Bounded and Divergent Orbits and Expanding Curves on Homogeneous Spaces

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The main objects

- G is a connected Lie group with Lie algebra $\mathfrak{g}$.
- g_t is Ad-diagonalizable over $\mathbb{R}$:

$$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \left\{ X \in \mathfrak{g} : g_t X g_{-t} = e^{\alpha(t)} X \right\}$$

- $u(Y) = \exp(Y)$ for $Y \in \mathfrak{g}$.
- X a topological space and $G \curvearrowright X$.

Admissible Curves

Definition

A map $\varphi : [0, 1] \to \mathfrak{g}$ is **$\mathbf{g_t}$-admissible** if:

1. φ is C^2 and $\dot{\varphi} \not\equiv 0$.
2. g_t normalizes $\dot{\varphi}$: the image of φ is contained in $\mathfrak{g}_\alpha$ for some $\alpha > 0$.
3. φ commutes with $\dot{\varphi}$: $[\varphi, \dot{\varphi}] \equiv 0$.

Central Questions

- Estimate the Hausdorff dimension of the set of parameters $s \in [0, 1]$:
  1. $g_t u(\varphi(s))x_0$ diverges on average in X: for any compact set $K \subseteq X$:
  $$\frac{1}{T}\int_0^T \chi_K(g_t u(\varphi(s))x_0)\, dt \to 0$$
  2. $g_t u(\varphi(s))x_0$ remains inside a compact subset of X for all $t > 0$.

Real Rank One Manifolds

- T^1M: unit tangent bundle of a rank 1, locally symmetric manifold of finite volume, $p \in M$.
- $g^t : T^1M \to T^1M$: the geodesic flow.
- $\varphi : [0, 1] \to T^1_pM$ a g^t-admissible map. (Automatic for $\mathbb{H}^n$) .

Theorem (K. '18)

The Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. *$g^t\varphi(s)$ diverges on average is at most $1/2$.*
2. *$g^t\varphi(s)$ is bounded is equal to 1. (This set is winning).*

- Remark: (2) was previously obtained by Aravinda and Leuzinger by different methods (ETDS '95).

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Diophantine Approximation in Number Fields

- $K = \mathbb{Q}(\alpha)$ a number field of degree d, e.g. $K = \mathbb{Q}(\sqrt{2})$.
- $\mathcal{O}_K$ its ring of integers, e.g. $\mathbb{Z}[\sqrt{2}]$.
- Σ the set of Galois embeddings of K into $\mathbb{R}$ and $\mathbb{C}$, e.g.

$$a + b\sqrt{2} \mapsto a + b\sqrt{2}, \qquad a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

Diophantine Approximation in Number Fields

- $K_\Sigma = \mathbb{R}^r \times \mathbb{C}^s$, $r + s = |\Sigma|$.
- $\mathbf{x} = (x_\sigma)_{\sigma \in \Sigma} \in K_\Sigma$ is *badly approximable* by K if there exists $c > 0$, for all $p, q \in \mathcal{O}_K$:

$$\max_{\sigma \in \Sigma} \{|\sigma(p) + x_\sigma \sigma(q)|\} \max_{\sigma \in \Sigma} \{|\sigma(q)|\} \geqslant c$$

Diophantine Approximation in Number Fields

- $G = \mathrm{SL}(2, \mathbb{R})^r \times \mathrm{SL}(2, \mathbb{C})^s$.
- Γ is image of diagonal Galois embedding of $\mathrm{SL}(2, \mathcal{O}_K)$.

$$g_t = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right)_{\sigma \in \Sigma}, \qquad u(\mathbf{x}) = \left(\begin{pmatrix} 1 & \mathbf{x}_\sigma \\ 0 & 1 \end{pmatrix} \right)_{\sigma \in \Sigma}$$

- Einsiedler-Ghosh-Lyttle (Dani's correspondence in number fields): $\mathbf{x} \in K_\Sigma$ is *badly approximable* iff $g_t u(\mathbf{x})\Gamma$ remains bounded in G/Γ.

