CS 501 / MCS 501 – Computer Algorithms II
Spring 2024
Problem Set 5

Lev Reyzin

Due: 4/26/24 by the beginning of class

Instructions: Atop your answer sheet, write your name and the names of all the students with whom you have collaborated on this problem set. All answers require explanation or proof.

1. [10 pts] Prove that the dual of the dual of a linear program is the original linear program itself. For the purposes of this question, assume that in order to take the dual of an LP, you need to first rewrite it in "primal form," according to program (P) in Appendix A of the textbook.

2. [10 pts] Consider the following formulation of the linear program for Vertex Cover on a graph $G = (V, E)$ from Question 1 on Problem Set 2:

$$\begin{array}{lll} \text{minimize} & \sum_{i \in V} x_i & \\ \text{subject to} & x_i + x_j \geq 1, & \forall (i,j) \in E, \\ & x_i \geq 0, & \forall i \in V. \end{array}$$

Derive its dual and use it to give a primal-dual algorithm with a 2-approximation guarantee for Vertex Cover. (This is indeed a special case of the Set Cover analysis that we went over in class, but you should make the argument and the notation as specific as possible to this particular case.)

3. [10 pts] Prove that the primal-dual algorithm for the shortest path problem in Section 7.3 adds edges to the shortest path tree in the same order that Dijkstra's algorithm would add on the same input. You may assume that "ties" are broken the same way for both algorithms.

4. [10 pts] Semidefinite programs, like linear programs, have duals. The dual of the MAX-CUT SDP is:

$$\begin{array}{ll} \text{minimize} & \displaystyle \frac{1}{2} \sum_{i<j} w_{ij} + \frac{1}{4} \sum_i \gamma_i \\ \text{subject to} & W + \text{diag}(\gamma) \succeq 0 \end{array}$$

where W is the (symmetric) matrix of the edge weights w_{ij} of G and diag(γ) is the matrix with γ_i as the ith entry on the diagonal and 0 everywhere else. Show that the value of any feasible solution for this dual is an upper bound on the cost of any cut in G.[1]

[1] Note that this statement follows from the fact that SDP duals satisfy weak duality, but do not appeal to that general result in doing this problem. It may be of interest that, unlike LP duals, SDP duals do not necessarily satisfy strong duality.