Horocycles in the level aspect & a question of Mahler

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SOME SUGGESTIONS FOR FURTHER RESEARCH

KURT MAHLER

At the age of 80 I cannot expect to do much more mathematics. I may, however, state a number of questions where perhaps further research might lead to interesting results.

Mahler's Problem (1984)

How well can irrational points on Cantor's middle thirds set $\mathcal{C}_3$ be approximated by:

1) $\mathbb{Q} \cap \mathcal{C}_3$?

2) $\mathbb{Q}$?

Khintchine's Theorem

- $\psi : \mathbb{N} \to \mathbb{R}_+$

- $W(\psi) = \left\{ \vec{x} \in \mathbb{R}^d : \left\| \vec{x} - \frac{\vec{p}}{q} \right\| < \frac{\psi(q)}{q} \text{ for } \infty\text{-many } (\vec{p}, q) \in \mathbb{Z}^d \times \mathbb{N} \right\}$

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- E.g.: $\psi_\varepsilon(q) = q^{-(1/d + \varepsilon)}$.

$$\bigcup_{\varepsilon > 0} W(\psi_\varepsilon) = \text{Very Well Approximables.}$$

Khintchine's Theorem

Theorem (Khintchine 1926)

$\psi : \mathbb{N} \longrightarrow \mathbb{R}_+$ non-increasing. Then,

$$\mathrm{Leb}(W(\psi)) = \begin{cases} 0 & \sum_{q \geq 1} \psi^d(q) < \infty, \\ \text{Full} & \sum_{q \geq 1} \psi^d(q) = \infty. \end{cases}$$

History

Defn $\alpha \in \mathbb{R}$ algebraic if $\exists f \in \mathbb{Z}[X]$, $f(\alpha) = 0$.

Observe:

$x \in \mathbb{R}$ is VWA by alg. numbers of degree $\leq d$

$\leftrightarrow (x, x^2, \dots, x^d) \in \text{VWA} \subseteq \mathbb{R}^d.$

History - 1900's

- **Conjecture**: (Mahler, Sprindžuk, Baker 30s-80s) measure of intersection of VWA with non-degenerate manifolds is zero.
  (e.g: $x \mapsto (x, x^2, \ldots, x^d)$)

- **Kleinbock-Margulis (Annals '98)**:
  resolved using homogeneous dynamics.

History – 2000's

- **Kleinbock – Lindenstrauss – Weiss (2004):**

Does an analog of Khintchine Theorem hold for friendly measures?

(e.g: manifolds, self-similar fractals)

More on fractals

- **Theorem** (Weiss '02, Pollington-Velani '06)

μ = measure on a Cantor set, $s = \dim \mathcal{C}$.

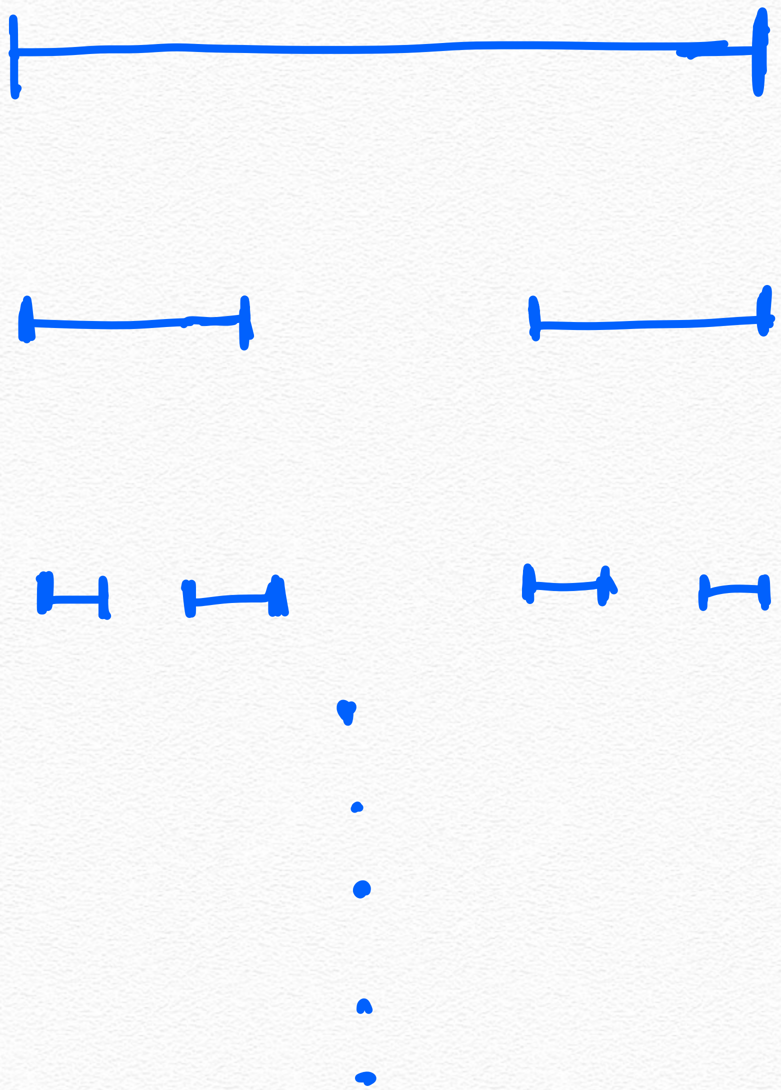
Then,
$$\sum_{q \geq 1} q^{s-1} \psi^{s}(q) < \infty \implies \mu(W(\psi)) = 0.$$

