Lectures on Infinitary Model Theory

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Part I
Classical Results in Infinitary Model Theory

1 Infinitary Languages

Let τ be a vocabulary, i.e., a set of function symbols, relation symbols and constant symbols. In the logic $\mathcal{L}_{\infty,\omega}$ we build formulas using the symbols from τ, equality, Boolean connectives $\neg$, $\bigwedge$ and $\bigvee$, quantifiers $\forall$ and $\exists$, variables $\{v_\alpha : \alpha \text{ an ordinal}\}$.[1]

- terms and atomic formulas are defined as in first order logic;
- if ϕ is a formula, then so is $\neg\phi$;
- if X is a set of formulas, then so are

$$\bigvee_{\phi\in X} \phi \text{ and } \bigwedge_{\phi\in X} \phi;$$

- if ϕ is a formula, so are $\exists v_\alpha\ \phi$ and $\forall v_\alpha\ \phi$.

We extend the usual definition of satisfaction by saying

$$\mathcal{M} \models \bigvee_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for some } \phi \in X$$

and

$$\mathcal{M} \models \bigwedge_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for all } \phi \in X.$$

For notational simplicity, we use the symbols $\wedge$ and $\vee$ as abbreviations for binary conjunctions and disjunctions. Similarly we will use the abreviations $\rightarrow$ and $\leftrightarrow$ when helpful.

We can inductively define the notions of *free variable*, *subformula*, *sentence*, *theory* and *satisfiability* in the usual ways.

[1] When no confusion arises we omit the τ and write $\mathcal{L}_{\infty,\omega}$ instead of $\mathcal{L}_{\infty,\omega}(\tau)$.

Exercise 1.1 Suppose ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and ψ is a subformula of ϕ. Prove that ψ has only finitely many free variables.

Let κ be an infinite cardinal. In the logic $\mathcal{L}_{\kappa,\omega}(\tau)$ we form formulas in a similar way but we only use variables $\{v_\alpha : \alpha < \kappa\}$ and restrict infinite conjunctions and disjunctions to $\bigvee_{\phi\in X}\phi$ and $\bigwedge_{\phi\in X}\phi$ where $|X| < \kappa$. Thus $\mathcal{L}_{\omega,\omega}$ is just the usual first order logic. Throughout these notes we be focusing primarily on $\mathcal{L}_{\omega_1,\omega}$, the logic where we allow countable conjunctions and disjunctions.[2]

Exercise 1.2 Show that if κ is a singular cardinal there are sentences of $\mathcal{L}_{\kappa,\omega}$ which have κ subformulas. For this reason, it is customary to restrict attention to $\mathcal{L}_{\kappa,\omega}$ for κ a regular cardinal.

Definition 1.3 We say $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}_{\omega_1,\omega}$ sentences ϕ. The notion $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$, is defined analogously.

Exercise 1.4 Show that if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

When studying the model theory of infinitary logics there is one fundamental and inescapable fact:

THE COMPACTNESS THEOREM FAILS FOR INFINITARY LANGUAGES.

Exercise 1.5 Let τ be the vocabulary with constant symbols $d, c_0, c_1, \ldots$ and let Γ be the set of sentences

$$\{d \neq c_i : i \in \omega\} \cup \{\forall v \bigvee_{i\in\omega} v = c_i\}.$$

Show that every finite subset of Γ is satisfiable, but Γ is not satisfiable. Thus the Compactness Theorem fails for $\mathcal{L}_{\infty,\omega}$.

[2]Logically it would make sense to call this logic $\mathcal{L}_{\aleph_1,\aleph_0}$ but we will follow the historical precedent and refer to it as $\mathcal{L}_{\omega_1,\omega}$.

The failure of compactness will lead to many new phenomena and force us to to find new approaches and develop new tools. [3]

Exercise 1.6 a) Given an example of structures $\mathcal{M}_0, \mathcal{M}_1, \ldots$ and $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M}_i \models \phi$ for all ϕ, but if $\mathcal{U}$ is a nonprincipal ultrafilter on ω then $\prod M_i/\mathcal{U} \models\neq \phi$.

b) Show that if $\mathcal{U}$ is a σ-complete ultrafilter on I, then

$$\prod_{i\in I} \mathcal{M}_i/\mathcal{U} \models \phi \leftrightarrow \{i \in I : \mathcal{M}_i \models \phi\} \in \mathcal{U}$$

for $\phi \in \mathcal{L}_{\omega_1,\omega}$.

If compactness fails, why then do we study the model theory infinitary languages? One reason is that we get new insights about first order model theory. But the simplest answer is that there are many natural classes that are axiomatized by $\mathcal{L}_{\omega_1,\omega}$-sentences.

Exercise 1.7 Show that the following classes are $\mathcal{L}_{\omega_1,\omega}$-axiomatizable for appropriate choices τ.

i) torsion abelian groups;

ii) finitely generated groups;

iii) non-finitely generated groups;

iii) linear orders isomorphic to $(\mathbb{Z}, <)$;

iv) archimedian fields;

v) connected graphs;

vi) ω-models of ZFC (i.e., models of ZFC where the integers are standard);

vii) models of T omitting p, where T is a first order theory and p is a type.

Exercise 1.8 Show by compactness that none on these classes is axiomatizable in first order logic.

Examples iii) and iv) can also be used to show failure of the Upward Löwenheim–Skolem Theorem in infinitary languages. There are no uncountable linear orders isomorphic to $(\mathbb{Z}, <)$ and any archimedian ordered field is

[3] Though we will not focus on it, another fruitful approach is to look for more general forms of the compactness theorem that hold in particular settings. One example of this is Barwise Compactness for countable admissible fragments. We will see an avatar of these results in Theorem **??**, but the interested reader should consult [5]. Another important example is studying compactness in languages $\mathcal{L}_{\kappa,\kappa}$ where κ is a large cardinal. See [16] or [15] for further information.

isomorphic to a subfield of the real numbers and hence has cardinality at most $2^{\aleph_0}$. We give another example in Exercise 1.26.

Exercise 1.9 Show by induction that for ordinal α there is an $\mathcal{L}_{\infty,\omega}$-sentence Φ_α describing $(\alpha, <)$ up to isomorphism.

Exercise 1.10 Let κ be a cardinal. Show there are $\alpha, \beta < (2^\kappa)^+$ such that $(\alpha, <) \equiv_{\kappa,\omega} (\beta, <)$.

Taken together these two exercises give examples of structures $\mathcal{M}, \mathcal{N}$ with $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$ but $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$ for all κ.[4]

We will also, from time to time, look at *pseudoelementary classes* which are simply reducts of elementary classes.

Definition 1.11 We say that a class $\mathcal{K}$ of τ-structures is a $\mathrm{PC}_{\omega_1,\omega}$-*class* if there is a vocabulary $\tau^* \supseteq \tau$ and $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau^*)$ such that

$$\mathcal{K} = \{\mathcal{M} : \text{ there is a } \tau\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \models \phi\}$$

i.e., $\mathcal{K}$ is the class of τ-reducts of models of ϕ.

Similarly we say that $\mathcal{K}$ is a PC-*class* if ϕ is a first order sentence and $\mathcal{K}$ is a PC_δ-*class* if ϕ is a first order theory.

Exercise 1.12 Show that the following classes are $\mathrm{PC}_{\omega_1,\omega}$.

a) orderable groups

b) free groups

c) 1-trainsitive linear orders, i.e. linear orders where for any a, b there is an order automorphism taking a to b;

d) fiields that are pure transcendental extensions of $\mathbb{Q}$;

e) incomplete ordered fields;

f) ordered fields with an integer part (i.e., an ordered subring with no element between 0 and 1 such that every element of the field is within distance at most one of some element of the subring);

Exercise 1.13 [Silver] Let $\tau = \{U\}$ where U is unary and let $\mathcal{K} = \{\mathcal{M} : |\mathcal{M}| \leq 2^{|U(\mathcal{M})|}\}$.

a) Show that $\mathcal{K}$ is PC-class.

b) Show that $\mathcal{K}$ is κ-categorical if and only if $\kappa = \beth_\alpha$ for some limit ordinal α.

[4]For a specific example. Suppose $\kappa < \lambda$ are uncountable cardinals. Show that$\kappa \equiv_{\omega_1,\omega} \lambda$, but $\kappa \not\equiv_{\infty,\omega} \lambda$. See [21] for details.

This example shows that the straightforward generalization of Morley's Categoricity Theorem to PC-classes fails

Exercise 1.14 Let $\mathcal{K}$ be a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that

$$\mathcal{K}_0 = \{\mathcal{M} \in \mathcal{K} : \mathcal{M} \text{ countable}\}$$

is also a $\mathrm{PC}_{\omega_1,\omega}$-class.

1.1 Fragments and Downward Löwenheim–Skolem

We will prove a useful and natural version of the Downward Löwenheim–Skolem Theorem. The next exercise shows that even here we will need to be careful.

Exercise 1.15 Give an example of a countable vocabulary τ and an $\mathcal{L}_{\omega_1,\omega}$-theory T such that every model of T has cardinality at least $2^{\aleph_0}$.

We will often restrict our attention to subcollections of the set of all $\mathcal{L}_{\omega_1,\omega}$-formulas. We will look at collections of formulas with some natural closure properties. One of these will be formal negation an operation that shows how we inductively move a negation inside a quantifier or a Boolean operation. Closure under formal negation isn't needed in this proof, but will be in later arguments.

Definition 1.16 For each formula ϕ we define $\sim \phi$, a *formal negation* of ϕ as follows:

i) for ϕ atomic, $\sim \phi$ is $\neg\phi$;

ii) $\sim (\neg\phi)$ is ϕ;

iii)

$$\sim \bigwedge_{\phi\in X} \phi \text{ is } \bigvee_{\phi\in X} \sim\phi \text{ and } \sim \bigvee_{\phi\in X} \phi \text{ is } \bigwedge_{\phi\in X} \sim\phi;$$

iv) $\sim \exists v\phi$ is $\forall v \sim \phi$ and $\sim \forall v\phi$ is $\exists v\neg\phi$.

Exercise 1.17 Show by induction that

$$\mathcal{M} \models \neg\phi \text{ if and only if } \mathcal{M} \models\sim \phi$$

for all $\phi \in \mathcal{L}_{\infty,\omega}$.

Definition 1.18 We say that a set of $\mathcal{L}_{\omega_1,\omega}$-formulas $\mathbb{A}$ is a *fragment* if there is an infinite set of variables V such that if $\phi \in \mathbb{A}$, then all variables occurring in ϕ are in V and A satisfies the following closure properties:

i) all atomic formulas using only the constant symbols of τ and variables from V are in $\mathbb{A}$;

ii) if $\phi \in A$ and ψ is a subformula of ϕ, then $\psi \in \mathbb{A}$;

iii) If $\phi \in \mathbb{A}$, v is free in ϕ, and t is a term where every variable is in V, then the formula obtained by substituting t into all free occurrences of v is in $\mathbb{A}$;

iv) $\mathbb{A}$ is closed under $\sim$;

v) $\mathbb{A}$ is closed under $\neg$, $\wedge$, $\vee$, $\exists v$, and $\forall v$ for $v \in V$.

Exercise 1.19 a) Suppose κ is regular (in particular this holds for $\mathcal{L}_{\omega_1,\omega}$). Prove that if T is a set of $\mathcal{L}_{\kappa,\omega}$-sentences with $|T| < \kappa$, then there is $\mathbb{A}$ a fragment of $\mathcal{L}_{\kappa,\omega}$ such that $T \subseteq \mathbb{A}$ and $|\mathbb{A}| < \kappa$.

b) Show that there is a smallest such fragment.

Exercise 1.20 Let T and $\mathbb{A}$ be as above. Show that every formula in $\mathbb{A}$ has only finitely many free variables. Give an example showing that even though are only finitely many free variables, there may be infinitely many free occurrences.

Exercise 1.21 We say that a formula is in *negation normal form* if the $\neg$ only occurs applied to atomic formulas. Show by induction that for any fragment $\mathbb{A}$ and any $\mathcal{L}_{\infty,\omega}$-formula $\phi \in \mathbb{A}$, there is $\psi \in \mathbb{A}$ such that ψ is equivalent to ϕ and ϕ is in negation normal form.

We write $\mathcal{M} \equiv_{\mathbb{A}} \mathcal{N}$ and $\mathcal{M} \prec_A \mathcal{N}$ for elementary equivalence and elementary submodels with respect to formulas in $\mathbb{A}$ (where $\mathbb{A}$ could be $\mathcal{L}_{\omega_1,\omega}$ or $\mathcal{L}_{\infty,\omega}$).

Theorem 1.22 (Downward Löwenheim–Skolem) *Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\infty,\omega}$ such that any formula $\phi \in \mathbb{A}$ has at most finitely many free variables. Let $\mathcal{M}$ be a τ-structure with $X \subseteq \mathcal{M}$. There is $\mathcal{N} \prec_{\mathbb{A}} \mathcal{M}$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}|, |X|)$.*

In particular, if τ is countable and $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$, then every τ-structure has a countable $\mathbb{A}$-elementary submodel.

The proof is a simple generalization of the proof in first-order logic. It is outlined in the following Exercise.

Exercise 1.23 a) Prove there is $\tau^* \supseteq \mathcal{L}$ and $\mathbb{A}^* \supseteq \mathbb{A}$ and $\mathcal{M}^*$ an τ^*-expansion of $\mathcal{M}$ such that $|\tau^*|, |\mathbb{A}^*| = |\mathbb{A}|$ and for each $\mathbb{A}^*$-formula $\phi(\overline{v}, w)$

with free variables from $v_1, \ldots, v_n, w$, there is an n-ary function symbol f_ϕ such that

$$\mathcal{M} \models \forall \overline{v}\ (\exists w \phi(\overline{v}, w) \to \phi(\overline{v}, f_\phi(\overline{v})))$$

b) Prove that if $\mathcal{N}$ is an τ^*-substructure of $\mathcal{M}^*$, then $\mathcal{N} \prec_{\mathbb{A}^*} \mathcal{M}^*$.

c) Prove that there is an τ^*-substructure $\mathcal{N}$ of $\mathcal{M}^*$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}^*|, |X|)$.

Exercise 1.24 For τ and $\mathbb{A}$ define the appropriate notion of *built-in Skolem functions*. Prove that for any τ and $\mathbb{A}$ and $\mathbb{A}$-theory T there are $\tau^* \supseteq \mathcal{L}$, $\mathbb{A}^* \supseteq \mathbb{A}$ and $T^* \supseteq T$ with $|\mathbb{A}^*| = |T^*| = |\mathbb{A}|$ where T^* has built-in Skolem functions.

Exercise 1.25 Let $\mathbb{A}$ be as above and let $\mathcal{K}$ is a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that if $\mathcal{M}$ is a τ-structure and $X \subset \mathcal{M}$ there is $\mathcal{N} \prec_A \mathcal{M}$ such that $|\mathcal{N}| = \max(|X|, |\mathbb{A}|)$ and $\mathcal{N} \in \mathcal{K}$.

Exercise 1.26 As mentioned above first order logic, the Upward Löwenheim-Skolem is an easy consequence of Compactness. In infinitary logics it is generally false. Let $\tau = \{U, S, E, c_1, \ldots, c_n, \ldots\}$ and let ϕ be the conjunction of

i) $\forall x\ U(x) \leftrightarrow \neg S(x)$,
ii) $\forall x \forall y\ (E(x, y) \to U(x) \wedge U(y))$;
iii) $c_i \neq c_j$, for $i \neq j$;
iv) $U(c_i)$ for all i,
v) $\forall y \forall z\ ([U(y) \wedge U(z) \wedge\ \forall x\ ((E(x, y) \leftrightarrow E(x, z))] \to y = z)$
vi) $\forall x(U(x) \to \bigwedge_{i=1}^{\infty} x = c_i$.

Prove that every model of ϕ has size at most $2^{\aleph_0}$.

Elementary chains behave as they do in first order logic.

Exercise 1.27 Suppose $\mathbb{A}$ is a fragment of $\mathcal{L}_{\infty,\omega}$, $(I, <)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain of τ-structures, i.e., $\mathcal{M}_i \prec_{\mathbb{A}} \mathcal{M}_j$ for $i < j$. Let $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Then $\mathcal{M}_i \prec_{\mathbb{A}} \mathcal{M}$ for all $i \in I$.

Exercise 1.28 Give an example showing that PC classes need not be preserved by elementary chains.

Exercise 1.29 Let τ be countable. Suppose $(\mathcal{M}_\alpha : \alpha < \omega_1)$ is a chain of countable τ-structures such that $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$ for β a limit ordinal. If $\phi \in \mathcal{L}_{\omega_1,\omega}$ and $\mathcal{M} \models \phi$, then $\{\alpha : \mathcal{M}_\alpha \models \phi\}$ is closed unbounded.

1.2 $\mathcal{L}_{\omega_1,\omega}$ and omitting first order types

In Exercise 1.7 we noted that the class of models of a first order theory omitting a type is expressible in $\mathcal{L}_{\omega_1,\omega}$. The next result, due to Chang, shows that any class axiomatizable by an $\mathcal{L}_{\omega_1,\omega}$-sentence is the reduct of a class of models of a first order theory omitting a set of types.

Theorem 1.30 *Let τ be a countable vocabulary and let T be a countable set of $\mathcal{L}_{\omega_1,\omega}$-sentences. There is a countable vocabulary $\tau^* \supseteq \tau$, a first order τ^*-theory T^* and a set of types Γ such that:*

i) if $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits all types in Γ, then the τ-reduct of $\mathcal{M}$ is a model of T;

ii) every model of T has an τ^*-expansion, that is a model of T^* omitting all types in Γ.

Proof Let $\mathbb{A}$ be the smallest fragment containing all sentences in T. We expand τ to τ^* so for each formula ϕ in $\mathbb{A}$ with free variables from $v_1, \ldots, v_n$ we have an n-ary relation symbol R_ϕ. T^* is formed by adding sentences to T.

i) if ϕ is atomic add

$$\forall \overline{v}\ R_\phi \leftrightarrow \phi$$

ii) if ϕ is $\neg\theta$ add

$$\forall \overline{v}\ R_\phi \leftrightarrow \neg R_\theta$$

iii) if ϕ is $\bigwedge_{\theta \in X} \theta$ add

$$\forall \overline{v}\ R_\phi \leftrightarrow R_\theta$$

for all $\theta \in X$ and let γ_ϕ be the type

$$\{\neg R_\phi\} \cup \{R_\theta : \theta \in X\}$$

iv) if ϕ is $\bigvee_{\theta_\in X} \theta$, add

$$\forall \overline{v} R_\theta \rightarrow R_\phi$$

for all θ in X and let γ_ϕ be the type

$$\{R_\phi\} \cup \{\neg R_\theta : \theta \in X\}$$

v) if ϕ is $\exists w\ \theta$, then add

$$\forall \overline{v}\ R_\phi \leftrightarrow \exists w R_\theta$$

vi) for each sentence $\phi \in T$, add R_ϕ to T^*.

Let Γ be the collection of types γ_ϕ described above.[5]

Exercise 1.31 a) Suppose $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits every type in Γ. Prove that

$$\mathcal{M} \models \forall \overline{v}\ \phi \leftrightarrow R_\phi$$

for all $\phi \in \mathbb{A}$.

Conclude that the τ-reduct of a model of T^* is a model of T.

b) Prove by induction that every $\mathcal{M} \models T$ has an expansion that is a model of T^* omitting all types in Γ by interpreting R_ϕ as ϕ.

This completes the proof. □

We will examine a refinement of this result in Theorem 2.29

[5]As described here for each sentence ϕ we have added a 0-ary predicate symbol. If you are not comfortable with this approach we could rephrase this by adding a single constant c, and for each sentence ϕ adding a unary predicate R_ϕ and making assertions about $R_\phi(c)$.

2 Back and Forth

2.1 Karp's Theorem

We next work on several characterizations of $\equiv_{\infty,\omega}$ due to Karp.

Definition 2.1 Let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. A *partial isomorphism system* between $\mathcal{M}$ and $\mathcal{N}$ is a non-empty collection P of partial τ-embeddings $f : A \to \mathcal{N}$ where $A \subseteq \mathcal{M}$ such that:

i) for all $f \in P$ and $a \in \mathcal{M}$ there is $g \in P$ such that $g \supseteq f$ and $a \in \text{dom}(g)$;

ii) for all $g \in P$ and $b \in \mathcal{N}$ there is $g \in P$ such that $g \supseteq f$ and $b \in \text{img}(g)$.

We write $\mathcal{M} \cong_p \mathcal{N}$ if there is a partial isomorphism system between $\mathcal{M}$ and $\mathcal{N}$.

Exercise 2.2 If $\mathcal{M}$ and $\mathcal{N}$ are countable and $\mathcal{M} \cong_p \mathcal{N}$, show by a back-and-forth argument that $\mathcal{M} \cong \mathcal{N}$.

Definition 2.3 Let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures. The game $G(\mathcal{M}, \mathcal{N})$ is played as follows. At stage n, Player I plays $a_n \in M$ or $b_n \in N$. In the first case Player II responds with $b_n \in N$ and in the later case Player II responds with $a_n \in M$. Player II wins the play of the game if the map $a_n \mapsto b_n$ is a partial τ-embedding.

Theorem 2.4 *The following are equivalent:*

i) $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$*;*

ii) $\mathcal{M} \cong_p \mathcal{N}$*;*

iii) there is partial isomorphism system P *between* $\mathcal{M}$ *and* $\mathcal{N}$ *where every* $p \in P$ *has finite domain;*

iv) Player II has a winning strategy in $G(\mathcal{M}, \mathcal{N})$*.*

Proof

iii) $\Rightarrow$ ii) is clear.

ii) $\Rightarrow$ i) Let P be a system of partial isomorphisms. We prove that for all $\phi(v_1, \ldots, v_n) \in \mathcal{L}_{\infty,\omega}$, $f \in P$ and $a_1, \ldots, a_n \in \text{dom}(f)$, then

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(f(\overline{a})).$$

We prove this by induction on formulas. This is clear for atomic formulas and the induction step is obvious for $\neg$, $\bigwedge$ and $\bigvee$. Suppose $\phi(\overline{v})$ is $\exists w\, \psi(\overline{v}, w)$.

Suppose $\mathcal{M} \models \exists w\, \psi(\overline{a}, c)$. There is $g \in P$ with $g \supseteq f$ and $c \in \text{dom}(P)$. By induction, $\mathcal{N} \models \phi(f(\overline{a}), f(c))$, so $\mathcal{N} \models \phi(f(\overline{a}))$.

On the other hand, if $\mathcal{N} \models \psi(f(\overline{a}), d)$. There is $g \in P$ with $g \supseteq f$ and $c \in \mathrm{dom}(g)$ such that $g(c) = d$. By induction, $\mathcal{M} \models \psi(\overline{a}, c)$. Thus $\mathcal{M} \models \exists\phi(\overline{a})$.

This proves the claim. In particular, if ϕ is a sentence $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$.

iv)$\Rightarrow$ iii) Let η be a winning strategy for Player II. Let P be the set of all maps $f(a_i) = b_i$ where $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are the results of some play of the game where at each stage Player I has played either a_n or b_n and Player II has responded using η. Since η is a winning strategy for Player II, each such f is a partial τ-embedding. Since Player I can at any stage play any element from $\mathcal{M}$ or $\mathcal{N}$, P satisfies i) and ii) in the definition of a partial isomorphism system.

i) $\Rightarrow$ iv) We need one fact.

claim Suppose $(\mathcal{M}, \overline{a}) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b})$ and $c \in M$, there is $d \in N$ such that

$$(\mathcal{M}, \overline{a}, c) \equiv_{\infty,\omega} (\mathcal{N}, \overline{b}, d).$$

Suppose not. Then for all $d \in \mathcal{N}$ there is ϕ_d such that $\mathcal{M} \models \phi_d(\overline{a}, c)$ and $\mathcal{N} \models \neg\phi_d(\overline{b}, d)$. But that

$$\mathcal{M} \models \exists v \bigwedge_{d \in N} \phi_d(\overline{a}, v)$$

and

$$\mathcal{N} \not\models \exists v \bigwedge_{d \in N} \phi_d(\overline{b}, v)$$

a contradiction.