Diophantine Approximation in Number Fields

- $K_\Sigma = \mathbb{R}^r \times \mathbb{C}^s$ can be identified with the full unstable manifold of g_t via $\mathbf{x} \mapsto u(\mathbf{x})$.
- $\varphi = (\varphi_\sigma)_{\sigma \in \Sigma} : [0,1] \to \mathbb{R}^r \times \mathbb{C}^s$ is $C^{1+\varepsilon}$.
- **Maximality Assumption**:

$$\dot{\varphi}_\sigma \not\equiv 0, \qquad \sigma \in \Sigma$$

Diophantine Approximation in Number Fields

Theorem (K. '18)

For all $x_0 \in G/\Gamma$, the Hausdorff dimension of the set of $s \in [0,1]$ such that

1. $g_t u(\varphi(s)) x_0$ *is divergent on average in* G/Γ *is at most* $1/2$.
2. $g_t u(\varphi(s)) x_0$ *is bounded in* G/Γ *is equal to* 1. *(The set is winning).*

The result for curves remains true for:

1. reducible lattices, or
2. any semisimple algebraic group G and $\Gamma < G$ is an arithmetic lattice of $\mathbb{Q}$-rank equal to 1 under an appropriate maximality condition.

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Remarks:

- $G = \mathrm{SL}(2,\mathbb{R})^r \times \mathrm{SL}(2,\mathbb{C})^s$, $\Gamma = \Delta(\mathrm{SL}(2,\mathcal{O}_K))$:
  Dimension of bounded orbits on curves was previously obtained by Einsiedler, Ghosh and Lyttle by different methods (ETDS '16).

- Y. Cheung (ETDS '07): the dimension of divergent orbits for g_t in the entire $\mathrm{SL}(2,\mathbb{R})^n/\mathrm{SL}(2,\mathbb{Z})^n$ is $3n - 1/2$ for $n \geq 2$.

Systems of Linear Forms

- $Y \in M_{m,n}(\mathbb{R})$ is **badly approximable** if there exists $c > 0$ for all $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$:

$$\|\mathbf{p} + Y \cdot \mathbf{q}\|_\infty^m \|\mathbf{q}\|_\infty^n \geqslant c$$

- Y is **singular** if for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$; for all $N \geqslant N_0$, there exists $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$:

$$\begin{cases} \|\mathbf{p} + Y\mathbf{q}\| \leqslant \varepsilon / N \\ 0 < \|\mathbf{q}\| \leqslant N^{n/m} \end{cases}$$

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Systems of Linear Forms

- $G = \mathrm{SL}(m+n,\mathbb{R}), \Gamma = \mathrm{SL}(m+n,\mathbb{Z})$,

$$g_t = \begin{pmatrix} e^{nt}\mathrm{I}_m & \mathbf{0} \\ \mathbf{0} & e^{-mt}\mathrm{I}_n \end{pmatrix}, \qquad u(Y) = \begin{pmatrix} \mathrm{I}_m & Y \\ \mathbf{0} & \mathrm{I}_n \end{pmatrix}$$

- Dani's Correspondence: Y is **badly approximable** iff $g_t u(Y)\Gamma$ remains bounded in G/Γ and **singular** iff $g_t u(Y)\Gamma$ diverges in G/Γ.

Systems of Linear Forms

Theorem (K. '18)

Suppose $A \in \mathrm{GL}(n, \mathbb{R})$, $B \in M_{n,n}(\mathbb{R})$ *and* $\varphi : [0,1] \to M_{n,n}(\mathbb{R})$ *is given by*

$$\varphi(s) = B + sA$$

Then, for any $x_0 \in G/\Gamma$, *the Hausdorff dimension of the set of* $s \in [0,1]$ *such that*

1. $g_t u(\varphi(s))x_0$ *diverges on average is at most* $1/2$.
2. $g_t u(\varphi(s))x_0$ *remains bounded in* G/Γ *is equal to* 1. *(This set is winning).*

Bounded orbits: a very brief history

- Schmidt 1969: the set of badly approximable matrices in $M_{m,n}(\mathbb{R})$ is winning (has full dimension).
- Beresnevich (Invent. Math. '15): (weighted) badly approximable points on non-degenerate curves in $M_{1,n}(\mathbb{R}) \cong \mathbb{R}^n$ have dimension 1.
- Kleinbock-Weiss (Adv. in Math. '10, JMD '13): the set of bounded orbits for a partially hyperbolic algebraic flow on a homogeneous space is winning.

