# Representation Theory

Drewtheph

## 1 Group ring and group algebra

### 1.1 $F$-algebra

If $F$ is a field, then an algebra over $F$ (or simply an $F$-algebra) is a set $A$ with a ring structure and an $F$-vector space structure that share the same addition operation, and with the additional property that $(Aa)b = A(ab) = a(Ab)$ for any $\lambda \in F$ and $a, b \in A$.

### 1.2 Semisimple

A module is said to be semisimple if it is a direct sum of simple modules.
We shall say that the algebra $A$ is semisimple if all non-zero $A-$modules are semisimple

### 1.3 Group ring

Let $K$ be a ring and let $G$ be a multiplicative group, not necessarily finite. Then the group ring $K[G]$ is a $K-$vector space with basis $G$

### 1.4 Group algebra $FG$

The group algebra of $G$ over $F(F$-algebra), denoted by $FG$, consists of all finite formal $F$-linear combinations of elements of $F$, i.e. a $F$-vector space with basis $G$, hence with dimension $|G|$.

$\mathbb{C}G$ is a $\mathbb{C}$-algebra

### 1.5 Dimension of $\mathbb{C}G$-modules

$\mathbb{C}G$ as a $\mathbb{C}$-algebra and $M$ an $\mathbb{C}G$-module. Then $M$ is also a $\mathbb{C}$-vector space, where defining scalar multiplication by: $\lambda \in \mathbb{C}, m \in M$, $\lambda m := (\lambda 1_{\mathbb{C}G}).m$, and addition is from the module structure. And the dimension of a $\mathbb{C}G$-module $M$, is its dimension as a $\mathbb{C}$-vector space.

### 1.6 One dimensional $FG$-module is simple

If $S$ is a nontrivial submodule of one dimensional $FG$-module $M$, since $S$ is also a $F$-vector space, its dimension and be 0 or 1. Since $S$ not trivial so its dimension is 1, therefore $S = M$. Hence $M$ is simple.

### 1.7 Projection

Let $V = W \oplus W'$ and say that $W'$ is a complement of $W$ in $V$. The mapping $p$ which sends each $x \in V$ to its component $w \in W$ is called the **projection** of $V$ onto $W$ associated with the decomposition $V = W \oplus W'$; the image of $p$ is $W$, and $p(x) = x$ for $x \in W$;

## 2 Representation

**Representation of a group $G$ associated with a vector space $V$, roughly speaking is a homomorphism $p : G \to GL(V)$, it can be 'identified' as $V$, and its subrepresentation is actually the representation of $G$ associated with a sub-vector space $W$ of $V$, i.e. $p(g)|W \in GL(W)$(we say W is $G$-invariant) in other words, this subrepresentation $p|W$ can be 'identified' as the sub-vector space $W$ of $V$. A irreducible representation is a representation without proper subrepresentation. And we will see each representation of a group $G$ associated with vector space $V$ can be decomposed as the sub-vector spaces and all of these sub-vector spaces are $G$-invariant. Moreover, we will construct a special representation(regular representation) for finite group by associating with the complex vector space & group algebra: $\mathbb{C}G$. We will focus on the regular representation of finite groups $S_n$**

### 2.1 Representation identified as a vector space

There are at least two ways really to define a linear representation of a finite group:
- A map $\phi : G \times V \to V$ on a vector space $V$ such that for every fixed $g \in G$, the assignment $v \mapsto \phi(g, v)$ is a linear map. In other words, a linear group action.
- A group homomorphism $\rho : G \to \mathrm{GL}(V)$.

The vector space $V$ may be called the carrier space of the representation. If the action of the group $G$ on the vector space is clear from context, or is an arbitrary action, then we may just call $V$ the representation even though technically the letter $V$ initially is reserved for the vector space.

So of course $\mathbb{C}S_n c_\lambda$ is not a group homomorphism, but it is a vector space and it is clear how $S_n$ acts on it: by left multiplication. Indeed, if we instead think of it as a

representation of the complex group algebra $\mathbb{C}S_n$, or in other words a left module over it, then this is automatic. Given any algebra $A$ and left ideal $I \triangleleft A$, we know $I$ is automatically a left $A$-module using left multiplication, and in particular $\mathbb{C}S_n c_\lambda$ is a principal left ideal of $\mathbb{C}S_n$.

