delta_+} \mu^u_{y_\rho}(W_\rho),$$

where Δ_+ is the constant in (3.1). Moreover, Propositions 3.1 and 4.3 allow us to use closeness of y_ρ and $x_{\rho,\ell}$ along with regularity of holonomy to deduce that

$$V(y_\rho)\mu^u_{y_\rho}(W_\rho) \asymp V(x_{\rho,\ell})\mu^u_{x_{\rho,\ell}}(W_\ell). \quad (8.52)$$

Here, we also use the fact that both $x_{\rho,\ell}$ and y_ρ belong to $N_1^-\Omega$; cf. (8.35).

Hence, we can use (8.40) to estimate the sum of the errors in (8.51) yielding the following estimate on the main term in (8.44):

$$\begin{aligned} &e^{-\delta\gamma(w+jT_0)} \sum_{\rho \in \mathcal{P}^0_j} \sum_{\ell \in I_{\rho,j}} \int_{\mathbb{R}} \int_{W_\rho} \left(\sum_{\ell \in I_{\rho,j}} e^{-ib(t-\tau_\ell(n))} \phi_{\rho,\ell}(t,n) \right) \mathscr{F}_\star(g_t n y_\rho) \, dndt \\ &+ O\left(e^{-(1-\gamma)(w+jT_0)} \|f\|_1 \mu^u_x(N_1^+) V(x) \iota_j^{-(2D+1+k+\Delta_+)} \right), \end{aligned}$$

where we used that the above integrands have uniformly bounded support in $\mathbb{R} \times N^+$, independently of ℓ (and ρ). Indeed, the boundedness in the $\mathbb{R}$ direction follows from that of the partition of unity p_j; cf. (8.6). We also used (8.37) to bound the C^0 norm of $\phi_{\rho,\ell}$. Summing the above error term over j and w using (8.12) and (8.45), taking α and ϵ small enough, and remembering (8.23), we obtain

$$O\left(\frac{\|f\|_1 \mu^u_x(N_1^+) V(x)}{(a+0.65)^m} \right).$$

Recall the norm $\|\cdot\|_{1,B}$ defined in (8.2) and note that $\|\cdot\|_1 \leq B \|\cdot\|_{1,B}$. Choosing $\varkappa > 0$ small enough and

$$a = 0.378, \quad (8.53)$$

one checks that $e^{1+\varkappa}/(a+0.65)$ is at most $1/(a+\sigma_0)$, for some $\sigma_0 > 0$. With these choices, taking $B = b^{1+\varkappa}$ yields an error term of the form:

$$O\left(\frac{\|f\|_{1,B} \mu^u_x(N_1^+) V(x)}{(a+\sigma_0)^m} \right). \quad (8.54)$$

8.3. **The role of oscillatory integrals.** We are left with estimating integrals of the form:

$$\int_{\mathbb{R}\times W_\rho} \Psi_\rho(t,n)\mathscr{F}_\star(g_t n y_\rho)\, dndt, \qquad \Psi_\rho(t,n) := \sum_{\ell\in I_{\rho,j}} e^{-z(t-\tau_\ell(n))}\phi_{\rho,\ell}(t,n). \tag{8.55}$$

We begin by collecting apriori bounds on Ψ_ρ and $\mathscr{F}_\star$. Denote by $J_\rho \subset \mathbb{R}$ the bounded support of the integrand in t coordinate of the above integrals. Note that (8.37) and the fact that $|t| \ll 1$ imply

$$\|\phi_{\rho,\ell}\|_{L^\infty(J_\rho\times W_\rho)} \ll 1, \qquad \|\Psi_\rho\|_{L^\infty(J_\rho\times W_\rho)} \ll \# I_{\rho,j}. \tag{8.56}$$

The following lemma estimates the L^2 norm of $\mathscr{F}_{\rho_0}$. Its proof is given in Section 9.2.

Lemma 8.11. *For all $y \in N_1^-\Omega$, and $s \geq 0$, we have*

$$e^{(D-\delta)s}\int_{W_\rho} |\mathscr{F}_{\rho_0}(g_s n y)|^2\, dn \ll_k \iota_j^{-2k} e_{k,0}(f)^2 V^2(y)\mu_y^u(N_1^+)\times V(y_{\rho_0})^{\delta/\beta},$$

where $N_{\rho_0}^+$ parametrizes local strong unstable leaves in the flow box B_{ρ_0} centered at y_{ρ_0}; cf. (8.33).

Recall that (8.17), (8.8) and (8.9) imply

$$e^s = e^{(1-\gamma)(w+jT_0)} \ll_{T_0} b^\eta. \tag{8.57}$$

We also have that $y_\rho \in N_1^-\Omega$, $|J_\rho| \ll 1$, $s \geq 1$ and $V(y_{\rho_0})^{\delta/\beta} \ll e^{\delta(2\alpha j+3)T_0}$; cf. (8.19). Hence, Lemma 8.11, the Cauchy-Schwarz inequality, the definition of $\mathscr{F}_\star$ in (8.43), and (8.22) yield

$$\begin{aligned}&\left|\int_{\mathbb{R}\times W_\rho} \Psi_\rho(t,n)\mathscr{F}_\star(g_t n y_\rho)\, dndt\right|^2\\ &\ll_{T_0,k} e^{(D-\delta)s}\left(e_{k,0}(f)V(y_\rho)\right)^2\mu_{y_\rho}^u(N_1^+)\times \iota_j^{-(2D+1+2k)}e^{2\delta\alpha jT_0}\times\int_{\mathbb{R}\times W_\rho}|\Psi_\rho(t,n)|^2\, dndt.\end{aligned} \tag{8.58}$$

We note that by (8.33) and our choice of W_ρ, we have

$$|N_{\rho_0}| \asymp |W_\rho|. \tag{8.59}$$

To proceed, we wish to make use of the oscillations due to the large phase ib to obtain cancellations. To that end, we need to make sure that $\tau_{\ell_1}(n) - \tau_{\ell_2}(n)$ has significant size compared to that of the size of the phase b, for most pairs $\ell_1,\ell_2 \in I_{\rho,j}$. On the set of pairs ℓ_1,ℓ_2 which fail this separation requirement, we use a trivial estimate combined with a counting argument for such pairs. Dolgopyat's insight, though in a completely different set up, was the realization that non-joint integrability of the strong stable and unstable foliations implies that the functions τ_ℓ are non-constant so that such a strategy may have a hope of succeeding; cf. [Dol98].

Recall the notation pertaining to the intersection points (8.34) with the transversals T_ρ of our flow boxes. Let $\kappa \in (0,1)$ be a parameter to be specified in Section 8.4. Recall from Section 2.5 the parametrization of N^- by its Lie algebra $\mathfrak{n}^- = \mathfrak{n}_\alpha^- \oplus \mathfrak{n}_{2\alpha}^-$ via the exponential map. Denote by $C_{\rho,j}(\kappa)$ the following subset of $I_{\rho,j}^2$:

$$C_{\rho,j}(\kappa) = \left\{(\ell_1,\ell_2)\in I_{\rho,j}^2 : n_{\rho,\ell_1}^-(n_{\rho,\ell_2}^-)^{-1} = \exp(u,s), \|u\|, \|s\| \leq b^{-\kappa}\right\}.$$

We also set

$$S_{\rho,j}(\kappa) = I_{\rho,j}^2 \setminus C_{\rho,j}(\kappa).$$

Then, $C_{\rho,j}(\kappa)$ parametrizes pairs of unstable manifolds which are too close to one another along the weak stable foliation.

Expanding the square and using (8.56), we obtain

$$\int_{J_\rho \times W_\rho} \left| \sum_{\ell \in I_{\rho,j}} e^{-ib(t-\tau_\ell(n))} \phi_{\rho,\ell}(t,n) \right|^2 dndt$$
$$\ll_a \#C_{\rho,j}(\kappa)|W_\rho| + \sum_{(\ell_1,\ell_2)\in S_{\rho,j}(\kappa)} \left| \int_{\mathbb{R}\times N^+} e^{-ib(\tau_{\ell_1}(n)-\tau_{\ell_2}(n))} \phi_{\rho,\ell_1}(t,n) \overline{\phi_{\rho,\ell_2}(t,n)}\, dndt \right|, \tag{8.60}$$

where for $\mathfrak{z} \in \mathbb{C}$, $\bar{\mathfrak{z}}$ denotes its complex conjugate.