Divergent orbits: a very brief history

- Y. Cheung (Annals '11): singular vectors in $M_{1,2}(\mathbb{R}) \cong \mathbb{R}^2$ has dimension $4/3$.
  - Cheung-Chevallier (Duke '16): singular vectors in $\mathbb{R}^n$ have dimension $n^2/n+1$.
- Kadyrov-Kleinbock-Lindenstrauss-Margulis (J. d'Analyse '17): singular matrices in $M_{m,n}(\mathbb{R})$ have dimension **at most** $mn - \frac{mn}{m+n}$.

Ingredients of the proof

1. Contraction Hypothesis $\Longrightarrow$ Dimension Estimates.

2. Establish the Contraction Hypothesis.

Recall our set up

- G is a connected Lie group with Lie algebra $\mathfrak{g}$.
- g_t is Ad-diagonalizable over $\mathbb{R}$:

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- $u(Y) = \exp(Y)$ for $Y \in \mathfrak{g}$.
- $G \curvearrowright X$, a topological space (not necessarily a homogeneous space for G).

Recall our set up

Definition

A map $\varphi : [0,1] \to \mathfrak{g}$ is $\mathbf{g_t}$**-admissible** if:

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Height Functions

$f : X \to [0, \infty]$ is a **height function**:

1. f is proper and finite on compact subsets of $X \setminus \{f = \infty\}$.
2. f is **log-smooth**: for every bounded set $\mathcal{O} \subset G$, there exists $C \geq 1$, for all $g \in \mathcal{O}$ and all $x \in X \setminus \{f = \infty\}$,

$$C^{-1} f(x) \leqslant f(gx) \leqslant C f(x)$$

3. $\{f = \infty\}$ is G-invariant.

Bounded and divergent orbits - another look

- For $M > 0$, χ_M indicator function of $\{f \leq M\}$.
- For $x \in X$, we say
  1. $g_t x$ diverges on average if for all $M > 0$:
  $$\frac{1}{T}\int_0^T \chi_M(g_t x)\, dt \to 0$$
  2. $g_t x$ is bounded if
  $$\sup_{t>0} f(g_t x) < \infty$$

The Contraction Hypothesis

Definition

φ satisfies the **first order β-contraction hypothesis** on X if there exists a height function f and $0 < \beta < 1$ such that for all $t > 0$:

$$\int_0^1 f(g_t u(r\dot{\varphi}(s))x)\, dr \leqslant c e^{-\beta\alpha(t)} f(x) + b$$

for some constants $c, b > 0$.

In words, g_t orbits starting from points on φ are biased towards sublevel sets of f: when $f(x) \gg 1$

$$\int_0^1 f(g_t u(r\dot{\varphi}(s))x)\, dr \ll e^{-\beta\alpha(t)} f(x)$$

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Contraction $\implies$ Dimension Estimates

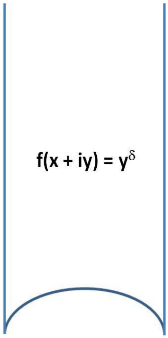
Theorem (K. '18)

Suppose φ is a g_t-admissible curve satisfying the 1^{st} order β-contraction hypothesis. Then, for all $x_0 \in X \setminus \{f = \infty\}$, the Hausdorff dimension of the set of $s \in [0, 1]$ such that

1. *$g_t u(\varphi(s))x_0$ is divergent on average is at most $1 - \beta$.*
2. *$g_t u(\varphi(s))x_0$ remains bounded in X is equal to 1.*

An example: $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$

$f : \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{R}_+$ is given by the y-coordinate in the upper half plane model.