We now describe Player II's strategy. Player II always has a play to ensure $(\mathcal{M}, a_1, \ldots, a_n) \equiv_{\infty,\omega} (\mathcal{N}, b_1, \ldots, b_n)$. As long as Player II does this the resulting map will be a partial $\mathcal{L}$-embedding. $\square$

Exercise 2.5 If $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$, then for any countably generated substructure $\mathcal{M}_0 \subseteq \mathcal{N}$ there is an isomorphic $\mathcal{N}_0 \subseteq \mathcal{N}$.

Exercise 2.6 We say that an abelian group is $\aleph_1$-*free* if every countable subgroup is free.

Prove that if G is free abelian and $H \equiv_{\infty,\omega} G$, then H is $\aleph_1$-free. We will show the converse in Corollary 2.20.

Exercise 2.7 For any abelian group A we say that a subgroup A_0 is thin if we there is a subgroup A_1 such that $A = A_0 \oplus A_1$ and A_1 is isomorphic to A.

Let G be the free abelian group on countably many generators. Let P be the set of isomorphisms between thin subgroups of G and thin subgroups of $\mathbb{Z}^\omega$.

a) Prove that P is a partial isomorphism system. Conclude that $G \equiv_{\infty,\omega} \mathbb{Z}^\omega$;

b) The group $\mathbb{Z}^\omega$ is known not to be free abelian (see for example ??). Conclude that the class of free abelian groups is $\mathrm{PC}_{\omega_1,\omega}$ but not $\mathcal{L}_{\infty,\omega}$-axiomatizable. In Appendix A.4 we will show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$.

Exercise 2.8 Prove that $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if and only if there is a forcing extension of the universe $\mathbb{V}[G]$ where $\mathcal{M} \cong \mathcal{N}$.

For another view, we begin by defining a deceptively simple sequence of equivalence relations. For each ordinal α, we will have a relation

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$$

where $\mathcal{M}$ and $\mathcal{N}$ are τ-structures, $\overline{a} \in \mathcal{M}^n$ and $\overline{b} \in \mathcal{N}^n$ and $n = 0, 1, 2, \ldots$.

- $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(\overline{b})$ for all atomic τ-formulas ϕ.
- For all ordinals α, $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$ if for all $c \in \mathcal{M}$ there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and for all $d \in \mathcal{N}$ there is $c \in \mathcal{M}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$.
- For all limit ordinals β, $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{N}, \overline{b})$ if and only if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ for all $\alpha < \beta$.

We show that $\sim_\alpha$ captures the more complicated notion of equivalence for sentences of quantifier rank at most α.

Definition 2.9 Define $\mathrm{qr}(\phi)$ the *quantifier rank* of $\mathcal{L}_{\infty,\omega}$-formulas as follows:

- If ϕ is atomic, then $\mathrm{qr}(\phi) = 0$;
- $\mathrm{qr}(\neg\phi) = \mathrm{qr}(\phi)$;
- $\mathrm{qr}(\bigwedge_{\phi\in X} \phi) = \mathrm{qr}(\bigvee_{\phi\in X} \phi) = \sup_{\phi\in X} \mathrm{qr}(\phi)$;

- $\mathrm{qr}(\exists v\ \phi) = \mathrm{qr}(\forall v\ \phi) = \mathrm{qr}(\phi) + 1$.

Definition 2.10 We say $\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$$

for any $\mathcal{L}_{\infty,\omega}$-sentence ϕ of quantifier rank at most α.

Theorem 2.11 *$\mathcal{M} \equiv_\alpha \mathcal{N}$ if and only if $\mathcal{M} \sim_\alpha \mathcal{N}$.*

We leave the proof as an exercise.

Exercise 2.12 Prove by induction on α that for all τ-structures $\mathcal{M}, \mathcal{N}$, $\overline{a} \in \mathcal{M}$ and $\overline{b} \in \mathcal{N}$, $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if

$$\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{b})$$

for all formulas $\phi(\overline{v})$ of quantifier rank at most α.

2.2 Scott's Theorem

If τ is any vocabulary and $\mathcal{M}$ is an τ-structure we define a sequence of $\mathcal{L}_{\infty,\omega}$-formulas $\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v})$ for $\overline{a} \in \mathcal{M}^{<\omega}$ and α is an ordinal as follows:

$$\Phi^{\mathcal{M}}_{\overline{a},0}(\overline{v}) = \bigwedge_{\psi \in X} \psi(\overline{v}),$$

where $X = \{\psi : \mathcal{M} \models \psi(\overline{a})$ and ψ is atomic or the negation of an atomic τ-formula$\}$.

If α is a limit ordinal, then

$$\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{\beta<\alpha} \Phi^{\mathcal{M}}_{\overline{a},\beta}(\overline{v}).$$

If $\alpha = \beta + 1$, then

$$\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \bigwedge_{b \in \mathcal{M}} \exists w\ \Phi^{\mathcal{M}}_{\overline{a},b,\beta}(\overline{v}, w) \wedge \forall w \bigvee_{b \in \mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},b,\beta}(\overline{v}, w).$$

Lemma 2.13 *Let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures, $\overline{a} \in \mathcal{M}^l$, and $\overline{b} \in \mathcal{N}^l$. Then, $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$.*

Proof We prove this by induction on α. Because $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if and only if they satisfy the same atomic formulas, the lemma holds for $\alpha = 0$.

Suppose that γ is a limit ordinal and the lemma is true for all $\alpha < \gamma$. Then

$$\begin{aligned}(\mathcal{M}, \overline{a}) \sim_\gamma (\mathcal{N}, \overline{b}) &\Leftrightarrow (\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b}) \text{ for all } \alpha < \gamma \\ &\Leftrightarrow \mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b}) \text{ for all } \alpha < \gamma \\ &\Leftrightarrow \mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\gamma}(\overline{b}).\end{aligned}$$

Suppose that the lemma is true for α. First, suppose that $\mathcal{N} \models \phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. Let $c \in \mathcal{M}$. Because

$$\mathcal{N} \models \bigwedge_{x \in \mathcal{M}} \exists w \; \Phi^{\mathcal{M}}_{\overline{a},x,\alpha}(\overline{b}, w),$$

there is $d \in N$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. By induction, $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. If $d \in \mathcal{N}$, then because

$$\mathcal{N} \models \forall w \bigvee_{c \in \mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, w)$$

there is $c \in \mathcal{M}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$ and $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. Thus $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$.

Suppose, on the other hand, that $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{b})$. Suppose that $c \in \mathcal{M}$, then there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. Similarly, if $d \in \mathcal{N}$, then there is $c \in \mathcal{M}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$. Thus, $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$, as desired. □

A similar induction shows that even more is true. The formula $\Phi^{\mathcal{M}}_{\overline{a},\alpha}$ depends only on the $\sim_\alpha$-class of $(\mathcal{M}, \overline{a})$.

Exercise 2.14 Prove that if $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b})$ if and only if $\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) = \Phi^{\mathcal{N}}_{\overline{b},\alpha}(\overline{v})$

Lemma 2.15 *For any infinite τ-structure $\mathcal{M}$, there is an ordinal $\alpha < |M|^+$ such that if $\overline{a}, \overline{b} \in \mathcal{M}^l$ and $(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ for all β. The least such α is called the* Scott rank *of $\mathcal{M}$ and denoted* $\mathrm{sr}(\mathcal{M})$.

Proof Let $\Gamma_\alpha = \{(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \mathcal{M}^l \text{ for some } l = 0, 1, \ldots \text{ and } (\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{M}, \overline{b})\}$. Clearly, $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$.

claim 1 If $\Gamma_\alpha = \Gamma_{\alpha+1}$, then $\Gamma_\alpha = \Gamma_\beta$ for all $\beta > \alpha$.

We prove this by induction on β. If β is a limit ordinal and the claim holds for all $\gamma < \beta$, then it also holds for β. Suppose that the claim is true for $\beta > \alpha$ and we want to show that it holds for $\beta + 1$. Suppose that $(\mathcal{M}, \overline{a}) \sim_\beta (\mathcal{M}, \overline{b})$ and $c \in \mathcal{M}$. Because $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{M}, \overline{b})$, there is $d \in \mathcal{N}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{M}, \overline{b}, d)$. By our inductive assumption, $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Similarly, if $d \in \mathcal{M}$, then there is $c \in \mathcal{M}$ such that $(\mathcal{M}, \overline{a}, c) \sim_\beta (\mathcal{M}, \overline{b}, d)$. Thus, $(\mathcal{M}, \overline{a}) \sim_{\beta+1} (\mathcal{M}, \overline{b})$ as desired.

claim 2: There is an ordinal $\alpha < |M|^+$ such that $\Gamma_\alpha = \Gamma_{\alpha+1}$.

Suppose not. Then, for each $\alpha < |\mathcal{M}|^+$, choose $(\overline{a}_\alpha, \overline{b}_\alpha) \in \Gamma_{\alpha+1} \setminus \Gamma_\alpha$. Because $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha < \beta$, the function $\alpha \mapsto (\overline{a}_\alpha, \overline{b}_\alpha)$ is one-to-one. Because there are only $|\mathcal{M}|$ finite sequences from $\mathcal{M}$ this is impossible. □

A more subtle notion of Scott rank is sometimes useful

Definition 2.16 For $\overline{a} \in \mathcal{M}$ define $r(\overline{a})$ as the least ordinal α such that

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{M}, \overline{b}) \Rightarrow (\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{M}, \overline{b}).$$

Then

$$sr(\mathcal{M}) = \sup\{r(\overline{a}) : \overline{a} \in \mathcal{M}\}.$$

We also define

$$\mathrm{SR}(\mathcal{M}) = \sup\{r(\overline{a}) + 1 : \overline{a} \in \mathcal{M}\}.$$

Exercise 2.17 Calculate SR for the following structures.

a) $(\omega, <)$

b) $(\omega + 1, <)$;

c) $(\omega + \omega, <)$;

d) $(\mathbb{Z}, +, 0)$.

We will see later that at limit ordinals α there can be interesting distinctions between $\mathrm{SR}(\mathcal{M}) = \alpha$ and $\mathrm{SR}(\mathcal{M}) = \alpha + 1$ while in both cases $\mathrm{sr}(\mathcal{M}) = \alpha$.

Exercise 2.18 Formulate and prove a version of Lemma 2.15 for finite τ-structures.

We conclude this section with Scott's Isomorphism Theorem

Let $\mathcal{M}$ be an infinite τ-structure of cardinality κ, and let α be the Scott rank of $\mathcal{M}$. Let $\Phi^{\mathcal{M}}$ be the sentence

$$\Phi^{\mathcal{M}}_{\emptyset,\alpha} \wedge \bigwedge_{l=0}^{\infty} \bigwedge_{\overline{a}\in M^l} \forall \overline{v}(\Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{v}) \to \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v})).$$

Because all of the conjunctions and disjunctions in $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ are of size κ, $\Phi^{\mathcal{M}}_{\overline{a},\beta} \in \mathcal{L}_{\kappa^+,\omega}$ for all ordinals $\beta < \kappa^+$. Thus $\Phi^{\mathcal{M}}$ is an $\mathcal{L}_{\kappa^+,\omega}$-sentence. We call $\Phi^{\mathcal{M}}$ the *Scott sentence* of $\mathcal{M}$. If τ is a countable vocabulary and $\mathcal{M}$ is countable τ-structure, then $\Phi^{\mathcal{M}} \in \mathcal{L}_{\omega_1,\omega}$.

Theorem 2.19 (Scott's Isomorphism Theorem) *Let $\mathcal{M}$ and $\mathcal{N}$ be τ-structures, and let $\Phi^{\mathcal{M}} \in \mathcal{L}_{\infty,\omega}$ be the Scott sentence of $\mathcal{M}$. Then, $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$ if and only if $\mathcal{N} \models \Phi^{\mathcal{M}}$.*

In particular, if $\mathcal{M}$ and $\mathcal{N}$ are countable then $\phi \in \mathcal{L}_{\omega_1,\omega}$ and, if $\mathcal{N} \models \Phi^{\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$.

Proof Clearly if $\mathcal{N} \equiv_{\infty,\omega} \mathcal{M}$, then $\mathcal{N} \models \Phi^{\mathcal{M}}$.

For the other direction, suppose $\mathcal{N} \models \Phi^{\mathcal{M}}$. We will show that $\mathcal{N} \cong_p \mathcal{M}$. Let α be the Scott rank of $\mathcal{M}$. Let P be the set of partial functions f with finite domain from $\mathcal{M}$ to $\mathcal{N}$ such that if $\overline{a}$ is the domain of f and $f(\overline{a}) = \overline{b}$ then $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$. Since $\mathcal{N} \models \Phi^{\mathcal{M}}$, the empty map is in P so $P \neq \emptyset$.

Suppose we have f in P with domain $\overline{a}$ with $f(\overline{a}) = \overline{b}$. Let $c \in \mathcal{M}$. Since $\mathcal{N} \models \Phi^{\mathcal{M}} \wedge \Phi^{\mathcal{M}}_{\overline{a},\alpha}(\overline{b})$, $\mathcal{N} \models \Phi^{\mathcal{M}} \wedge \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$. But $\Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{v}) \to \exists w\ \Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{v}, w)$. Thus there is $d \in \mathcal{N}$ such that $\mathcal{N}$ models $\Phi^{\mathcal{M}}_{\overline{a},c,\alpha}(\overline{b}, d)$ and we can extend f by sending c to d.

On the other hand suppose we are given $\widehat{d} \in \mathcal{N}$. Since $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\alpha+1}(\overline{b})$,

$$\mathcal{N} \models \forall w \bigvee_{m\in\mathcal{M}} \Phi^{\mathcal{M}}_{\overline{a},m,\alpha}(\overline{b}, w).$$

Thus there is $\widehat{c} \in \mathcal{N}$ such that $\mathcal{N} \models \Phi^{\mathcal{M}}_{\overline{a},\widehat{c},\alpha}(\overline{b}, d)$ and we can extend f be sending $\widehat{c}$ to $\widehat{d}$.

Thus P is a partial isomorphism system and $\mathcal{M} \cong_p \mathcal{N}$. By Theorem 2.4 $\mathcal{M} \equiv_\infty \mathcal{N}$ and be Exercise 2.2, if $\mathcal{M}$ and $\mathcal{N}$ are countable, then $\mathcal{M} \cong \mathcal{N}$. □

We use this to prove a converse to Exercise 2.6

Corollary 2.20 *Let G be the free abelian group on countably many generators and let H be an $\aleph_1$-free group that is not free on a finite set of generators. Then $G \equiv_{\infty,\omega} H$.*

Proof Let $\Phi^G \in \mathcal{L}_{\omega_1,\omega}$ be the Scott sentence of G. Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\infty,\omega}$ containing Φ^G and a sentence asserting H is not free on finitely many generators. By Downward Löwenheim–Skolem we can find $H_1 \prec_{\mathbb{A}} G$ containing countably many independent elements with $|H_1| = \aleph_0$. Since H is $\aleph_1$-free then H is free abelian on countably many generators and hence isomorphic to G. But then $H_1 \models \Phi^G$ and hence $H \models \Phi^G$. Thus $H \equiv_{\infty,\omega} G$. □

A stronger version of this result is proved in Exercise 2.52.

Exercise 2.21 Show that the class of $\aleph_1$-free abelian groups is $\mathcal{L}_{\omega_1,\omega}$-axiomatizable.

2.2.1 Complete Sentences

Definition 2.22 A sentence $\phi \in \mathcal{L}_{\infty,\omega}$ is *complete* if for all $\psi \in \mathcal{L}_{\infty,\omega}$

$$\phi \models \psi \text{ or } \phi \models \neg\psi.$$

Corollary 2.23 *For any $\mathcal{M}$ the Scott sentence $\Phi^{\mathcal{M}}$ is complete.*

Exercise 2.24 For any satisfiable sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$ there is a complete sentence ϕ such that $\phi \models \psi$.

Exercise 2.25 A sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ is complete if and only if it is $\aleph_0$-categorical.

Exercise 2.26 Suppose $\psi \in \mathcal{L}_{\omega_1,\omega}$ is κ-categorical for some infinite cardinal κ. Show that if $\mathcal{M}, \mathcal{N} \models T$ of cardinality at least κ then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. (Compare this to Exercise 5.10.)

Exercise 2.27 Give an example of an $\aleph_1$-categorical $\phi \in \mathcal{L}_{\omega_1,\omega}$ that is not complete.

Exercise 2.28 Let σ be a complete $\mathcal{L}_{\omega_1,\omega}$-sentence. and let $\mathcal{M}$ be the unique countable model. Show that the following are equivalent.

i) There is an uncountable model of σ.

ii) There is a non-surjective $\mathcal{L}_{\omega_1,\omega}$-elementary embedding of $\mathcal{M}$ into itself.

iii) There is a non-surjective $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into itself, where $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$-containing ϕ.

Gao [11] has shown these conditions are equivalent to the non-existence of a complete left-invariant metric on $\mathrm{Aut}(\mathcal{M})$ the automorphism group of $\mathcal{M}$.

For complete sentences in $\mathcal{L}_{\omega_1,\omega}$ we can give an improvement of Chang's Theorem 1.30.

Theorem 2.29 *Let ϕ be a complete sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$. There is $\tau^* \supseteq \tau$ and T^* a first order τ^*-theory such that $\mathcal{M} \models \phi$ if and only $\mathcal{M}$ is the τ-reduct of an atomic model of T^*.*

Proof Let $\mathcal{M}$ be the unique countable model of ϕ and let α be its Scott rank. Let $\mathbb{A}$ be a countable fragment containing all formulas $\Phi^{\mathcal{M}}_{\overline{a},\alpha}$ for $\overline{a} \in \mathcal{M}$. Now follow the proof of Theorem 1.30 to obtain $\tau^* = \tau \cup \{R_\psi : \psi \in \mathbb{A}\}$ and Γ a collection of τ^*-types such that models of $\Phi^{\mathcal{M}}$ are exactly that reducts of models of T^* that omit all types in Γ. Let T be the complete τ^*-theory of $\mathcal{M}$. As $\mathcal{M}$ omits all of the types in Γ, no type in Γ is implied by a principal type. Thus all types in Γ are omitted in any atomic model of T^* and the reduct of an atomic model of T^* is a model of ϕ.

For any $\mathcal{N} \models \phi$ and $\overline{a} \in \mathcal{N}$ there is $\overline{b} \in \mathcal{M}$ such that $R_{\Phi^{\mathcal{M}}_{\overline{b},\alpha}}(\overline{v})$ isolates $\mathrm{tp}^{\mathcal{M}}(\overline{a})$. □

2.3 Countable Approximations

In this section we give another characterization of $\equiv_\infty$ due to Kueker [20]. We begin with some set theoretic preliminaries. Kueker, and independently, Jech., introduced the following generalization of the the filter of closed unbounded subsets of a regular cardinal.

For any set X let $\mathcal{P}_{\omega_1}(X)$ be the collection of countable subsets of X.

Definition 2.30 $A \subseteq \mathcal{P}_{\omega_1}(X)$ is *closed* if $\bigcup a_n \in X$ for any $a_1 \subset a_2 \subset \ldots$ in X and $A \subseteq \mathcal{P}_{\omega_1}(X)$ is *unbounded* if for all $a \subset X$ countable, there is $b \in A$ with $a \subseteq b$.

Lemma 2.31 *Suppose $A_0, A_1, \ldots \in \mathcal{P}_{\omega_1}(X)$ are closed and unbounded. Then so is $\bigcap A_i$ is closed and unbounded.*

Proof It is clear that $\bigcap A_i$ is closed, so we need only show it i unbounded.

Let $a \in \mathcal{P}_{\omega_1}(X)$. Let $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$ be a bijective pairing function such that $\langle m, i \rangle < \langle m, j \rangle$ if $i < j$. We build

$$a = a_{-1} \subset a_0 \subset a_1 \subset \ldots \in \mathcal{P}_{\omega_1}(X)$$

such that if $n = \langle i, j \rangle$ we choose $a_n \in A_i$ with $a_{n-1} \subset a_n$. This is always possible since each A_i is unbounded. Then

$$b = \bigcap_{n \in \omega} a_n = \bigcap_{j \in \omega} a_{\langle i,j \rangle} \in A_i$$

for all $i \in \omega$. Thus $b \in \bigcap A_i$ and $\bigcap A_i$ is unbounded. □

Let $D(X) = \{A \subseteq \mathcal{P}_{\omega_1} : A \text{ has a closed, unbounded subset}\}$.

Exercise 2.32 Show that $D(X)$ is a countably complete filter on $\mathcal{P}_{\omega_1}(X)$.

We also have closure under diagonal intersections.

Lemma 2.33 *Suppose $I \subseteq X$ and A_i is closed and unbounded for all $i \in I$ then so is*

$$\triangle A_i = \{a \in \mathcal{P}_{\omega_1}(X) : a \in A_i \text{ for all } i \in a \cap I\}.$$

Proof Suppose $a_0 \subset a_1 \subset \ldots$ are in $\triangle A_i$ and $a = \bigcup a_i$. Let $j \in a \cap I$. Then there is n_0 such that $j \in a_n$ for $n \geq n_0$. But then $a_n \in A_j$ for $n \geq n_0$ and, since A_j is closed, $a \in A_i$. Thus $a \in \triangle A_i$ and $\triangle A_i$ is closed.

Let $a \in \mathcal{P}_{\omega_1}(X)$. Let $a_0 = a$ Given a_n, we know from Lemma 2.31 that $\bigcap_{i \in a_n \cap I} A_i$ is closed and unbounded. Thus there is

$$a_n \subset a_{n+1} \in \bigcap_{i \in a_n \cap I} A_i.$$

Let $a = \bigcup a_n$. Suppose $i \in a$. There is n_0 such that $i \in a_n$ for $n \geq n_0$. Thus $a_{n+1} \in A_i$ for $n \geq n_0$ and, since A_i is closed, $a \in A_i$. Thus $a \in \triangle A_i$ and $\triangle A_i$ is unbounded. □

Kueker also gave a useful game theoretic formulation of membership in $D(X)$.

Exercise 2.34 For $A \subseteq \mathcal{P}_{\omega_1}(X)$ consider the game $G(A)$ where Players I and II alternate playing elements of X.

Player I	Player II
c_0	
	d_0
c_1	
	d_1
$\vdots$	$\vdots$

Player II wins if $\{c_i, d_i : i \in \omega\}$ is in A. Prove that Player II has a winning strategy in $G(A)$ if and only if $A \in D(X)$.

We use the filter $D(X)$ as a notion of "almost all". Let X be a set. Let τ be a countable vocabulary and let $\mathcal{M}$ be a τ-structure. We say that X is *large enough to approximate* $\mathcal{M}$ if the universe of $\mathcal{M}$ is contained in X.

Exercise 2.35 Suppose X is large enough to approximate $\mathcal{M}$. Show that $\{a \in \mathcal{P}_{\omega_1}(X) : \mathcal{M} \cap a \text{ is a substructure of } \mathcal{M}\} \in D(X)$.

Let $\mathcal{M}^a$ denote $\mathcal{M} \cap a$. We say that $\mathcal{M}^a$ is a substructure *almost everywhere*.