We first estimate the first term in (8.60). The following proposition provides the key counting estimate on $\#C_{\rho,j}(\kappa)$. Its proof is given in Section 10.1. In what follows, we use the following notation to distinguish the real hyperbolic case:

$$\kappa_0 := \begin{cases} \kappa, & \mathfrak{K} = \mathbb{R}, \\ \kappa/2, & \mathfrak{K} = \mathbb{C}, \mathbb{H}, \text{ or } \mathbb{O}, \end{cases}$$

where we recall that our underlying manifold is a geometrically finite quotient of $\mathbb{H}^d_{\mathfrak{K}}$, for $\mathfrak{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.

Proposition 8.12. *For all $\kappa > 0$ and $\ell \in I_{\rho,j}$,*

$$\#\left\{\ell' \in I_{\rho,j} : (\ell,\ell') \in C_{\rho,j}(\kappa)\right\} \ll 1 + e^{\Delta_+\gamma(w+jT_0)} b^{-\kappa_0\Delta_+} \iota_j^{-\Delta_+}.$$

In what follows, we will select η, γ and κ such that

$$\gamma\eta \le \kappa_0. \tag{8.61}$$

In light of (8.8) and (8.9), this choice combined with Proposition 8.12 imply that for all $\ell \in I_{\rho,j}$,

$$\#\left\{\ell' \in I_{\rho,j} : (\ell,\ell') \in C_{\rho,j}(\kappa)\right\} \ll 1. \tag{8.62}$$

For all $\rho \in \mathcal{P}_j^0$, since W_ρ has radius $\asymp \iota_j$, cf. (8.19), we have by Proposition 3.1 and (8.52) that for all $\ell \in I_{\rho,j}$,

$$\mu^u_{y_\rho}(N_1^+) \ll \iota_j^{-\Delta_+} \mu^u_{y_\rho}(W_\rho) \asymp \iota_j^{-\Delta_+} \mu^u_{x_{\rho,\ell}}(W_\ell).$$

Hence, 8.62 and the Cauchy-Schwarz inequality yield the following estimate on the sum of the first term in (8.60):

$$\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\sqrt{\mu^u_{y_\rho}(N_1^+)\#C_{\rho,j}(\kappa)} \ll \iota_j^{-\Delta_+}\sqrt{\#\mathcal{P}_j^0} \times \sqrt{\sum_{\rho\in\mathcal{P}_j^0, \ell\in I_{\rho,j}} V^2(x_{\rho,\ell})\mu^u_{x_{\rho,\ell}}(W_\ell)}.$$

The terms $V(y_\rho)\sqrt{\mu^u_{y_\rho}(N_1^+)}$ come from (8.58) and we used the estimate (8.59). We estimate $\#\mathcal{P}_j^0$ using (8.22) and bound the sum using (8.41) to get, for $A = 2D + 1 + 2\Delta_+$,

$$\sum_{\rho\in\mathcal{P}_j} V(y_\rho)\sqrt{\mu^u_{y_\rho}(N_1^+)\#C_{\rho,j}(\kappa)} \ll V(x)\mu^u_x(N_1^+) \times \iota_j^{-A/2} \times e^{\delta\gamma(w+jT_0)/2}. \tag{8.63}$$

We now turn our attention to the second term in (8.60). The following proposition gives the oscillation estimate on separated pairs appearing in that sum. Its proof is given in Section 10.3.

Proposition 8.13. *For all $\ell_1, \ell_2 \in S_{\rho,j}(\kappa)$, we have*

$$\left| \int_{\mathbb{R}} \int_{N^+} e^{-ib(\tau_{\ell_1}(n)-\tau_{\ell_2}(n))} \phi_{\rho,\ell_1}(t,n) \overline{\phi_{\rho,\ell_2}(t,n)}\, dndt \right| \ll_k b^{-k(1-\kappa)} \iota_j^{-2k} m^{2k} |W_\rho|.$$

Remark 8.14. The proof of Proposition 8.13 is based on integration by parts k-times, where $k \in \mathbb{N}$ is the order of regularity of test functions used for our seminorm $e_{k,0}$; cf. (6.3). In particular, the proof requires that the "temporal distance functions" τ_ℓ to be at least of class C^k. In our setting, this allows us to choose the parameter κ close to 1, which broadens the applicability of the second assertion in Theorem 8.2. We note however that only Hölder regularity of τ_ℓ is needed to obtain an estimate in Proposition 8.13 with a weaker power of b^{-1}. Such weaker estimate suffices to establish the first assertion of Theorem 8.2.

From the formula of the measures $\mu^u_{y_\rho}$ in (2.2) and Lemma 4.8, we see that

$$\mu^u_{y_\rho}(W_\rho) \gg e^{-\delta \operatorname{dist}(y_\rho,o)} \gg V^{-\delta/\beta}(y_\rho) \gg_{T_0} e^{-2\delta\alpha j T_0},$$

where we also used the fact that y_ρ belongs to the unit neighborhood of K_j to bound its height; cf. (8.19). Thus, arguing as in the proof of (8.63), using Proposition 8.13, along with (8.40), and (8.52), we obtain for $k \in \mathbb{N}$ to be chosen in Section 8.4 the following estimate:

$$\begin{aligned}
&\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\left(\mu^u_{y_\rho}(N_1^+)\sum_{(\ell_1,\ell_2)\in S_{\rho,j}(\kappa)}\left|\int_{\mathbb{R}\times N^+} e^{-z\tau_{\ell_1}(n)-\bar{z}\tau_{\ell_2}(n)}\phi_{\rho,\ell_1}(t,n)\overline{\phi_{\rho,\ell_2}(t,n)}\,dndt\right|\right)^{1/2}\\
&\ll_{T_0,k} b^{-k(1-\kappa)/2}\iota_j^{-(k+\Delta_+/2)}m^k e^{\delta\alpha jT_0}\times\sum_{\rho\in\mathcal{P}_j^0} V(y_\rho)\mu^u_{y_\rho}(W_\rho)\#I_{\rho,j}\\
&\ll b^{-k(1-\kappa)/2}\iota_j^{-(k+\Delta_+/2)}m^k\times e^{\delta(\gamma+\alpha)(w+jT_0)}\times\mu^u_x(N_1^+)V(x).
\end{aligned}$$

Combining this estimate with (8.58), (8.60), (8.57), and (8.63), we obtain the following estimate on the integrals in (8.55):

$$\begin{aligned}
&e^{-\delta\gamma(w+jT_0)}\sum_{\rho\in\mathcal{P}_j^0}\int_{J_\rho\times W_\rho}\Psi_\rho(t,n)\mathscr{F}_\star(g_tny_\rho)\,dndt\\
&\ll_k e_{k,0}(f)V(x)\mu^u_x(N_1^+)\times\iota_j^{-(A+3k)}e^{\delta\alpha(w+jT_0)}\left(e^{((D-\delta)(1-\gamma)-\delta\gamma)(w+jT_0)/2}+b^{(\eta(D-\delta)-k(1-\kappa))/2}m^k\right),
\end{aligned}$$

where we used the elementary inequality $\sqrt{x+y}\le\sqrt{x}+\sqrt{y}$ for any $x,y\ge 0$ along with the fact that $|J_\rho\times W_\rho|\ll 1$. Let $L = 2A+6k+2\delta$. Summing the above error terms over j and w, taking α and γ small enough, and recalling (8.22), we obtain an error term of the form

$$O_{T_0,k}\left(e_{k,0}(f)V(x)\mu^u_x(N_1^+)(1+\epsilon)^m\times b^{\eta(D-\delta)/2}\times\left[\frac{1}{(a+\delta\gamma/2-4\alpha L)^m}+\frac{b^{-k(1-\kappa)/2}m^k}{(a-4\alpha L)^m}\right]\right). \tag{8.64}$$

In the large critical exponent regime, i.e. when hypothesis (8.3) is satisfied, we use do not use the bound (8.57) and instead obtain the following estimate:

$$\begin{aligned}
O_{T_0,k}\Bigg(&e_{k,0}(f)V(x)\mu^u_x(N_1^+)(1+\epsilon)^m\\
&\times\left[\frac{1}{(a+(\delta\gamma-(D-\delta)(1-\gamma))/2-4\alpha L)^m}+\frac{b^{(\eta(D-\delta)-k(1-\kappa))/2}m^k}{(a-4\alpha L)^m}\right]\Bigg).
\end{aligned}\tag{8.65}$$