- **Question**: Is this optimal?

Missing Digit Sets:

- $D \in \mathbb{N}$, $\Delta \subseteq \{0, 1, \dots, D-1\}$.
- $\mathcal{C}_\Delta := \left\{ x \in [0,1] : \begin{array}{l} D\text{-adic expansion of } x \\ \text{belongs to } \Delta \end{array} \right\}$
- E.g. $D=3$, $\Delta = \{0, 2\} \rightsquigarrow \mathcal{C}_\Delta =$ Cantor's Set.

Missing Digit Measures



- $\mu_\Delta :=$ limit of restrictions of Lebesgue measure

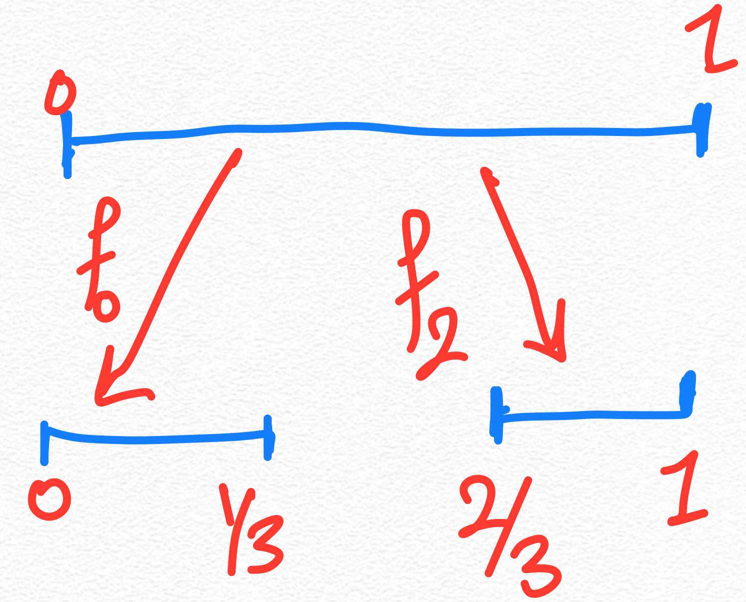
$$\boxed{\dim \mathcal{C}_\Delta = \frac{\log \#\Delta}{\log D}}$$

Self-Similarity

- $f_i(x) = \dfrac{x+i}{D}$, $i \in \Delta$.

- $\mathcal{C}_\Delta = \bigcup_{i \in \Delta} f_i(\mathcal{C}_\Delta)$.

- $\mu_\Delta = \dfrac{1}{\#\Delta} \sum_{i \in \Delta} (f_i)_* \mu_\Delta$.



Khintchine on Fractals

Theorem (K.-Luethi '21):

Assume D is prime & $\dim \mathcal{C}_\Delta \geq 0.84$.

Then,

$$\mu_\Delta(W(\psi)) = \begin{cases} 0 & \sum_{q\geq 1} \psi(q) < \infty, \\ 1 & \sum_{q \geq 1} \psi(q) = \infty. \end{cases}$$

Remarks

- $\dim(\text{Cantor's set}) = 0.6\ldots$
- $D=5$, $\#\Delta=4$ works.
- Even the convergence part is new.
- Intuition: $\mathbb{E}(W(\psi)\,|\,e_\Delta) = \mathbb{E}(W(\psi))$.