## 2.2 Representation(linear representation)

Let $V$ be a vector space over the field $\mathbb{C}$ of complex numbers and let $GL(V)$ be the group of isomorphisms of $V$ onto itself. An element a of $GL(V)$ is, by definition, a linear mapping of $V$ into $V$.
A **linear representation** of $G$ in $V$ is a homomorphism $p$ from the group $G$ into the group $GL(V)$. In other words, we associate with each element $s \in G$ an element $p(s)$ of $GL(V)$ in such a way that we have the equality $p(st) = p(s).p(t)$ for $s, t \in G$. When $p$ is given, we say that $V$ is a representation space of $G$ (or even simply, by abuse of language, a representation of $G$).

### 2.2.1 $GL(V)$ and matrices

When $V$ has a finite basis $(e_j)$ of $n$ elements, each linear map $a : V \to V$ is defined by a square matrix $(a_{ij})$ of order $n$. The coefficients $a_{ij}$ are complex numbers; they are obtained by expressing the images $a(e_j)$ in terms of the basis $(e_j)$:

$$a(e_j) = \sum a_{ij} e_i$$

Saying that $a$ is an isomorphism is equivalent to saying that the determinant $det(a) = det(a_{ij})$ of a is not zero. The group $GL(V)$ is thus identifiable with the group of invertible square matrices of order $n$.

## 2.3 Subrepresentation

Let $p : G \to GL(V)$ be a linear representation and let $W$ be a vector subspace of $V$. Suppose that $W$ is stable under the action of $G$ (we say also "$G$-invariant"), or in other words, suppose that $x \in W$ implies $P_s x \in W$ for all $s \in G$. The restriction $P_s|W$ of $P_s$ to $W$ is then an isomorphism of $W$ onto itself, and we have $P_{st}|W = P_s|W \cdot P_t|W$, Thus $P|w : G \in GL(W)$ is a linear representation of $G$ in $W$; $W$ is said to be a subrepresentation of $V$.

## 2.4 Irreducible representation

Let $p : G \to GL(V)$ be a linear representation of $G$. We say that it is irreducible or simple if $V$ is not 0 and if no vector subspace of $V$ is stable under $G$, except of course 0 and $V$.

### 2.4.1 Irreducible representation and representation

Every representation is a direct sum of irreducible representations(Linear Representation of Finite groups, Theorem 2).The proof refers to Theorem 1 of the book (which says $Letp : G \to GL(V)$ be a linear representation of $G$ in $V$ and let $W$ be a vector subspace of $V$-stable under $G$. Then there exists a complement $W^O$ of $W$ in $V$ which is stable under $G$) hence these irreducible representations are all $G$-invariant, i.e. each of these irreducible representations $G$-invariant appears as a subrepresentation of the regular representation.

## 2.5 Regular representation(this is unique for each group)

For a finite group $G$, the group algebra $\mathbb{C}G$ can be also seen as a $\mathbb{C}$-vector space with the basis $(e_t)_{t\in G}$ indexed by the elements $t$ of $G$. For $s \in G$, let $p_s$ be the linear map of $\mathbb{C}G$ into $\mathbb{C}G$ which sends $e_t$ to $e_{st}$; this defines a linear representation: , which is called the **regular representation of** $G$, in other words, the regular representation of $G$ is the $G$-action defined by left-multiplication.

### 2.5.1 Regular representation and irreducible representation

Since every representation is a direct sum of irreducible representations. Each irrep of $G$ appears as a subrepresentation of the regular representation i.e. each irrep of $G$ is associated with an $G$-invariant subspace of $\mathbb{C}G$.

## 2.6 Representation of $\mathbb{C}S_n$

If $W$ is subrepresentation, then it's $S_n$-invariant, and since $W$ is itself a $\mathbb{C}$-vector space, we can say $W$ is $\mathbb{C}S_n$-invariant.