Definition 2.36 Suppose $\phi \in \mathcal{L}_{\infty,\omega}$ and $a \in \mathcal{P}_{\omega_1}(X)$ we define the approximation ϕ^a to ϕ inductively as follows:

- $\phi^a = \phi$ if ϕ is atomic;
- $(\neg\phi)^a = \neg\phi^a$;
- $(\forall v\ \phi)^a = \forall v\ \phi^a$ and $(\exists v\ \phi)^a = \exists v\ \phi^a$;
- $\left(\bigwedge_{i\in I} \phi_i\right)^a = \bigwedge_{i\in I\cap a} \phi_i^a$ and $\left(\bigvee_{i\in I} \phi_i\right)^a = \bigvee_{i\in I\cap a} \phi_i^a$.

We say that X is *large enough to approximate* ϕ is $I \subset X$ whenever $\bigvee_{i\in I} \psi_i$ or $\bigwedge_{j\in psi} i$ is in the smallest fragment of $\mathcal{L}_{\infty,\omega}$ containing ϕ.

Exercise 2.37 Show that if $\phi \in \mathcal{L}_{\infty,\omega}$ and ψ is the canonical formula in negation normal form (see Exercise 1.21) such that $\models \phi \leftrightarrow \psi$ and X is large enough to approximate ϕ, then

$$\models \phi^a \leftrightarrow \psi^a \text{ a.e..}$$

We work in an X large enough to approximate $\mathcal{M}$ and ϕ. Intuitively, we think of X as some sufficiently rich substructure of the universe of sets $\mathbb{V}$ containing $\mathcal{M}$ and ϕ.

Theorem 2.38 *For any $\mathcal{M}$ and $\phi \in \mathcal{L}_{\infty,\omega}$*

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{M}^a \models \phi^a \text{ a.e.}$$

i.e., $\{a \in \mathcal{P}_{\omega_1}(X) : \mathcal{M}^a \models \phi^a\} \in D(X)$.

Proof First note that, since $D(X)$ is a filter, it is impossible that

$$\mathcal{M}^a \models \phi^a \text{ a.e. and } \mathcal{M}^a \models \neg\phi^a \text{ a.e..}$$

Thus it suffices to prove $\Rightarrow$ direction.

Next note that, by Exercise 2.37, we may assume that ϕ is in negation normal form. We prove by induction that if ψ is a subformula of ϕ, $\bar{b} \in \mathcal{M}$ and $\mathcal{M} \models \psi(\bar{b})$, then $\mathcal{M}^a \models \psi^a(\bar{b})$ a.e.. Note that $\bar{b} \in \mathcal{M}^a$ a.e., so this makes sense.

- If ψ is atomic or negated atomic this is clear since $\mathcal{M}^a$ is a substructure of $\mathcal{M}$. Note that these are the only negations we need to deal with.
- If ψ is $\bigvee_{i\in I} \theta_i(\bar{b})$, then there is $j \in I$ such that $\mathcal{M} \models \theta_j(\bar{b})$. Note that $j \in a$ for almost every $a \in P_{\omega_1}(X)$. By induction $\mathcal{M}^a \models \theta_j^a(\bar{b})$ a.e.. Thus $\mathcal{M}^a \models \psi^a(\bar{b})$ a.e..
- Suppose ψ is $\bigwedge_{i\in I} \theta_i(\bar{b})$ and $\mathcal{M} \models \psi(\bar{b})$. By induction,

  $$A_i = \{a : \mathcal{M}^a \models \theta_i^a(\bar{b})\} \in D(X)$$

  for all $i \in I$. By Lemma 2.33, $\triangle A_i \in D(X)$. Let $a \in \triangle D(X)$. If $i \in a \cap I$, then $a \in A_i$ so $\mathcal{M}^a \models \theta_i^a(\bar{b})$. Thus $\mathcal{M}^a \models \psi^a$ a.e..
- If ψ is $\exists v\ \theta(v, \bar{b})$, this is similar to the $\bigvee$-case.
- if ψ is $\forall v\ \theta(v, \bar{b})$ this is similar to the $\bigwedge$-case.

□

Exercise 2.39 Fill in the details in the $\exists v\ \theta$ and $\forall v\ \theta$ cases.

Corollary 2.40 *i) If $\phi \in \mathcal{L}_{\omega_1,\omega}$, then $\mathcal{M} \models \phi$ if and only if $\mathcal{M}^a \models \phi$ a.e..*
ii) If $\mathcal{M}$ is countable and $\phi \in \mathcal{L}_{\infty,\omega}$, $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \phi^a$ a.e..

Theorem 2.41 *i)* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N} \Leftrightarrow \mathcal{M}^a \cong \mathcal{N}^a$ *a.e..*
ii) $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N} \Leftrightarrow \mathcal{M}^a \ncong \mathcal{N}^a$ *a.e..*

Proof Note that filter $D(X)$ is not an ultrafilter, thus ii) is not an immediate consequence of i). Though, since it is a filter, it is enough to prove the $\Rightarrow$ directions of both i) and ii).

Suppose $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$. Let $\phi \in \mathcal{L}_{\infty,\omega}$ such that $\mathcal{M} \models \phi$ and $\mathcal{N} \models \neg\phi$. By the previous theorem, $\mathcal{M}^a \models \phi^a$ a.e. and $\mathcal{N}^a \models \neg\phi^a$ a.e. Since $D(X)$ is a filter $\{a : \mathcal{M}^a \models \phi^a \text{ and } \mathcal{N}^a \models \neg\phi^a\} \in D(X)$ and, thus, $\{a : \mathcal{M}^a \ncong \mathcal{N}^a\} \in D(X)$.

Suppose $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$. By Theorem 2.4 there is a non-empty system P of partial isomorphism with finite domains from $\mathcal{M}$ to $\mathcal{N}$. Let $A = \{a \in \mathcal{P}(\omega_1(X)) : \text{such that } \{f \in P : \text{dom}(f) \subset \mathcal{M}^a \text{ and } \text{img}(f) \subset \mathcal{N}^a\}$ is a nonempty system of partial isomorphism between $\mathcal{M}^a$ and $\mathcal{N}^a\}$.

Exercise 2.42 Show that $A \in D(X)$ and prove that $\mathcal{M}^a \cong \mathcal{N}^a$ for $a \in A$. □

Corollary 2.43 *If* $\mathcal{M}$ *is countable, then* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ *if and only if* $\mathcal{M} \cong \mathcal{N}^a$ *a.e..*

The corollary leads to another proof that an abelian group is $\equiv_{\infty,\omega}$ to a free abelian group if and only if every countable subgroup is free.

2.4 Larger Infinitary Languages

For completeness, we mention the the logics $\mathcal{L}_{\kappa,\lambda}$. In these logics we allow a more general quantification rule. We will say very little in this section and refer to reader to [9] for a more detailed survey.

Suppose ϕ is a formula and $\vec{v}$ is a sequence of variables freely occuring in ϕ with $|\vec{v}| < \lambda$, then $\exists\vec{v}\phi$ is a formula.

Thus we are allowed existential and, by taking negations in the usual way, universal quantification over sequences of variables of cardinality less than λ.

Exercise 2.44 Show there is $\phi \in \mathcal{L}_{\omega_1,\omega_1}$ such that $\mathcal{M} \models \phi$ if and only if $|M| > \aleph_0$.

Exercise 2.45 Show there is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence ϕ such that $(A, <) \models \phi$ if and only if $(A, <)$ is a well-order.

Exercise 2.46 Show that is an $\mathcal{L}_{\omega_1,\omega_1}$-sentence ϕ such that $(A,<) \models \phi$ if and only if $(A,<) \cong (\mathbb{R},<)$.

Exercise 2.47 Show that there is an $\mathcal{L}_{\omega_1,\omega_1}$ sentence ϕ such that ϕ has a model of cardinality κ if and only if the cofinality of κ is ω.

Exercise 2.48 Recall that a linear ordering $(A,<)$ is $\aleph_1$-*like* if $|A| = \aleph_1$ and $|\{b : b < a\}| \leq \aleph_0$ for all $a \in A$. Show that the class of $\aleph_1$-like linear orders is $\mathcal{L}_{\omega_1,\omega_1}$-axiomatizable.

Definition 2.49 We say $\mathcal{M} \cong_p^\kappa \mathcal{N}$ if and only if there is a nonempty system of partial embeddings P with the property that:

i) if $f \in P$, $A \subset \mathcal{M}$ and $|A| < \kappa$, then there is $g \in P$, $f \subseteq g$ and $A \subseteq \text{dom}(g)$ and

ii) if $f \in P$, $B \subset \mathcal{N}$ and $|B| < \kappa$, then there is $g \in P$, $f \subseteq g$ and $B \subseteq \text{img}(g)$.

Our original notion $\cong_p$ is $\cong_p^{\aleph_0}$.

Exercise 2.50 a) Prove $\mathcal{M} \cong_p^\kappa \mathcal{N} \Rightarrow \mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ by induction on complexity of formulas.

b) Prove $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N} \Rightarrow \mathcal{M} \cong_p^\kappa \mathcal{N}$. [Hint: Let P be the set of $f : A \to \mathcal{N}$ where $|A| < \kappa$ and f is $\mathcal{L}_{\infty,\kappa}$-elementary.]

Exercise 2.51 Let $\mathcal{M}$ and $\mathcal{N}$ be $\aleph_1$-like dense linear orders with out endpoints. Prove $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$. [Hint: Consider $P = \{f : A \to B : A$ an initial segment of $\mathcal{M}$, B is an initial segment of $\mathcal{N}$, $\mathcal{M} \setminus A$ and $\mathcal{N} \setminus B$ has no least element and f is order preserving$\}$.]

Note in A.2 we show there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-like dense linear orders.

Exercise 2.52 Suppose G and H are uncountable $\aleph_1$-free abelian groups. Prove that $G \equiv_{\infty,\omega_1} H$. [Hint: Say $A_0 \subseteq A$ is smooth if there is a basis of S of A and $S_0 \subseteq S$ countable such that A_0 is the group generated by S_0. Consider $P = \{f : A \to B : A \subset G$ and $B \subset H$ are smooth, and f is a bijective homomorphism$\}$.]

In Exercise A.4 we show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$.

3 The Space of Countable Models

There are many connections between the model theory of $\mathcal{L}_{\omega_1,\omega}$ and Descriptive Set Theory. Suppose τ is a countable vocabulary. The first observation is that the set of countably infinite τ-structures can naturally be given the structure of a Polish space. The reader should consult [17] for any unexplained notions or results from Descriptive Set Theory.

3.1 Spaces of τ-structures

To each countable vocabulary τ we will define a Polish space on the set of τ-structures with universe ω. The notation in the general case makes this seem more mysterious then it is, so we first illustrate with a simple example. Let τ_0 be the language with one binary relation symbol R. In this case we will let $\mathbb{X}_{\tau_0} = 2^{\omega\times\omega}$. If $\mathcal{M}$ is a τ_0-structure with universe ω, define $f_{\mathcal{M}} : \omega \times \omega \to 2$ by $f_{\mathcal{M}}(i,j) = 1$ if and only if $\mathcal{M} \models R(i,j)$. Similarly, for any $f : \omega^2 \to 2$ we can define $\mathcal{M}_f$ as the τ_0-structure on ω with binary relation $R_f = \{(i,j) : f(i,j) = 1\}$. The maps $\mathcal{M} \mapsto f_{\mathcal{M}}$ and $f \mapsto \mathcal{M}_f$ are easily seen to be inverses. Thus we can identify the set τ_0-structures with universe ω with elements of $\mathbb{X}_{\tau_0}$. Of course $\mathbb{X}_{\tau_0}$ is a Polish space, indeed it is homeomorphic to the Cantor space.

Suppose we instead consider τ_1 where we have one unary function symbol. Then we could identify τ_1-structures with ω^ω.

Let τ be a countable vocabulary with constant symbols $(c_i : i < \alpha_C)$, relations symbols $(R_i : i < \alpha_R)$ and function symbols $(f_i : i < \alpha_F)$ where $\alpha_C, \alpha_R, \alpha_F \leq \omega$. For each $i < \alpha_R$ and $j < \alpha_F$ let $m(i)$ and $n(j)$ be the arities of R_i and f_j respectively. Define

$$\mathbb{X}_\tau = \omega^{\alpha_C} \times \prod_{i<\alpha_R} 2^{\omega^{m(i)}} \times \prod_{j<\alpha_F} \omega^{\omega^{n(j)}}.$$

Arguing as above it is easy to see that there is a natural bijection between $\mathbb{X}_\tau$ and the set of all τ-structures with universe ω. The ω^{α_C} part tells how to interpret the constant symbols, the $\prod_{i<\alpha_R} 2^{\omega^{m(i)}}$ part tells how to interpret the relation symbols and the $\prod_{j<\alpha_F} \omega^{\omega^{n(j)}}$ part tells how to interpret the function symbols. [6]

[6]Note that this definition still makes sense if one or several of α_C, α_R or α_F are zero. In particular if τ is the empty vocabulary, then $\mathbb{X}_\tau$ is the one element space.

Suppose $\phi(v_{i_1}, \ldots, v_{i_m})$ is an $\mathcal{L}_{\omega_1,\omega}$-formula with free variables from $v_{i_1}, \ldots, v_{i_m}$ and $n_1, \ldots, n_m \in \omega$. Define

$$E_{\phi,\overline{n}} = \{\mathcal{M} \in \mathbb{X}_\tau : \mathcal{M} \models \phi(n_1, \ldots, n_m).\}$$

Lemma 3.1 *Each $E_{\phi,\overline{n}}$ is a Borel subset of $\mathbb{X}_\tau$.*

Proof We prove this by induction on the complexity of $\mathcal{L}_{\omega_1,\omega}$-formulas.

claim If ϕ is atomic, then $E_{\phi,\overline{n}}$ is clopen.

The formula ϕ is either $R(t_1, \ldots, t_l)$ or $t_1 = t_2$ where $t_1, \ldots, t_l$ are terms and R is a relation symbol. If each of the t_i is a variable, this is clear from the construction of $\mathbb{X}_\tau$.

Unwinding the terms, we can find atomic formulas $\psi_1(\overline{v}, \overline{w}), \ldots, \psi_k(\overline{v}, \overline{w})$ each of the form a relation symbol or $=$ applied to variables or a function symbol applied to variables is equal to a variable and

$$\begin{array}{rcl} \phi(\overline{v}) & \Leftrightarrow & \exists \overline{w}\ (\psi_1(\overline{v}, \overline{w}) \wedge \ldots \wedge \psi_k(\overline{v}, \overline{w})) \\ & \Leftrightarrow & \forall \overline{w}\ [(\psi_1(\overline{v}, \overline{w}) \wedge \ldots \wedge \psi_{k-1}(\overline{v}), \overline{w}) \rightarrow \psi_k(\overline{v}, \overline{w})]. \end{array}$$

For example, suppose $\phi(v_1, v_2)$ is $R(v_1, g(f(v_1, v_2)))$. Then

$$\begin{array}{rcl} \phi(v_1, v_2) & \Leftrightarrow & \exists w_1 \exists w_2\ [f(v_1, v_2) = w_1 \wedge g(w_1) = w_2 \wedge R(v_1, w_2)] \\ & \Leftrightarrow & \forall w_1 \forall w_2 [(f(v_1, v_2) = w_1 \wedge g(w_1) = w_2) \rightarrow R(v_1, w_2)]. \end{array}$$

Thus we have $\phi(\overline{v}) \Leftrightarrow \exists \overline{w}\ \psi(\overline{v}, \overline{w}) \Leftrightarrow \forall \overline{w}\ \theta(\overline{v}, \overline{w})$, where each $E_{\psi,\overline{a}}$ and $E_{\theta,\overline{a}}$ is clopen. But then

$$\begin{array}{rcl} E_{\phi,\overline{m}} & = & \bigcup\limits_{\overline{a}} E_{\psi,\overline{m},\overline{b}} \text{ an open set} \\ & = & \bigcap\limits_{\overline{b}} E_{\theta,\overline{m},\overline{b}} \text{ a closed set} \end{array}$$

Thus $E_{\phi,\overline{m}}$ is clopen.

The rest of the induction is easy

- $E_{\neg\phi,\overline{m}} = X_\tau \setminus E_{\phi,\overline{m}}$;
- $E_{\bigvee \phi,\overline{m}} = \bigcup E_{\phi,\overline{m}}$;
- $E_{\bigwedge \phi,\overline{m}} = \bigcap E_{\phi,\overline{m}}$;

- $E_{\exists v\phi,\overline{m}} = \bigcup_a E_{\phi,\overline{m},a}$;
- $E_{\forall v\phi,\overline{m}} = \bigcap_a E_{\phi,\overline{m},a}$;

So, by induction on complexity, each $E_{\phi,\overline{m}}$ is Borel. □

Definition 3.2 If ϕ is a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$, let

$$\mathrm{Mod}(\phi) = \{\mathcal{M} \in \mathbb{X}_\tau : \mathcal{M} \models \phi\}.$$

Then $\mathrm{Mod}(\phi)$ is a Borel subset of $\mathbb{X}_\tau$ and can be viewed as a standard Borel space. Not every Borel subset of $\mathbb{X}_\tau$ arises as $\mathrm{Mod}(\phi)$ for some $\mathcal{L}_{\omega_1,\omega}$-sentence.

There is a natural action of S_∞,the set of all permutations of ω, on X_τ by homeomorphisms. If $\mathcal{M} \in \mathbb{X}_\tau$ and $\sigma \in S_\infty$ let $\sigma(\mathcal{M})$ be the τ-structure induced by σ, i.e.,

$$\sigma(\mathcal{M}) \models \phi(\overline{a}) \Leftrightarrow \mathcal{M} \models \phi(\sigma^{-1}(\overline{a})).$$

We say that $A \subseteq \mathbb{X}_\tau$ is *invariant* if $\sigma(A) = A$ for all $\sigma \in S_\infty$.

Corollary 3.3 *For any sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, $\mathrm{Mod}(\phi)$ is an invariant Borel subset of $\mathbb{X}_\tau$.*

Proof If ϕ is a sentence, then $\mathcal{M} \models \phi$ if and only if $\sigma(\mathcal{M}) \models \phi$. □

In Theorem 4.25 we will prove the converse namely that for every invariant Borel set $A \subseteq \mathbb{X}_\tau$ there is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence ϕ such that $A = \mathrm{Mod}(\phi)$.

Corollary 3.4 *For any $\mathcal{M} \in \mathbb{X}_\tau$, $\{\mathcal{N} : \mathcal{N} \cong \mathcal{M}\}$ is invariant Borel.*

Proof By Scott's Theorem the isomorphism class of $\mathcal{M}$ is $\mathrm{Mod}(\Phi^{\mathcal{M}})$, where $\Phi^{\mathcal{M}}$ is the Scott sentence of $\mathcal{M}$. □

Exercise 3.5 Prove that if $\mathcal{K}$ is a $\mathrm{PC}_{\omega_1,\omega}$-class then $\mathcal{K} \cap \mathbb{X}_\tau$ is invariant $\mathbf{\Sigma}^1_1$.

3.2 The Number of Countable Models

One of the nagging long standing open problems in the model theory of first order logic is *Vaught's Conjecture* that if T is a complete first order theory in a countable language, then the number of non-isomorphic models countable of T is either at most $\aleph_0$ or exactly $2^{\aleph_0}$. It is natural to extend the conjecture from first order logic to models of an $\mathcal{L}_{\omega_1,\omega}$-sentence. Thus, *Vaught's Conjecture for* $\mathcal{L}_{\omega_1,\omega}$ asserts that for any sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ the number of non-isomorphic countable models is either at most $\aleph_0$ or exactly $2^{\aleph_0}$. Of course, if the Continuum Hypothesis holds, Vaught's Conjecture is a trivial consequence, but we will later give a rephrasing of Vaught's Conjecture that does not depend on the continuum hypothesis. The first progress toward Vaught's Conjecture is still the strongest general result known.

Theorem 3.6 (Morley) *Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. Then the number of non-isomorphic countable models of ϕ is either at most $\aleph_1$ or exactly $2^{\aleph_0}$.*

We will give Morley's original proof below, but first it worth noting that, from the point of view of modern Descriptive Set Theory, Morley's Theorem is an easy consequence of core theorems on equivalence relations. Consider the isomorphism relation $\cong$ on $\mathbb{X}_\tau$. Then $\mathcal{M} \cong \mathcal{N}$ if and only if

$\exists f \in \omega^\omega$ f is a bijection and f preserves all of the constant, relation and function symbols of τ.

This is easily seen to be $\mathbf{\Sigma}^1_1$. For example, let τ be the vocabulary with a single binary relation symbol R. Then $\mathcal{M} \cong \mathcal{N}$ if and only if there exists $f \in \omega^\omega$ such that f is a bijection and

$$\forall m, n \; \mathcal{M} \models R(m,n) \leftrightarrow \mathcal{M} \models (f(m), f(n)).$$

Thus for any ϕ, isomorphism is a $\mathbf{\Sigma}^1_1$-equivalence relation on $\mathrm{Mod}(\phi)$.

Theorem 3.7 *i)* **(Silver)** *If E is a $\mathbf{\Pi}^1_1$-equivalence relation on standard Borel space X with uncountably many equivalence classes, then there is a perfect set of E-inequivalent elements. In particular, this is true for any Borel equivalence relation.*

ii) **(Burgess)** *If E is a $\mathbf{\Sigma}^1_1$-equivalence relation on standard Borel space X with at least $\aleph_2$ equivalence classes, then there is a perfect set of E-inequivalent elements.*

Morely's Theorem is, of course, an easy consequence of Burgess' result. It is worth noting that Burgess's result can't be improved. For example define an equivalence relation on E on ω^ω such that xEy if and only if either neither codes a well ordering or there both code linear orders and there is an isomorphism between the orders they code. This is easily seen to be a Σ^1_1-equivalence relation with exactly $\aleph_1$-classes.

In the isomorphism equivalence relation on $\mathrm{Mod}(\phi)$ we have added information for Corollary 3.4 that every equivalence class is Borel. This is also not enough.

Exercise 3.8 Define an equivalence relation E on ω^ω such that xEy if and only if $\omega_1^x = \omega_1^y$ (see §**??** for definitions). Show that E is $\mathbf{\Sigma}^1_1$-equivalence relation with exactly $\aleph_1$ classes, all of which are Borel.

Further examples get closer to Vaught's Conjecture. In particular there are $\mathrm{PC}_{\omega_1,\omega}$ classes with exactly $\aleph_1$-countable models.

Exercise 3.9 (Friedman) a) Suppose $\mathcal{M}$ is a countable ω-model of ZFC and the ordinals of $\mathcal{M}$ are ill-founded. Show that there is an ordinal α such that the ordinals of $\mathcal{M}$ have order type $\alpha + (\mathbb{Q} \times \alpha)$.

b) Show that a) gives rise to a $\mathrm{PC}_{\omega_1,\omega}$-class with exactly $\aleph_1$ non-isomorphic countable models.

Exercise 3.10 (Kunen) A linear order is *1-transitive* if for all a, b there is an order preserving automorphism taking a to b.

For $\alpha < \omega$, we say that the support of $f : \alpha \to \mathbb{Z}$ is $\{\beta < \alpha : f(\beta) = 0\}$ and let $\mathbb{Z}^\alpha$ be the set of finite support functions from α to $\mathbb{Z}$ ordered lexicographically. [Note $\mathbb{Z}^0 = \{\emptyset\}$.]

a) Prove that each $\mathbb{Z}^\alpha$ is 1-transitive.

b) Prove that every countable 1-transitive linear order is isomorphic to $\mathbb{Z}^\alpha$ or $\mathbb{Q} \times \mathbb{Z}_\alpha$ ordered lexicographically.

c) Use b) describe a $\mathrm{PC}_{\omega,\omega}$-class with exactly $\aleph_1$ non-isomorphic countable models.