A Counter-example

Prop: $\exists$ a prob. measure μ on $\mathbb{R}$:

1) $\dim(\operatorname{supp}\mu)$ arbitrarily close to 1.

2) Ahlfors-regular: $\mu(B(x,r)) \asymp r^{\delta}$.

3) Fourier transform of μ decays polynomially.

4) μ is self-conformal (non-linear)

But, μ fails both theorems.

Homogeneous Dynamics

- $G = SL_{d+1}(\mathbb{R})$, $\Gamma = SL_{d+1}(\mathbb{Z})$.

- $X := G/\Gamma$

- $g_t = \begin{pmatrix} e^t & & & \\ & e^t & & \\ & & \ddots & \\ & & & e^{-dt} \end{pmatrix}$, $t \in \mathbb{R}$

$u(\vec{x}) = \begin{pmatrix} 1 & & & \\ & 1 & & \vec{x} \\ & & \ddots & \\ & & & 1 \end{pmatrix}$, $\vec{x} \in \mathbb{R}^d$

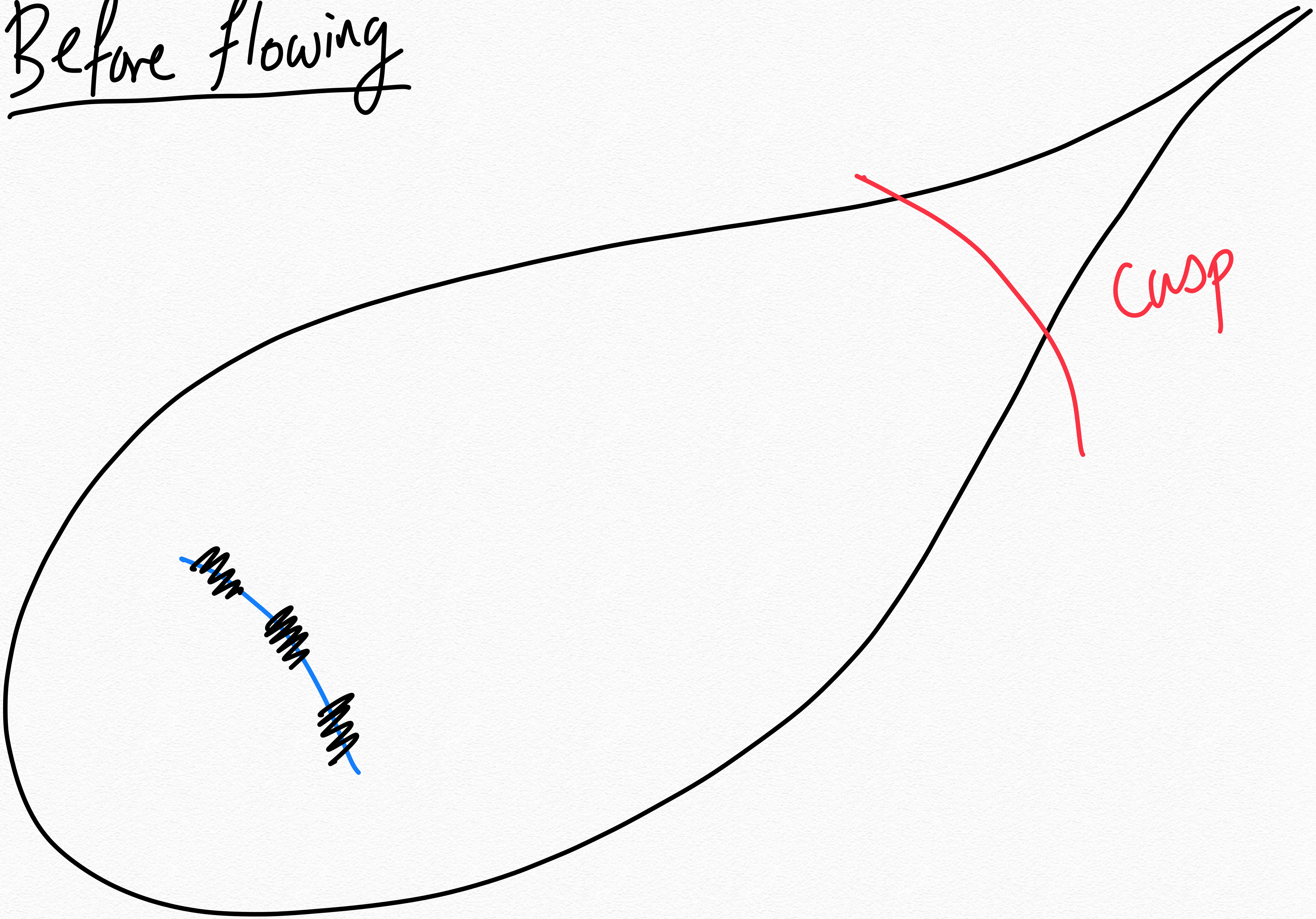
Dynamics – Shrinking targets

Prop (Kahzdan, Sullivan, Dani, Kleinbock-Margulis)