8.4. **Parameter selection and conclusion of the proof.** In this subsection, we finish the proof of Theorem 8.2. First, we handle the case of small critical exponent, i.e.

$$\delta \le \begin{cases} 2D/3, & \mathfrak{K}=\mathbb{R},\\ 5D/6, & \mathfrak{K}=\mathbb{C},\mathbb{H},\text{ or }\mathbb{O}.\end{cases}\tag{8.66}$$

We begin by simplifying the error expression in (8.64). As before, we will absorb the dependence on T_0 in (8.64) by taking b large enough at the cost of replacing ϵ with 2ϵ in the denominators of the above expression. In this case, we take κ to be any fixed constant in $(0,1)$[11]. Taking k large enough and α and ϵ small enough, we can ensure that

$$\frac{b^{(\eta(D-\delta)-k(1-\kappa))/2}m^k(1+\epsilon)^m}{(a-4\alpha L)^m} \leq \frac{1}{(a+\sigma_1)^m}, \tag{8.67}$$

for some fixed constant $\sigma_1 > 0$ and for all large enough b. Note that, once σ_1 is fixed, the above inequality remains valid after further decreasing α and ϵ. Then, we can take η, α and ϵ small enough so that

$$\frac{(1+\epsilon)^m b^{\eta(D-\delta)/2}}{(a+\delta\gamma/2-4\alpha L)^m} \leq \frac{1}{(a+\sigma_2)^m}, \tag{8.68}$$

for some constant $\sigma_2 > 0$. Hence, the error term in (8.64) becomes

$$e_{k,0}(f)V(x)\mu_x^u(N_1^+) \times O\left(\frac{1}{(a+\sigma_2)^m} + \frac{1}{(a+\sigma_1)^m}\right). \tag{8.69}$$

Note that the parameter α (and ϵ) remain unconstrained. We let $\sigma_3 > 0$ be such that the error terms in (8.13) and (8.27) satisfy

$$\frac{1}{(a+\beta\alpha-\epsilon)^m} + \frac{1}{(a+\beta\alpha/2-2\epsilon)^m} \leq \frac{2}{(a+\sigma_3)^m}. \tag{8.70}$$

Let $\sigma_\star = \min\{\sigma_i : 0 \leq i \leq 3\}$. Making η smaller if necessary, we may assume that $a\eta < 1$. Recall the parameter $\xi \in (0,1)$ provided by Lemma 8.7 in the case $a\eta < 1$. Collecting the error terms in (8.10), (8.13), (8.27), (8.46), (8.54), (8.69), and Lemma 8.7 and taking ϵ small enough, we obtain

$$e_{k,0}(R(z)^m f) \ll \frac{e_{k,0}(f)}{(a+\xi^m)^m} + \frac{\|f\|_{1,B}}{(a+\sigma_\star)^m}.$$

Letting C_Γ denote the implied constant and choosing $\varrho > 0$ so that $\xi^m \geq |b|^{-\varrho}$, this estimate concludes the proof of the first assertion in Theorem 8.2.

In the large critical exponent case, i.e. when (8.66) does not hold, we use the bound in (8.65) instead. First, we take

$$\eta = 2.6. \tag{8.71}$$

In this case, one checks that by taking $\kappa < 1$ to be close enough to 1, this choice of η satisfies (8.61). Then, the estimate (8.67) will hold for all large b by taking k large enough and α and ϵ small enough. Moreover, for our choice of γ in (8.23), we have

$$\delta\gamma - (D-\delta)(1-\gamma) > 0$$

in this case. Hence, further decreasing α and ϵ as necessary, we obtain

$$\frac{(1+\epsilon)^m}{(a+(\delta\gamma-(D-\delta)(1-\gamma))/2-4\alpha L)^m} \leq \frac{1}{(a+\sigma_2)^m}$$

for a possibly smaller constant $\sigma_2 > 0$. The estimate (8.70) can also be arranged to hold for a possibly smaller constant $\sigma_3 > 0$ depending on α.

Finally, in light of (8.53), we see that $a\eta > 1$ in this case. Thus, Lemma 8.7 implies that we instead get a resolvent bound of the form

$$e_{k,0}(R(z)^m f) \ll \frac{e_{k,0}(f)}{(a+\sigma_4)^m} + \frac{\|f\|_{1,B}}{(a+\sigma_\star)^m},$$

[11]In low regularity settings, κ will have to be taken small since one cannot do integration by parts many times as in Proposition 8.13 to compensate for a choice of κ close to 1.

for some constant $\sigma_4 > 0$. Since $e_{k,0}(f) \leq \|f\|_{1,B}$, making $\sigma_\star$ smaller if necessary proves the second assertion of the theorem, which is a stronger bound than the bound in the first assertion.

We note that our choice of $\sigma_\star$ depends only on the critical exponent δ and the ranks of the cusps of Γ (if any) through its dependence on Δ_+ and Δ.

Remark 8.15. It is worth noting that the above arguments allowed us to avoid issues related to the mismatch in the doubling exponents Δ_+ and Δ in Proposition 3.1 in the case the manifold has cusps.

9. Transverse intersections and smooth conditionals

In this section, we provide the proofs of the auxiliary results stated in Section 8 pertaining the conversion from the Patterson-Sullivan conditionals to integrals against the Lebesgue measure; namely Proposition 8.9 and Lemmas 8.10 and 8.11.

9.1. **Transverse intersections and Lebesgue Conditionals.** Proposition 8.9 follows at once from the following lemma.

Lemma 9.1. *Let $0 < r \leq 1$ and ϕ in the unit ball of $C_c^1(N_r^+)$ be given. For all $y \in N_1^-\Omega$, $t \geq 0$ and $\rho \in \mathcal{P}_j^0$, we have*

$$\int_{N_r^+} \phi(n)(\rho f)(g_t n y)\, d\mu_y^u = e^{(D-\delta)t} \int_{N_r^+} \phi(n)\mathscr{F}_\rho(g_t n y) dn + O(e^{-t}(r\iota_j)^{-\Delta_+}) e_{k,0}(f) V(y) \mu_y^u(N_r^+),$$

where $D = \dim N^+$ and Δ_+ is given in (3.1).

Proof of Lemma 9.1. We begin by proving an analog of (8.32), rewriting the integral as a sum of integrals over strong unstable leaves. We let $N_r^+(t)$ denote a neighborhood of N_r^+ defined by the property that the intersection

$$B_\rho \cap \left(\mathrm{Ad}(g_t)(N_r^+(t)) \cdot g_t y\right)$$

consists entirely of full local strong unstable leaves in B_ρ. We set $\varphi_t(n) := \phi(g_{-t} n g_t)$, $\mathcal{A}_t := \mathrm{Ad}(g_t)(N_r^+(t))$, and denote by $\mathcal{W}_{\rho,t}$ the collection of connected components of the set

$$\{n \in \mathcal{A}_t : n g_t y \in B_\rho\}.$$

Let $I_{\rho,t}$ be an index set for $\mathcal{W}_{\rho,t}$. For each $W \in \mathcal{W}_{\rho,t}$ with index $\ell \in I_{\rho,t}$, let $n_\ell \in W \subset N^+$ be such that $x_\ell := n_\ell g_t y$ belongs to the transversal $T_\rho = P_\rho^- \cdot y_\rho$. Define $W_\ell := W n_\ell^{-1}$ and note that

$$W_\ell = N_\rho^+ \tag{9.1}$$

in view of our choice of $N_r^+(t)$. Moreover, since the support of ρ is properly contained in B_ρ, setting

$$\rho_\ell(n) := \chi_{W_\ell}(n)\rho(n x_\ell), \qquad \forall n \in N^+, \tag{9.2}$$

we see that ρ_ℓ is in fact a smooth function on N^+. Finally, since $y_\rho \in N_{1/2}^-\Omega$ and $x_\ell \in T_\rho$, cf. (8.33), we see that

$$x_\ell \in N_1^-\Omega, \tag{9.3}$$

where we used the fact that $\rho \in \mathcal{P}_j^0$.