We also know that the isomorphism equivalence relation on $\mathrm{Mod}(\phi)$ is also the orbit equivalence relation of the action of the Polish group S_∞ on the standard Borel space $\mathrm{Mod}(\phi)$. This leads to the Topological Vaught Conjecture which is also still open.

Topological Vaught Conjecture Suppose G is a Polish group and there is a Borel measurable action of G on a standard Borel space $\mathbb{X}$. If G has

uncountably many orbits, then there is a perfect set of elements in distinct orbits.

Although Morely's Theorem follow's immediately from Burgess' Theorem, it is instructive though to see how to prove it using only Silver's Theorem. For each $\alpha < \omega_1$ let $\sim_\alpha$ be as in Definition 1.16. These are Borel equivalence relations, that is for each n the equivalence relation $(\mathcal{M}, \overline{a}) \sim (\mathcal{N}, \overline{b})$, where $\overline{a}, \overline{b}$ are of length n, is a Borel equivalence relation on $\mathbb{X}_\tau \times \omega^n$. This is an easy induction since

- $\sim_0$ is closed since $(\mathcal{M}, \overline{a}) \sim_0 (\mathcal{N}, \overline{b})$ if and only if $\mathcal{M} \models \phi(\overline{a}) \Leftrightarrow \mathcal{N} \models \phi(\overline{b})$ for all quantifier free ϕ ;
- for α a limit, $\sim_\alpha = \bigcap_{\beta<\alpha} \sim \beta$;
- $(\mathcal{M}, \overline{a}) \sim_{\alpha+1} (\mathcal{N}, \overline{c}) \Leftrightarrow \forall c \exists d \, (\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$ and $\forall d \exists c \, (\mathcal{M}, \overline{a}, c) \sim_\alpha (\mathcal{N}, \overline{b}, d)$. Thus $\sim_\alpha$ is a countable intersection of countable unions of Borel sets and, hence, Borel.

For each $\mathcal{M} \models \phi$, if α is the Scott rank of $\mathcal{M}$, then

$$\mathcal{M} \sim_\alpha \mathcal{N} \Leftrightarrow \mathcal{M} \cong \mathcal{N}.$$

Suppose there are $\aleph_2$ non-isomorphic countable models of ϕ. Then for some $\alpha < \omega_1$ there are $\aleph_2$ models of Scott rank at most α. But then, by Silver's Theorem, there must be a perfect set of $\sim_\alpha$-inequivalent elements and hence a perfect set of non-isomorphic elements.

One consequence of Silver's Theorem is that if $\mathrm{Mod}(\phi)$ is Borel, then Vaught's Conjecture holds for ϕ. When does this happen? The arguments above show that if there is $\alpha < \omega_1$ such that all models have Scott rank at most α, then the isomorphism relation is just $\sim_\alpha$, a Borel equivalence relation. We will argue in Theorem **??** that $\cong$ is Borel if and only if there is a countable bound on Scott rank.

3.3 Scattered Sentences and Morley's Proof

Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. If $\mathcal{M}$ is a τ-structure and $a_1, \ldots, a_n \in \mathcal{M}$ then the $\mathbb{A}$-*type* of $\overline{a}$ is

$$\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}) = \{\phi(\overline{v}) \in \mathbb{A} : \mathcal{M} \models \phi(\overline{a})\}$$

and $S_n(\mathbb{A}, \phi)$ is the set of $\mathbb{A}$-types of n-tuples realized in some model of T.

Definition 3.11 We say that ϕ is *scattered* if $S_n(\mathbb{A}, \phi)$ is countable for all countable fragments $\mathbb{A}$.

Non-scattered theories have many non-isomorphic models.

Lemma 3.12 *If $S_n(\mathbb{A}, \phi)$ is uncountable, then $|S_n(\mathbb{A}, \phi)| = 2^{\aleph_0}$.*

Proof We know similar results for type spaces in first order logic. Those proofs though always use the compactness of first order logic to prove the compactness of the type spaces. Here we can not rely on compactness.

Let X be the set of $\mathbb{A}$-formulas with n-free variables. We identify $S_n(\mathbb{A}, \phi)$ as a subset of the Polish space 2^X. Consider the standard Borel space $\mathrm{Mod}(\phi) \times \omega^n$ and the map F from $\mathrm{Mod}(\phi) \times \omega^n$ onto $S_n(\mathbb{A}, \phi)$ given by $(\mathcal{M}, \overline{a}) \mapsto \mathrm{tp}^{\mathcal{M}}(\overline{a}, \overline{A})$. By the arguments above this map is Borel. Thus $S_n(\mathbb{A}, \phi)$ is $\mathbf{\Sigma}^1_1$ and, by the Perfect Set Theorem for $\mathbf{\Sigma}^1_1$-sets (see [17] 14.13), must contain a non-empty perfect subset of 2^X. □

Corollary 3.13 *If ϕ is not scattered there are $2^{\aleph_0}$ non-isomorphic countable models of ϕ. Indeed, there are a perfect set of non-isomorphic models.*

Proof There is a countable fragment $\mathbb{A}$ such that $|S_n(\mathbb{A}, \phi)| = 2^{\aleph_0}$. As each model only realizes countably many types, there must be $2^{\aleph_0}$ non-isomorphic countable models.

To find a perfect set of non-isomorphic models, consider the equivalence relation on $\mathrm{Mod}(\phi)$ where $\mathcal{M} \approx \mathcal{N}$ if and only $\{\mathrm{tp}^{\mathcal{M}}(\overline{a}) : \overline{a} \in \mathcal{M}\} = \{\mathrm{tp}^{\mathcal{N}}(\overline{a}) : \overline{a} \in \mathcal{N}\}$, i.e., $\mathcal{M} \approx \mathcal{N}$ if and only if they realize the same $\mathbb{A}$-types. This is a Borel equivalence relation and if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \approx \mathcal{N}$. But if $|S_N(A, \phi)|$ is uncountable, $\approx$ has uncountably many equivalence class. Thus, by Silver's Theorem, there is a perfect set of $\approx$-inequivalent, and hence non-isomorphic models. □

The last step of Morely's original proof is to show that if ϕ is scattered, then there are at most $\aleph_1$ non-isomorphic models. This proof uses a variant of the type of analysis used in the proof of Scott's Isomorphism Theorem.

We build a sequence of countable fragments $(\mathbb{A}_\alpha : \alpha < \omega_1)$ of $\mathcal{L}_{\omega_1,\omega}$. Let $\mathbb{A}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$-containing ϕ. If α is a limit ordinal, let $\mathbb{A}_\alpha = \bigcup_{\beta<\alpha} \mathbb{A}_\beta$. Given $\mathbb{A}_\alpha$, let $\mathbb{A}_{\alpha+1}$ be the smallest fragment of $\mathcal{L}_{\omega_1,\omega}$ containing all formulas

$$\Theta_p(\overline{v}) = \bigwedge_{\psi \in p} \psi(\overline{v})$$

for all $p \in S_n(\mathbb{A}, \phi)$ and $n \in \omega$. Since ϕ is scattered, $\mathbb{A}_{\alpha+1}$ is a countable fragment.

Lemma 3.14 *For each $\mathcal{M} \models \phi$ and $\overline{a} \in \mathcal{M}$ there is $\alpha < \omega_1$ such that if $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\alpha)$, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\beta) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\beta)$ for all $\overline{b} \in \mathcal{M}$.*

We call the least such α the Morley height *of $\overline{a} \in M$.*

Proof Let $\alpha = \sup\{\beta + 1 : \exists \overline{b} \in \mathcal{M}\ \mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\beta) = \mathrm{tp}^{\mathcal{M}}(\overline{b}, \mathbb{A}_\beta)$ but $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_{\beta+1}) \neq \mathrm{tp}^{\mathcal{M}}(\overline{b}, A_{\beta+1})\}$. □

We call the least such α the *Morley height* of $\overline{a} \in M$. The Morley height of $\mathcal{M}$ is the sup of all Morley height's of tuples in $\mathcal{M}$.

Lemma 3.15 *Suppose $\mathcal{M}$ has Morley height α and $\mathcal{N} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{M}$. If $\overline{a}, \overline{b} \in \mathcal{N}$ and $\mathrm{tp}^{\mathcal{N}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_\alpha)$, then $\mathrm{tp}^{\mathcal{N}}(\overline{a}, \mathbb{A}_{\alpha+1}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_{\alpha+1})$. Thus $\mathcal{N}$ has Morley height α as well.*

Proof Let $p = \mathrm{tp}^{\mathcal{N}}(\overline{a}) = \mathrm{tp}^{\mathcal{N}}(\overline{b})$. Then, for any $\psi \in \mathbb{A}_{\alpha+1}$

$$\mathcal{M} \models \forall \overline{v} \forall \overline{w}\ [(\Theta_p(\overline{v}) \leftrightarrow \Theta_p(\overline{w})) \rightarrow (\psi(\overline{v}) \leftrightarrow \psi(\overline{w}))].$$

But this sentence is in $\mathbb{A}_{\alpha+1}$ and, hence, is also true in $\mathcal{N}$. □

Lemma 3.16 *If $\mathcal{M}$ and $\mathcal{N}$ are countable models of ϕ, $\mathcal{M}$ has Morley height α and $\mathcal{M} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{N}$, then $\mathcal{M} \cong \mathcal{N}$.*

Proof Via a back and forth argument, we build a sequence of finite functions $f_0 \subseteq f_1 \subseteq \ldots$ from $\mathcal{M}$ to $\mathcal{N}$ so that $f = \bigcup_{n \in \omega} f_n$ is the desired isomorphism. We will always have that if $\overline{a}$ is the domain of f_i, then $\mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(f_i(\overline{a}), \mathbb{A}_\alpha)$. Since $\mathcal{M} \equiv_{\mathbb{A}_{\alpha+1}} \mathcal{N}$, we can take $f_0 = \emptyset$. Suppose we have f_i with domain $\overline{a}$ and $f(\overline{a}) = \overline{b}$ and we want to add c to the domain of f_α. Let p be the $\mathbb{A}_\alpha$ type of $(\overline{a}, c)$. Then

$$\mathcal{M} \models \exists \overline{v} \exists w \Theta_p(\overline{v}, w).$$

Thus there is $(\overline{s}, t) \in \mathcal{N}$ such that $\mathrm{tp}^{\mathcal{N}}(\overline{s}, t, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{M}}(\overline{a}, c)$. In particular, $\mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_\alpha) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_\alpha)$ and, by the previous lemma $\mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_{\alpha+1}) = \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}_{\alpha+1})$. But the formula $\exists w \Theta_p(\overline{v}, w) \in \mathrm{tp}^{\mathcal{N}}(\overline{s}, \mathbb{A}_{\alpha+1})$. Thus we can find d such that $\mathrm{tp}^{\mathcal{N}}(\overline{b}, d) = p$.

Similarly, we can add elements of $\mathcal{N}$ to the domain of any f_i. Thus by the usual back-and-forth bookkeeping we can build $f_0 \subset f_1 \subset \ldots$ so that $\bigcup f_n$ is the desired isomorphism. □

Corollary 3.17 *If ϕ is scattered, then ϕ has at most $\aleph_1$ countable models.*

Proof Suppose for contradiction that T has at least $\aleph_2$ models. For some $\alpha < \omega_1$ there are uncountably many models of Morley height at most α. But any model of Morley height α is determined up to isomorphism by its $\mathbb{A}_{\alpha+1}$-theory and there are only countably many choices for the theory, a contradiction. □

This finishes Morley's proof. Either ϕ is scattered and there are at most $\aleph_1$ countable models or ϕ is not scattered and there are continuum many models.

We will examine several equivalents of being scattered.

Proposition 3.18 *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$ then ϕ is scattered if and only if $\sim_\alpha$ has countably many equivalence classes of models of ϕ for all $\alpha < \omega_1$;*

Proof ($\Leftarrow$) Suppose $\mathbb{A}$ is a countable fragment such that $S_n(\phi, \mathbb{A})$ is uncountable. Let α be a bound on the quantifier rank of formulas. Since

$$(\mathcal{M}, \overline{a}) \sim_\alpha (\mathcal{N}, \overline{b}) \Leftrightarrow (\mathcal{M}, \overline{a}) \equiv_\alpha (\mathcal{N}, \overline{b})$$

there are uncountably many $\sim_\alpha$-classes.

($\Rightarrow$) Suppose α is least such that for some n there are uncountably many $\sim_\alpha$ classes of $(\mathcal{M}, a_1, \ldots, a_n) \models \phi$.

If $\alpha = 0$ then there are uncountably many quantifier free types and ϕ is not scattered.

If $\alpha = \beta + 1$, let $\mathbb{A}$ be the smallest fragment containing ϕ and all Scott sentences $\Phi^{\mathcal{M}}_{\overline{m},\beta}$ for $(\mathcal{M}, \overline{m}) \models \phi$ for $\overline{m} \in \mathcal{M}^{<\omega}$. Since there are only countably many $\sim_\beta$-classes, $\mathbb{A}$ is countable. Suppose $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{N}, \overline{b})$. Then, without loss of generality, there is a $c \in \mathcal{M}$ such $(\mathcal{M}, \overline{a}, c) \not\sim_\beta (\mathcal{N}, barb, d)$ for any $d \in \mathcal{N}$. Thus the formula

$$\exists w \ \Phi^{\mathcal{M}}_{\overline{a},c,\beta}(\overline{v}, w) \in \mathrm{tp}^{\mathcal{M}}(\overline{a}, \mathbb{A}) \setminus \mathrm{tp}^{\mathcal{N}}(\overline{b}, \mathbb{A}).$$

Since the are uncountably many $\sim_\alpha$ classes there are uncountably many $\mathbb{A}$-types and ϕ is not scattered.

If α is a limit ordinal, let $\mathbb{A}$ be the countable fragment generated by all formulas $\Phi^{\mathcal{M}}_{\overline{a},\beta}$ for $\beta < \alpha$. If $(\mathcal{M}, \overline{a}) \not\sim_\alpha (\mathcal{N}, \overline{b})$, then $(\mathcal{M}, \overline{a}) \not\sim_\beta (\mathcal{N}, \overline{b})$ for some $\beta < \alpha$. □

Morley's result shows that if ϕ is not scattered there is a perfect set of non-isomorphic models. We show that the converse is also true.

Proposition 3.19 *If there is a perfect set of non-isomorphic models, then ϕ is not scattered.*

Proof We will give a different proof in Proposition **??**. We quickly sketch here an argument using forcing and absoluteness.

claim "ϕ is scattered" and "there is a perfect set of non-isomorphic models of ϕ" are absolute.

ϕ is scattered if and only if for all countable sets of codes A for $\mathcal{L}_{\omega_1,\omega}$ formulas $\mathbb{A}$ there is no perfect set of elements $(\mathcal{M}, \overline{a}) \models \phi$ where all $\text{tp}^{\mathcal{M}}(\overline{a}, A)$ are distinct. This can be shown to be $\Pi^1_2(\phi)$ and hence absolute by Shoenfield's Absoluteness Theorem (see [15] 25.20). Note, one ingredient of a full proof would be noting that the set of codes for $\mathcal{L}_{\omega_1,\omega}$-formulas is a Π^1_1 condition (see [26] 8.2).

There is a perfect set of non-isomorphic models is easily seen to be by $\Sigma^1_2(\phi)$ and hence absolute.

Suppose ϕ is scattered and there is a perfect set of non-isomorphic models of ϕ. By Morley's argument, this is only possible if $2^{\aleph_0} = \aleph_1$. Consider a forcing extension of the universe $\mathbb{V}[G] \models 2^{\aleph_0} > \aleph_1$. By absolutness, ϕ is still scattered in $\mathbb{V}[G]$ and there is a perfect set of non-isomorphic elements but this contradicts Morley's results. □

Putting this all together.

Corollary 3.20 *The following are equivalent*

i) ϕ is scattered;

ii) $\sim_\alpha$ has countably many equivalence classes of models of ϕ for all $\alpha < \omega_1$;

iii) there is no perfect set of non-isomorphic countable models of ϕ.

When studying Vaught's Conjecture we often look instead at the *Strong Vaught Conjecture*: If $\phi \in \mathcal{L}_{\omega_1,\omega}$, then either there are countably many models of ϕ or there is a perfect set of non-isomorphic models of ϕ. Clearly the strong version implies the usual one. Moreover, if the original version of Vaught's Conjecture is provable for ϕ in ZFC, then so is strong form. Suppose not. If there is no perfect set of non-isomorphic models, then ϕ is scattered. Suppose we can find $(\mathcal{M}_\alpha : \alpha < \omega_1)$ a sequence of pairwise non-isomorphic countable models of ϕ. Pass to a forcing extension $\mathbb{V}[G]$ of the universe making $2^{\aleph_0} > \aleph_1$ without collapsing cardinals (for example, add $\aleph_2$

Cohen-reals). By Shoenfield Absoluteness $\mathbb{V}[G] \models \phi$ is scattered and there are uncountably many non-isomorphic models of ϕ. But then, by Morely's analysis there are exactly $\aleph_1$ non-isomorphic countable models, contradicting the fact that Vaught's Conjecture is true in all models of ZFC.

For this reason we tend to focus on the strong conjecture. This way it makes sense to study counterexamples without worrying about whether the Continuum Hypothesis is true. This leads to the following definition.

Definition 3.21 A *Vaught counterexample* is a scattered ϕ with uncountably many non-isomorphic countable models. Equivalently, it is a sentence with uncountably many non-isomorphic countable models but no perfect set of non-isomorphic models.

We will examine uncountable models of Vaught counterexamples in §**??**.

4 The Model Existence Theorem and Applications

4.1 Consistency Properties and Model Existence

Although we don't have the Compactness Theorem or a useful Completeness Theorem, Henkin-style arguments can still be used in some contexts to build models. In this section we describe the general framework and give several applications.

Throughout this section we will assume that τ is a countable signature and that C is an infinite set of constant symbols of τ (though perhaps not all of the constant symbols of τ). The following idea, due to Makkai, is the key. It tells us exactly what we need to do a Henkin argument.

Definition 4.1 A *consistency property* Σ is a collection of countable sets σ of $\mathcal{L}_{\omega_1,\omega}$-sentences with the following properties. For $\sigma \in \Sigma$:

C0) if $\mu \subseteq \sigma$, then $\mu \in \Sigma$;

C1) if $\phi \in \sigma$, then $\neg\phi \notin \sigma$;

C2) if $\neg\phi \in \sigma$, then $\sigma \cup \{\sim \phi\} \in \Sigma$;

C3) if $\bigwedge_{\phi\in X} \phi \in \sigma$, then $\sigma \cup \{\phi\} \in \Sigma$ for all $\phi \in X$;

C4) if $\bigvee_{\phi\in X} \phi \in \sigma$, then $\sigma \cup \{\phi\} \in \Sigma$ for some $\phi \in X$;

C5) if $\forall v\ \phi(v) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$ for all $c \in C$;

C6) if $\exists v\ \phi(v) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$ for some $c \in C$;

C7) let t be a term with no variables and let $c, d \in C$,

- a) if $c = d \in \sigma$, then $\sigma \cup \{d = c\} \in \Sigma$;
- b) if $c = t, \phi(t) \in \sigma$, then $\sigma \cup \{\phi(c)\} \in \Sigma$;
- c) $\sigma \cup \{e = t\} \in \Sigma$ for some $e \in C$.

Lemma 4.2 *Let Σ be a consistency property with $\sigma \in \Sigma$, $c, d, e \in C$.*

i) There is $\mu \in \Sigma, \sigma \subseteq \mu$ with $\{c = c\} \in \mu$.

ii) Suppose $c = d, d = e \in \sigma$. Prove that $\sigma \cup \{c = e\} \in \Sigma$.

iii) If $\phi, \phi \to \psi \in \sigma$, then there is $\mu \in \Sigma$, $\sigma \subseteq \mu$ with $\psi \in \mu$.

Proof i) By C7c) there is $d \in C$ such that $\sigma \cup \{d = c\} \in \Sigma$. Then by C7a) $\sigma \cup \{c = d, d = c\} \in \Sigma$. Applying C7b) with the "v=c" as $\phi(v)$, $\sigma \cup \{c = d, d = c, c = c\} \in \Sigma$. □

Exercise 4.3 Prove ii) and iii) of the Lemma.

The next Exercise shows we really only need to verify C1)–C7).

Exercise 4.4 Suppose Σ_0 satisfies C1)–C7). Let

$$\Sigma = \{\Delta : \exists \mu \in \Sigma \ \Delta \subseteq \mu\}.$$

Prove that Σ is a consistency property.

Exercise 4.5 Suppose Σ_0 is a consistency property.

a) Show that $\Sigma_1 = \{\sigma \in \Sigma_0 : \sigma \text{ is finite}\}$ is a consistency property.

b) Show that $\Sigma_2 = \{\sigma \in \Sigma_0 : \text{only finitely many constants from } C \text{ occur in } \sigma\}$ is a consistency property.

Theorem 4.6 (Model Existence Theorem) *If Σ is a consistency property and $\sigma \in \Sigma$, there is a countable $\mathcal{M} \models \sigma$.*

Proof Let Λ be the smallest set of $\mathcal{L}_{\infty,\omega}$ sentences such that:

- $\sigma \subseteq \Lambda$;
- if ϕ is a subsentence of a sentence in Λ, then $\phi \in \Lambda$;
- if $\phi(\overline{v})$ is a subformula of a sentence in Λ and $\overline{c} \in C$, then $\phi(\overline{c}) \in \Lambda$;
- If $\neg\phi \in \Lambda$, then $\sim \phi \in \Lambda$;
- if $c, d \in C$, then $c = d \in \Lambda$.

Let $\phi_0, \phi_1, \ldots$ list all sentences in Λ We assume that each sentence is listed infinitely often. Let $t_0, t_1, \ldots,$ list all τ-terms with no variables.

Using the fact that Σ is a consistency property, we build

$$\sigma = \sigma_0 \subseteq \sigma_1 \subset \ldots$$

such that each $\sigma_i \in \Sigma$ and

A) if $\sigma_n \cup \{\phi_n\} \in \Sigma$, then $\phi_n \in \sigma_{n+1}$, in this case:

a) if ϕ_n is $\bigvee_{\phi \in X} \phi$, then $\phi \in \sigma_{n+1}$ for some $\phi \in X$;

b) if ϕ_n is $\exists v \phi(v)$, then $\phi(c) \in \sigma_{n+1}$ for some $c \in C$;

B) $c = t_n \in \sigma_{n+1}$ for some $c \in C$.

Let $\Gamma = \bigcup_{n=1}^{\infty} \sigma_n$. We will build a model of Γ. For $c, d \in C$ we say $c \sim d$ if $c = d \in \Gamma$.

claim 1 σ is an equivalence relation.

Let $c \in C$. The sentence $c = c$ is ϕ_n for some n. By Lemma 4.2, and A) $c = c \in \sigma_{n+1} \subseteq \Gamma$. Thus $c \sim c$.

If $c = d \in \Gamma$, then we can find n such that $c = d \in \sigma_n$ and $d = c$ is ϕ_n. Then by C7a) and condition A) $d = c \in \Gamma$.

If $c = d, d = e \in \Gamma$, choose n such that $c = d, d = e \in \sigma_n$ and $c = e$ is ϕ_n. Then by Lemma 4.2 and A) $c = e \in \Gamma$.

Let $[c]$ denote the $\sim$-class of c. Let $M = \{[c] : c \in C\}$.

If d is a constant symbol of τ (perhaps not in C), then by B) there is C in C such that $c \in C$ such that $c = d \in \Gamma$. We interpret $d^{\mathcal{M}}$ as $[c]$.

