Flattening, Mixing, & Fourier Decay

Q: μ = Borel prob meas. on $\mathbb{R}^n$

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi \cdot x} \, d\mu(x)$$

What can be said about the set

$$\mathcal{E}(\mu) = \{ \|\xi\| \leq R : |\hat{\mu}(\xi)| \text{ is large} \} ?$$

size, dimension, distribution, ...

$$\mu \xleftarrow[\text{transform}]{\text{Fourier}} \mathcal{E}$$

I) Examples:

① Diophantine approximation on manifolds:

M = bounded subfld of $\mathbb{R}^n$,

$q \in \mathbb{N}$, $\varepsilon(q) > 0$

$\mathcal{U} = \varepsilon(q)$- nbhd of M.

$$\sum_{\vec{p} \in \mathbb{Z}^d} \mathbb{1}_{\mathcal{U}}\left(\vec{p}/q\right) \underset{\text{Poisson summation}}{\approx} \sum_{\vec{p} \in \mathbb{Z}^d} \hat{\mu}(\vec{p}),$$

where μ = Lebesgue measure on qM

Approaches: decompose M into

flat part & curved part

has small measure

use rapid decay of $\hat{\mu}$

② Dirichlet Polynomials:

$$D_N(t) = \sum_{N}^{2N} a_n n^{it}, \quad |a_n| \leq 1$$

$$= \widehat{\mu}(t), \text{ where}$$

$$\mu = \sum_{N}^{2N} a_n \delta_{\log n}$$

Guth–Maynard '24:

Strong bounds on $|\mathcal{E}(\mu)|$

$\Longrightarrow$ Counts of Riemann zeros in critical strips

corollaries

& Counting of primes in short intervals

Cartoon of Proof:

$$\mathcal{E} \to \mathcal{E} \text{ is structured (large additive energy)}$$

↳ use "algebraic methods", properties of $\log n$

$$\mathcal{E} \to \mathcal{E} \text{ is not structured}$$

↳ use "analytic methods" stationary phase arguments

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(3) Rigidity of Random Walks:

ν = finitely supp. prob. meas. on $SL(d, \mathbb{Z})$

$\langle \operatorname{supp} \nu \rangle$ = Zariski-dense

Thm (Bourgain–Furman–Lindenstrauss–Mozes)

Let $x \in \mathbb{T}^d$, $\mu_n = \nu^{*n} * \delta_x$

Then,

$$x \notin \mathbb{Q}^d/\mathbb{Z}^d \implies \mu_n \xrightarrow[n\to\infty]{} \mathrm{Leb}_{\mathbb{T}^d}$$

with a rate

Cartoon of Proof: $\hat{\mu}_n(\xi) = \sum_{\gamma \in SL(d,\mathbb{Z})} \nu^{*m}(\gamma)\, \hat{\mu}_{n-m}(\gamma^* \xi)$

$\mathcal{E}(\mu_n) \neq \emptyset$

$\underset{\text{hard work}}{\implies}$ $\mathcal{E}(\mu_{n-m})$ has large covering number at most scales.

$\Downarrow$

μ_{n-m} is supported on a small cardinality

$\Downarrow$

x rational with bounded denominator

④ Quantum Unique Ergodicity

Conj: $|\varphi_j|^2 dx \xrightarrow[\text{eigenval} \to \infty]{} dx$

Dyatlov-Jin: reduce the question of whether limit measures have full support to Fractal Uncertainty Principle (FUP)

Thm (Bourgain-Dyatlov, Cohen)

Let X, Y porous fractals in $\mathbb{R}$

Let $f \in L^2(\mathbb{R})$, $\operatorname{supp} \hat{f} \subseteq Y$.

Then $\operatorname{supp} f \not\subseteq X$ quantitatively.