# 3 Character

**Like regular representation, Character of a group $G$ is also induced by a map from $G$ to a $\mathbb{C}$-vector space(here the space usually is a $\mathbb{C}G$-module), but character is $f : g \in G \to tr(p(g)) \in \mathbb{C}$ . A character of group $G$(associated with a $\mathbb{C}G$-module) basically is a function $f : G \to \mathbb{C}$, s.t. $f(g) = f(hgh^{-1})$. Here, we will use $\mathbb{C}G$-modules to define characters of group $G$(associated with some $\mathbb{C}G$-module $U$). (The construction is: the linear map $p : G \to Aut(U)$, sending $u \in U$ to $gu$, induces a map $p : G \to GL(U)$(See $U$ as a $\mathbb{C}$-vector space) we define the the character of $G$ associated with $U$ by the map $f : G \to \mathbb{C}$, where $f := tr(p(g)$. Another key point is: for a group $G$, there are specific $r_G$($r_G$ is a number) simple $\mathbb{C}G$-modules unique up to isomorphism, any $\mathbb{C}G$-module can be written uniquely in the combination of the $r_G$ simple $\mathbb{C}G$-modules. Then the character of any**

**$\mathbb{C}G$-module is the corresponding combination of characters of the corresponding simple $\mathbb{C}G$-modules(namely, irreducible characters), i.e. the characters of group $G$ can be generated by a basis consisting of finite $r_G$($r_G$ is a number) characters of group $G$ associated with the simple $\mathbb{C}G$-modules(namely, irreducible characters), which are uniquely determined by isomorphism.**
**Note, by the property of character: $f : G \to \mathbb{C}$, in order to compute $f$, we just need to focus on the value of $f$ on the conjugacy classes of $G$, in addition, since every $f$ is a combination of the irreducible characters, we just need to focus on the value of the irreducible characters(associated with the $r_G$ simple $\mathbb{C}G$-modules) on the conjugacy classes of $G$. Moreover, $r_G$ equals to the number of conjugation classes of $G$, thus we can determine the characters of group $G$ by considering a $r_G \times r_G$ array giving the values of the $r_G$ irreducible characters on the $r_G$ conjugacy classes of $G$. From now on, when we consider the character of a group element $g$, by first figuring out the corresponding conjugacy class of $g$**

**In this chapter $G$ denotes a finite group, and all $\mathbb{C}G$-modules are finitely generated.**

## 3.1 Simple $\mathbb{C}G$-modules

Assume there are exactly $r$ isomorphism classes of simple $\mathbb{C}G$-modules, and if we let $S_1, ..., S_r$ be representatives of these $r$ classes. The number $r$ of simple $\mathbb{C}G$-modules is equal to the number of conjugacy classes of $G$.

## 3.2 Decomposition of $\mathbb{C}G$

1.$CG \cong f_1S_1 \bigoplus ... \bigoplus f_rS_r$, where $dim_{\mathbb{C}}S_i = f_i$ for each $i$, are called the degrees of $G$.

2.Any $\mathbb{C}G$-module can be written uniquely in the form $a_1S_1 \bigoplus ... \bigoplus a_rS_r$ where the $a_i$ are non-negative integers.

## 3.3 Relation of degrees $f_i$

$\sum f_i^2 = |G|$

## 3.4 Character of the $\mathbb{C}G$-module and the induced representation

1.If $U$ is a $\mathbb{C}G$-module, to each $g \in G$, by sending $u \in U$ to $gu$, we can define an invertible linear transformation of $U$, namely $\rho(g)$. We have required $U$ as a $\mathbb{C}G$-module to be finitely generated, thus **the linear transformation $\rho(g)$ can be viewed as a matrix**, by $\rho(g)(x_i) = g_{i,1}x_1 + ...g_{i,n}x_n$, where $x_1, ..x_n$ is the basis of $U$, i.e. $\rho(g)(u) =$

$(u_1, ...u_n)(g_{i,j})_{n\times n}$. We define the character of $U$ to be the function $\chi_u : G \to \mathbb{C}$, where $\chi_u(g)$ is the trace of $\rho(g)$, i.e. $tr((g_{i,j})_{n\times n})$.
2. Here the $\rho(g)$, induces a representation:$\rho : G \to GL(U)$(see $U$ as a $\mathbb{C}$-vector space), called the representation.

### 3.4.1 Definition of Character of $G$

By the context on textbook(AB), p.139, a Character of $G$ means a character of $\mathbb{C}G$-module $U$.

### 3.4.2 Linear character

A character of a one-dimensional $\mathbb{C}G$-module is called a linear character.