Let f be an n-ary function symbol of τ and let $c_1, \ldots, c_n \in C$. By B) there is $d \in C$ such that $d = f(c_1, \ldots, c_n) \in \Gamma$. Suppose $d_1 = f(c_1, \ldots, c_n)$ is also in Γ, using C7b), Lemma 4.2 and A) we see that $d = d_1 \in \Gamma$ and $d \sim d_1$. Also note that if $c_0 = f(c_1, \ldots, c_n) \in \Gamma$, and $d_i \sim c_i$ for $i = 0, \ldots, n$ then, a similar argument shows $d_0 = f(d_1, \ldots, d_n) \in \Gamma$. Thus we can define $f^{\mathcal{M}} : M^n \to M$ by

$$f([c_1], \ldots, [c_n]) = [d] \Leftrightarrow f(c_1, \ldots, c_n) = d \in \Gamma.$$

claim 2 Let $t(v_1, \ldots, v_n)$ be a term, $c_1, \ldots, c_n, d \in C$. If $d = t(c_1, \ldots, c_n) \in \Gamma$, then $t^{\mathcal{M}}([c_1], \ldots, [c_n]) = [d]$.

We prove this by induction on complexity of terms. Suppose $t = f(t_1, \ldots, t_m)$. For $i \leq m$ there is $d_i \in C$ such that $d_i = t_i(\overline{c}) \in \Gamma$. By induction, $[d_i] = t_i^{\mathcal{M}}([\overline{c}])$. By C7b) $d = f(d_1, \ldots, d_m) \in \Gamma$. Thus

$$t^{\mathcal{M}}([\overline{c}]) = f^{\mathcal{M}}([d_1], \ldots, [d_m]) = [d]$$

as desired.

For $c_1, \ldots, c_n$ and R an n-ary relation symbol of τ, we define

$$R^{\mathcal{M}}([c_1], \ldots, [c_n]) \Leftrightarrow R(c_1, \ldots, c_n) \in \Gamma.$$

As above, we can show this does not depend on the choice of c_i.

claim 3 If $\phi \in \Gamma$, then $\mathcal{M} \models \phi$.

We prove this by induction on complexity.[7]

[7] We need to be slightly careful how we define "complexity". For example, $\exists v \psi(v)$ is more complex than any $\psi(c)$ and $\neg \bigvee_{\psi \in X} \psi$ is more complicated than $\neg\psi$ for any $\psi \in X$.

- ϕ is $t_1 = t_2$

  There are $c_1, c_2 \in C$ such that $c_1 = t_1, c_2 = t_2 \in \Gamma$. Then $c_1 = c_2 \in \Gamma$ and by claim 2, $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$, so $\mathcal{M} \models \phi$.

- ϕ is $R(t_1, \ldots, t_n)$

Exercise 4.7 Prove $\mathcal{M} \models R(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$.

- ϕ is $\bigwedge_{\psi \in X} \phi$

  By C3) and A), $\phi \in \Gamma$ for each $\psi \in X$. By induction $\mathcal{M} \models \psi$ for all $\psi \in X$. Thus $\mathcal{M} \models \phi$.

- ϕ is $\bigvee_{\psi \in X} \psi$

  By C4) and A), there is $\psi \in X$ such that $\psi \in \Gamma$. By induction $\mathcal{M} \models \psi$. Thus $\mathcal{M} \models \phi$.

- v) ϕ is $\forall v\ \psi(v)$

  By C5) and A), $\psi(c) \in \Gamma$ for all $c \in C$. By induction $\mathcal{M} \models \psi(c)$ for all $c \in C$. Since every element of M is named by a constant, $\mathcal{M} \models \phi$.

- vi) ϕ is $\exists v\ \psi(v)$

  By A) there is $c \in C$ such that $\psi(c) \in \Gamma$. By induction, $\mathcal{M} \models \psi(c)$. Thus $\mathcal{M} \models \phi$.

- ϕ is $\neg\psi$

  By C2) and A, $\sim \psi \in \Gamma$. This now breaks into cases depending on ψ.

  - ψ is $t_1 = t_2$

    There are constants $c_1, c_2 \in C$ such that $c_i = t_i \in \Gamma$. Since $\sim \psi$ is $t_1 \neq t_2$, by C7) and A), $c_1 \neq c_2 \in \Gamma$. Suppose, for contradiction, that $\mathcal{M} \models \psi$. By claim 2 $[c_i] = t_i^{\mathcal{M}}$. Thus $\mathcal{M} \models c_1 = c_2$. Hence $c_1 \sim c_2$ and $c_1 = c_2 \in \Gamma$. This contradicts C1).

  - ψ is $R(t_1, \ldots, t_m)$

    Then $\neg R(t_1, \ldots, t_m) \in \Gamma$. There are $c_1, \ldots, c_m \in C$ such that $c_i = t_i \in \Gamma$. Then $\neg R(c_1, \ldots, c_m) \in \Gamma$. By C1), $R(c_1, \ldots, c_m) \notin \Gamma$. Thus

    $$\mathcal{M} \models \neg R(c_1, \ldots, c_m).$$

By claim 2,

$$\mathcal{M} \models \neg R(t_1, \ldots, t_m).$$

Thus $\mathcal{M} \models \phi$.

- ψ is $\neg\theta$

  Then $\theta \in \Gamma$. My induction $\mathcal{M} \models \theta$ and $\mathcal{M} \models \phi$.

- ψ is $\bigwedge_{\theta \in X} \theta$

  Then $\bigvee_{\theta \in X} \neg\theta \in \Gamma$ and $\neg\theta \in \Gamma$ for some $\theta \in X$. By induction, $\mathcal{M} \models \neg\theta$. Thus $\mathcal{M} \models \phi$.

- ψ is $\bigvee_{\theta \in X} \theta$

  Then $\bigwedge_{\theta \in X} \neg\theta \in \Gamma$ and $\neg\theta \in \Gamma$ for all $\theta \in X$. By induction $\mathcal{M} \models \neg\theta$ for all $\theta \in X$. Thus $\mathcal{M} \models \phi$.

- ψ is $\exists v \theta(v)$

  Then $\forall v \neg\theta(v) \in \Gamma$ and $\theta(c) \in \Gamma$ for all $c \in C$. By induction, $\mathcal{M} \models \theta(c)$ for all $c \in C$. Thus $\mathcal{M} \models \phi$.

- ψ is $\forall v \theta(v)$.

  Then $\exists v \neg\theta(v) \in \Gamma$ and $\theta(c) \in \Gamma$ for some $c \in C$. But then $\mathcal{M} \models \theta(c)$ and $\mathcal{M} \models \phi$.

This completes the induction. Thus $\mathcal{M} \models \Gamma$. □

We can easily modify the Henkin argument above to prove the following useful variant.

Theorem 4.8 (Extended Model Existence Theorem) *Let τ be as above. Let T be a countable set of $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentences. Suppose Σ is a consistency property such that for all $\sigma \in \Sigma$ and $\phi \in T$, $\sigma \cup \{\phi\} \in \Sigma$. Then there is $\mathcal{M} \models T$.*

Proof Let $\theta_0, \theta_1, \ldots$ enumerate T. Modify the construction of $\sigma_0 \subseteq \sigma_1 \subseteq \ldots$ such that $\theta_n \in \sigma_{n+1}$ for all n. Then $T \subset \Gamma$ and $\mathcal{M} \models T$. □

4.2 Omitting Types

As a first application of the Model Existence Theorem we will show how to prove a generalization of the Omitting Types Theorem of first order logic.

Theorem 4.9 (Omitting Types Theorem) *Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$, let $T \subset \mathbb{A}$ be a satisfiable theory and let $\Theta_n(v_1, \ldots, v_{m_n})$ be a set of $\mathbb{A}$-formulas for each $n \in \omega$. Suppose that for all $\mathbb{A}$-formulas $\psi(v_1, \ldots, v_{m_n})$ such that $T + \exists \overline{v}\ \psi(\overline{v})$ is satisfiable, there is $\theta \in \Theta_n$ such that $T + \exists \overline{v}\ (\psi(\overline{v}) \wedge \theta(\overline{v}))$ is satisfiable.*

Then

$$T + \bigwedge_{n \in \omega} \forall \overline{v} \bigvee_{\theta \in \Theta_n} \theta(\overline{v})$$

is satisfiable.

Proof Let $\tau^* \supset \tau$ be obtained by adding a countable set of new constant symbols C and let $\mathbb{A}^*$ be the smallest fragment containing $\mathbb{A}$ such that if $\phi(\overline{v})$ is a formula in $\mathbb{A}$ and $\overline{c} \in C$ then $\phi(\overline{c}) \in \mathbb{A}^*$.

Let

$$\Delta = \{ \bigvee_{\theta \in \Theta_n} \theta(c_1, \ldots, c_{m_n}) : n \in \omega, \overline{c} \in C \}$$

and Σ be the set of all sets σ where

$$\sigma = \sigma_0 \cup T \cup \Delta$$

where σ_0 is a finite set of $\mathbb{A}^*$-sentences such that $T \cup \sigma_0$ is satisfiable. Note that σ_0 contains only finitely many constants from C.

We claim that Σ is a consistency property. Once we have shown this, the Model Existence Theorem will give us a model of $T \cup \Delta$ where every element is the interpretation of a constant in C. This is the desired model.

We will show that C4) holds. The rest of the verification is routine and left as an exercise. Suppose σ is as above and $\chi = \bigvee_{\psi \in X} \psi$.

<u>case 1</u> $\chi \in \sigma_0 \cup T$.

In this case we know there is $\mathcal{M} \models \sigma_0 \cup T$ and $\mathcal{M} \models \psi$ for some $\psi \in X$. Then

$$\sigma_0 \cup \{\psi\} \cup T \cup \{ \bigvee_{\theta \in \Theta_n} \theta(c_1, \ldots, c_{m_n}) : n \in \omega, \overline{c} \in C \} \in \Sigma$$

as desired.

<u>case 2</u> $\chi = \bigvee_{\theta \in \Theta_n} \theta(\overline{c})$ for some n and $\overline{c} \in C$.

Let $\overline{d}$ be the constants from C occurring in σ_0 but not in $\overline{c}$. Let

$$\Phi(\overline{c}, \overline{d}) = \bigwedge_{\phi \in \sigma_0} \phi.$$

Then

$$T + \exists \overline{x} \exists \overline{y}\ \Phi(\overline{x}, \overline{y})$$

is satisfiable. By our assumptions, there is $\theta \in \Theta_n$ such that

$$T + \exists \overline{x}\ \ (\theta \wedge \exists \overline{y}\ \Phi(\overline{x}, \overline{y}))$$

is satisfiable. Thus $T + \Phi(\overline{c}, \overline{d}) \wedge \theta(\overline{c})$ is satisfiable and

$$s_0 \cup \{\theta(\overline{c})\} \cup T \cup \Delta \in \Sigma.$$

Thus C4) holds. □

Exercise 4.10 Complete the verification that Σ is a consistency property.

Exercise 4.11 Show that Theorem 4.9 is a generalization of the classical Omitting Types Theorem for first order logic.

4.2.1 Atomic and Prime Models

Once we have the Omitting Types Theorem, many of Vaught's (proofs about prime and atomic models can easily be adapted in the infinitary setting.

Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. Let $T \subset \mathbb{A}$ be a satisfiable theory that is $\mathbb{A}$-complete, i.e., $T \models \phi$ or $T \models \neg\phi$ for any $\mathbb{A}$-sentence ϕ.

Definition 4.12 A satisfiable formula $\theta(\overline{v})$ is $\mathbb{A}$-*complete* if for any $\phi(\overline{v})$ either

$$T \models \theta(\overline{v}) \to \phi(\overline{v}) \text{ or } T \models \theta(\overline{v}) \to \neg\phi(\overline{v}).$$

A formula $\phi(\overline{v})$ is $\mathbb{A}$-*completable* if there is a complete $\theta(\overline{v}) \in A$ with $T \models \theta(\overline{v}) \to \phi(\overline{v})$. We say that ϕ is *incompletable if it is satisfiable but non completable.*

Definition 4.13 $\mathcal{M} \models T$ is $\mathbb{A}$-*atomic* if for all $\overline{a} \in \mathcal{M}$ there is an $\mathbb{A}$-complete formula $\theta(\overline{v})$ such that $\mathcal{M} \models \theta(\overline{a})$

Definition 4.14 $\mathcal{M} \models T$ is $\mathbb{A}$-*prime* if for every $\mathcal{N} \models T$ there is an $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Theorem 4.15 *$\mathcal{M} \models T$ is $\mathbb{A}$-prime if and only if it is countable and $\mathbb{A}$-atomic.*

Proof ($\Rightarrow$) Suppose $\mathcal{M}$ is a prime model of T. Since T has a countable model, $\mathcal{M}$ must be countable. Let $\overline{a} \in \mathcal{M}$ and suppose that $\overline{a}$ satisfies no complete formula. Let $\Delta = \{\phi(\overline{v}) : \mathcal{M} \models \phi(\overline{a})\}$. We apply the Omitting Types Theorem to T and $\bigvee_{\phi \in \Delta} \neg\phi(\overline{v})$. Suppose $T + \psi(\overline{v})$ is satisfiable.

If $\mathcal{M} \models \neg\psi(\overline{a})$, then $\neg\neg\psi(\overline{a}) \in \Delta$ and $\mathcal{M} \models \exists v\ (\psi(\overline{v}) \wedge \neg\neg\psi(\overline{v})$, as in the hypothesis of the Omitting Types Theorem.

If $\mathcal{M} \models \psi(\overline{a})$, then ψ is not complete, so we can find ϕ such that $\mathcal{M} \models \phi(\overline{a})$ and

$$T \not\models \psi(\overline{v}) \rightarrow \phi(\overline{v}) \text{ and } T \not\models \psi(\overline{v}) \rightarrow \phi(\overline{v}).$$

Thus $T + \exists\overline{v}\ (\psi(\overline{v}) \wedge \neg\phi(\overline{v}))$ is satisfiable and we again satisfy the hypothesis of the Omitting Types Theorem.

Thus there is a

$$\mathcal{N} \models T + \forall v \bigvee_{\mathcal{M} \models \phi(\overline{a})} \neg\phi(\overline{a})$$

but there is no $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

($\Leftarrow$) Suppose $\mathcal{M} \models T$ is countable and atomic and $\mathcal{N}$ models T. We must build an $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Let $a_0, a_1, \ldots$ enumerate the elements of $\mathcal{M}$. We build a family of partial $\mathbb{A}$-elementary maps

$$f_0 \subset f_1 \subset \ldots$$

maps into $\mathcal{N}$ where $\text{dom}(f_i) = \{a_0, \ldots, a_{i-1}\}$. Since T is complete, we can start with $f_0 = \emptyset$.

Given f_n. Let $b_i = f_n(a_i)$ for $i < n$. There is an $\mathbb{A}$-complete formula $\theta(v_0, \ldots, v_{n-1})$ such that $\mathcal{M} \models \theta(a_0, \ldots, a_{n-1})$ and a complete formula $\psi(v_0, \ldots, v_n)$ such that $\mathcal{M} \models \psi(a_0, \ldots, a_n)$. Since θ_0 is $\mathbb{A}$-complete,

$$\mathcal{M} \models \theta_0(v_0, \ldots, v_{n-1}) \rightarrow \exists v_n\ \theta(v_0, \ldots, v_n).$$

Since f_n is $\mathbb{A}$-elementary, $\mathcal{N} \models \theta_0(b_0, \ldots, b_{n-1})$. Thus we can find $b_n \in \mathcal{N}$ such that $\mathcal{N} \models \theta(b_0, \ldots, b_n)$. Extend f_n to f_{n+1} by sending a_n to b_n. Since $\theta(\overline{v})$ is complete, this is $\mathbb{A}$-elementary. The map $f = \bigcup f_n$ is the desired $\mathbb{A}$-elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. $\square$

Exercise 4.16 Show that any two $\mathbb{A}$-prime models of T are isomorphic.

Definition 4.17 T is $\mathbb{A}$-*atomic* if every satisfiable formula in $\mathbb{A}$ is completable.

Theorem 4.18 *T has a countable atomic model if and only if T is atomic.*

Proof ($\Rightarrow$) If $\psi(\overline{v})$ is satisfiable, then, since T is complete, there is $\overline{a} \in \mathcal{M}$ such that $\mathcal{M} \models \psi(\overline{a})$. But $\mathcal{M}$ satisfies an $\mathbb{A}$-complete formula $\theta(\overline{v})$ and we must have $T \models \theta(\overline{v}) \rightarrow \phi(\overline{v})$. Thus ϕ is completable.

($\Leftarrow$) For each n let Δ_n be the set of all complete formulas $\theta(v_1, \ldots, v_n)$. If $\psi(v_1, \ldots, v_n)$ is satisfiable, then there is $\theta(\overline{v}) \in \Delta_n(\overline{v})$ such that $T \models \theta(\overline{v}) \rightarrow \psi(\overline{v})$. In particular,

$$T + \exists \overline{v}\ (\theta(\overline{v}) \wedge \psi(\overline{v})$$

is satisfiable and, by the Omitting Types Theorem, there is an atomic $\mathcal{M} \models T$. $\square$

We next show that if there are only countably many $\mathbb{A}$-types then there is an $\mathbb{A}$-atomic model. One way to do this in the first order case is to assume there is an incompletable formula and then build a perfect tree of inconsistent incompletable formulas. By compactness, each branch gives rise to a different type. So we have that an non-atomic theory has $2^{\aleph_0}$ types. Here we use the Omitting Types Theorem directly to prove that an $\mathbb{A}$-theory with countably many types is atomic. We could combined this with Morley's argument from Lemma 3.12 to conclude that a non-$\mathbb{A}$-atomic theory has $2^{\aleph_0}$ $\mathbb{A}$-types.

Theorem 4.19 *If for all $n \in \omega$there are only countably many $\mathbb{A}$-types, then T is $\mathbb{A}$-atomic and has an $\mathbb{A}$-prime model.*

Proof Suppose $\phi(\overline{v})$ is satisfiable but not completable. Let $\gamma_0(\overline{v}), \gamma_1(\overline{v}), \ldots$ list all $\mathbb{A}$-types containing ϕ. Since none of those types contains a complete formula for any satisfiable formula $\psi(\overline{v})$ and any $i \in \omega$ we can find $\theta \in \gamma_i$ such that $T + \exists \overline{v}\ (\psi(\overline{v}) \wedge \neg\theta(\overline{v}))$ is satisfiable. By the Omitting Types Theorem we can find

$$\mathcal{M} \models T + \forall \overline{v} \bigwedge_{n \in \omega} \bigvee_{\theta \in \gamma_i} \neg\theta(\overline{v}).$$

Clearly $\mathcal{M}$ omits each γ_i. But these are all $\mathbb{A}$-types containing ϕ. But T is complete and ϕ is satisfiable, thus $\mathcal{M} \models \exists \overline{v} \phi(\overline{v})$, a contradiction. $\square$

4.3 The Interpolation Theorem

The next application of the Model Existence Theorem is the $\mathcal{L}_{\omega_1,\omega}$ version of Craig's Interpolation Theorem from first order logic. This result was first proved by Lopez-Escobar by different means.

Theorem 4.20 *Suppose ϕ_1 and ϕ_2 are $\mathcal{L}_{\omega_1,\omega}$-sentences with $\phi_1 \models \phi_2$. There is an $\mathcal{L}_{\omega_1,\omega}$-sentence θ such that $\phi_1 \models \theta$, $\theta \models \phi_2$ and every relation, function and constant symbol occurring in θ occurs in both ϕ_1 and ϕ_2.*

Proof Let C be a countably infinite collection of new constant symbols. Let $\mathbb{A}_i$ be the smallest countable fragment containing ϕ_i, where every relation and function symbol and every constant symbol not in C occurs in ϕ_i and every formula contains only finitely many constants from C. Let $\mathbb{A} = \mathbb{A}_1 \cap \mathbb{A}_2$.

Let Σ be the collection of finite $\sigma = \sigma_1 \cup \sigma_2$ where σ_i is a set of $\mathbb{A}_i$-sentences and if ψ_1, ψ_2 are $\mathbb{A}$-sentences such that $\sigma_1 \models \psi_1$ and $\sigma_2 \models \psi_2$, then $\psi_1 \wedge \psi_2$ is satisfiable.

claim Σ is a consistency property.

We will verify properties C3) and C4) and leave the remainder as an exercise.

C3) Suppose $\bigwedge_{\psi\in X} \psi \in \sigma \in \Sigma$, where $\sigma = \sigma_1 \cup \sigma_2$ as above. Suppose $\bigwedge_{\psi\in X} \psi \in \sigma_1$. Let $\sigma_1' = \sigma_1 \cup \{\psi\}$. We claim that $\sigma_1' \cup \sigma_2 \in \Sigma$. Suppose $\sigma_1' \models \theta_1$ and $\sigma_2 \models \theta_2$. Then $\sigma_1 \cup \theta_1$. By assumption, $\theta_1 \wedge \theta_2$ is satisfiable. Thus $\sigma_1' \cup \sigma_2 \in \Sigma$.

This is similar if $\bigwedge_{\psi\in X} \psi \in \sigma_2$.

C4) Suppose $\bigvee_{\psi\in X} \psi \in \sigma_1$. Let $\sigma_{1,\psi} = \sigma_1 \cup \{\psi\}$. We claim that some $\sigma_{1,\psi} \cup \sigma_2 \in \Sigma$. Suppose not. Then for each ψ there are $\theta_{1,\psi}$, $\theta_{2,\psi} \in \mathbb{A}$ such that $\sigma_{1,\psi} models \theta_{1,\psi}$, $\sigma_2 \models \theta_{2,\psi}$ and $\theta_{1,\psi} \wedge \theta_{2,\psi}$ is unsatisfiable. Then $\theta_{1,\psi} \models \neg\theta_{2,\psi}$. Since

$$\sigma_1 \models \bigvee_{\psi\in X} \psi$$

,

$$\sigma_1 \models \bigvee_{\psi\in X} \theta_{1,\psi}.$$

But

$$\sigma_2 \models \bigwedge_{\psi\in X} \theta_{2,\psi}$$

and

$$\bigvee_{\psi\in X} \theta_{1,\psi} \models \neg \bigwedge_{\psi\in X} \theta_{2,\psi}$$

contradicting that $\sigma \in \Sigma$.

Exercise 4.21 Finish the proof that Σ is a consistency property.

We now finish the proof of the Interpolation Theorem. Since $\phi_1 \to \phi_2$, by the Model Existence Theorem, $\{\phi_1, \neg\phi_2\} \not\in \Sigma$. Thus there are θ_1 and $\theta_2 \in \mathbb{A}$ such that $\phi_1 \models \theta_1$, $\neg\phi_2 \models \theta_2$ and $\theta_1 \cap \theta_2$ is unsatisfiable.

Thus

$$\phi_1 \models \theta_1, \theta_1 \models \neg\theta_2, \text{ and } \neg\theta_2 \models \phi_2.$$

It follows that

$$\phi_1 \models \theta_1 \text{ and } \theta_1 \models \phi_2.$$

We would be done except θ_1 may contain constants from C. Let $\theta_1 = \psi(\overline{c})$, where $\psi(\overline{v})$ is a $\mathbb{A}$-formula with no constants from C. Then

$$\phi_1 \models \forall\overline{v}\psi(\overline{v}) \text{ and } \exists\overline{v}\psi(\overline{v}) \models \phi_2.$$

Thus we can take

$$\forall\overline{v}\psi(\overline{v})$$

as the interpolant. □

The Interpolation Theorem gives rise to a separation theorem for $\mathrm{PC}_{\omega_1,\omega}$-classes, analogous to the Separation Theorem for $\mathbf{\Sigma}^1_1$-sets in Descriptive Set Theory.

