M = geometrically finite quotient of a rank one symmetric space $\tilde{M}$

$(\mathrm{Isom}(\tilde{M}) = SO(n,1), SU(n,1), Sp(n,1), F_4^{-20})$

g_t = geodesic flow on T^1M

μ = measure of maximal entropy

<u>Thm (K.)</u>

g_t is exponentially mixing w.r.t. μ:

$\exists \delta > 0$: $\forall \varphi, \psi \in C_c^1(T^1M)$, $t \geq 0$

$$\int \psi \cdot \varphi \circ g_t \, d\mu = \int \psi \int \varphi + O_{\varphi,\psi}(e^{-\delta t}).$$

History:

1) Naud, Stoyanov, Sarkar–Winter:

- no cusps
- symbolic dynamics + Dolgopyat method

2) Mohammadi–Oh, Edwards–Oh:

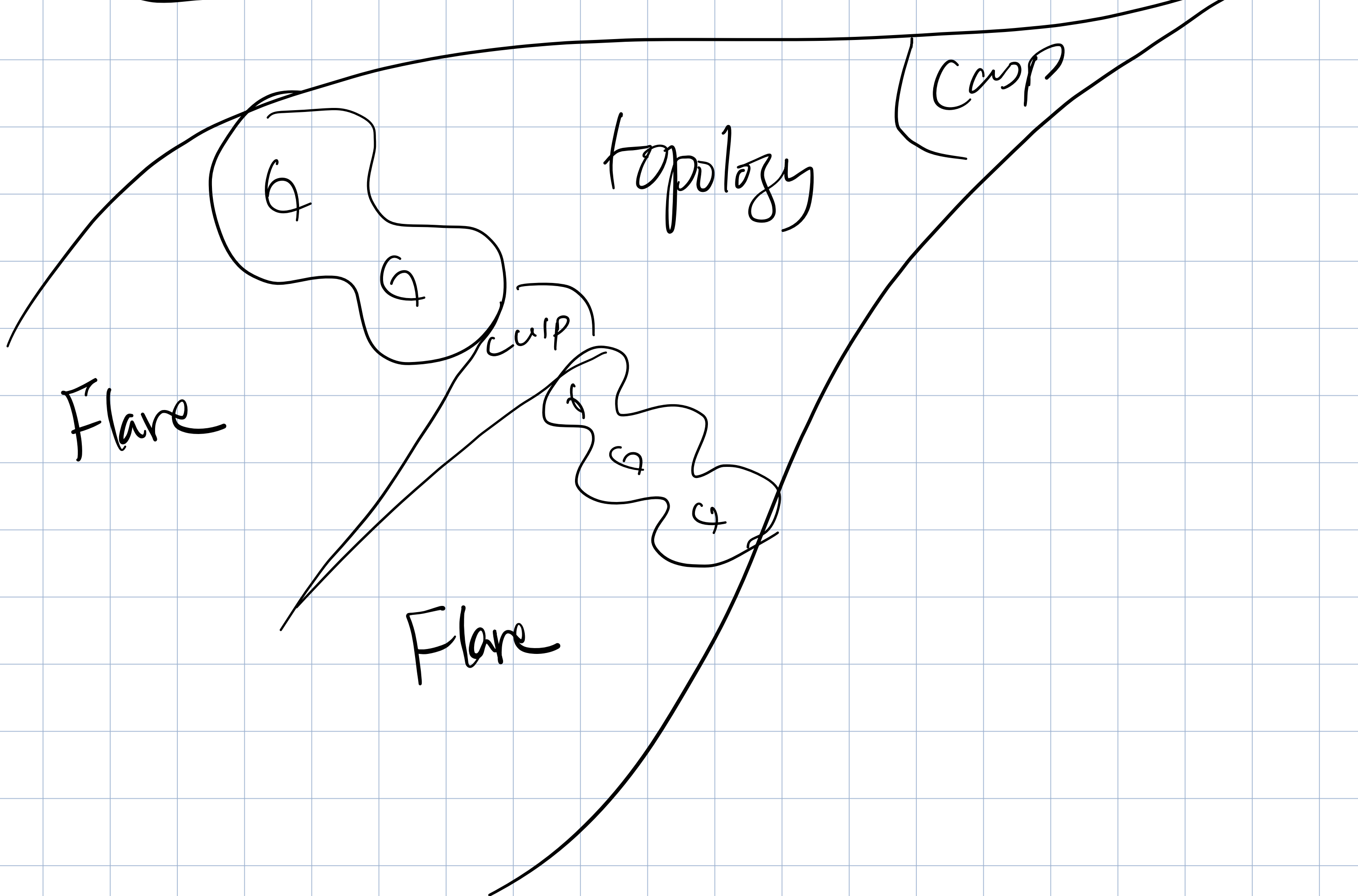
- real hyperbolic
- top entropy > $\frac{1}{2}$ volume entropy (optimal range)
- rep. theory, optimal rates

3) Jialun Li – Wenyu Pan:

- any real hyperbolic with cusps

Rmk: result new for non-real hyp. with cusps, proof is new in all cases.

