Factorials

We define $\boldsymbol{n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$ if n is a nonnegative integer.

An empty product is normally defined to be 1.

With this convention, $0! = 1$.

An alternative is to define $n!$ recursively on the nonnegative integers.

$$\mathbf{n!} = \begin{cases} \mathbf{1} & \textbf{if } \boldsymbol{n = 0}, \\ \boldsymbol{n(n-1)!} & \textbf{if } \boldsymbol{n \geq 1}. \end{cases}$$

As n increases, $n!$ increases *very* rapidly (exponentially).

n	$n!$
5	120
10	3628800
15	1.307674×10^{12}
20	2.432902×10^{18}
30	2.652529×10^{32}
40	8.159153×10^{47}
50	3.041409×10^{64}
60	8.320987×10^{81}
70	1.197857×10^{100}
80	7.156946×10^{118}

For any *fixed* number a, $\boldsymbol{n! > a^n}$ for all n sufficiently large.

On the other hand, $\boldsymbol{n! < n^n}$ for all n.

Stirling's Formula provides a good approximation to $n!$ in closed form:

$$\boldsymbol{n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}}$$

If $S_0(n)$ denotes $\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, then $\lim_{n\to\infty} S_0(n)/n! = 1$.

In fact, the limit approaches 1quite rapidly as n increases.

When $n = 5$, $S_0(n)/n! = 0.9835$.

When $n = 10$, $S_0(n)/n! = 0.9917$.

When $n = 50$, $S_0(n)/n! = 0.9983$.

An even better approximation is obtained by multiplying $S_0(n)$ by $1+1/(12n)$.

$$\boldsymbol{n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{12n}\right)}$$

If $S_1(n)$ denotes $\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{12n}\right)$, then

When $n = 1$, $S_1(n)/n! = 0.998982$.

When $n = 5$, $S_1(n)/n! = 0.999883$.

When $n = 10$, $S_1(n)/n! = 0.999968$.

When $n = 50$, $S_1(n)/n! = 0.999999$.

Here are the approximations to $n!$ for the values of n in the previous table.

n	$n!$	$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$	$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{12n}\right)$
5	120	118.019	119.986
10	3628800	3598696	3628685
15	1.307674×10^{12}	1.300431×10^{12}	1.307655×10^{12}
20	2.432902×10^{18}	2.422787×10^{18}	2.432882×10^{18}
30	2.652529×10^{32}	2.645171×10^{32}	2.652519×10^{32}
40	8.159153×10^{47}	8.142173×10^{47}	8.159136×10^{47}
50	3.041409×10^{64}	3.036345×10^{64}	3.041405×10^{64}
60	8.320987×10^{81}	8.309438×10^{81}	8.320979×10^{81}
70	1.197857×10^{100}	1.196432×10^{100}	1.197856×10^{100}
80	7.156946×10^{118}	7.149494×10^{118}	7.156942×10^{118}

Previously, we mentioned that $n!$ grows more rapidly than a^n (a fixed) but less rapidly than n^n.

By Stirling's formula, $n!$ grows about as rapidly as $(n/e)^n$.

Stirling's formula also gives a good approximation to $\lg(n!)$:

$$\lg(n!) \approx n\lg(n) - n\lg(e) + 0.5\lg(n) + 0.5\lg(2\pi) + \lg(e)/(12n)$$

or

$$\mathbf{\lg(n!) \approx n\lg(n) - 1.44\,n}$$

$\ln(1+x) \approx x$ for $|x|$ small. Let $x = 1/(12n)$.

We sometimes write $\lg(n!) \approx n\lg(n)$, but the $1.44n$ term never becomes negligible for practical values of n.

Why is $n!$ important in algorithms?

$n!$ is the number of permutations of an n-element sequence with distinct elements. In other words, it is the number of ways to arrange n distinct objects.

For example, there are $4! = 24$ ways to arrange the letters a, b, c, d:

abcd	bacd	cabd	dabc
abdc	badc	cadb	dacb
acbd	bcad	cbad	dbac
acdb	bcda	cbda	dbca
adbc	bdac	cdab	dcab
adcb	bdca	cdba	dcba

Any algorithm that looks at every possible arrangement of n objects would take time at least proportional to $n!$ (and thus be practical only for very small n — say n less than 15 or 20).

What if we have n elements that are not distinct? Say there are k distinct elements, occurring with frequencies $n_1, n_2, \ldots, n_k$, where $n_1+n_2+..+n_k = n$. The number of arrangements is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Thus there are $5! / (3!\,1!\,1!) = 20$ ways to arrange a, a, a, b, c:

aaabc	aacab	abcaa	baaac	caaab
aaacb	aacba	acaab	baaca	caaba
aabac	abaac	acaba	bacaa	cabaa
aabca	abaca	acbaa	bcaaa	cbaaa

We have defined $n!$ only on the nonnegative integers, but we can extend to the nonnegative real numbers (as well as certain negative real numbers).

Consider $\int_0^\infty t^x e^{-t}\,dt$, where x is any nonnegative real number. (Actually, we only need $x > -1$.)

The value of the integral depends on x, so denote it by $g(x)$.

$$g(0) = \int_0^\infty t^0 e^{-t}\,dt = -e^{-t}\Big]_0^\infty = -0 - (-1) = 1$$

For $x > 0$,

$$g(x) = \int_0^\infty t^x e^{-t}\,dt = \int_0^\infty u(t)\,v'(t)\,dt \qquad (u(t) = t^x,\ v(t) = -e^{-t})$$

$$= u(\infty)v(\infty) - u(0)v(0) - \int_0^\infty u'(t)\,v(t)\,dt \qquad (u'(t) = xt^{x-1})$$

$$= 0 - 0 - x\int_0^\infty t^{x-1}(-e^{-t})\,dt$$

$$= x\int_0^\infty t^{x-1} e^{-t}\,dt = x\,g(x-1).$$

Now $g(0) = 1$ and $g(x) = x\,g(x-1)$ for all $x > 0$ implies $g(x) = x!$ whenever x is a nonnegative integer. So it is natural to define

$$\boldsymbol{x! = \int_0^\infty t^x e^{-t}\,dt} \text{ for all nonnegative real numbers } x.$$

Actually, this definition makes sense for $x > -1$. When $x = -1$, the integral diverges.

One can show that

$$(1/2)! = \sqrt{\pi}/2 \approx 0.8862$$

$$(3/2)! = (3/2)\,(1/2)! = 3\sqrt{\pi}/4 \approx 1.3293$$

$$(5/2)! = (5/2)\,(3/2)! = 15\sqrt{\pi}/8 \approx 3.3234$$

$$(-1/2)! = (1/2)!\,/\,(1/2) = \sqrt{\pi} = 1.7724$$

Note: The function we defined as $g(x)$ is essentially the Gamma function $\Gamma(x)$, introduced by Euler.

However, $\Gamma(x)$ is defined as $\int_0^\infty t^{x-1} e^{-t}\,dt$ whenever $x > 0$.

So $\boldsymbol{x! = \Gamma(x+1)}$ whenever $x > -1$.