Corollary 4.22 *Suppose $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint $\mathrm{PC}_{\omega_1,\omega}$-classes of τ-structures. There is $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M} \models \phi$ for $\mathcal{M} \in K_0$ and $\mathcal{M} \models \neg\phi$ for $\mathcal{M} \in K_0$.*

Proof Let $\tau_0, \tau_1 \supseteq \tau$ and let $\phi_i \in \mathcal{L}_{\omega_1,\omega}(\tau_i)$ such that $\mathcal{K}_i$ is the class of τ-reducts of models of ϕ_i. We may assume that $\tau_1 \cap \tau_2 = \tau$. Since $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint we have that $\phi_0 \models \neg\phi_1$, i.e., if there is any expansion of $\mathcal{M}$ that makes ϕ_0 true, then there is no expansion of $\mathcal{M}$ making ϕ_1 true. By the Interpolation Theorem there is $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ such that $\phi_0 \models \phi$ and $\phi \models \neg\phi_1$. Then every structure in $\mathcal{K}_0$ is a model of ϕ and no structure in $\mathcal{K}_1$ is a model of ϕ. □

Indeed we can use these results to prove an invariant version of the Separation Theorem for $\mathbf{\Sigma}^1_1$-sets. In Exercise 3.5 we noted that every $\mathrm{PC}_{\omega_1,\omega}$-classes of countable models is invariant $\mathbf{\Sigma}^1_1$-sets. The converse is also true.

Lemma 4.23 *Every invariant $\mathbf{\Sigma}^1_1$ subset A of $\mathbb{X}_\tau$ is the set of countable models in a $\mathrm{PC}_{\omega_1,\omega}$-class.*

Proof For notational simplicity we assume $\tau = \{R\}$, where R is a binary relation symbol Fix $p : \omega^2 \to \omega$ a bijective pairing function. For $R \subseteq \omega \times \omega$, let $f_R : \omega \to 2$ be defined by $f_R(p(i,j)) = 1$ if and only if $R(i,j)$.

Since A is analytic, there is a tree $T \subseteq \{(\eta, \nu) : \eta \in 2^{<\omega}, \nu \in \omega^{<\omega}, |\eta| = |\nu|\}$ such that $(\omega, R) \in A$ if and only if there is g such that (f_R, g) is a path thru T.

Let $\tau^* = \{s, c, f, g, S_n : n = 1, 2, \ldots\}$ where s, f, g are unary functions, c is a constant symbol and S_n is a $2n$-ary relation symbol. Let Θ be a sentence asserting:

- s is one-to-one, every element except c has a preimage, there are no cycles and $\forall x \ \bigvee_{i=0}^{n} s^{(i)}(c) = x$;
- $\forall x \ f(x) = c \vee f(x) = s(c)$;
- $\bigwedge\limits_{i,j} [R(s^{(i)}(c), s^{(j)}(c)) \leftrightarrow f(s^{(p(i,j))}(c)) = s(c)]$;
- $$\bigwedge_{n} \bigwedge_{\eta \in 2^{<\omega}, \nu \in \omega^{<\omega}} \Big[\bigwedge_{(\eta,\nu) \in T} S_n(s^{\eta(0)}(c), \ldots, s^{\eta(n-1)}(c), s^{\nu(0)}(c), \ldots, s^{\nu(n-1)}(c)) \wedge \bigwedge_{(\eta,\nu) \notin T} \neg S_n(s^{\eta(0)}(c), \ldots, s^{\eta(n-1)}(c), s^{\nu(0)}(c), \ldots, s^{\nu(n-1)}(c)) \Big];$$
- $\bigwedge\limits_{n=1}^{\infty} S_n(f(c), f(s(c)), \ldots, f(s^{n-1}(c)), g(c), g(s(c)), \ldots, g(s^{n-1}(c)))$

Suppose $(\omega, R) \in A$. Let f_R be as above. Interpret c as 0 and s as $x \mapsto x + 1$. Choose g such that (f_R, g) is a path thru T. Interpret S_n by $S_n(i_0, \ldots, i_{n-1}, j_0, \ldots, j_{n-1})$ if and only if $(\bar{i}, \bar{j}) \in T$. The resulting structure is a model of Θ.

Suppose $\mathcal{M}^* = (\omega, R, s, f, g, S_1, \ldots) \models \Theta$. Define $\eta_0 \subset \eta_1 \subset \ldots$ and $\nu_0 \subset \nu_1 \subset \ldots$ such that $f(s^i(c)) = s^{\eta(i)}(c)$ and $g(s^i(c)) = s^{\nu(i)}(c)$. Let $\widehat{f} = \bigcup \eta_i$ and $\widehat{g} = \bigcup \nu_i$. Then $(\widehat{f}, \widehat{g})$ is a path thru T. Define $\widehat{R} \subseteq \omega \times \omega$ such that $f_{\widehat{R}} = \widehat{f}$ and let $\mathcal{N} = (\omega, R^*)$. Then $\mathcal{N} \in A$. But s gives an isomorphism between $\mathcal{N}$ and $\mathcal{M} = (\omega, R)$. Thus, since A is invariant, $\mathcal{M} \in A$. $\square$

Corollary 4.24 *If A and B are disjoint invariant $\mathbf{\Sigma}^1_1$-sets there is an invariant Borel C with $A \subset C$ and $B \cap C = \emptyset$.*

We next prove the converse to Corollary 3.3

Theorem 4.25 *Every invariant Borel subset of $\mathbb{X}_\tau$ is of the form* $\mathrm{Mod}(\phi)$ *for some $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence ϕ.*

Proof If A is invariant Borel, then A and $\mathbb{X}_\tau \setminus A$ are disjoint invariant $\mathbf{\Sigma}^1_1$ sets. We can find $\mathrm{PC}_{\omega_1,\omega}$-classes $\mathcal{K}_0$ and $\mathcal{K}_1$ such that $A = \mathbb{X}_\tau \cap \mathcal{K}_0$ and $\mathbb{X}_\tau \setminus A = \mathbb{X}_\tau \cap \mathcal{K}_1$. Since there are no countable structures in $\mathcal{K}_0 \cap \mathcal{K}_1$, by Löwenheim-Skolem, $\mathcal{K}_0$ and $\mathcal{K}_1$ are disjoint classes. Thus by Corollary 4.22 there is an $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence ϕ such that $A \subset \mathrm{Mod}(\phi)$ and $\mathrm{Mod}(\phi) \cap (\mathbb{X}_\tau \setminus A) = \emptyset$. Clearly $A = \mathrm{Mod}(\phi)$.[8] □

4.4 The Undefinability of Well-Ordering

Let $\tau = \{<, \ldots\}$.

Theorem 4.26 *Suppose ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and for all $\alpha < \omega_1$ there is $\mathcal{M} \models \phi$ where $(\alpha, <)$ embeds into $<^{\mathcal{M}}$. Then there is $\mathcal{N} \models \phi$ where $(\mathbb{Q}, <)$ embeds into $<^{\mathcal{N}}$.*

Corollary 4.27 *If ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence and $<^{\mathcal{M}}$ is well-ordered for all $\mathcal{M} \models \phi$, then there is $\alpha < \omega_1$ such that $<^{\mathcal{M}}$ has order type at most α for all $\mathcal{M} \models \phi$.*

Proof Let τ_0 be our original vocabulary. Form τ by adding a countable set of new constants C and distinct constants $D = \{d_q : q \in \mathbb{Q}\}$. Let Σ be all σ of the form

$$\sigma_0 \cup \{\phi\} \cup \{d_q < d_r : q < r\}$$

where σ is a finite set of $\mathcal{L}_{\omega_1,\omega}$-sentences using only finitely many constants from $C \cup D$ such that if we let

$$\theta(\overline{c}, d_{i_1}, \ldots, d_{i_m}) = \bigwedge_{\psi \in \sigma_0} \psi$$

where $i_1 < \ldots < i_m$ then for all $\alpha < \omega_1$ there is an $\mathcal{M}$ such that the following property (*) holds:

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ \theta(\overline{x}, b_1 \ldots, b_m)$$

[8] Alternatively using Exercise 1.14 we could just assume all models in $\mathcal{K}_0$ and $\mathcal{K}_1$ are countable to begin with.

and $A \subset \mathcal{M}$ well ordered by $<^{\mathcal{M}}$, $\overline{b} \in A$ and

$$\alpha \leq b_1, b_1 + \alpha \leq b_2, \ldots, b_{m-1} + \alpha \leq b_m.$$

In particular, taking $\sigma_0 = \emptyset$, we see that, by assumption, $\{\phi\} \cup \{d_q < d_r : q < r\} \in \Sigma$. We claim that Σ is a consistency property once we have shown this we will know there is a model of ϕ containing a densely ordered set.

We do several of the non-routine claims and leave the rest of the verification that Σ is a consistency property as an exercise.

C4) Suppose $\bigvee_{\psi \in X} \psi \in \sigma$. Then for each α there is $\psi_\alpha \in X$ and $\mathcal{M}$ such that (*) holds and $\mathcal{M} \models \psi_\alpha$. There is $\psi \in X$ such that $\psi = \psi_\alpha$ for uncountably many α. Note that if ψ works for α it works for all $\beta < \alpha$. Thus $\sigma \cup \{\psi\} \in \Sigma$.

C7c) Suppose $t = d_r$ and σ_0 uses $d_{i_1}, \ldots, d_{i_m}$ where $i_0 < \ldots < i_m$. Suppose $i_s < r < i_{s+1}$. Let c be element of C not yet used. We claim that $\sigma \cup \{c = d_r\} \in \Sigma$.

Let $\alpha < \omega_1$. Pick $\beta > \alpha + \alpha$. By (*) there is

$$\mathcal{M} \models \phi \wedge \exists \overline{x}\ \Theta(\overline{x}, \overline{b})$$

where

$$\beta \leq b_1, b_1 + \beta \leq b_2, \ldots, b_{m-1} + \beta \leq b_m.$$

Let $b = b_s + \alpha$. Then $b_s + \alpha \leq b$ and $b + \alpha \leq b_{s+1}$ as desired. □

Exercise 4.28 Complete the proof that Σ is a consistency property.

We will give a sharper version of this result in Theorem **??** and useful extension to uncountable models in Corollary 4.34.

4.5 Elementary End Extensions

The failure of compactness leaves us with no general tool for building elementary extensions and elementary chains. In this section we will give a criteria which will allow us to use the Model Existence Theorem to build elementary extensions in some special circumstances. Iterating this construction will be a useful tool for building models of cardinality $\aleph_1$.

Definition 4.29 Let $\tau = (<, \ldots)$. Let $\mathcal{M}$ be linearly ordered by $<$. We say that $\mathcal{N} \supset \mathcal{M}$ is an *end extension* if $\mathcal{M} \neq \mathcal{N}$ and $\mathcal{N} \models\ a < b$ for all $a \in M$ and $b \in N \setminus M$.

The following theorem of Keisler gives necessary and sufficient conditions for constructing elementary end extensions. For this section we let $\exists^* x\ \phi$ abbreviate $\forall y \exists x > y\ \phi$.

Theorem 4.30 *Let $\tau = \{<, \ldots\}$ be a countable vocabulary, let $\mathcal{M}$ be a countable τ-structure such that $<$ is a linear order, and let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. The following are equivalent:*

1. *$\mathcal{M}$ has a countable $\mathbb{A}$-elementary strong end extension;*
2. *Each of the following is true in $\mathcal{M}$.*
   - *(a) $\forall x \exists y\ x < y$*
   - *(b) $\forall \overline{w} \left[\exists^* x\ \bigvee_{n\in\omega} \phi_n(x, \overline{w}) \to \bigvee_{n\in\omega} \exists^* x\ \phi_n(x, \overline{w})\right]$, for $\bigvee_{n\in\omega} \phi_n \in \mathbb{A}$.*
   - *(c) $\forall \overline{w} \left[\exists^* x \exists y\ \phi(x, y, \overline{w}) \to (\exists y \exists^* x\ \phi(x, y, \overline{w}) \vee \exists^* y \exists x\ \phi(x, y, \overline{w}))\right]$, for $\phi \in \mathbb{A}$.*

Proof $(1 \Rightarrow 2)$ Suppose $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ and $\mathcal{N}$ is an end extension of $\mathcal{M}$. The main idea is the following overspill principle: if $b \in \mathcal{N} \setminus \mathcal{M}$, $\overline{a}, c \in \mathcal{M}$ and $\mathcal{N} \models \phi(b, \overline{a})$, then $\mathcal{M} \models \exists x > c\ \phi(x, y)$. Thus $\mathcal{M} \models \exists^* x\ \phi(x, \overline{a})$.

(a) If $a \in \mathcal{M}$ and $b \in \mathcal{N} \setminus \mathcal{M}$, $a < b$. Since $\mathcal{M} \prec \mathcal{N}$, there is $c \in \mathcal{M}$ such that $a < c$.

(b) If $\mathcal{M} \models \exists^* x \bigvee \phi_n(x)$, then the same is true in $\mathcal{N}$ and we can find $b \in \mathcal{N} \setminus \mathcal{M}$ and $n \in \omega$ such that $\mathcal{N} \models \phi_n(b)$. By overspill $\mathcal{M} \models \exists^* x \phi_n(x)$.

(c) If $\mathcal{M} \models \exists^* x \exists y\ \phi(x, y)$, then $\mathcal{N} \models \exists^* x \exists y\ \phi(x, y)$. Thus there is $b \in \mathcal{N} \setminus \mathcal{M}$ such that $\mathcal{N} \models \exists y\ \phi(y)$. Suppose $\mathcal{N} \models \phi(b, c)$. If $c \in \mathcal{M}$, then, by overspill, $\mathcal{M} \models \exists^* x\ \phi(x, c)$. If $c \notin \mathcal{M}$, then, by overspill, $\mathcal{M} \models \exists^* y \exists x\ \phi(x, y)$.

$(2 \Rightarrow 1)$ We expand the vocabulary τ to τ' by adding a constant symbol for each element of $\mathcal{M}$ and a new constant symbol d. Let $\mathbb{A}'$ be the smallest fragment containing $\mathbb{A}$ and the new constant symbols. Let

$$T = \{\chi(d, \overline{m}) : \mathcal{M} \models \exists y \forall x > y\ \chi(x, \overline{m}) \text{ for } \chi \in \mathbb{A}, \overline{m} \in \mathcal{M}\}.$$

Note that if $\mathcal{N} \models T$ then $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$. For $a \in \mathcal{M}$ let Θ_a be the sentence

$$\forall y \bigvee_{b\in M} (y = b \vee a < y).$$

If $\mathcal{N} \models T + \bigwedge_{a \in \mathcal{M}} \Theta_a$, then $\mathcal{M} \prec_{\mathbb{A}} \mathcal{N}$ and $\mathcal{N}$ is an end extension of $\mathcal{M}$. We will build $\mathcal{N}$ using the Omitting Types Theorem 4.9. We must show that T is satisfiable and that if $\exists y\ \phi(y) \in \mathbb{A}'$, $T \cup \exists y\ \phi(y)$ is satisfiable and $a \in \mathcal{M}$, then there is $b \in M$ such that $T + \exists y\ [\phi(y) \wedge (y = b \vee a < y)]$ is satisfiable.

claim T is satisfiable.

Let τ'' be obtained by adding C a countable set of new constant symbols and let $\mathbb{A}''$ be the smallest fragment containing $\mathbb{A}'$ and the new constants. Let Σ be all finite sets σ of $\mathbb{A}''$-sentences such that if $\theta_\sigma(c_1, \ldots, c_m, d)$ is the conjunction of all sentences in σ, then $\mathcal{M} \models \exists^* x \exists y_1 \ldots \exists y_m \theta_\sigma(y_1, \ldots, y_m, x)$.

We claim that Σ is a consistency property. Note that, by condition (a), $\exists^* x\ x = x$. Thus Σ is nonempty. As usual we will only check C4). Suppose $\bigvee_{\psi \in X} \psi(c_1, \ldots, c_m, d) \in \sigma \in \Sigma$. Then

$$\mathcal{M} \models \exists^* x \exists v_1 \ldots \exists v_m\ \theta_\sigma(v_1, \ldots, v_m, x).$$

But then

$$\mathcal{M} \models \exists^* x \bigvee_{\psi \in X} \exists v_1 \ldots \exists v_m\ (\psi(\overline{v}, x) \wedge \theta_\sigma(\overline{v}, x))$$

and, by condition (b), there is $\psi \in X$ such that

$$\mathcal{M} \models \exists^* x \exists v_1 \ldots \exists v_m\ (\psi(\overline{v}, x) \wedge \theta_\sigma(\overline{v}, x)).$$

Thus $\sigma \cup \{\psi(\overline{c}, d)\} \in \Sigma$.

To show that T is satisfiable, we will show that if $\psi \in T$ and $\sigma \in \Sigma$, then $\sigma \cup \{\psi\} \in T$. The Extended Model Existence Theorem 4.8 will then guarantee the existence of Σ.

Suppose $\mathcal{M} \models \exists y \forall x > y\ \chi(x))$ and $\sigma \in \Sigma$. Then

$$\mathcal{M} \models \exists^* x (\chi(x) \wedge \exists \overline{v}\ \theta_\sigma(\overline{v}, x)).$$

Hence $\sigma \cup \{\chi(d)\} \in \Sigma$ as desired.

Thus T is satisfiable.

claim For $\overline{m} \in \mathcal{M}$, $\phi(x, \overline{y}) \in \mathbb{A}$, $T + \phi(d, \overline{m})$ is satisfiable if and only if $\mathcal{M} \models \exists^* x\ \phi(x, \overline{m})$.

($\Rightarrow$) Since $\neg\phi(d, \overline{m}) \notin T$,

$$\mathcal{M} \not\models \exists x \forall y > x \neg\phi(y, \overline{m}).$$

Thus $M \models \exists^* y\ \phi(y, \overline{m})$.

($\Leftarrow$) We use the consistency property Σ. If $\mathcal{M} \models \exists^* x\ \phi(x, \overline{m})$, then $\{\psi(d, \overline{m})\} \in \Sigma$. By the arguments above and the Extended Model Existence Theorem 4.8, $T + \psi(d, \overline{m})$ is satisfiable.

claim If $\exists y\ \phi(y) \in \mathbb{A}'$, $T \cup \exists y\ \phi(y)$ is satisfiable and $a \in \mathcal{M}$, then there is $b \in M$ such that $T + \exists y\ [\phi(y) \wedge (y = b \vee a < y)]$ is satisfiable.

Suppose $T + \exists y\ \psi(y, d, \overline{m})$ is satisfiable and $a \in \mathcal{M}$. By the claim above, $\mathcal{M} \models \exists^* x \exists y \psi(y, x, \overline{m})$. Thus

$$\mathcal{M} \models \exists^* x\ \left[(\exists y\ (\psi(y, x, \overline{m}) \wedge a < y)) \vee (\exists y\ (\psi(y, x, \overline{m}) \wedge a \geq y))\right].$$

By condition b) we are in at least one of two cases.

<u>case 1</u> $\mathcal{M} \models \exists^* x \exists y (\psi(y, x, \overline{m}) \wedge a < y)$

By the previous claim $T + \exists y(\psi(y, d, \overline{m}) \wedge a < y)$ is satisfiable, so certainly

$$T + \exists y(\psi(y, d, \overline{m}) \wedge (y = b) \vee a < y)$$

is satisfiable for any $b \in \mathcal{M}$.

<u>case 2</u> $\mathcal{M} \models \exists^* x \exists y\ (\psi(y, x, \overline{m}) \wedge y \leq a)$

By now apply condition c). Clearly

$$\mathcal{M} \not\models \exists^* y \exists x\ ((\psi(y, x, \overline{m}) \wedge y \leq a).$$

Thus by c) there is $b \in \mathcal{M}$ such that

$$\mathcal{M} \models \exists^* x\ (\psi(b, x, \overline{m}) \wedge b \leq a).$$

But the $T + \exists y(\psi(y, d, \overline{m} \wedge y = b)$ is satisfiable, so

$$T + \exists y(\psi(y, d, \overline{m}) \wedge (y = b) \vee a < y)$$

is satisfiable for some $b \in \mathcal{M}$.

We have verified the conditions of the Omitting Types Theorem. Thus there is

$$\mathcal{N} \models T \cup \bigwedge_{a \in \mathcal{M}} \forall y \bigvee_{b \in \mathcal{M}} (y = b \vee a < y).$$

Then $\mathcal{M}$ is an $\mathbb{A}$-elementary end extension of $\mathcal{M}$. $\square$

We can iterated this method to build elementary extensions of size $\aleph_1$.

Corollary 4.31 *Let $\mathcal{M}$ be a countable τ-structure and let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$. If $\mathcal{M}$ has an $\mathbb{A}$-elementary strong end extension, then $\mathcal{M}$ has an $\mathbb{A}$-elementary strong end extension of cardinality $\aleph_1$.*

Proof We can build an $\mathbb{A}$-elementary chain of countable strong end extensions $(\mathcal{M}_\alpha : \alpha < \omega_1)$. The conditions (a)–(c) of Theorem 4.30 are $\mathbb{A}$-elementary, so given $\mathcal{M} \prec_{\mathbb{A}} \mathcal{M}_\alpha$ we can build an $\mathbb{A}$-elementary strong end extension of $\mathcal{M}_\alpha$. $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is the desired $\mathbb{A}$-elementary strong end extension of cardinality $\aleph_1$. □

Keisler's Theorem works in a slightly more general setting than linear orders. Let $\tau\{R, \ldots\}$ be a countable vocabulary, where R is a binary relation symbol.

Definition 4.32 We say that $\mathcal{N}$ is an end extension of $\mathcal{M}$ if whenever $a \in \mathcal{N}$, $b \in \mathcal{M}$ and aRb, then $a \in \mathcal{M}$.

We say that $\mathcal{N}$ is a *strong end extension* if $M \neq N$ and $\mathcal{N} \models aRb \wedge \neg bRa$ for all $a \in M$ and $b \in N \setminus M$.

If R is a linear ordering, then any end extension is strong.

Exercise 4.33 Modify the proof of Theorem 4.30 to show that if $\mathcal{M}$ is a countable τ-structure such that R is transitive and irreflexive then $\mathcal{M}$ has a strong $\mathbb{A}$-elementary end extension if and only if a′),b) c) hold where

a′) is $\forall x \forall y \exists z\ (xRy \wedge xRz)$

b) and c) are as above but we interpret $\exists^* x\ \phi$ as $\forall y \exists x\ (yRx \wedge \phi)$.

To give an illustration of the usefulness of this construction, consider the following uncountable version of the undefinability of well order (Theorem 4.26).

Corollary 4.34 *Let τ be a countable similarity type containing a binary relation symbol $<$, let $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, and suppose $\mathcal{M} = (M, <, \ldots) \models \phi$ where $<$ has order type ω_1. Then there is $\mathcal{N} \models \phi$ of cardinality $\aleph_1$ and an order preserving embedding of $(\mathbb{Q}, <)$ into $\mathcal{N}$.*

Proof Let $\mathbb{A}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$ containing ϕ. We expand the signature by adding a new unary predicate P. Let Ψ assert that $\mathcal{M}$ is an $\mathbb{A}$-elementary end extension of $P(\mathcal{M})$. For any $\alpha < \omega_1$ we can find P_α such that

$$(\mathcal{M}, P_\alpha) \models \phi \wedge \Psi$$

and the order type of P_α is at least α.