Geom. finite:



Key features:

1. supp(μ) can have arbitrary Hdim (very <u>not</u> abs. cts).
2. Locally

$$g_t \sim \begin{pmatrix} e^{2t} & & & & \\ & e^{t} & & & \\ & & 1 & & \\ & & & e^{-t} & \\ & & & & e^{-2t} \end{pmatrix} \begin{matrix} \} \text{ unstable} \\ \} \text{ center} \\ \} \text{ stable} \end{matrix}$$

Proof: $\int \varphi \, d\mu = 0$, $\rho(t) = \int \psi \cdot \varphi \circ g_t \, d\mu$

Goal: $|\rho(t)| \lesssim e^{-\delta t}$, $\mathcal{L}_t \varphi = \varphi \circ g_t$

Dream: Find $\|\cdot\|$ on C_c^1:

① $|\rho(t)| \lesssim_\psi \|\mathcal{L}_t \varphi\|$

② $\|\mathcal{L}_t\|_{op} \lesssim e^{-\delta t}$

Candidate (Blank, Keller, Liverani, Gouëzel, Baladi, ...)

$$\|\varphi\| \overset{\text{def}}{=} \sup_{x, \psi, D^{cs}} \underbrace{\int_{W^u_{loc}(x)} \psi \, D^{cs} \varphi \, d\mu^u_x}_{\text{height}(x)}$$

D^{cs}: Center-stable derivatives

* No contraction of $D^c \Rightarrow$ no (2) ☹

Paley-Wiener Thm (Pollicott)

For $z \in \mathbb{C}$, $Re(z) > 0$,

$$\hat{\rho}(z) = \int_0^\infty e^{-zt} \rho(t)\, dt$$

$\hat{\rho}$ extends analytically to $Re(z) > -\varepsilon$ $\Longleftarrow$ $|\rho(t)| \lesssim e^{-\delta t}$

(Precise statement used due to Butterley)

Def: $R(z) = \displaystyle\int_0^\infty e^{-zt} \mathcal{L}_t \, dt$

$z \longmapsto$ Bounded operators on $\|\cdot\|$

$Re(z) > 0$

$$\hat{\rho}(z) = \int \psi \; R(z)(\varphi)\, d\mu$$

Goal: extend $R(z)$ analytically to $Re(z) > -\varepsilon$

Lemma $X \stackrel{def}{=} \frac{d}{dt}\Big|_{t=0} \mathcal{L}_t$ exists

semigroup property $\Rightarrow \mathcal{L}_t \text{ "=" } e^{tX}$, $X|_{C^1} = D^c$

$$R(z) = \int_0^\infty e^{-zt+tX}\, dt = (z\,Id - X)^{-1}$$

$$\{\text{poles of } z \mapsto R(z) \text{ in } Re(z) > -\varepsilon\} = Spec(X) \cap \{Re\, z > -\varepsilon\}$$