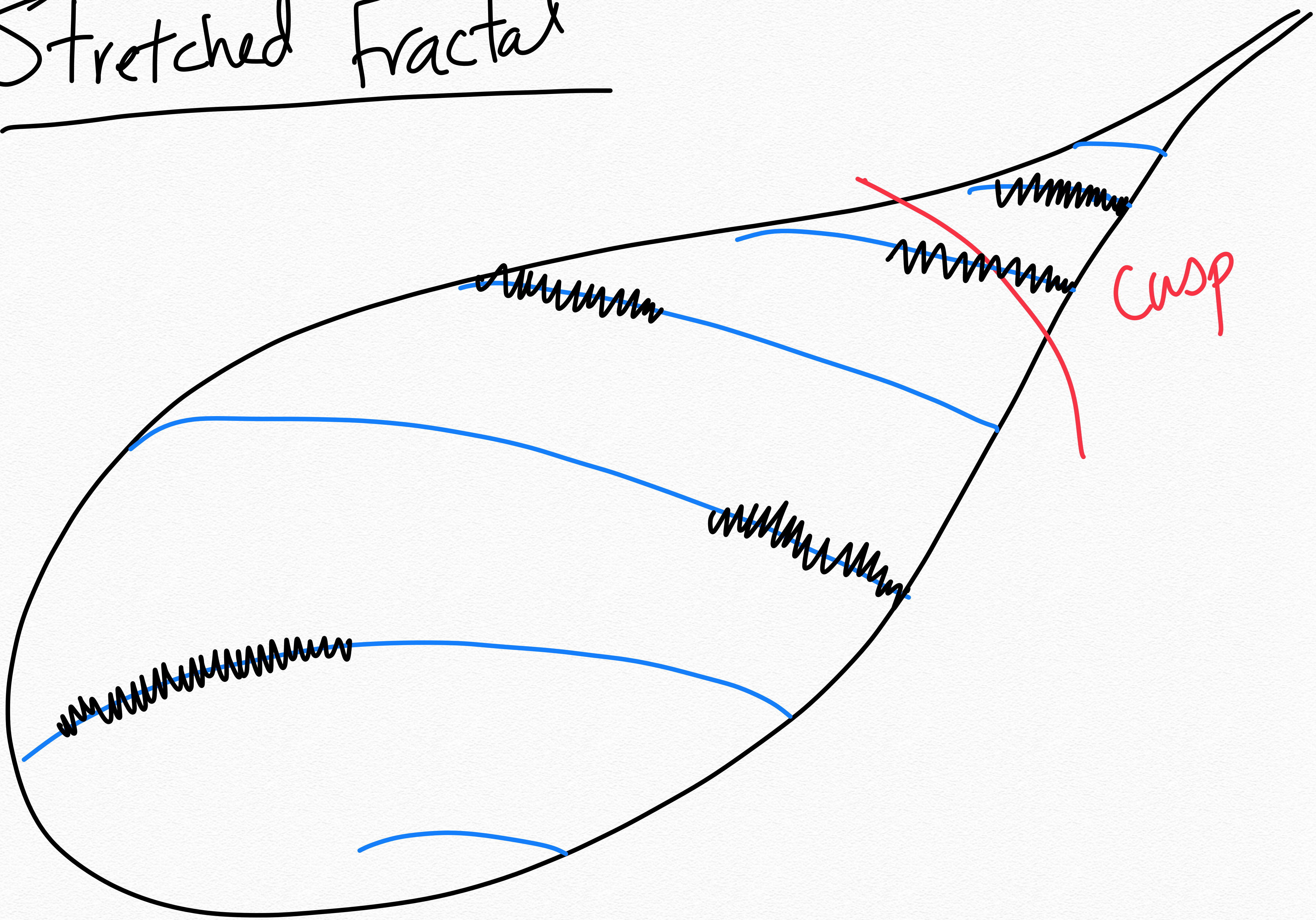
$x \in W(\psi) \longleftrightarrow$

$\left(g_t u(x)\Gamma\right)_{t \geq 0}$ Visits a shrinking sequence