Changing variables using (2.3) and (2.4), since ρF is supported inside B_ρ, it follows that

$$\begin{aligned}\int_{N_r^+} \phi(n)(\rho f)(g_t n y) d\mu_y^u = \int_{N_1^+(t)} \phi(n)(\rho f)(g_t n y) d\mu_y^u &= e^{-\delta t} \sum_{W \in \mathcal{W}_{\rho,t}} \int_{n \in W} \varphi_t(n)(\rho f)(n g_t y) d\mu_{g_t y}^u \\ &= e^{-\delta t} \sum_{\ell \in I_{\rho,t}} \int \varphi_t(n n_\ell)\rho_\ell(n) F(n x_\ell)\, d\mu_{x_\ell}^u.\end{aligned}$$

Since ϕ has C^1 norm at most 1 and each W_ℓ has diameter ι_j, where ι_j denotes the radius of B_ρ, we obtain

$$|\varphi_t(nn_\ell) - \varphi_t(n_\ell)| \ll e^{-t}\iota_j, \qquad \forall n \in W_\ell, \tag{9.4}$$

where we used the fact that $\mathrm{Ad}(g_t)$ expands N^+ by at least e^t. Hence, since ρ has C^1 norm $O(\iota_j^{-1})$, we see that the function

$$n \mapsto \rho_\ell(n)(\varphi_t(nn_\ell) - \varphi_t(n_\ell))$$

has C^1 norm $\ll e^{-t}$. Hence, by definition of the coefficient $e_{k,0}$, we obtain

$$\left| \int \varphi_t(nn_\ell)\rho_\ell(n)F(nx_\ell)\, d\mu^u_{x_\ell} - \varphi_t(n_\ell)\int \rho_\ell(n)F(nx_\ell)\, d\mu^u_{x_\ell} \right| \ll e^{-t}e_{k,0}(f)V(x_\ell)\mu^u_{x_\ell}(N_1^+), \tag{9.5}$$

where we used (9.3). To estimate the sum of the above errors, we note that Propositions 3.1 and 4.3 yield

$$V(x_\ell)\mu^u_{x_\ell}(N_1^+) \ll \iota_j^{-\Delta_+}V(x_\ell)\mu^u_{x_\ell}(W_\ell) \ll \iota_j^{-\Delta_+}\int_{W_\ell} V(nx_\ell)\, d\mu^u_{x_\ell}.$$

Reversing our changes of variables, and using Theorem 4.1, along with positivity of V, we obtain

$$e^{-\delta t}\sum_{\ell\in I_{\rho,t}}\int_{W_\ell} V(nx_\ell)\, d\mu^u_{x_\ell} \leq \int_{N_3^+} V(g_tny)\, d\mu^u_y \ll (e^{-\beta t}V(y)+1)\mu^u_y(N_1^+) \ll V(y)\mu^u_y(N_1^+), \tag{9.6}$$

where we used the fact that $V(\cdot) \gg 1$ on bounded neighborhoods of Ω, $N_r^+(t) \subseteq N_3^+$, and the doubling estimates of Proposition 3.1.

These estimates, together with the definition of $\mathscr{F}_\rho$ in (8.42), yield

$$\int_{N_r^+} \phi(n)(\rho f)(g_tny)d\mu^u_y = e^{-\delta t}\sum_{\ell\in I_{\rho,t}}\varphi_t(n_\ell)\int_{n\in W_\ell}\mathscr{F}_\rho(nx_\ell)dn + O(e^{-t}\iota_j^{-\Delta_+})e_{k,0}(f)V(y)\mu^u_y(N_1^+).$$

Note that $\mathscr{F}_\rho(nx_\ell)$ is constant as n varies in W_ℓ. Using (9.4) and the same argument as above, we can put φ_ℓ back inside the integral to get

$$\sum_{\ell\in I_{\rho,t}}\varphi_t(n_\ell)\int_{n\in W_\ell}\mathscr{F}_\rho(nx_\ell)dn = \sum_{\ell\in I_{\rho,t}}\int_{n\in W_\ell}\varphi_t(nn_\ell)\mathscr{F}_\rho(nx_\ell)dn + O(e^{-t}\iota_j^{-\Delta_+})e_{k,0}(f)V(y)\mu^u_y(N_1^+).$$

Finally, we note that the Jacobian of the change of variables $n \mapsto \mathrm{Ad}(g_t)(n)$ with respect to the Haar measure is e^{-Dt}. Thus, reversing our change of variables to integrate over N_1^+, but with respect to the Haar measure in place of μ^u_x, and using the estimate $\mu^u_y(N_1^+) \ll r^{-\Delta_+}\mu^u_y(N_r^+)$ supplied by Proposition 3.1, we obtain the lemma. ∎

9.2. **Transverse regularity.** In this section, we give estimates on the regularity of $\mathscr{F}_\rho$ which imply Lemmas 8.10 and 8.11. The main step in the proof is the following lemma.

Lemma 9.2. *For all $\rho \in \mathcal{P}_j^0$, $u^- \in N^-_{1/10}$ and $y \in X$, we have*

$$|\mathscr{F}_\rho(y)| \ll_k \iota_j^{-k}|N_\rho|^{-1}e_{k,0}(f)V(y_\rho)\mu^u_{y_\rho}(N_\rho^+),$$
$$|\mathscr{F}_\rho(u^-y) - \mathscr{F}_\rho(y)| \ll_k \mathrm{dist}(u^-,\mathrm{Id})|N_\rho^+|^{-1}\iota_j^{-k}\mu^u_{y_\rho}(N_\rho^+)V(y_\rho)\left\|f\right\|_1.$$

Proof. Since $\mathscr{F}_\rho$ is supported in B_ρ, we may assume that $y \in B_\rho$. Since $\mathscr{F}_\rho$ depends only on the transversal coordinate in B_ρ, we may further assume $y \in T_\rho$.

Since ρ has C^k norm $O(\iota_j^{-k})$, cf. (8.20), we obtain by definition of the seminorm $e_{k,0}$ that

$$|\mathscr{F}_\rho(y)| \ll_k |N_\rho|^{-1}\iota_j^{-k}e_{k,0}(f)V(y)\mu^u_y(N_\rho^+). \tag{9.7}$$

Similarly to (8.52), using the doubling results of Proposition 3.1, we further obtain

$$\mu_y^u(N_\rho^+) \asymp \mu_{y_\rho}^u(N_\rho^+), \qquad \forall y \in T_\rho. \tag{9.8}$$

Here, we use the fact that $T_\rho \subset N_2^- \cdot \Omega$ since $\rho \in \mathcal{P}_j^0$ so that $y_\rho \in N_1^-\Omega$. Moreover, since y and y_ρ are at a uniformly bounded distance apart, Proposition 4.3 gives that $V(y_\rho) \asymp V(y)$, thus concluding the proof of the first estimate.

For the second estimate, we note that since the support of ρ is properly contained inside B_ρ, we may replace u^- with an element closer to identity if necessary so as to ensure that both y and u^-y belong to B_ρ. We may further assume that y (and hence u^-y) belongs to T_ρ so that

$$|\mathscr{F}_\rho(u^-y) - \mathscr{F}_\rho(y)| = |N_\rho^+|^{-1} \left| \int_{N_\rho^+} (\rho f)(nu^-y)\, d\mu_{u^-y}^u(n) - \int_{N_\rho^+} (\rho f)(ny)\, d\mu_y^u(n) \right|.$$

Recall that $\rho_*(n) := \rho(nx)\chi_{N_\rho^+}(n)$ is in fact a smooth function on N^+ with C^1 norm $\ll \iota_j^{-1}$; cf. (9.2) and the discussion preceding it. Arguing similarly to the proof of Proposition 6.6, there is a map $p^- : N_1^+ \longrightarrow P^- = MAN^-$ such that changing variables via weak stable holonomy, denoted Φ, yields

$$\int_{N_\rho^+} (\rho f)(nu^-y)\, d\mu_{u^-y}^u(n) = \int \rho_*(n) F(nu^-y)\, d\mu_{u^-y}^u(n) = \int \rho_*(\Phi^{-1}(n)) F(p^-(n)ny) J\Phi(n)\, d\mu_y^u,$$

where $J\Phi$ is the Jacobian of Φ; cf. (2.9). In particular, we have for all $n \in N_1^+$.

$$\mathrm{dist}(p^-(n), \mathrm{Id}) \ll \mathrm{dist}(u^-, \mathrm{Id}).$$

Recalling (2.9) and (8.20), we have that

$$\|\rho_*\|_{C^0}, \|J\Phi\|_{C^0} \ll 1, \qquad \|J\Phi - 1\|_{C^0} \ll \mathrm{dist}(u^-, \mathrm{Id}).$$

Hence, in view of (9.7) and following a similar argument to the proof of Proposition 6.6, we obtain

$$|\mathscr{F}_\rho(u^-y) - \mathscr{F}_\rho(y)| \ll \mathrm{dist}(u^-, \mathrm{Id}) |N_\rho^+|^{-1} \iota_j^{-1} \mu_y^u(N_\rho^+) V(y) \|f\|_1 .$$

Here, we are using the fact y belongs to $N_{3/4}^-\Omega$. Indeed, this follows since y_ρ belongs to $N_{1/2}^-\Omega$ and y belongs to T_ρ. The desired estimate now follows since $\mu_y^u(N_\rho^+)V(y) \asymp \mu_{y_\rho}^u(N_\rho^+)V(y_\rho)$; cf. (9.8). □

This lemma yields the following immediate corollary by reversing the argument in Lemma 9.1. The corollary is a slightly stronger version of Lemmas 8.10 and 8.11.