II) Results

Thm A (K. '23+)

Let μ = Prob. meas., cpt'ly supp on $\mathbb{R}^d$. Then,

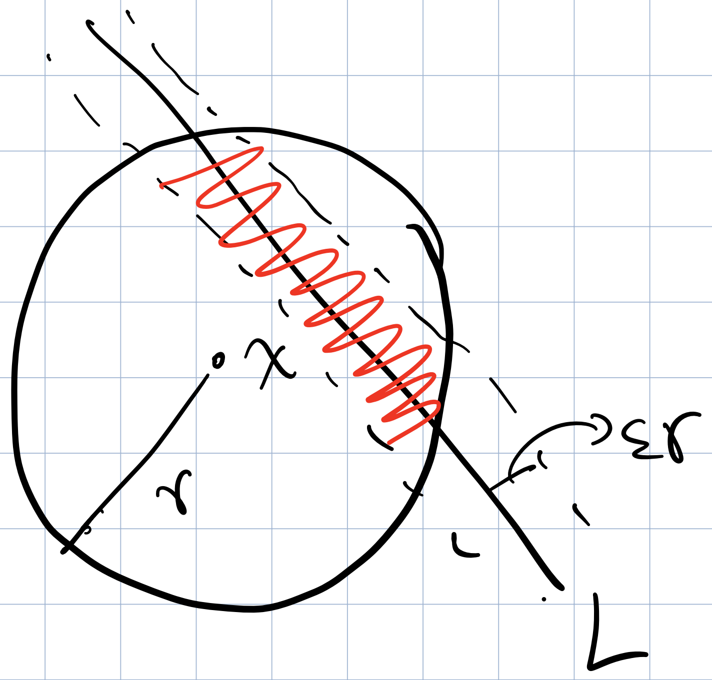
① $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall R > 1$:

$$\left| \left\{ \|\xi\| < R : |\hat{\mu}(\xi)| > \|\xi\|^{-\delta} \right\} \right|$$

$$\lesssim R^{\varepsilon}$$

has small covering number

② for many pts $x \in \mathrm{Supp}(\mu)$, & many scales $R^{-1} \leq r \leq 1$



most of the mass of $\mu|_{B(x,r)}$ $\subseteq B(L, \varepsilon r)$

Consequences of Thm A:

① Exponential Mixing:

Thm B (K. '23+)

X = unit tangent bundle of a geometrically finite, negatively curved, locally symmetric space.

μ = measure of maximal entropy

for g_t = geodesic flow on X.

Then, g_t is exp. mixing w.r.t. μ; i.e.

$\exists \delta > 0, \forall \varphi, \psi \in C_c^1(X)$,

$$\int_X \varphi \circ g_t \, \psi \, d\mu = \int_X \varphi \, d\mu \int_X \psi \, d\mu + O_{\varphi, \psi}(e^{-\delta t}).$$

Previous Work:

① Naud, Stoyanov, Sarkar-Winter, Chow-Sarkar

* Convex cocpt mflds
* Method: thermodynamic formalism + Dolgopyat's method

② Mohammadi–Oh '12:

* Frame Flow
* geometrically finite quotients of $\mathbb{H}^n_{\mathbb{R}}$
* $\delta_\Gamma > n-2$ if $n \geq 4$

  $\delta_\Gamma > (n-1)/2$ if $n = 2, 3$

* Method: representation theoretic

③ Edwards–Oh '21:

* geodesic flow, geom. finite $\mathbb{H}^n_{\mathbb{R}}/\Gamma$
* $\delta_\Gamma > (n-1)/2$ $\forall n$ (optimal)
* rep. theory

* optimal rates of mixing

(4) <u>Jialun Li – Wenyu Pan '19</u>:

* settle longstanding case of $\mathbb{H}^n_{\mathbb{R}}/\Gamma$, Γ = geom. finite & arbitrary δ_Γ.

* Constructed a coding of g_t using ∞ alphabet + significant generalization of Dolgopyat method

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<u>Rmk</u>: The pf of Thm B is new in all cases, the new cases are cusped non-real hyp. mflds.