of cusp nbhds ∞-often as $t \to \infty$.

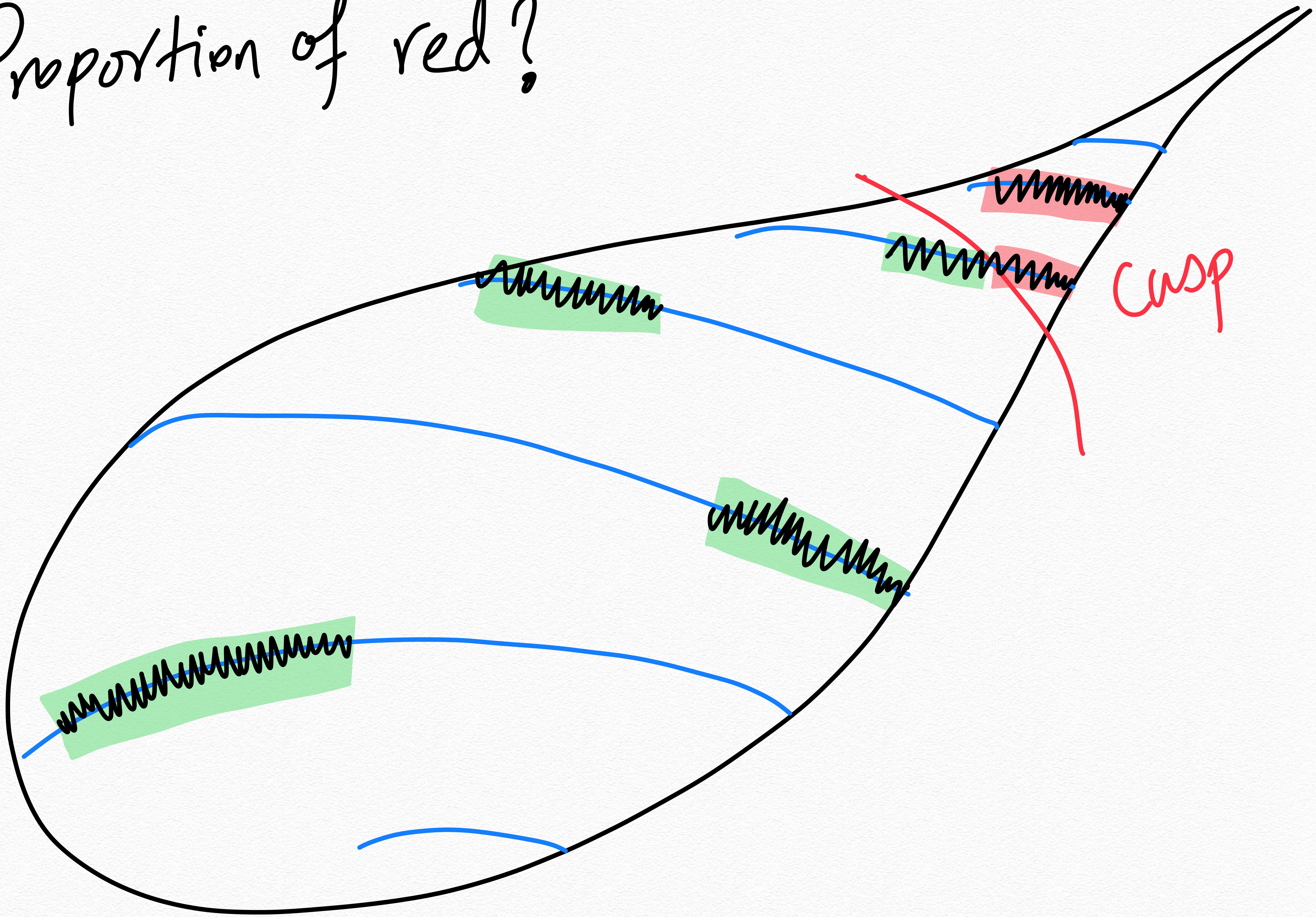
Before flowing



Stretched Fractal



Proportion of red?