Corollary 9.3. *For all $0 < r \ll 1$, $\rho \in \mathcal{P}_j^0$, $u^- \in N_{1/10}^-$, $y \in N_{1/2}^-\Omega$ and $t \geq 0$, we have*

$$e^{(D-\delta)t} \int_{N_r^+} |\mathscr{F}_\rho(g_t ny)|^2\, dn \ll_k \iota_j^{-2k} e_{k,0}(f)^2 V^2(y) \mu_y^u(N_1^+) \times V(y_\rho)^{\delta/\beta},$$

$$e^{(D-\delta)t} \int_{N_1^+} |\mathscr{F}_\rho(u^- g_t ny) - \mathscr{F}_\rho(g_t ny)|\, dn \ll_k \mathrm{dist}(u^-, \mathrm{Id}) \iota_j^{-k} \|f\|_1 \mu_y^u(N_1^+) V(y).$$

Proof. Recall the notation in the proof of Lemma 9.1. Then, changing variables and arguing as in the proof of the lemma, we obtain

$$e^{(D-\delta)t} \int_{N_r^+} |\mathscr{F}_\rho(g_t ny)|^2\, dn \leq e^{-\delta t} \int_{\mathcal{A}_t} |\mathscr{F}_\rho(g_t ny)|^2\, dn = e^{-\delta t} \sum_{\ell \in I_{\rho,t}} \int_{W_\ell} |\mathscr{F}_\rho(n x_\ell)|^2\, dn.$$

Note that the first inequality follows by non-negativity since $\mathrm{Ad}(g_t)(N_r^+) \subseteq \mathcal{A}_t$.

Recall that $N_\rho^+ = W_\ell$ for all ℓ; cf. (9.1). Hence, by Lemma 9.2, we obtain

$$\int_{W_\ell} |\mathscr{F}_\rho(nx_\ell)|^2 \, dn \ll (\iota_j^{-k} e_{k,0}(f) V(y_\rho) \mu_{x_\ell}^u(W_\ell))^2 \\ \ll \iota_j^{-2k} e_{k,0}(f)^2 \mu_{x_\ell}^u(W_\ell) \int_{W_\ell} V^2(nx_\ell) \, d\mu_{x_\ell}^u(n),$$

where we also used the fact that $V(y_\rho) \asymp V(z)$ for all $z \in B_\rho$; cf. Proposition 4.3. Using the formula for the measures $\mu_\bullet^u$ in (2.2) and Lemma 4.8, we see that

$$\mu_{x_\ell}^u(W_\ell) \ll e^{\delta \operatorname{dist}(x_\ell, o)} \ll V(y_\rho)^{\delta/\beta},$$

where o is our fixed basepoint. Here, we also used the estimate $V(x_\ell) \ll V(y_\rho)$.

To estimate the sum of this estimate over ℓ, we argue as in the proof of (9.6), using the integrability of V^2 provided by Theorem 4.1 and Remark 8.1, to obtain

$$e^{-\delta t} \sum_{\ell \in I_{\rho,t}} \int_{W_\ell} V^2(nx_\ell) \, d\mu_{x_\ell}^u \ll V^2(y) \mu_y^u(N_1^+).$$

For the second estimate, arguing as above, we obtain via Lemma 9.2

$$e^{(D-\delta)t} \int_{N_1^+} |\mathscr{F}_\rho(u^- g_t ny) - \mathscr{F}_\rho(g_t ny)| \, dn = e^{-\delta t} \sum_{\ell \in I_{\rho,t}} \int_{W_\ell} |\mathscr{F}_\rho(u^- nx_\ell) - \mathscr{F}_\rho(nx_\ell)| \, dn \\ \ll \operatorname{dist}(u^-, \mathrm{Id}) \iota_j^{-k} \|f\|_1 \times e^{-\delta t} \sum_{\ell \in I_{\rho,t}} \mu_{x_\ell}^u(W_\ell) V(x_\ell).$$

The second estimate then follows by (9.6). □

10. Counting and Uniform Non-integrability

In this section, we provide the proofs of Propositions 8.12 and 8.13, thus completing the proof of Theorem 8.2. The key property that we use for the proof of the latter result relies on the uniform joint non-integrability of these foliations.

10.1. **Counting close pairs and proof of Proposition 8.12.** The idea of the proof is the same as that of [Liv04, Lemma 6.2].

Recall our definition of the points $x_{\rho,\ell}$ in (8.34) and of $N_1^+(j)$ in the paragraph above (8.31). For each $\ell \in I_{\rho,j}$, fix some $u_\ell \in N_1^+(j) \subseteq N_3^+$ such that

$$x_{\rho,\ell} = g^\gamma p_\ell^+ \cdot x, \qquad p_\ell^+ := m_{\rho,\ell} g_{t_{\rho,\ell}} u_\ell. \tag{10.1}$$

Here, we are using that the groups $A = \{g_t\}$ and M commute. Denote by P^+ the parabolic subgroup N^+AM of G. Since M is compact, $|t_{\rho,\ell}| < 1$, and $N_1^+(j)$ is contained in N_3^+, there is a uniform constant $C > 0$ such that

$$\left\{p_\ell^+ : \ell \in I_{\rho,j}\right\} \subset P_C^+, \tag{10.2}$$

where P_C^+ denotes the ball of radius C around identity in P^+.

Let $\mathfrak{C}(\ell_0)$ denote the set of $\ell \in I_{\rho,j}$ such that $(\ell_0, \ell) \in C_{\rho,j}(\kappa)$. Recalling the definition of the Carnot metric in (2.7), the definition of $C_{\rho,j}(\kappa)$ implies that

$$d_{N^-}(n_{\rho,\ell}^-, n_{\rho,\ell_0}^-) \ll \begin{cases} b^{-\kappa}, & \mathfrak{K} = \mathbb{R}, \\ b^{-\kappa/2}, & \mathfrak{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}, \end{cases}$$

since $\mathfrak{n}_{2\alpha}^- = 0$ in the real hyperbolic case. Set $\epsilon = b^{-\kappa}$ in the real case and $\epsilon = b^{-\kappa/2}$ in the other cases. Then, we can find $\tilde{u}_\ell^- \in N_\epsilon^-$, such that $g^\gamma p_\ell^+ \cdot x = \tilde{u}_\ell^- \cdot g^\gamma p_{\ell_0}^+ \cdot x$ for all $\ell \in \mathfrak{C}(\ell_0)$. In particular, for $t_\star := \gamma(w + jT_0)$ and $u_\ell^- = \mathrm{Ad}(g^\gamma)^{-1}(\tilde{u}_\ell^-)$, since $g^\gamma = g_{t_\star}$ by (8.24), we have that

$$p_\ell^+ x = u_\ell^- \cdot p_{\ell_0}^+ x \in N_{e^{t_\star}\epsilon}^- \cdot p_{\ell_0}^+ x, \qquad \forall \ell \in \mathfrak{C}(\ell_0). \tag{10.3}$$

Our counting estimate will follow by estimating from below the separation between the points $p_\ell^+ x$, combined with a measure estimate on the ball $N_{e^{t_\star}\epsilon}^- \cdot p_{\ell_0}^+ x$.