Cartoon of Pf: $\int \psi \, d\mu = 0.$

$$\rho(t) = \int \varphi \circ g_t \, \psi \, d\mu$$

Paley-Wiener Thm: $z \in \mathbb{C}$, $\operatorname{Re}(z) > 0$

$$\hat{\rho}(z) = \int_0^\infty e^{-zt} \rho(t) \, dt$$

Laplace transform

$\hat{\rho}(z)$ admits analytic extension to $\operatorname{Re}(z) > -\varepsilon$ $\Longleftarrow$ $|\rho(t)| \lesssim e^{-\delta t}$

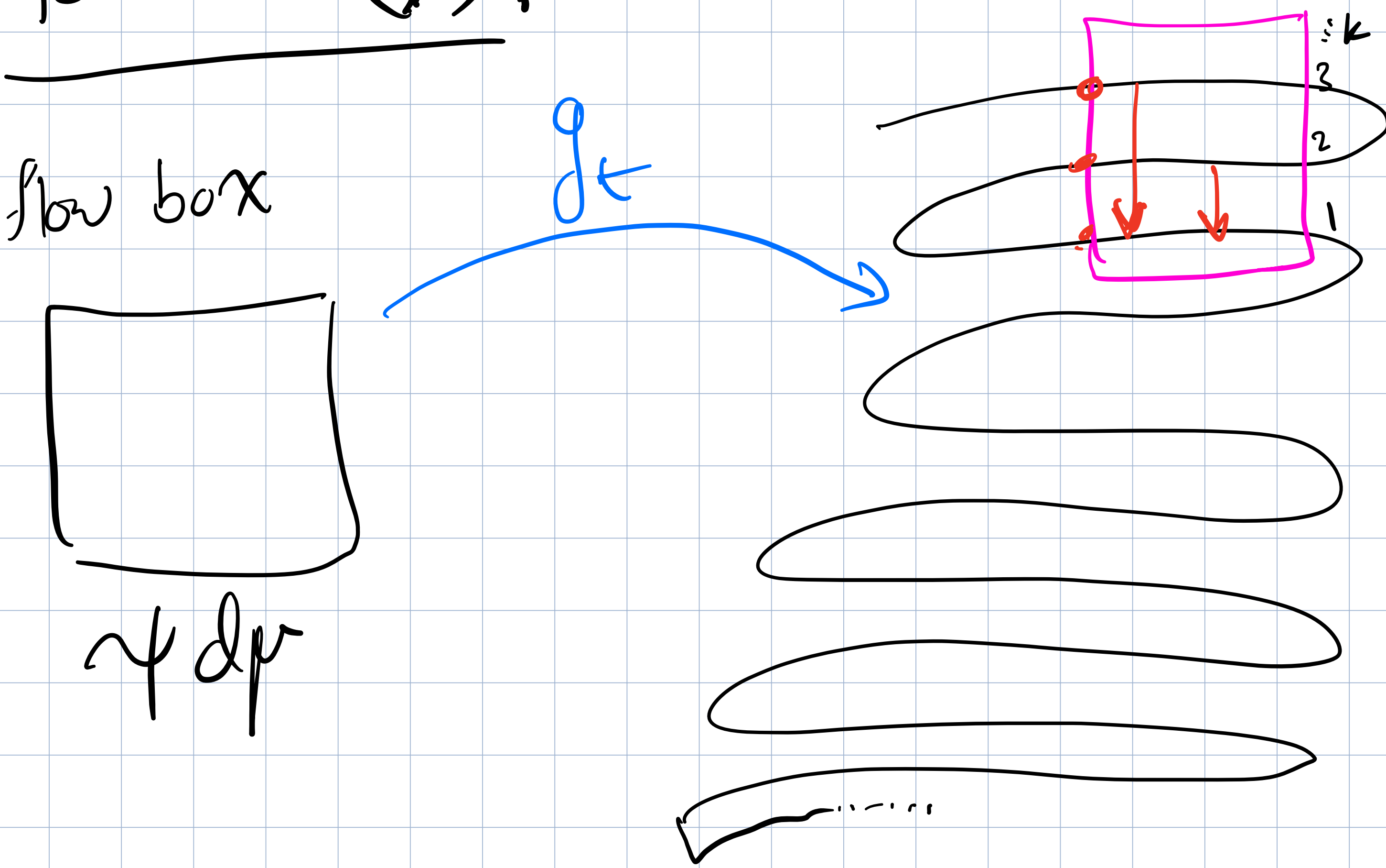
Let $z = a + ib$.

anisotropic Banach spaces $\rightsquigarrow$

$$\left| \int_T^{T+1} e^{-zt} \rho(t) \, dt \right| \lesssim e^{-aT} |b|^{-\varepsilon} \quad (*)$$

for $T \approx \log b$.