Key Ingredient

Theorem (K.-Luethi '21):

Assume D is prime and $\dim \mathcal{C}_\Delta \geqslant 0.84$.

Then, $\exists\, \sigma > 0$, $\forall\, t \geqslant 0$, $\varphi \in C_c^\infty(X)$,

$$\int \varphi(g_t u(x)\Gamma)\, d\mu_\Delta(x) = \int_X \varphi + O_\varphi(e^{-\sigma t}).$$

Random Walks

- Recall: $f_i(x) = \frac{x+i}{D}$, $\mu_\Delta = \frac{1}{\#\Delta} \sum_{i\in\Delta} (f_i)_* \mu_\Delta$.

- **Key Idea:**

$$\begin{pmatrix} 1/D & 0 \\ 0 & 1 \end{pmatrix} g_t u(x) \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i/D \\ 0 & 1 \end{pmatrix}$$

$$= g_t u(f_i(x)).$$

Random Walks

- Define $\mathcal{P}_\Delta : C(X) \longrightarrow C(X)$ by

$$\mathcal{P}_\Delta(\varphi)(g\Gamma) = \frac{1}{\#\Delta} \sum_{i \in \Delta} \varphi\left(\begin{pmatrix} 1/D & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i/D \\ 0 & 1 \end{pmatrix} \Gamma \right).$$

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- <u>Self-similarity:</u>

$$\int P_\Delta(\varphi)(g_t u(x)\Gamma)\, d\mu_\Delta = \int \varphi(g_t u(x)\Gamma)\, d\mu_\Delta$$

Random Walks

- Not well-defined ----

Random Walks

- $\Gamma = SL_2\mathbb{Z}$ & operator supported on rational matrices.
- P_Δ is well-defined on a finite cover

$$P_\Delta : C(G/\Gamma) \longrightarrow C(G/\underbrace{\Gamma_1}_{X_1})$$

Random Walks

$$\begin{array}{c} \vdots \\ \downarrow \\ X_2 \\ \downarrow \\ X_1 \\ \downarrow \\ X_0 \end{array}$$

- Pass to inverse limit:

$$SL_2(\mathbb{R}\times\mathbb{Q}_D)/SL_2(\mathbb{Z}[1/D])$$

- **Prop (K.-Luethi):** $P_\Delta \curvearrowright L^2$ has a spectral gap

- Venkatesh's trick.

Equidistribution of Horocycles

Thm $\forall \varphi \in C_c^\infty(X_n)$, $t \geq 0$, $n \geq 1$:

$$\int_0^1 \varphi(g_t u(x) \Gamma_n)\, dx = \int_{X_n} \varphi + O_\varphi\left(\mathrm{Vol}(X_n)^\delta e^{-\sigma t}\right).$$

We can take $\delta = 1/2$, σ "=" spectral gap.

Subconvexity

Question 1:

Can δ be improved (without sacrificing σ)?

Effective Joinings?

- **Fact:** X_n embed into $G \times G / \Gamma \times \Gamma$ as closed orbits of $\Delta(G)$.

Effective Joinings?

- **Fact:** X_n embed into $G \times G/_{\Gamma \times \Gamma}$ as closed orbits of $\Delta(G)$.

Question 2:

Can δ be improved for $\varphi \in C_c^\infty(G^2/\Gamma^2)$?

Sup norm problem

Sobolev Embedding Thm: $\forall\, \varphi \in C_c^\infty(X_n)$:

$$\|\varphi\|_\infty \lesssim \mathrm{Vol}(X_n)^{\alpha} \cdot H^s(\varphi),$$

for $\alpha = 1/2$.

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Thm (Burger, Flaminio-Forni, Strombergsson)

We can take $\delta = \alpha$.

Sup norm problem

Thm (Iwaniec-Sarnak, Blomer-Holowinsky, ...)

α can be improved when φ is a Hecke-Maass eigenform on Hecke congruence covers

Thanks!