Example 2: $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$

$$\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z}) \leftrightarrow \{\text{unimodular lattices in } \mathbb{R}^n\}$$

$$f(x) = \max_{1 \leqslant i \leqslant n} \max \left\{ \frac{1}{\|\Lambda\|} : \Lambda \text{ is a subgroup of } x \text{ of rank } i \right\}$$

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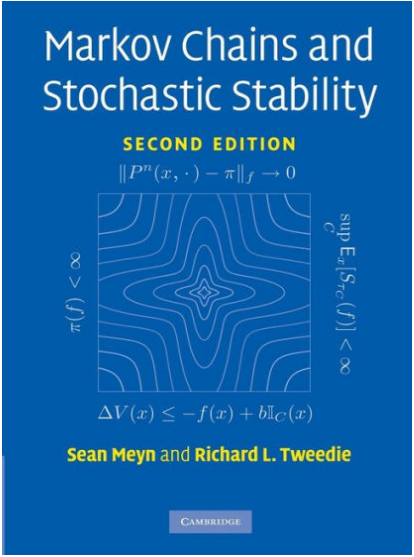
A history of contraction



A history of contraction

- Eskin-Margulis-Mozes: averaging over $\mathrm{SO}(p) \times SO(q) < SL(p+q, \mathbb{R})$.
- Eskin-Margulis, Benoist-Quint: random walks on homogeneous spaces.
- Eskin-Masur: recurrence of Teichmüller flow orbits in strata of quadratic differentials.
- Eskin-Mirzakhani-Mohammadi: recurrence away from proper affine submanifolds.

Contraction in higher rank: the enemy

- $G = \mathrm{SL}(3, \mathbb{R})$ and $\Gamma = \mathrm{SL}(3, \mathbb{Z})$:

$$g_t = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}, \qquad u_s = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Mahler's compactness criterion: a subset K of unimodular lattices inside G/Γ is bounded iff for all lattices $\Lambda \in K$, $\Lambda \cap B_\varepsilon(\mathbf{0}) = \{\mathbf{0}\}$ for some $\varepsilon > 0$.

$$g_t u_s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix} \xrightarrow{t \to \infty} \mathbf{0}$$

Higher order contraction

- A uniform first order contraction hypothesis is not possible!
- A higher order form of the contraction hypothesis can be established:

$$\int_0^1 f(g_t \Phi(r) x)\, dr \leqslant a f(x) + b$$

for some $0 < a < 1$ and $b > 0$ and Φ a certain Taylor polynomial for the curve φ.

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Higher order contraction

- $G = SL(m+n, \mathbb{R})$, $\Gamma = SL(m+n, \mathbb{Z})$ and $X = G/\Gamma$.
- $Y \in M_{m,n}$, $(\mathbf{r}, \mathbf{s}) = (r_1, \ldots, r_m, s_1, \ldots, s_n) \in \mathbb{R}^n_+$ with $\sum r_i = 1 = \sum s_j$:

$$u(Y) = \begin{pmatrix} \mathrm{I}_m & Y \\ 0 & \mathrm{I}_n \end{pmatrix}, \qquad g_t^{\mathbf{r},\mathbf{s}} = \mathrm{diag}(e^{r_1 t}, \ldots, e^{r_m t}, e^{-s_1 t}, \ldots, e^{-s_n t})$$

Theorem (K. 18)

Suppose $\varphi : [0,1] \to M_{m,n}$ is a strongly non-planar curve and $(\mathbf{r}, \mathbf{s})$ is any weight with $\sum r_i = 1 = \sum s_j$. Then, for all $x \in X$,

$$\sup_{t>0} \int_0^1 f\left(g_t^{\mathbf{r},\mathbf{s}} u(\varphi(s)) x\right)\, ds < \infty$$

Moreover, the supremum can be taken to be uniform as x varies in compact subsets of X.

Higher order contraction

1. This implies very well approximable points have measure 0. (Kleinbock-Margulis 1998, Kleinbock-Margulis-Wang 2010).

2. The approach uses the (C, α)-good theory of polynomials only.

Thanks!