Towards (✱):



localize & local stable holonomy:

$$\int_{T}^{T+1} e^{-zt} \int \varphi \circ g_t \, \psi \, d\mu(x) \, dt$$

weak unstable mfld

$$(x,t) \xrightarrow[\text{stable hol}]{} \tau_k(x,t)$$

<u>Ignore φ & ψ</u> & $\int dt$ & e^{-at}

$$\rightsquigarrow \quad \sum_k \int_{\substack{\text{local}\\ \text{unstable}\\ \text{leaf}}} e^{-ib\tau_k(x,t)} \, d\mu(x)$$

<u>Linearize τ_k</u>: $\quad \xi_k = d\tau_k$

$$\sum_k \int e^{-ib\xi_k \cdot x} \, d\mu(x)$$

$$= \sum_k \hat{\mu}(b\xi_k)$$

<u>Prop</u>: μ doesn't satisfy alternative (2) in Thm <u>A</u>.

<u>Prop ^{2}</u>: ξ_k are well-separated

Prop 1 + 2 + Thm A $\Longrightarrow$ we're done.

<u>Thm C</u> (Baker–K.–Sahlsten '24+)

Let μ be one of the following:

(1) Diophantine self-similar measure

(2) Zariski-dense convex cocpt Patterson–Sullivan measure

(3) Non-integrable self-conformal meas.

(4) many more examples ...

Then, $\hat{\mu}(\xi) \longrightarrow 0$ as $\|\xi\| \to \infty$

with a rate

<u>Rmk</u>: The red words are conditions that allow to verify separation of the freqs that arise when

$\hat{\mu}(\xi) =$ avg over $\hat{\mu}(\xi_n)$

using the defining dynamics

<u>E.g.</u>: $f_1(x) = \frac{1}{2}x$, $f_2(x) = \frac{1}{3}x + \pi$

$$\mu = \frac{1}{2}(f_1)_* \mu + \frac{1}{2}(f_2)_* \mu$$

$$f_\omega(x) = r_\omega x + t_\omega$$

$$r_\omega = \left(\frac{1}{2}\right)^a \left(\frac{1}{3}\right)^b$$

<u>Thm</u> (Li-Sahlsten) $|\hat{\mu}(\xi)| \lesssim (\log \xi)^{-k}$, $k>0$.

Sketch (Baker - Sahlsten - K.)

$$\mathcal{W} = \{ \overset{\text{words}}{\omega} : r_\omega \xi \asymp (\log \xi)^c \}$$

$$|\hat{\mu}(\xi)| \leq \sum_{\omega \in \mathcal{W}} \mathbb{P}(\omega) \, |\hat{\mu}(r_\omega \xi)| \quad \text{(averaging)}$$

$$\# \mathcal{W} \asymp ((\log \xi)^{O(1)})$$

$$|r_{\omega_1} \xi - r_{\omega_2} \xi| \geq |\xi r_{\omega_1}| \left| \frac{r_{\omega_2}}{r_{\omega_1}} - 1 \right|$$

$$\gtrsim (\log \xi)^c \log \left| \frac{r_{\omega_2}}{r_{\omega_1}} \right|$$

$$\left| \log \frac{r_{\omega_2}}{r_{\omega_1}} \right| = |p \log 2 + q \log 3| \quad (*) \; p, q \in \mathbb{Z}$$

Diophantine $\Longrightarrow$ $(*) \gtrsim |q|^{-\lambda}$, $\lambda > 0$.

(Separation)

Apply flattening $\Longrightarrow$ $|\hat{\mu}(r_\omega \xi)|$ small for most ω.