By the undefinability of well order (Theorem 4.26), there is a countable model $(\mathcal{N}, P(\mathcal{N})) \models \phi \wedge \Psi$ where $(\mathbb{Q}, <)$ can be embedded into $(\mathcal{N}, <)$. Since $P(\mathcal{N}) \prec_{\mathbb{A}} \mathcal{N}$ and $P(\mathcal{N})$ has an $\mathbb{A}$-elementary end extension conditions (a)–(c) of Theorem 4.30 hold in $P(\mathcal{N})$ and hence in $\mathcal{N}$. Thus $\mathcal{N}$ has an $\mathbb{A}$-elementary end extension, which, by Corollary 4.31 we may assume has cardinality $\aleph_1$. $\square$

5 Hanf Numbers and Indiscernibles

In Exercise 1.26 we showed that the Upward Löwenheim–Skolem Theorem fails in $\mathcal{L}_{\omega_1,\omega}$, by giving a sentence with models of size $2^{\aleph_0}$ but no larger models. In this section we will show that there is a cardinal κ such that if ϕ has models of cardinality κ, then ϕ has arbitrarily large models. We call the least such cardinal the *Hanf number* of $\mathcal{L}_{\infty,\omega}$. It is general nonsense that there is an Hanf number.

Exercise 5.1 Let I be a set. For each $i \in I$, let $\mathcal{K}_i$ be a class of structures. Let $\mathcal{K} = \{\mathcal{K}_i : i \in I\}$. Prove there is a cardinal κ such that for all i, if $\mathcal{K}_i$ has a structure of size κ, then $\mathcal{K}_i$ contains arbitrarily large structures. The least such κ is the *Hanf number* for $\mathcal{K}$.

Recall that for κ an infinite cardinal and α an ordinal, we inductively define $\beth_\alpha(\kappa)$ by $\beth_0(\kappa) = \kappa$ and

$$\beth_\alpha(\kappa) = \sup_{\beta<\alpha} 2^{\beth_\beta(\kappa)}.$$

In particular $\beth_1(\kappa) = 2^\kappa$. We let $\beth_\alpha = \beth_\alpha(\aleph_0)$. Under the Generalized Continuum Hypothesis $\beth_\alpha = \aleph_\alpha$.

The main theorem, due to Morley is that the Hanf number for $\mathcal{L}_{\omega_1,\omega}$ is $\beth_{\omega_1}$.

Theorem 5.2 *Let $\phi \in \mathcal{L}_{\omega_1,\omega}$. If for all $\alpha < \omega_1$ there is $\mathcal{M} \models \phi$ with $|M| \geq \beth_\alpha$, then ϕ has models of all infinite cardinalities.*

The next exercise generalizes Exercise 1.26 and shows that $\beth_{\omega_1}$ is optimal.

Exercise 5.3 Let $\alpha < \omega_1$. Let $\tau = \{U_\beta : \beta \leq \alpha + 1\} \cup \{E\} \cup \{c_0, c_1, \ldots\}$ where U_β is a unary relation symbol, E is a binary relation symbol and $c_0, c_1, \ldots$ are constants. Let ϕ assert:

i) $U_0 = \{c_0, c_1, \ldots\}$ and $\forall x\ U_{\alpha+1}(x)$;

ii) $U_\gamma \subseteq U_\beta$ for $\gamma < \beta$;

iii) $\forall x\ (U_\beta(x) \leftrightarrow \bigvee_{\beta<\alpha} U_\beta(x))$ for β a limit ordinal;

iv) $\forall x \forall y\ [(U_{\beta+1}(x) \wedge E(y,x)) \rightarrow U_\beta(y)]$;

v) if $\{x : E(x,y)\} = \{x : E(x,z)\}$, then $y = z$.

a) Show that there is $\mathcal{M} \models \phi$ with $|M| = \beth_{\alpha+1}$.

b) Show that every model of ϕ has cardinality at mode $\beth_{\alpha+1}$.

Here is the main idea of the proof of Theorem 5.2.

- By expanding the signature we may assume that we have $\phi \in T$ where T is a satisfiable theory in a countable fragment with build in Skolem functions.
- Under the assumptions of Theorem 5.2 we can find a model of T with an infinite set of indiscernibles.
- Taking Skolem hulls we get models of all infinite cardinalities.

The first and third steps are exactly as in first order model theory. In first order model theory the second step is accomplished using Ramsey's Theorem and a compactness argument. We will need a stronger partition theorem and an application of the Model Existence Theorem.

5.1 The Erdös–Rado Partition Theorem

The material in this section is verbatim from §5 of [25].

For X a set and κ, λ (possibly finite) cardinals, we let $[X]^\kappa$ be the collection of all subsets of X of size κ. We call $f : [X]^\kappa \to \lambda$ a *partition* of $[X]^\kappa$. We say that $Y \subseteq X$ is *homogeneous* for the partition f if there is $\alpha < \lambda$ such that $f(A) = \alpha$ for all $A \in [Y]^\kappa$ (i.e., f is constant on $[Y]^\kappa$). Finally, for cardinals κ, η, μ, and λ, we write $\kappa \to (\eta)^\mu_\lambda$ if whenever $|X| \geq \kappa$ and $f : [X]^\mu \to \lambda$, then there is $Y \subseteq X$ such that $|Y| \geq \eta$ and Y is homogeneous for f.

Stated in this notation Ramsey's Theorem becomes $\aleph_0 \to (\aleph_0)^n_m$ for all $n, m \in \omega$.

When we begin partitioning sets into infinitely many pieces it becomes harder to find homogeneous sets.

Proposition 5.4 $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$.

Proof We define $F : [2^\omega]^2 \to \omega$ by $F(\{f, g\})$ is the least n such that $f(n) \neq g(n)$. Clearly, we cannot find $\{f, g, h\}$ such that $f(n) \neq g(n)$, $g(n) \neq h(n)$, and $f(n) \neq h(n)$. □

On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2_{\aleph_0}$. This is a special case of an important generalization of Ramsey's Theorem.

Theorem 5.5 (Erdös–Rado Theorem) $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.

Proof We prove this by induction on n. For $n = 0$, $\kappa^+ \to (\kappa^+)^1_\kappa$ is just the Pigeonhole Principle.

Suppose that we have proved the theorem for $n-1$. Let $\lambda = \beth_n(\kappa)^+$, and let $f : [\lambda]^{n+1} \to \kappa$. For $\alpha < \lambda$, let $f_\alpha : [\lambda \setminus \{\alpha\}]^n \to \kappa$ by $f_\alpha(A) = f(A \cup \{\alpha\})$.

We build $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_\alpha \subseteq \ldots$ for $\alpha < \beth_{n-1}(\kappa)^+$ such that $X_\alpha \subseteq \beth_n(\kappa)^+$ and each X_α has cardinality at most $\beth_n(\kappa)$. Let $X_0 = \beth_n(\kappa)$. If α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$.

Suppose we have X_α with $|X_\alpha| = \beth_n(\kappa)$. Because

$$\beth_n(\kappa)^{\beth_{n-1}(\kappa)} = (2^{\beth_{n-1}(\kappa)})^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa),$$

there are $\beth_n(\kappa)$ subsets of X_α of cardinality $\beth_{n-1}(\kappa)$. Also note that if $Y \subset X_\alpha$ and $|Y| = \beth_{n-1}(\kappa)$, then there are $\beth_n(\kappa)$ functions $g : [Y]^n \to \kappa$ because

$$\kappa^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa).$$

Thus, we can find $X_{\alpha+1} \supseteq X_\alpha$ such that $|X_{\alpha+1}| = \beth_n(\kappa)$, and if $Y \subset X_\alpha$ with $|Y| = \beth_{n-1}(\kappa)$ and $\beta \in \lambda \setminus Y$, then there is $\gamma \in X_{\alpha+1} \setminus Y$ such that $f_\beta | [Y]^n = f_\gamma | [Y]^n$.

Let $X = \bigcup_{\alpha<\beth_{n-1}(\kappa)^+} X_\alpha$. If $Y \subset X$ with $|Y| \leq \beth_{n-1}(\kappa)$, then $Y \subset X_\alpha$ for some $\alpha < \beth_n(\kappa)^+$. If $\beta \in \lambda \setminus Y$, then there is $\gamma \in X \setminus Y$ such that $f_\beta | [Y]^n = f_\gamma | [Y]^n$.

Fix $\delta \in \lambda \setminus X$. Inductively construct $Y = \{y_\alpha : \alpha < \beth^+_{n-1}(\kappa)\} \subseteq X$. Let $y_0 \in X$. Suppose that we have constructed $Y_\alpha = \{y_\beta : \beta < \alpha\}$. Choose $y_\alpha \in X$ such that $f_{y_\alpha} | [Y_\alpha]^n = f_\delta | [Y_\alpha]^n$.

By the induction hypothesis, there is $Z \subseteq Y$ such that $|Z| \geq \kappa^+$ and Z is homogeneous for f_δ. Say $f_\delta(B) = \gamma$ for all $B \in [Z]^n$. We claim that Z is homogeneous for f. Let $A \in [Z]^{n+1}$. There are $\alpha_1 < \ldots < \alpha_{n+1}$ such that $A = \{y_{\alpha_1}, \ldots, y_{\alpha_{n+1}}\}$. Then

$$f(A) = f_{y_{\alpha_{n+1}}}(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = f_\delta(\{y_{\alpha_1}, \ldots, y_{\alpha_n}\}) = \gamma.$$

Thus, Z is homogeneous for f. □

We will use the following corollary.

Corollary 5.6 $\beth^+_{\alpha+n} \to (\beth^+_\alpha)^{n+1}_{\beth_\alpha}$.

Proof This follows from Erdös–Rado as $\beth_{\alpha+n} = \beth_n(\beth_\alpha)$. □

5.2 The Hanf Number of $\mathcal{L}_{\omega_1,\omega}$

We now prove Theorem 5.2. We assume that the reader is used to construction and use of indiscernibles in first order logic as in §5 of [25].

Exercise 5.7 Suppose $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and T is an $\mathbb{A}$-theory with built-in Skolem functions. Suppose $\mathcal{M} \models T$ contains an infinite set of indiscernibles for the fragment $\mathbb{A}$. Then T has arbitrarily large models.

We begin by expanding the signature we may assume that:

- there is a countable fragment $\mathbb{A}$ of $\mathcal{L}_{\omega_1,\omega}$ and $T \subset \mathbb{A}$ a theory with built in Skolem functions such that $\phi \in T$ and T has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$;
- we have two disjoint countably infinite sets of constant symbols $C = \{c_0, c_1, \ldots\}$ and $D = \{d_0, d_1, \ldots\}$, but formulas in $\mathbb{A}$ use only finitely many constant symbols from $C \cup D$.

Let $\Gamma = \{d_i \neq d_j : i < j\} \cup \{\theta(d_{i_1}, \ldots, d_{i_n}) \leftrightarrow \theta(d_{j_1}, \ldots, d_{j_n}) : \theta(v_1, \ldots, v_n) \in \mathbb{A}, i_1 < \ldots < i_n, j_1 < \ldots < j_n$ and no constants from $C \cup D$ occur in $\theta\}$.

If we can find $\mathcal{M} \models T \cup \Gamma$, then the interpretation of D in $\mathcal{M}$ gives an infinite set of indiscernibles. We can then stretch the indiscernibles to build arbitrarily large models.

If σ is a finite set of $\mathbb{A}$-sentences, we let Θ_σ be the $\mathbb{A}$-formula with no constants from $C \cup D$ such that $\Theta_\sigma(\overline{c}, \overline{d})$ is the conjunction of σ.

Let Σ be the set of all finite sets σ of $\mathbb{A}$-sentences such that for all $\alpha < \omega$ there is $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$ with $|A| = \beth_\alpha$, $<$ a linear order of A and

$$\mathcal{M} \models \exists \overline{v}\ \Theta_\sigma(\overline{v}, a_1, \ldots, a_n)$$

for all $a_1 < \ldots < a_n$ in A.

We need to show that Σ is a consistency property and if $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$. Once we have established these claims we can use the Extended Model Existence Theorem 4.8 to conclude that there is $\mathcal{M} \models T \cup \Gamma$.

claim Σ is satisfiable.

The only tricky case is C4). Suppose $\bigvee_{\psi \in X} \psi \in \sigma \in \Sigma$. Let $\Theta(\overline{c}, d_1, \ldots, d_n)$ be conjunction of σ. For all $\alpha < \omega_1$ we can find $\mathcal{M}_\alpha \models T$, $A_\alpha \subset \mathcal{M}_\alpha$ of cardinality at least $\beth_{\alpha+n}$ and $<$ a linear order of A such that

$$\mathcal{M}_\alpha \models \exists \overline{v}\ \Theta(\overline{v}, a_1, \ldots, a_n).$$

Let $f_\alpha : [A_\alpha]^n \to X$ such that if $f_\alpha(\overline{a}) = \psi(\overline{c}, d_1, \ldots, d_n)$, then

$$\mathcal{M}_\alpha \models \exists \overline{v}[\Theta(\overline{v}, \overline{a}) \wedge \psi(\overline{v}, \overline{a})].$$

Since $\beth_{\alpha+n} \geq \beth^+_{\alpha+n-1}$, by Erdos–Rado we can find $A'_\alpha \subseteq A_\alpha$ of cardinality $\beth_\alpha$ and $\psi \in X$ such that

$$\mathcal{M}_\alpha \models \exists \overline{v}\ [\Theta(\overline{v}, a_1, \ldots, a_n) \wedge \psi(\overline{v}, a_1, \ldots, a_n)]$$

for all $a_1 < \ldots < a_n$ in A'_α. Thus $\sigma \cup \{\psi(\overline{c}, d_1, \ldots, d_n)\} \in \Sigma$.

claim If $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$.

Suppose $\sigma \in \Sigma$. It is clear that if $\chi \in T$, then $\sigma \cup \{\chi\} \in \Sigma$. Suppose χ is

$$\psi(d_{i_1}, \ldots, d_{i_m}) \leftrightarrow \psi(d_{j_1}, \ldots, d_{j_m})$$

where $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ and no constants from $C \cup D$ occur in ψ.

Let $\Theta(\overline{c}, d_1, \ldots, d_n)$ be the conjunction of σ. We may assume all $i_k, j_k \leq n$. For all $\alpha < \omega_1$ there is $\mathcal{M}_\alpha \models T$, $A_\alpha \subset M_\alpha$ of cardinality $\beth_{\alpha+m}$ such that

$$\mathcal{M}_\alpha \models \exists \overline{v}\ \Theta(\overline{v}, a_1, \ldots, a_n)$$

for $a_1 < \ldots < a_n$ in A_α.

Let $f_\alpha : [A_\alpha]^m \to \{0, 1\}$ with $f_\alpha(\overline{a}) = 1$ if $\mathcal{M}_\alpha \models \psi(\overline{a})$. By Erods–Rado we can find $A'_\alpha \subset A_\alpha$ of cardinality $\beth_\alpha$ homogeneous for f_α. Then

$$\mathcal{M}_\alpha \models \exists v\ [\Theta(\overline{v}, a_1, \ldots, a_n) \wedge (\psi(a_{i_1}, \ldots, a_{i_m}) \leftrightarrow \psi(a_{j_1}, \ldots, a_{j_n})]$$

for increasing sequences from A'_α. Thus $\sigma \cup \{\chi\} \in \Sigma$.

This completes the Proof of Theorem 5.2. □

The following Corollary is proven directly in Theorem 5.2.14 of [25].

Corollary 5.8 *Suppose T is a first order theory in a countable signature and $p_1, p_2, \ldots$ are partial types. If for all $\alpha < \omega_1$ there are models of T of cardinality $\beth_\alpha$ omitting all of the p_i, then there are arbitrarily large models of T omitting all of the p_i.*

The proof given in [25] is essentially the same as the proof given here.

Exercise 5.9 Show that Theorem 5.2 follows immediately from Corollary 5.8 and the characterization in Theorem 1.30 of $\mathcal{L}_{\omega_1,\omega}$-elementary classes as reducts of classes of models of a first order theory omitting a set of types.

Exercise 5.10 Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ is κ-categorical for some $\kappa \geq \beth_{\omega_1}$. Prove that if $\mathcal{M}, \mathcal{N} \models \phi$ have cardinality at least $\beth_{\omega_1}$ then $\mathcal{M} \equiv_{\omega_1,\omega} \mathcal{N}$. (Compare this to Exercise 2.26.)

We can refine the Hanf number question by looking at complete sentences.

Exercise 5.11 Let $\tau = \{+, 0, G_1, G_2, \ldots\}$ and let ϕ be the $\mathcal{L}_{\omega_1,\omega}$ sentence asserting:

i) we have an Abelian group where every element has order 2;

ii) G_1 is an index 2 subgroup and G_{n+1} is an index 2 subgroup of G_n for each n;

iii) $\bigcap G_n = \{0\}$.

Prove that ϕ is complete and every model of ϕ has cardinality at most $2^{\aleph_0}$.

Baumgartner [8], building on work of Malitz [23], showed that Hanf number for complete $\mathcal{L}_{\omega_1,\omega}$-sentences is still $\beth_{\omega_1}$. Knight [19] showed there is a complete sentence with models of size $\aleph_1$ but no larger. Hjorth [13] extended this by showing that for all $\alpha < \omega_1$ there is a complete sentence with models of size $\aleph_1$ but no larger. We will give Hjorth's proof for $\aleph_1$ and an application to Vaught's Conjecture in §5.4. Without assumptions on cardinal exponentiation Hjorth's result is as far as we can go as Shelah [35] has shown that if ϕ has a model of size $\aleph_{\omega_1}$ then ϕ has a model of cardinality $2^{\aleph_0}$.

5.3 Morley's Two Cardinal Theorem

We give one more application of using Erdös–Rado to build useful indiscernibles.

Definition 5.12 Let $\tau = \{U, \ldots\}$ where U is a unary predicate. We say that a τ-structure $\mathcal{M}$ is a (κ, λ)-model if $|\mathcal{M}| = \kappa$ and $|U(\mathcal{M})| = \lambda$.

Theorem 5.13 (Morley's Two Cardinal Theorem) *Let ϕ be an $\mathcal{L}_{\omega_1,\omega}$-sentence. Suppose for arbitrarily large $\alpha < \omega_1$ there is an infinite κ and $\mathcal{M}_\alpha$ a $(\beth_\alpha(\kappa), \kappa)$-model of ϕ. Then for all κ there is a $(\kappa, \aleph_0)$-model of ϕ.*

Proof We can extend the signature to find a countable fragment $\mathbb{A}$, an $\mathbb{A}$-theory T with built in Skolem functions with $\phi \in T$ such that there are arbitrarily large α there a $(\beth_\alpha(\kappa), \kappa)$-model of T for some infinite κ.

Suppose $\mathcal{M} \models T$. Suppose $I \subseteq M$ and $<$ linearly orders I. We say that I is *indiscernible over* U if

$$\mathcal{M} \models \psi(x_1, \ldots, x_n, \overline{a}) \leftrightarrow \psi(y_1, \ldots, y_n, \overline{a})$$

for all $\psi(v_1, \ldots, v_n, \overline{u}) \in \mathbb{A}$, $\overline{a} \in U(\mathcal{M})$, $x_1 < \ldots < x_n, y_1 < \ldots < y_n$ in I.

Exercise 5.14 Suppose there is $\mathcal{M} \models T$ countable with $I \subseteq \mathcal{M}$ an infinite set of indiscernibles over U. Show that T has $(\kappa, \aleph_0)$ models for all infinite κ. [Hint: Prove that if $x_1 < \ldots < x_n \in I$, $\overline{a} \in U(\mathcal{M})$ and $f(\overline{x}, \overline{a}) \in U(\mathcal{M})$, then $f(\overline{y}, \overline{a}) = f(\overline{x}, \overline{a})$ for all $y_1 < \ldots < y_n$ in I.]

Add two new countable infinite sets of constant symbols C and D.

Let $\Gamma = \{d_i \neq d_j : i < j\} \cup \{\forall \overline{u}\, [U(\overline{u}) \rightarrow (\psi(d_{i_1}, \ldots, d_{i_m}, \overline{u}) \leftrightarrow \psi(d_{j_1}, \ldots, d_{j_m}, \overline{u}) :$ for $\phi \in \mathbb{A}$ (with no contants from $C \cup D$), $i_1 < \ldots < i_m$, $j_1 < \ldots < j_m\}$.

We define a consistency property Σ. If σ is a finite set of $\mathbb{A}$-sentences with finitely many constants from $C \cup D$, then $\sigma \in \Sigma$ if and only if for arbitrarily large $\alpha < \omega_1$ there is $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| \geq \beth_\alpha(U(\mathcal{M}))$ and $<$ an ordering of A such that

$$\mathcal{M} \models \exists \overline{v} \Theta(\overline{v}, \overline{a})$$

for all $a_1 < \ldots < a_n$ in A, where $\Theta(\overline{c}, d_1, \ldots, d_n)$ is the conjunction of all formulas in σ.

Exercise 5.15 Prove that Σ is a consistency property.

Exercise 5.16 Prove that if $\sigma \in \Sigma$ and $\psi \in T \cup \Gamma$, then $\sigma \cup \{\psi\} \in \Sigma$.

Combining the exercises as in the proof of Theorem 5.2 we conclude that for all infinite κ, there is a $(\kappa, \aleph_0)$-model of T. □

The next exercise shows that the assumptions are optimal.

Exercise 5.17 Modify Exercise 5.3 to show that for all $\alpha < \omega_1$ there is a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ with a $(\beth_\alpha, \aleph_0)$-model that does not have $(\kappa, \aleph_0)$-models for all infinite κ.

By a similar argument Morley proved a sharper result for first order theories. For a proof see Theorem 4.2.15 of [25].

Theorem 5.18 *Let T be a first order theory in a countable language. Suppose that T has a $(\beth_n, \aleph_0)$-model for all $n \in \omega$. Then for all $\kappa \geq \lambda$, T has a (κ, λ)-model.*

5.4 Characterizing $\aleph_1$

Theorem 5.19 (Knight [19]) *There is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ such that ϕ has models of cardinality $\aleph_1$ but no larger models, indeed no model of size $\aleph_1$ has a proper extension that is a model of ϕ.*

We will give Hjorth's proof of a refinement on Knight's theorem and his application to Vaught Counterexamples. Hjorth's construction uses a variation of the Fraïssé construction. We begin with a quick review of the method.

Definition 5.20 Let $\mathcal{K}$ be a class of finite structures closed under isomorphism. The class $\mathcal{K}$ has the *joint embedding property* if whenever $A_0, A_1 \in \mathcal{K}$ there is $B \in \mathcal{K}$ and an embedding $f_i : A_i \to B$.

The class $\mathcal{K}$ has the *amalgamation property* if whenever $A, B_0, B_1 \in \mathcal{K}$ and $f_i : A \to B_i$ are embeddings, there is $C \in \mathcal{K}$ and embeddings $g_i : B_i \to C$. We say that $\mathcal{K}$ has the *disjoint embedding property* if we can choose g_0, g_1, and C such that $g_0(B_0 \setminus f_0(A)) \cap g_1(B_1 \setminus f_1(A)) = \emptyset$.

Definition 5.21 We say that $\mathcal{M}$ is $\mathcal{K}$*-generic* if

i) for every finite $A \subset \mathcal{M}$ there is $B \in \mathcal{K}$ such that $A \subseteq B \subset \mathcal{M}$;

ii) for every $A \in \mathcal{K}$, there is an embedding of A into $\mathcal{M}$;

iii) if $A, B \in \mathcal{K}$, $A \subset B$ and $A \subset \mathcal{M}$ there is $A \subset B_0 \subset \mathcal{M}$ such that there is an isomorphism $f : B \to B_0$ such that $f|A$ is the identity.

Proposition 5.22 *Suppose there are $\aleph_0$ isomorphism types of structures in $\mathcal{K}$ and $\mathcal{K}$ has the joint embedding property and the amalgamation property. Then there is a $\mathcal{K}$-generic model of cardinality $\aleph_0$ and any two such models are isomorphic.*

Proof We build a countable $\mathcal{K}$-generic model $\mathcal{M}$ as a union of $A_0 \subset A_1 \subset \ldots$ of elements of $\mathcal{K}$. We can take $A_0 \in \mathcal{K}$ arbitrary. At stage s of our construction we are trying to insure an instance of ii) or iii).