$$= \bigcup_{R(z) > -\varepsilon} \left\{ z - \frac{1}{spec(R(z))} \right\}$$

* Show spectral radius of $R(z)$ is $\frac{1}{Re(z)+\varepsilon}$

when $|Im(z)| \gg 1$

$\Uparrow$

$$\| R(z)^m \| \lesssim \frac{1}{(a+\varepsilon)^m} \quad (\cancel{A})$$

$z = a+ib$,

$m = \log|b|$

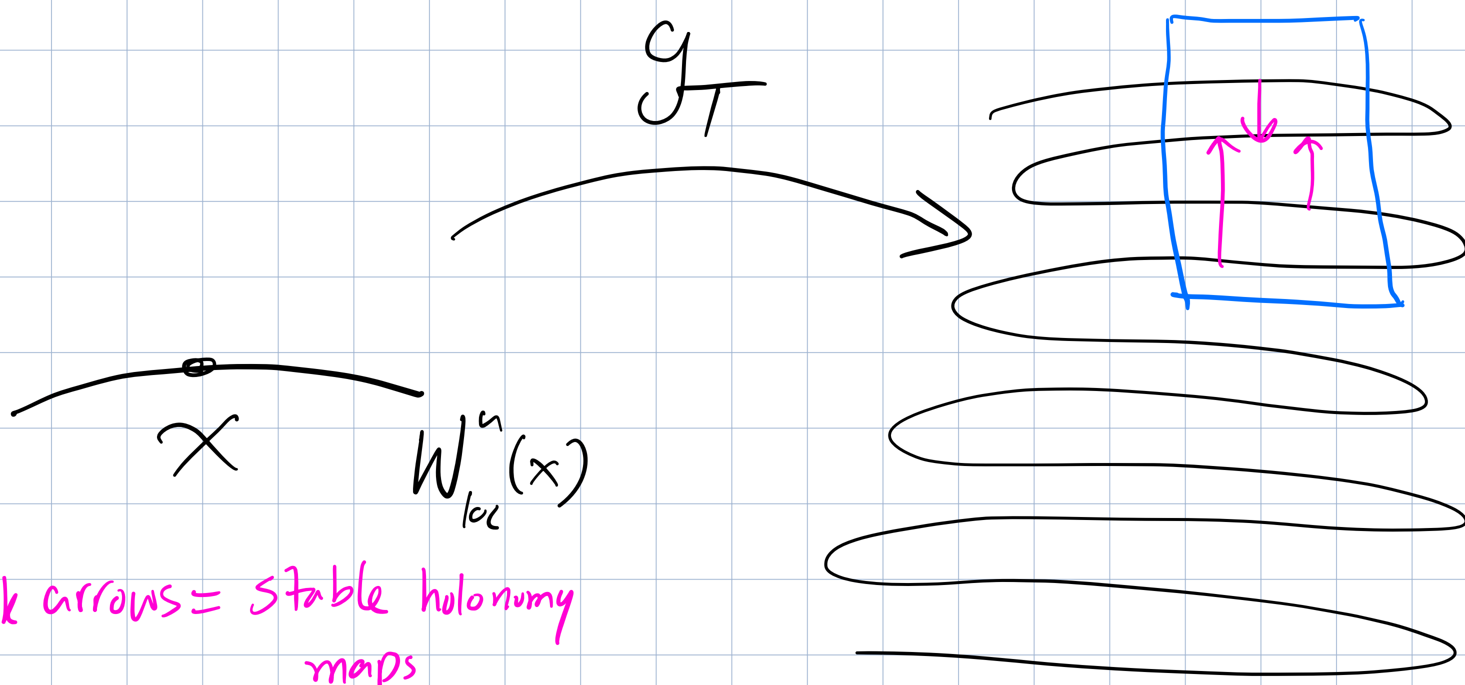
$$R(z)^{m+1} = \int_0^\infty \frac{t^m}{m!} e^{-zt} \mathcal{L}_t \, dt$$

$\rightarrow$ bulk at $t \sim m$

$\Uparrow$

For $T \sim m$, show

$$\left| \int_T^{T+1} e^{-ibt} \int_{W^u_{loc}(x)} \varphi \circ g_t \, d\mu^u_x \, dt \right| \lesssim |b|^{-\eta} \|\varphi\|, \text{ for some } \eta > 0$$



pink arrows = stable holonomy maps

localize in flow boxes

$$\sim \sum_{y_k} \int_T^{T+1} e^{-ibt} \int_{W^u(y_k)} \mathcal{L}_t \varphi \, d\mu^u_{y_k} \, dt$$

Stable holonomy (ignore φ & $\int_T^{T+1} dt$)

$$\sim \sum_k \int_{W^u_{loc}(y_\star)} e^{ib\tau_k(n)} \, d\mu^u_{y_\star}(n)$$

$$(t, n) \longmapsto \left(t + \tau_k(n), \underset{(t,n)}{\text{function of}} \right)$$

Linearize τ_k:

$$\sim \sum_k \widehat{\mu^u_{y_\star}}(b \xi_k)$$

$\widehat{\ }$ → Fourier transform

ξ_k ↪ $\nabla \tau_k$

Thm (K.)

$\nu =$ cpt supp. prob. meas. on $\mathbb{R}^n$:

Assume

$\forall \varepsilon > 0,\ r > 0$

$$(\#) \quad \nu\left(W^{(\varepsilon r)} \cap B_r\right) \leq \delta(\varepsilon)\, \nu(B_{cr}),$$

where $\delta(\varepsilon) \to 0$ w/ ε.

Then $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall N \geq 1,$

$$\#\left\{ \|\xi\| \leq N \;\middle|\; |\hat{\nu}(\xi)| > N^{-\delta} \right\} \lesssim N^{\varepsilon}$$

* Uses ideas from:

Shmerkin, Hochman, Bourgain

End of Proof:

1) Verify (*) for $\mu_{y_k}^u$:

→ fails in presence of cusps, but a suitable sufficient weaker form holds allowing for small exceptions

2) Verify that the frequencies ξ_k are well-separated:

→ Consequence of non-divergence estimates + polynomial decay of PS-mass of ε-nbhds of proper subvarieties on boundary

→ Latter estimate is very delicate in cusped, non-real hyp. case & is in fact itself a consequence of the flattening Thm.