To this end, recall the sublevel set K_j and the injectivity radius ι_j in (8.19). Recall also by (8.14) that x belongs to K_j. It follows that the injectivity radius of the weak unstable ball $P_C^+ \cdot x$ is $\gg \iota_j$. This implies that there is a radius r_j with $\iota_j \ll r_j \leq \iota_j$ such that for every $\ell \in \mathfrak{C}(\ell_0)$, the map $n^- \mapsto n^- \cdot p_\ell^+ x$ is an embedding of $N_{r_j}^-$ into X and the disks

$$\left\{ N_{r_j}^- \cdot p_\ell^+ x : \ell \in \mathfrak{C}(\ell_0) \right\}$$

are disjoint. Recalling (10.3), it follows that the disks $N_{r_j}^- \cdot u_\ell^-$ form a disjoint collection of disks inside $N_{e^{t_\star}\epsilon+\iota_j}^-$. In particular,

$$\#\mathfrak{C}(\ell_0) \leq \frac{\mu^s_{p_{\ell_0}^+ x}(N^-_{e^{t_\star}\epsilon+\iota_j})}{\min_{\ell\in\mathfrak{C}(\ell_0)} \mu^s_{p_{\ell_0}^+ x}(N^-_{r_j} \cdot u_\ell^-)},$$

where $\mu^s_\bullet$ denote the Patterson-Sullivan conditional measures on N^-, defined analogously to the unstable conditionals in (2.2).

Fix some arbitrary $\ell \in \mathfrak{C}(\ell_0)$ and recall (10.1) and (10.3). Then, changing variables using (2.4) and (2.3), the doubling results in Proposition 3.1 imply that for $\varsigma = 2(e^{t_\star}\epsilon + \iota_j)$, we have

$$\frac{\mu^s_{p_{\ell_0}^+ x}(N_\varsigma^- \cdot u_\ell^-)}{\mu^s_{p_{\ell_0}^+ x}(N_{r_j}^- \cdot u_\ell^-)} = \frac{\mu^s_{p_\ell^+ x}(N_\varsigma^-)}{\mu^s_{p_\ell^+ x}(N_{r_j}^-)} = \frac{\mu^s_{x_{\rho,\ell}}(N^-_{e^{-t_\star}\varsigma})}{\mu^s_{x_{\rho,\ell}}(N^-_{e^{-t_\star}r_j})} \ll \left(\frac{\epsilon + e^{-t_\star}\iota_j}{e^{-t_\star}r_j}\right)^{\Delta_+} \ll \left(e^{t_\star}\epsilon\iota_j^{-1} + 1\right)^{\Delta_+}.$$

To conclude the proof, note that u_ℓ^- is at distance at most $e^{t_\star}\epsilon$ from identity so that

$$N^-_{e^{t_\star}\epsilon+\iota_j} \subseteq N^-_{2(e^{t_\star}\epsilon+\iota_j)} \cdot u_\ell^-.$$

The result now immediately follows if $\Delta_+ \leq 1$ and by Hölder's inequality otherwise.

10.2. **Explicit formula for the temporal function.** In this section, we give explicit formulas for the commutation relations between stable and unstable subgroups of G. These formulas will be used in obtaining estimates on oscillatory integrals involving the temporal functions τ_ℓ in the proof of Proposition 8.13.

Let $\mathfrak{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Consider the following quadratic form on $\mathfrak{K}^{d+1}$: for $x = (x_i) \in V$,

$$Q(x) = 2\mathrm{Re}(\bar{x}_0 x_n) - |x_1|^2 - \cdots - |x_{d-1}|^2.$$

Then, we can realize G as the orthogonal group $O_{\mathfrak{K}}(Q)$; i.e. the subgroup of $\mathrm{SL}(\mathfrak{K}^{d+1})$ preserving Q. We take

$$A = \left\{ g_t = \mathrm{diag}(e^t, \mathrm{I}_{d-1}, e^{-t}) : t \in \mathbb{R} \right\},$$

where I_{d-1} denotes the identity matrix in dimension $d-1$. Denote by M the centralizer of A inside the standard maximal compact subgroup $K \cong O(n; \mathfrak{K})$ of G.

For $u \in \mathfrak{K}^m$, viewed as a row vector, we write u^t for its transpose and $\bar{u}$ for the component-wise conjugate. We let $\|u\|^2 := u \cdot \bar{u}^t$, and $u \cdot \bar{u}^t$ denotes the standard Euclidean dot product. Hence,

N^+ takes the form

$$N^+ = \left\{ n^+(u,s) := \begin{pmatrix} 1 & u & s + \frac{\|u\|^2}{2} \\ \mathbf{0} & \mathrm{I}_{d-1} & \bar{u}^t \\ 0 & \mathbf{0} & 1 \end{pmatrix} : u \in \mathfrak{K}^{d-1}, s \in \mathrm{Im}\mathfrak{K} \right\}. \tag{10.4}$$

The group N^- is parametrized by the transpose of the elements of N^+ follows

$$N^- = \left\{ n^-(u,s) := (n^+(u,s))^t : u \in \mathfrak{K}^{d-1}, s \in \mathrm{Im}\mathfrak{K} \right\}.$$

Note that the product map $M \times A \times N^+ \times N^- \to G$ is a diffeomorphism near identity. In the above parametrizations, given $t \in \mathbb{R}$ and small enough $u, v \in \mathfrak{K}^{d-1}$ and $r, s \in \mathrm{Im}\mathfrak{K}$, we would like to find the A component of the matrix $n^-(u,s)g_t n^+(v,r)$, in its unique decomposition as mau^+u^-, for some $u^+ \in N^+, u^- \in N^-, a \in A, m \in M$. Explicit computation shows that the top left entry of $n^-(u,s)n^+(v,r)$ is given by

$$1 + u \cdot \bar{v} + \left(s + \frac{\|u\|^2}{2}\right)\left(r + \frac{\|v\|^2}{2}\right).$$

Thus, letting

$$\tau(t,(v,r)) := t + \log \mathrm{Re}\left(1 + e^{-t} u \cdot \bar{v} + e^{-2t}\left(s + \frac{\|u\|^2}{2}\right)\left(r + \frac{\|v\|^2}{2}\right)\right), \tag{10.5}$$

we see that the A component of $n^-(u,s)g_t n^+(v,r)$ is given by $g_{\tau(t,(v,r))}$. The function $\tau(t,(v,r))$ in (10.5) is known as the **temporal function**.

The above constructions do not work for the Octonions $\mathbb{O}$ due to non-associativity. In this case, we will reduce the computations to the case $G \cong \mathrm{SU}(2,1)$ or $\mathrm{SL}_2(\mathbb{R})$.

10.3. **Oscillatory integrals and proof of Proposition 8.13.** Fix $(\ell_1, \ell_2) \in S_{\rho,j}(\kappa)$ and let

$$\psi_{1,2}(t,n) := \phi_{\rho,\ell_1}(t,n)\overline{\phi_{\rho,\ell_2}(t,n)}.$$

We wish to estimate

$$\int_{\mathbb{R}} \int_{N^+} e^{-ib(\tau_{\ell_1}(n) - \tau_{\ell_2}(n))} \psi_{1,2}(t,n) \, dndt.$$

We can interpret this integral as taking place over the (local) weak unstable manifold of y_ρ. Moreover, by definition of the identity neighborhood $W_\rho \subset N^+$, the integrand is supported inside W_ρ; cf. (8.49). Recall the change of variables map Φ_ℓ in (8.48), which we viewed as a strong stable holonomy map from the weak unstable manifold of $x_{\rho,\ell}$ to that of y_ρ. It is convenient to reverse the change of variables Φ_{ℓ_1} to integrate over W_{ℓ_1} instead of W_ρ. We do so by composing the integrand with $\Phi_{\ell_1}^{-1}$ (which is well-defined on $\mathbb{R} \times W_\rho$) to obtain

$$\int_{\mathbb{R}} \int_{W_\rho} e^{-ib(\tau_{\ell_1}(n) - \tau_{\ell_2}(n))} \psi_{1,2}(t,n) \, dndt = \int_{\mathbb{R}} \int_{N^+} e^{-ib(t - \hat{\tau}_2(n))} \hat{\psi}_{1,2}(t,n) J\Phi_{\ell_1}^{-1}(n) \, dndt,$$

where $J\Phi_{\ell_1}$ is the Jacobian of the change of variables with respect to the Haar measure and

$$\hat{\psi}_{1,2} := \psi_{1,2} \circ \Phi_{\ell_1}, \qquad \hat{\tau}_2(n) = \tau_{\ell_2} \circ \Phi_{\ell_1}.$$

Fix some $t \in \mathbb{R}$ in the support of $\psi_{1,2}$. It will also be convenient to use the Lebesgue measure on the Lie algebra $\mathfrak{n}^+ := \mathrm{Lie}(N^+)$ instead of the Haar measure dn. Let dx denote the Lebesgue measure on $\mathfrak{n}^+$, which is induced from some fixed volume form on G. Denote by J_0 the Radon-Nikodym derivative of the pushforward of dn under the inverse of the exponential map with respect to dx. Hence, we can rewrite the above integral as

$$\int_{N^+} e^{ib\hat{\tau}_2(n)}\hat{\psi}_{1,2}(t,n)J\Phi_{\ell_1}^{-1}(n)\,dn = \int_{\mathfrak{n}^+} e^{ib\hat{\tau}_2(x)}\hat{\psi}_{1,2}(t,x)J\Phi_{\ell_1}^{-1}(x)J_0(x)dx, \tag{10.6}$$

where we suppress the implicit composition with the exponential map.