Suppose we are given $A_s \in \mathcal{K}$ and $B \in \mathcal{K}$. Since $\mathcal{K}$ has the joint embedding property we can find $A_{s+1} \in \mathcal{K}$ such that A_s and B both embedd in A_{s+1}.

Suppose $A \subset A_s$, $A \subset B$ where A_s, A and $B \in \mathcal{K}$. Since $\mathcal{K}$ has the amalgamation property we can find $A_{s+1} \supset A_s$ and $B' \subset A_{s+1}$ such that B and B' are isomorphic over A.

Let $\mathcal{M} = \bigcup_{s\in\omega} A_s$. Clearly $\mathcal{M}$ satisfies i) and by carefully organizing our construction we can ensure $\mathcal{M}$ satisfies ii) and iii).

Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{K}$-generic models of cardinality $\aleph_0$. We build a sequence of partial embeddings

$$f_0 \subset f_1 \subset \dots$$

such that $A_s = \mathrm{dom}(f_s) \in \mathcal{K}$.

Let $A_0 \subset \mathcal{M}$ be an arbitrary element of $\mathcal{K}$. By ii) there is an embedding of A_0 into $\mathcal{N}$. Suppose we have $f_s : A_s \to \mathcal{N}$ where $A_s \in \mathcal{K}$ and $A_s \subset \mathcal{M}$. Let $a \in \mathcal{M} \setminus A_s$. By i) there is $A_{s+1} \subset \mathcal{M}_s$ such that $A_s \cup \{b\} \subseteq A_{s+1}$ and $A_{s+1} \in \mathcal{K}$. By iii) we can find $f_{s+1} \supset f_s$ such that $f_{s+1} : A_{s+1} \to \mathcal{N}$. Similarly, we can extend f_s to add any element of $\mathcal{N}$ to the image. Thus we can construct the sequence of partial embedding such that $f = \bigcup f_s$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

In our examples we will have $\emptyset \in \mathcal{K}$. In this case the amalgamation property implies the joint embedding property, so we can ignore the later.

Exercise 5.23 Show that conditions i), ii) and iii) give a simple description of $\Phi_\mathcal{K}$ a Scott sentence of the countable $\mathcal{K}$-generic model.

Exercise 5.24 Suppose $(\mathcal{M}_\alpha : \alpha < \beta)$ is a chain $\mathcal{M}_0 \subset \dots \subset \mathcal{M}_\alpha \subset \dots$ of models of Φ_K. Show that $\bigcup_{\alpha<\beta} \mathcal{M}_\alpha \models \Phi_\mathcal{K}$.

Proposition 5.25 *Suppose in addition that $\mathcal{K}$ has the disjoint amalgamation property and $\emptyset \in \mathcal{K}$. If $\mathcal{M}$ is the countable $\mathcal{K}$-generic model there is an embedding $f : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{img}(f) \subset \mathcal{M}$.

If $\Phi_\mathcal{K}$ is the Scott sentence of $\mathcal{M}$, then $\Phi_\mathcal{K}$ has a model of size $\aleph_1$.

Proof Let $\tau' = \tau \cup \{P\}$ where P is a unary function symbol. Let $\mathcal{K}'$ be the class of all expansions of structures in $\mathcal{K}$ to τ'.

Exercise 5.26 Show that since $\mathcal{K}$ has the disjoint amalgamation property and $\emptyset \in \mathcal{K}$, $\mathcal{K}'$ has joint embedding property and the amalgamation property.

Let $\mathcal{M}'$ be the unique $\mathcal{K}'$-generic model and let $\mathcal{M}$ be it's reduct to τ. We claim that there is a τ-embedding $f : \mathcal{M} \to P(\mathcal{M}')$. Then $\mathcal{M}$ is a proper embedding of $\mathcal{M}$ into itself.

Suppose $A \in \mathcal{K}$, $A \subset \mathcal{M}$ and $f : A \to P(\mathcal{M}')$ is a τ-embedding. Let $a \in \mathcal{M} \setminus A$ and let $B_1 \in \mathcal{K}', B_1 \subset \mathcal{M}'$ such that $A \cup \{a\} \subseteq B_1$. Let B

be the τ-reduct of B_1. Let B_2 be the τ'-structure such that the τ-reduct is isomorphic to B and $P(B_2) = B_2$. Since $\mathcal{M}'$ is $\mathcal{K}'$-generic, we can extend f to $\widehat{f} : B \to P(\mathcal{M}')$. Carefully iterating this construction we can build a τ-embedding $f : \mathcal{M} \to P(\mathcal{M})$.

If $\mathcal{M}$ is the countable $\mathcal{K}$-generic model, we can build a chain $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_\alpha \subset \ldots$ for $\alpha < \omega_1$ of countable models of Φ_K. At successor stages we use the construction above to build $\mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1}$ and at limit stages we use Exercise 5.24. Then $\mathcal{N} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is the desired model of size $\aleph_1$. $\square$

For Hjorth's example we let $\tau_0 = \{S_i, R_i : i \in \omega\}$ where S_i is a binary relation symbol and R_i is an $i+2$-ary relation symbol. We will consider the class $\mathcal{K}_0$ of finite τ-structures with the following properties:

i $\forall a, b \ \bigvee_{i\in\omega} S_i(a, b)$;

ii $\forall a, b \ \bigwedge_{i\neq j}(S_i(a, b) \to \neg S_j(a, b))$;

iii $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \to (a_0 \neq a_1 \wedge b_i \neq b_j)]$, for all $i < j < k$;

iv $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \leftrightarrow R_k(a_0, a_1, b_{\sigma(0)}, \ldots, b_{\sigma(k-1)})]$ for all k and σ a permutation of $\{0, \ldots, k-1\}$;

v $\forall a_0, a_1, b_0, \ldots, b_{k-1} \ [R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \to (S_i(a_0, b_i) \leftrightarrow S_i(a_1, b_i))]$ for all k and i;

vi $\forall a_0, a_1, b_0, \ldots, b_{k-1}, c \ \ [(R_k(a_0, a_1, b_0, \ldots, b_{k-1}) \ \wedge \ \bigwedge_{i<k} c \ \neq \ b_i) \ \to \ \bigwedge_{i\in\omega}(S_i(a_0, c) \to \neg S_i(a_1, c))]$ for all k;

vii $\forall a_0, a_1 [a_0 \neq a_1 \to \bigvee_{k\in\omega} \exists b_0, \ldots, b_{k-1} \ R_k(a_0, a_1, b_0, \ldots, b_{k-1})]$.

We can think of structures in $\mathcal{K}_0$ as being complete graphs where we color the directed edge (a, b) with the unique color i such that $S_i(a, b)$. For each a_0 and a_1 there is a maximal set $\{b_0, \ldots, b_{k-1}\}$ such that each edge (a_0, b_i) has the same color as the edge (a_1, b_i) and $R_k(a_0, a_1, b_0, \ldots, b_1)$.

Exercise 5.27 a) Show that there are only countably many isomorphism types in $\mathcal{K}_0$.

b) Let A be a set of cardinality n. Color the complete directed graph on A such that each edge gets a distinct color. Let R_0 hold of all pairs (a, b) where $a \neq b$ and no other R_i relation holds. Show that $A \in \mathcal{K}_0$.

c) Conclude that $\mathcal{K}_0$ is a countably infinite set of isomorphism types and $\emptyset \in \mathcal{K}_0$.

Lemma 5.28 *$\mathcal{K}_0$ has the disjoint amalgamation property. Thus there is a countable $\mathcal{K}_0$-generic model $\mathcal{M}$. If $\Phi_{\mathcal{K}_0}$ is the Scott sentence of $\mathcal{M}$, then $\Phi_{\mathcal{K}_0}$ has models of size $\aleph_1$.*

Proof Without loss of generality, we may suppose we have $A, B_0, B_1 \in \mathcal{K}_0$ with $A = B_0 \cap B_1$. We make $B_0 \cup B_1$ into a τ_0-structure as follows.

- If $a_0 \in B_l$ and $a_1 \in B_{1-l}$ let $b_0, \ldots, b_{k-1}$ be the elements of A such that $S_i(a_0, b_j)$ and $S_i(a_1, b_j)$ for some i. Make $R_k(a_0, a_1, b_0, \ldots, b_{k-1})$ hold.
- For each pair $a_0 \in B_l, a_1 \in B_{1-l}$, choose a new i and make $S_i(a_0, a_1)$ hold.

It is easy to check that this makes $B_0 \cup B_1$ into a structure in $\mathcal{K}_0$. □

Lemma 5.29 *If $\mathcal{M} \models \Phi_{\mathcal{K}_0}$ is uncountable, then there is no proper extension $\mathcal{M} \subset \mathcal{N}$ where $\mathcal{N} \models \Phi_{\mathcal{K}_0}$. In particular, Φ_{K_0} has no models of size $\aleph_2$.*

Proof Suppose $\mathcal{M} \subset \mathcal{N}$ and $a_0, a_1 \in \mathcal{M}$. For some k there are $b_0, \ldots, b_{k-1} \in \mathcal{M}$ such that $\mathcal{M} \models R_k(a_0, a_1, b_0, \ldots, b_{k-1})$. If $b \in \mathcal{N} \setminus \mathcal{M}$, by vi) if $S_i(a_0, b)$, then $\neg S_i(a_1, b)$. Thus the function that sends $a \in \mathcal{M}$ to the unique i such that $S_i(a, b)$ is injective and $\mathcal{M}$ must be countable. □

Thus $\Phi_{\mathcal{K}_0}$ is the desired complete $\mathcal{L}_{\omega_1,\omega}$-sentence with models of size $\aleph_1$ but no models of size $\aleph_2$.

5.4.1 Vaught counterexamples with no models of size $\aleph_2$

Let τ_0 and $\mathcal{K}_0$ be as above. Let $\tau_1 = \tau_0 \cup \{P, Q, F\}$ where P and Q are unary relation symbols and F is a binary relation symbol. Let $\mathcal{K}_1$ be the class of τ_1 structures A such that:

- A is the disjoint union of $P(A)$ and $Q(A)$;
- F defines a function from $P(A)$ onto $Q(A)$;
- the relation symbols from τ_0 only hold of elements from $P(A)$;
- $P(A) \in \mathcal{K}_0$.

Exercise 5.30 a) Show that $\mathcal{K}_1$ has the amalgamation and joint embedding properties (this argument should only need the fact that $\mathcal{K}_0$ has the disjoint embedding property and $\emptyset \in \mathcal{K}_0$).

b) Let $\mathcal{M}$ be the countable $\mathcal{K}_1$-generic model. Then $P(\mathcal{M}) \models \Phi_{K_0}$.

c) Let Φ_{K_1} be the Scott sentence of $\mathcal{M}$. Then Φ_{K_1} has models of size $\aleph_1$, but no uncountable model has a proper extension that is a model of Φ_{K_1}.

Definition 5.31 Let $\mathcal{M}$ be a model and let $X \subset \mathcal{M}$ be $\mathcal{L}_{\omega_1,\omega}$-definable. We say that X is a set of *absolute indiscernibles* in $\mathcal{M}$ if for every permutation σ_0 of X there is an automorphism σ of $\mathcal{M}$ such that $\sigma_0 \subset \sigma$.

Lemma 5.32 *Let $\mathcal{M}$ be the countable $\mathcal{K}_1$-generic model. Then $Q(\mathcal{M})$ is a set of absolute indiscernibles.*

Proof Let σ be a permutation of $Q(\mathcal{M})$. Suppose we have $A \subset \mathcal{M}$ such that A and $f : A \to \mathcal{M}$ such that $A \in \mathcal{K}_1$ and $f|Q(A) = \sigma|Q(A)$.

Suppose $b \in \mathcal{M} \setminus A$. We show how to add b into the domain of f. Choose $A \subset B \subset \mathcal{M}$ with $b \in B$. Let $c_1, \ldots, c_m$ be the elements of $Q(B) \setminus Q(A)$. Let $D = f(A) \cup \{\sigma(c_1), \ldots, \sigma(c_n)\}$. Since $\mathcal{M}$ is $\mathcal{K}_1$-generic, we can find $E \subset \mathcal{M}$ and an isomorphism $g : B \to E$ such that $f \subset g$ and $f(c_i) = \sigma(c_i)$ for $i = 1, \ldots, n$.

Similarly, we can elements of $\mathcal{M} \setminus f(A)$ to the image of f. It follows that there is an automorphism f of $\mathcal{M}$ such that $f \supset \sigma$. □

Corollary 5.33 (Hjorth[14]) *If there is a counterexample $\psi \in \mathcal{L}_{\omega_1,\omega}$ to Vaught's Conjecture, then there is a counterexample with no model of size $\aleph_2$.*

Proof Suppose ψ is a Vaught counterexample in $\mathcal{L}_{\omega_1,\omega}(\tau)$ where we may assume $\tau \cap \tau_1 = \emptyset$ and τ is relational. Consider the $\tau \cup \tau_1$-sentence Ψ with models $\mathcal{N}$ such that: that

- $\mathcal{N} \models \Phi_{\mathcal{K}_1}$;
- all of the τ-structure is on $Q(\mathcal{N})$;
- $Q(\mathcal{N}) \models \psi$.

Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are countable models of Ψ. We claim that if $Q(\mathcal{N}_1) \cong Q(\mathcal{N}_2)$, then $\mathcal{N}_1 \cong \mathcal{N}_2$. Suppose $f_0 : Q(\mathcal{N}_1) \to Q(\mathcal{N}_2)$ is a τ-isomorphism.

Since the τ_1-reducts of $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\mathcal{K}_1$-generic, there is $f : \mathcal{N}_1 \to \mathcal{N}_2$ a τ_1-isomorphism. Then $\sigma_0 = f^{-1} \circ f_0$ a permutation of $Q(\mathcal{N}_1)$. Since $Q(\mathcal{N}_1)$ is a set of absolute indiscernibles, there is $\sigma \supset \sigma_0$ an automorphism of the τ_1-reduct of $\mathcal{N}_1$. But then $f \circ \sigma$ is a τ_1-isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$. But $f \circ \sigma \supset \sigma_0$, so this is the desired isomorphism between $\mathcal{N}_1$ and $\mathcal{N}_2$

For any countable $\mathcal{A} \models \psi$, there is a countable $\mathcal{N} \models \Psi$ with $Q(\mathcal{N}) \cong \mathcal{A}$. Since ψ has uncountably many non-isomorphic countable models but no perfect set of non-isomorphic models the same is true of Ψ. But if $\mathcal{M} \models \Psi$, $|Q(\mathcal{M})| \leq |P(\mathcal{M})| \leq \aleph_1$. Thus Ψ is a counterexample of Vaught's Conjecture with no models of size $\aleph_2$. □

Hjorth's original proof was somewhat different. He used a slight variant of $\mathcal{K}_1$ and showed that the automorphism group of the $\mathcal{K}_1$-generic countable model had a closed subgroup with a surjective homomorphism onto S_∞. He was then able to construct the counterexample with no model of size $\aleph_2$ using some basic facts about the dynamics of Polish group actions. In [4] the authors notice that Hjorth's construction can be easily adapted to get a model with a set of absolute indiscernibles and from this the main result quickly follows.

Exercise 5.34 Show that the example Ψ has models of size $\aleph_1$.

We will see in §**??** that every Vaught counterexample has models of size $\aleph_1$.

A $\aleph_1$-Free Abelian Groups

We will prove that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free abelian groups of cardinality $\aleph_1$. The proof works by coding stationary sets into the groups. This is a standard trick in many model arguments. We need the following set theoretic fact.

Lemma A.1 *There is a family $(X_\alpha : \alpha < \omega_1)$ of disjoint stationary subsets of ω_1. For $A \subseteq \omega_1$, let $X_A = \bigcup_{\alpha \in A} X_\alpha$. Then $(X_A : A \subset \omega_1)$ is a family of $2^{\aleph_1}$ stationary subsets of ω_1 such that $X_A \triangle X_B$ is stationary for $A \neq B$.*

See, for example, [25] 5.3.10. The following exercise gives a simple example of how we can use stationary sets to build many non-isomorphic models.

Exercise 1.2 Recall that a linear order $(X, <)$ is $\aleph_1$-*like* if and only if $|X| = \aleph_1$ and $|\{y : y < x\}| \leq \aleph_0$ for all $x \in X$. We will show that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-like dense linear orders.

Let $(S_A : A \subset \omega_1)$ be a family of stationary subsets of ω_1 such that $S_A \triangle S_B$ is stationary for $A \neq B$. Fix $(L, <)$ a countable dense linear order with least element but no greatest element. For $A \subseteq \omega_1$ define a dense linear order $(X^A, <)$ as follows. For $\alpha < \omega_1$, let

$$\mathbb{X}^A_\alpha = \begin{cases} (\mathbb{Q}, <) & \text{if } \alpha \in S_A \\ (L, <) & \text{if } \alpha \notin S_A \end{cases}$$

and let $X^A = \mathbb{Q} + \sum_{\alpha < \omega_1} X^A_\alpha$. Let $X^A_{<\alpha} = \mathbb{Q} + \sum_{\beta < \alpha} X^A_\beta$. We may assume each X^A has underlying set ω_1.

a) Show that each X^A is an $\aleph_1$-like dense linear order.

b) Show that $\{\alpha < \omega_1 :$ the underlying set of $X^A_{<\alpha} = \alpha\}$ is closed unbounded.

c) Suppose that $f : \omega_1 \to \omega_1$ is a bijection. Then $\{\alpha : f|\alpha$ is a bijection onto $\alpha\}$ is closed unbounded.

d) Show that $X^A \ncong X^B$ for A and B distinct subsets of ω_1. [Hint: Suppose f is an isomoprhim. Find $\alpha \in S_A \triangle S_B$ such that $f|X^A_{<\alpha}$ is an isomorphism onto $f|X^B_{<\alpha}$. Find a contradiction.

We say that an abelian group is is $\aleph_1$-free if every countable subgroup is free.

We describe the construction of abelian groups of size $\aleph_1$ that are $\aleph_1$-free but not free. Throughout this appendix all groups are abelian.

Throughout we use the basic fact that a subgroup of a free abelian group is free.

We will need the following algebraic lemma.

Lemma A.3 *There are free abelian groups $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$, $K = \bigcup K_n$, and $F \supset K$ such that:*

i) F/K_m is free for all m, but F/K is not;

ii) each K_m, F/K, K_{m+1}/K_m has rank $\aleph_0$.

Proof Let G be the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{2^n} : n \in \omega\}$. Let $\widehat{F}$ be the free abelian group on generators $x_0, x_1, \ldots$ and let $f : \widehat{F} \to G$ be the surjective homomorphism $x_i \mapsto \frac{1}{2^i}$. Let $\widehat{K}$ be the kernel of F. Then $\widehat{K}$ is the free abelian group on generators $\{x_0 - 2x_1, x_1 - 2x_2, \ldots\}$. Then $F/K \cong G$ is not free.

Let $\widehat{K}_m$ be the free abelian group on $\{x_0 - 2x_1, \ldots, x_{m-1} - 2x_m\}$. Then $\widehat{F}/\widehat{K}_m$ is isomorphic to the free abelian group on $x_m, x_{m+1}, \ldots$.

The groups we have constructed satisfy i), but not ii). It is easy to make ii) true by adding in large free factors. Here are the details. Let H be a free abelian group on $\aleph_0$ generators.

Let

$$K_m = \widehat{K}_m \oplus \oplus_{i=0}^{m} H,$$

let

$$K = \bigcup_{m=0}^{\infty} K_m = \widehat{K} \oplus \oplus_{i=0}^{\infty} H$$

and let $F = \widehat{F} \oplus \oplus_{i=0}^{\infty} H \oplus H$. These groups have the desired properties. $\square$

Theorem A.4 *There are $2^{\aleph_1}$ nonisomorphic $\aleph_1$-free groups of cardinality $\aleph_1$.*

Proof We fix a family $\mathcal{F}$ of $2^{\aleph_1}$ stationary subsets of ω_1 such that if $S_1, S_2 \in \mathcal{F}$ are distinct, then $S_1 \triangle S_2$ is stationary. We may assume each $S \in \mathcal{F}$ is a set of limit ordinals. Let $S \in \mathcal{F}$. We construct a sequence of countable free abelian groups

$$G_0 \subset G_1 \subset \ldots G_\alpha \subset \ldots$$

for $\alpha < \omega_1$. We will do this so that:

$(*)$ if $\beta < \alpha$, then G_α is free over $G_{\beta+1}$.

i) $G_0 = \mathbb{Z}$.

ii) If α is a limit ordinal, let $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$. Choose $\gamma_0 < \gamma_1 < \dots$ with $\sup \gamma_n = \alpha$ such that each γ_n is a successor ordinal. By $(*)$ $G_{\gamma_{n+1}}$ is free over G_{γ_n}. Thus G_α is free. Indeed G_α is free over each G_{γ_n}. If $\beta < \alpha$, choose n such that $\beta < \gamma_n$. Then $G_{\gamma(n)}$ is free over $G_{\beta+1}$ and G_α is free over $G_{\gamma(n)}$. Thus G_α is free over $G_{\beta+1}$ and $(*)$ holds.

iii) If $\alpha \notin S$, then $G_{\alpha+1} = G_\alpha \oplus H$, where H is free abelian on $\aleph_0$-generators. Clearly $(*)$ holds.

iv) Suppose $\alpha \in S$. Choose $\gamma_0 < \gamma_1 < \dots$ successor ordinals with $\alpha = \sup \gamma_n$.

Let $F, K, K_0, K_1, \dots$ be as in the Lemma. Since G_{γ_0} and K_0 are both free abelian on $\aleph_0$-generators, we can find an isomorphism $\phi_0 : G_{\gamma_0} \to K_0$. Since $G_{\gamma_{n+1}}$ is free over G_γ of rank $\aleph_0$ and K_{n+1}/K_n is free of rank $\aleph_0$, we can extend ϕ_n to an isomorphism $\phi_{n+1} : G_{\gamma_n} \to K_n$. Then $\phi = \bigcup \phi_n$ is an isomorphism from G_α to K. We then define $G_{\alpha+1} \supset G_\alpha$ such that ϕ extends to an isomorphism from $G_{\alpha+1} \to F$. Thus we have

$(**)$ if $\alpha \in S$, then $G_{\alpha+1}/G_\alpha$ is not free.

On the other hand if $\beta + 1 < \alpha + 1$, there is an n such that $\beta + 1 < \gamma_n$. Then G_{γ_n} is free over $G_{\beta+1}$ and, by construction, $G_{\alpha+1}$ is free over G_{γ_n}. Thus $G_{\alpha+1}$ is free over $G_{\beta+1}$. Thus $(*)$ holds.

This concludes the construction. If $A \subseteq G$ is countable, there is an α such that $A \subseteq G_\alpha$. Since G_α is free abelian so is A. Thus G is $\aleph_1$-free.

Suppose S_1 and S_2 are distinct elements of $\mathcal{F}$. Let G_1 and G_2 be groups we constructed. We claim that $G_1 \ncong G_2$. Suppose not. Let $f : G_1 \cong G_2$ be an isomorphism. Then $C = \{\alpha : f \text{ is an isomorphism between } G_{1,\alpha} \text{ and } G_{2,\alpha}\}$ is closed unbounded.

Since $S_1 \triangle S_2$ is stationary, we can, without loss of generality, find $\alpha \in (C \cap S_1) \setminus S_2$.

Then f is an isomorphism between G_1 and G_2 such that f maps $G_{1,\alpha}$ onto $G_{2,\alpha}$. But $G_1/G_{1,\alpha}$ is not free while $G_2/G_{2,\alpha}$ is free, a contradiction. $\square$

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