If $U$ is one-dimensional, for any $u \in U$ we have $gu = \chi(g)u$ and $hu = \chi(h)u$, and thus $\chi(gh)u = (gh)u = \chi(g)\chi(h)u$;

### 3.4.3 Irreducible characters and Irreducible representations of $G$

We will denote the characters of the $r$ simple $\mathbb{C}G$-modules by $\chi_1, ...\chi_r$, these characters will be referred to as the irreducible characters of $G$. And the corresponding $\rho$ are called the **irreducible representation of** $G$, i.e. $\chi_i(g) = tr(\rho_i(g))$.

### 3.4.4 Irreducible characters on the $r$ conjugacy classes

Since $gh \cdot u = g(h \cdot u) = ((u_1, ...u_n)(g_{i,j})_{n\times n})(h_{i,j})_{n\times n}$, we have $gh = (g_{i,j})_{n\times n}(h_{i,j})_{n\times n}$, hence $hgh^{-1}$ corresponds to $(h_{i,j})_{n\times n}(g_{i,j})_{n\times n}(h_{i,j}^{-1})_{n\times n} = (h_{i,j})_{n\times n}(g_{i,j})_{n\times n}(h_{i,j})_{n\times n}^{-1}$. Now, for any $g, h \in G$, the linear transformations of $U$ defined by $g$ and $hgh^{-1}$ are similar and hence have the same trace. Therefore, any character of $G$ is constant on each conjugacy class of $G$

### 3.4.5 Class function on $G$

1.A class function on $G$ is a function from $G$ to $\mathbb{C}$ whose value within any conjugacy class is constant, for example, characters of $\mathbb{C}G$-modules are class functions.

2.The irreducible characters of $G$ form a basis for the space of class functions on $G$. **Thus any character of $G$ is a $Z$-linear combination of the $r$ irreducible characters $\chi_1, ...\chi_r$ of $G$.** And similarly, there is a decomposition of the representation of $G$ into the irreducible representations.

### 3.5 The Character Table

Any character of $G$ is a $Z$-linear combination of the $r$ irreducible characters $\chi_1, ...\chi_r$ of $G$, where $r$ is by Theorem 14.3 from textbook(AB) equal to the number of conjugacy classes of $G$.
Since by 2.4.3, we know each irreducible character is specified by its value on each conjugacy class of $G$, it follows that the characters of $G$ are completely determined by an $r \times r$ array giving the values of the $r$ irreducible characters on the $r$ conjugacy classes of $G$. This array is called the character table of $G$.

## 4 Conjugacy classes and the irreducible representations

**As we showed above, in any finite group, the number of conjugacy classes equals the number of irreps(irreducible representations). But, in most groups, to a irrep, we may not find a associated conjugacy classes. However, the symmetric group $S_n$ is one of the few where there is a natural one-to-one correspondence between the two. We will use Young diagram to illustrate the correspondence because each Young diagram determines a conjugacy class**

### 4.1 The correspondence between conjugacy classes and the irreducible representations in $S_n$

#### 4.1.1 Construction of Young diagram and each Young diagram determines a conjugacy class of $S_n$

As a first step in defining this correspondence, we define a partition of a positive integer $n$ to be a nonincreasing sequence of positive integers summing to n. That is, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is a partition of $n$, if $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 1$ and $\sum \lambda_i = n$. The partitions of $n$ are in one-to-one correspondence with conjugacy classes of $S_n$: the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is associated with the conjugacy class of permutations which, written in cycle notation, contain cycles of length $\lambda_1, \lambda_2, ..., \lambda_m$. For example, (3,2,1) of $S_6$ is just (123)(45)(6); (1,1,1) of $S_3$ is just (1)(2)(3)(Note (123)(45) and (124)(35) are in the same conjugacy class by conjugating (34), so a Young diagram determines a conjugacy class). If $\lambda$ is a partition of $n$, we write $|\lambda| = n$.

The next step is to associate each partition with a Young diagram. A Young diagram consists of rows of square boxes, with each row left-justified, and with each row no longer than the row above. The Young diagram associated with a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is the one with $\lambda_i$ boxes in the $i^{th}$ row from the top. Abusing notation, we use $\lambda$ to denote the Young diagram in addition the partition, when there is no risk of confusion.