The next step is to select a convenient line in $\mathfrak{n}^+$ to compute the integral over, and estimate trivially in the other directions. Recall that $\mathfrak{n}^+$ and $\mathfrak{n}^- := \mathrm{Lie}(N^-)$ are parametrized by $\mathfrak{K}^{d-1}\oplus \mathrm{Im}(\mathfrak{K})$; cf. Section 10.2. We also recall the elements $n^-_{\rho,\ell}\in N^-$ which were defined by the displacement of the points $x_{\rho,\ell}$ from y_ρ along N^- inside the flow box B_ρ; cf. (8.34).

Let $u\in\mathfrak{K}^{d-1}$ and $s\in\mathrm{Im}(\mathfrak{K})$ be such that

$$(n^-_{\rho,\ell_1})^{-1}\cdot n^-_{\rho,\ell_2} = n^-(u,s). \tag{10.7}$$

First, we suppose that $\mathfrak{K}=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$ so that we may use the formula for the temporal function in (10.5). For $x=(v,r)\in\mathfrak{n}^+$ which is close enough to the origin, we have by (10.5) that

$$\hat{\tau}_2(x) = t + \log\mathrm{Re}\left(1+e^{-t}u\cdot\bar{v} + e^{-2t}\left(s+\frac{\|u\|^2}{2}\right)\left(r+\frac{\|v\|^2}{2}\right)\right).$$

Moreover, by definition of $S_{\rho,j}(\kappa)$, we have that either $\|u\|\gg b^{-\kappa}$ or $\|s\|\gg b^{-\kappa}$. In the first case, set $\hat{u}=u/\|u\|$ and $y:=(\hat{u},0)$. In the case where $\|s\|\gg b^{-\kappa}$, we let $\hat{s}=\bar{s}/\|s\|$ and $y:=(0,\hat{s})$. On the support of our integrals, we have[12] the following elementary estimate in both cases:

$$|\partial_y\hat{\tau}_2(w)|\gg b^{-\kappa}. \tag{10.8}$$

Fix some t and note that the function

$$a(x) := \hat{\psi}_{1,2}(t,x)J\Phi_{\ell_1}^{-1}(x)J_0(x)$$

is C^k with norm satisfying

$$\|a\|_{C^k(\mathfrak{n}^+)} = \left\|\hat{\psi}_{1,2}(t,\cdot)J\Phi_{\ell_1}^{-1}J_0\right\|_{C^k(\mathfrak{n}^+)} \ll_k \left\|\hat{\psi}_{1,2}(t,\cdot)\right\|_{C^k(\mathfrak{n}^+)},$$

where the second inequality follows since the support of $\hat{\psi}_{1,2}(t,\cdot)$ is uniformly bounded in all parameters, and the Jacobians $J\Phi_{\ell_1}^{-1}$ and J_0 have C^k norms $\ll_k 1$ near the origin. Hence, recalling (8.38), we get

$$\|a\|_{C^k(\mathfrak{n}^+)} \ll_k \iota_j^{-2k}m^{2k}. \tag{10.9}$$

We wish to perform integration by parts k times. Denote by M the operator on $C^0(\mathfrak{n}^+)$ given by multiplication by $1/\partial_y\hat{\tau}_2$ and let T denote the operator $\partial_y\circ M$. Then, we observe that

$$\begin{aligned}\int_{\mathfrak{n}^+} e^{ib\hat{\tau}_2(x)}a(x)\,dx &= -\int_{\mathfrak{n}^+} e^{ib\hat{\tau}_2(x)}\partial_y\left(\frac{a(x)}{ib\partial_y\hat{\tau}_2(x)}\right)\,dx\\ &= \cdots\\ &= (-ib)^{-k}\int_{\mathfrak{n}^+} e^{ib\hat{\tau}_2(x)}T^k(a)(x)\,dx \ll b^{-k}|W_\rho|\left\|T^k(a)\right\|_{C^0}.\end{aligned}$$

The following elementary lemma provides the desired estimate on $\left\|T^k(a)\right\|_{C^0}$ and concludes the proof in the case $\mathfrak{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$. Its proof is given at the end of the section .

Lemma 10.1. *We have the following bound on $T^k(a)$:*

$$\left\|T^k(a)\right\|_{C^0} \ll_k \iota_j^{-2k}m^{2k}b^{k\kappa}.$$

[12] Up to scaling down the radius of our flow boxes by an absolute amount if necessary so that $\hat{\tau}_2(w)$ is well-defined on such supports.

Now, suppose $\mathfrak{K}$ is the Octonion algebra $\mathbb{O}$. Denote by θ a Cartan involution of the Lie algebra $\mathfrak{g}$ sending ω to $-\omega$, where $g_t = \exp(t\omega)$. In particular, θ sends $\mathfrak{n}^+$ onto $\mathfrak{n}^-$. Let $(u,s) \in \mathfrak{n}^-$ be as in (10.7). If either u or s is 0, then setting f equal to the non-zero component, one verifies that $(f, \omega, \theta(f))$ span a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Indeed, note that f and $\theta(f)$ are both eigenvectors for $\mathrm{ad}(\omega)$ with the same eigenvalue. Moreover, $Y := [f, \theta(f)]$ has eigenvalue 0 with respect to $\mathrm{ad}(\omega)$ (i.e. Y commutes with ω), while $\theta(Y) = -Y$. This implies that Y is a (non-zero) multiple of ω and completes the verification of the isomorphism $\mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{R})$.

If both u and s are non-zero, then the subalgebra $\mathfrak{h}$ generated by $u, s, \theta(u)$, and $\theta(s)$ is isomorphic to $\mathfrak{su}(2,1)$ by [Hel78, Theorem IX.3.1]. Moreover, $\mathfrak{h}$ contains ω by the argument in the previous case. In either case, the formula (10.5) holds along $\mathfrak{h} \cap \mathfrak{n}^+$, so that we may pick the direction y inside $\mathfrak{h} \cap \mathfrak{n}^+$ and carry out the estimates as above.

Proof of Lemma 10.1. To estimate $\left\|T^k(a)\right\|_{C^0}$, for each $r \in \mathbb{N}$, let $C^r_y(\mathfrak{n}^-)$ denote the space of C^0 functions h on $\mathfrak{n}^-$ so that $\partial^r_y h$ is continuous. We endow this space with the usual C^r norm but where we only measure derivatives using powers of the operator ∂_y. Then, we note that the Leibniz rule (cf. (6.2)) gives

$$\left\|T^k(a)\right\|_{C^0} \leq \left\|M(T^{k-1}(a))\right\|_{C^1_y} \leq \left\|(\partial_y \hat{\tau}_2)^{-1}\right\|_{C^1_y} \left\|T^{k-1}(a)\right\|_{C^1_y}.$$

Thus, estimating $\left\|T^{k-1}(a)\right\|_{C^1_y} \leq \left\|M(T^{k-2}(a))\right\|_{C^2_y}$ and continuing by induction, we obtain

$$\left\|T^k(a)\right\|_{C^0} \leq \left\|(\partial_y \hat{\tau}_2)^{-1}\right\|^k_{C^k_y} \|a\|_{C^k} \ll_k \left\|(\partial_y \hat{\tau}_2)^{-1}\right\|^k_{C^k_y} \times \iota_j^{-2k} m^{2k},$$

where the last inequality follows by (10.9).

It remains to show that $\left\|(\partial_y \hat{\tau}_2)^{-1}\right\|_{C^k_y} \ll_k b^\kappa$. Indeed, the bound on the C^0 norm follows from (10.8). Fix a line $L = v + \mathbb{R} \cdot y \subseteq \mathfrak{n}^+$. Let $g(t) := \partial_y \hat{\tau}_2(v + ty)$ and $f(t) := 1/t$. Then, it suffices to show that $f \circ g$ satisfies the desired bound (on the subset of $t \in \mathbb{R}$ where $v + ty$ belongs to the support of the integrals in question). The latter estimate then follows readily from Faá di Bruno's formula for derivatives of composite functions. In fact, the formula shows that all the higher derivatives are $O_k(1)$, using the explicit shape of $\hat{\tau}_2$.

□

References

[AG13] Artur Avila and Sébastien Gouëzel, *Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow*, Ann. of Math. (2) **178** (2013), no. 2, 385–442. MR 3071503

[AGY06] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz, *Exponential mixing for the Teichmüller flow*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 104, 143–211. MR 2264836

[AM16] Vitor Araújo and Ian Melbourne, *Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor*, Ann. Henri Poincaré **17** (2016), no. 11, 2975–3004. MR 3556513

[Bab02] Martine Babillot, *On the mixing property for hyperbolic systems*, Israel J. Math. **129** (2002), 61–76. MR 1910932

[BD17] Jean Bourgain and Semyon Dyatlov, *Fourier dimension and spectral gaps for hyperbolic surfaces*, Geom. Funct. Anal. **27** (2017), no. 4, 744–771. MR 3678500

[BDL18] Viviane Baladi, Mark F. Demers, and Carlangelo Liverani, *Exponential decay of correlations for finite horizon Sinai billiard flows*, Invent. Math. **211** (2018), no. 1, 39–177. MR 3742756

[BGK07] Jean-Baptiste Bardet, Sébastien Gouëzel, and Gerhard Keller, *Limit theorems for coupled interval maps*, Stoch. Dyn. **7** (2007), no. 1, 17–36. MR 2293068

[BKL02] Michael Blank, Gerhard Keller, and Carlangelo Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity **15** (2002), no. 6, 1905–1973. MR 1938476

[Bow93] B.H. Bowditch, *Geometrical finiteness for hyperbolic groups*, Journal of Functional Analysis **113** (1993), no. 2, 245–317.

[BQ11] Yves Benoist and Jean-François Quint, *Random walks on finite volume homogeneous spaces*, Inventiones mathematicae **187** (2011), no. 1, 37–59.

[BQ16] ———, *Random walks on reductive groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 62, Springer, Cham, 2016. MR 3560700

[BT65] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 55–150. MR 0207712

[But16a] Oliver Butterley, *A note on operator semigroups associated to chaotic flows*, Ergodic Theory Dynam. Systems **36** (2016), no. 5, 1396–1408. MR 3519417

[But16b] ———, *A note on operator semigroups associated to chaotic flows—corrigendum [MR3519417]*, Ergodic Theory Dynam. Systems **36** (2016), no. 5, 1409–1410. MR 3517543

[Cor90] Kevin Corlette, *Hausdorff dimensions of limit sets. I*, Invent. Math. **102** (1990), no. 3, 521–541. MR 1074486

[DG16] Semyon Dyatlov and Colin Guillarmou, *Pollicott-Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146. MR 3556517

[DG18] ———, *Afterword: dynamical zeta functions for Axiom A flows*, Bull. Amer. Math. Soc. (N.S.) **55** (2018), no. 3, 337–342. MR 3803156

[Dol98] Dmitry Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. (2) **147** (1998), no. 2, 357–390. MR 1626749

[EMM98] Alex Eskin, Gregory Margulis, and Shahar Mozes, *Upper bounds and asymptotics in a quantitative version of the oppenheim conjecture*, Annals of Mathematics **147** (1998), no. 1, 93–141.

[EO21] Samuel C. Edwards and Hee Oh, *Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds*, Duke Math Journal, to appear (2021), arXiv:2001.03377.

[GBW22] Yannick Guedes Bonthonneau and Tobias Weich, *Ruelle–pollicott resonances for manifolds with hyperbolic cusps*, J. Eur. Math. Soc. **24** (2022), no. 3, 851–923.

[GL06] Sébastien Gouëzel and Carlangelo Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 189–217. MR 2201945

[GL08] ———, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, J. Differential Geom. **79** (2008), no. 3, 433–477. MR 2433929

[GLP13] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178** (2013), no. 2, 687–773. MR 3071508

[Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR 514561

[Hen93] Hubert Hennion, *Sur un théorème spectral et son application aux noyaux lipchitziens*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 627–634. MR 1129880

[JN12] Dmitry Jakobson and Frédéric Naud, *On the critical line of convex co-compact hyperbolic surfaces*, Geom. Funct. Anal. **22** (2012), no. 2, 352–368. MR 2929068

[Kle10] Dmitry Kleinbock, *Quantitative nondivergence and its Diophantine applications*, Homogeneous flows, moduli spaces and arithmetic, Clay Math. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 2010, pp. 131–153. MR 2648694

[KR95] A. Korányi and H. M. Reimann, *Foundations for the theory of quasiconformal mappings on the Heisenberg group*, Adv. Math. **111** (1995), no. 1, 1–87. MR 1317384

[Liv04] Carlangelo Liverani, *On contact Anosov flows*, Ann. of Math. (2) **159** (2004), no. 3, 1275–1312. MR 2113022

[LNP21] Jialun Li, Frédéric Naud, and Wenyu Pan, *Kleinian Schottky groups, Patterson-Sullivan measures, and Fourier decay*, Duke Math. J. **170** (2021), no. 4, 775–825, With an appendix by Li. MR 4280090

[LP82] Peter D. Lax and Ralph S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*, J. Functional Analysis **46** (1982), no. 3, 280–350. MR 661875

[LP20] Jialun Li and Wenyu Pan, *Exponential mixing of geodesic flows for geometrically finite hyperbolic manifolds with cusps*, arXiv e-prints (2020), arXiv:2009.12886.

[MN20] Michael Magee and Frédéric Naud, *Explicit spectral gaps for random covers of Riemann surfaces*, Publ. Math. Inst. Hautes Études Sci. **132** (2020), 137–179. MR 4179833

[MN21] Michael Magee and Frédéric Naud, *Extension of Alon's and Friedman's conjectures to Schottky surfaces*, arXiv e-prints (2021), arXiv:2106.02555.

[MO15] Amir Mohammadi and Hee Oh, *Matrix coefficients, counting and primes for orbits of geometrically finite groups*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 4, 837–897. MR 3336838

[MO20] Amir Mohammadi and Hee Oh, *Isolations of geodesic planes in the frame bundle of a hyperbolic* 3*-manifold*, arXiv e-prints (2020), arXiv:2002.06579.

[Mos73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. MR 0385004

[Nau05] Frédéric Naud, *Expanding maps on Cantor sets and analytic continuation of zeta functions*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 116–153. MR 2136484

[New03] Florence Newberger, *On the Patterson-Sullivan measure for geometrically finite groups acting on complex or quaternionic hyperbolic space*, Geom. Dedicata **97** (2003), 215–249, Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999). MR 2003699

[Pat76] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), no. 3-4, 241–273. MR 450547

[PPS15] Frédéric Paulin, Mark Pollicott, and Barbara Schapira, *Equilibrium states in negative curvature*, Astérisque (2015), no. 373, viii+281. MR 3444431

[Rob03] Thomas Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. (N.S.) (2003), no. 95, vi+96. MR 2057305

[Sch04] Barbara Schapira, *Lemme de l'ombre et non divergence des horosphères d'une variété géométriquement finie*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 4, 939–987. MR 2111017

[Sch05] ______, *Equidistribution of the horocycles of a geometrically finite surface*, Int. Math. Res. Not. (2005), no. 40, 2447–2471. MR 2180113

[Sma67] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR 228014

[Sto11] Luchezar Stoyanov, *Spectra of Ruelle transfer operators for axiom A flows*, Nonlinearity **24** (2011), no. 4, 1089–1120. MR 2776112

[Sul79] Dennis Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202. MR 556586

[SV95] B. Stratmann and S. L. Velani, *The Patterson measure for geometrically finite groups with parabolic elements, new and old*, Proc. London Math. Soc. (3) **71** (1995), no. 1, 197–220. MR 1327939

[SW20] Pratyush Sarkar and Dale Winter, *Exponential mixing of frame flows for convex cocompact hyperbolic manifolds*, arXiv e-prints (2020), arXiv:2004.14551.

[Yan20] Pengyu Yang, *Equidistribution of expanding translates of curves and Diophantine approximation on matrices*, Invent. Math. **220** (2020), no. 3, 909–948. MR 4094972

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