#### 4.1.2 Young symmetrizer

Starting with a Young diagram(**it represents a conjugacy class of** $S_n$) $|\lambda| = n$, number the boxes 1 to n, left-to-right top-to-bottom. Let $S_n$ permute the numbers in the boxes, and define the subgroups $A_\lambda, B_\lambda \subset S_n$ to be the permutations preserving rows(the elements of every same are preserved but the position may permuted) and the permutations preserving columns, respectively. Now define two elements of $\mathbb{C}[S_n]$ by $a_\lambda = \sum_{g \in A_\lambda} g$ and $b_\lambda = \sum_{g \in B_\lambda} (sign(g))g$. The Young symmetrizer of $\lambda$ is defined to be $c_\lambda = a_\lambda b_\lambda$.

#### 4.1.3 Theorem 4.3 of Fulton

For $|\lambda| = n$, the image of the map $\phi_\lambda : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$, $x \to xc_\lambda$ is an irrep of $S_n$. As $\lambda$ varies over partitions of $n$, each irrep of $S_n$ (up to isomorphism) is generated exactly once.(Since the number of of conjugacy classes equals the number of irreps, we get the correspondence). Proof is just HW5.

#### 4.1.4 Some propositions of Young diagram

1. If $T$ is the tableau corresponding to partition $\lambda$, then $T' = gT$, where $g \in S_n$, is the tableau obtained by replacing each entry $i$ of $T$ by $g(i)$ corresponding to the partition named as $\lambda'$
2. **Conjugation of** $t \in S_n$**:** Assume $t$ refers to the entries/integers $t_1, ...t_n$, with positions $x_1, ...x_n$ in $T$, then $gtg^{-1}$ refers to the entries/integers in $T'$ with the same positions as in $T$, i.e positions $x_1, ...x_n$ in $T'$. Thus we can get the corollary that if $t \in A_\lambda$, then $gtg^{-1} \in A_{\lambda'}$.

### 4.2 Representation of $\mathbb{C}S_n$

We can also view $\mathbb{C}S_n$ as a complex vector space with basis elements indexed by the $e_\sigma$, where the $e_\sigma$ are basis elements indexed by the $\sigma \in S_n$. Any element $d \in \mathbb{C}S_n$ gives us a subspace $\mathbb{C}S_n d$ of $\mathbb{C}S_n$ which is invariant under the (left) action of $\mathbb{C}S_n$.

## 5 Frobenius' Theorem

### 5.1 Definitions

#### 5.1.1 Permutation group of $X$

Permutation group of $X$ is a subgroup of $Sym(X)$.

#### 5.1.2 Transitive permutation group of $X$

If $G$ is a transitive permutation group of $X$, then it mean to any $x, y \in X$, there exits a $g \in G$, s.t. $g \cdot x = y$.

### 5.1.3 Frobenius group

A Frobenius group is a group $G$ having a subgroup $H$ such that $H \cap gHg^{-1} = 1$ for every $g \in G - H$. We call $H$ a Frobenius complement of $G$.

### 5.1.4 Faithful Group Action

A group action $\phi : G \times X \to X$ is called faithful if there are no group elements $g$ (except the identity element) such that $g \cdot x = x$ for all $x$ in $X$. Equivalently, the map $\phi$ induces an injection of $G$ into the symmetric group $S_X$. So $G$ can be identified with a permutation subgroup.

## 5.2 Frobenius' Theorem

Let $G$ be a transitive permutation group on a finite set $X$, and suppose that each non-identity element of $G$ fixes at most one element of $X$. Then the union of the identity element and the set of elements of $G$ that have no fixed points is a normal subgroup of $G$.

# 6 Induction

**Indcution**
Let $H$ be subgroup of $G$, and let $V$ be an $FH$-module. Since the group algebra $FG$ is an $(FG, FH)$-bimodule, we can construct the $FG$-module $FG \otimes_{FH} V$. We call this $FG$-module the induced $FG$-module of $V$ (or the induction of $V$ to $G$), and we denote it by $Ind_H^G V$.

**Restriction**
Let $U$ be an $FG$-module, and let $H$ be subgroup of $G$. When regarding $U$ as an $FH$-module, we will write $Res_H^G U$ in place of $U$; this $FH$-module is called the restriction of $U$ to $H$.