Lectures on Large Stable Fields I

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Our main goal

Theorem (Johnson, Tran, Walsberg, Ye)

A large stable field is separably closed.

Theorem (Walsberg, Pillay)

An infinite large simple field is bounded (i.e., has only finitely many Galois extensions of each finite degree).

Poizat: Every infinite bounded stable field is separably closed.

Outline

1. Introduction
2. Stable groups–chain conditions
3. Stable groups–generic types
4. Stable fields–early results
5. Basics on large fields
6. the work of Johnson, Tran, Walsberg and Yi
7. Generalizations?? Simple fields? NIP fields? dp-minimal?

Review–Separably Closed Fields

Definition

A field K is *separably closed* if it has no proper separable algebraic extensions.

If char(K)=0, then separably closed $\Rightarrow$ algebraically closed.

We fix $p > 0$ prime. Suppose K is separably closed. Then K is algebraically closed iff $K = K^p$.

Assume $K \neq K^p$. $x \mapsto x^p$ is a field isomorphism. Thus we have

$$K \supset K^p \supset K^{p^2} \supset \dots.$$

K is a K^p vector space which is either infinite dimensional or of dimension p^e, $e = 1, 2, \dots.$
We call e or ∞ the *degree of imperfection* of K.

Model Theory of Separably Closed Fields

Let $\mathrm{SCF}_{p,e}$ be the theory of separably closed fields of characteristic p where the degree of imperfection is e.

Theorem (Eršov '67)

$\mathrm{SCF}_{p,e}$ *is a complete theory.*

e finite we get a model complete theory by adding a basis $b_1, \ldots, b_{p^e}$ for K over K^p.

[Delon] we get quantifier by adding functions $\lambda_1, \ldots, \lambda_{p^e}$ such that

$$a = \sum_{i=1}^{p^e} \lambda_i(a) b_i.$$

The infinite invariant case is slightly trickier.

See Delon "Separably closed fields" or Messmer "Some model theory of separably closed fields" for surveys.

Stability of Separably Closed Fields

Theorem (Wood '79)

The theory $\mathrm{SCF}_{p,e}$ is stable for all $e \leq \infty$.

Proof.

Suppose $K \models \mathrm{SCF}_{p,e}$ where $e < \infty$.

For $\sigma \in \{1, \dots, p^e\}^n$, let $\lambda_\sigma(x) = \lambda_{\sigma(0)}(\dots(\lambda_{\sigma(n-1)}(x))\dots,\dots)$.

$\mathrm{tp}(x/K)$ is determined by the sequence of ideals $I_1, I_2, \dots$ where

$$I_n = \{f(X_\sigma : \sigma \in \{1, \dots, p^e\}^n) : f(\lambda_\sigma(x) : \sigma \in \{1, \dots, p^e\}^n) = 0\}.$$

There are $|K|$ choices for each I_n and thus at most $|K|^{\aleph_0}$ types over K. □

$\mathrm{SCF}_{p,e}$ is not superstable.

Consider the type of an element x where all $I_n = \{0\}$.

We get a forking extension by naming b such that $x \in bK^p$.
Thus $K \supset K^p \supset K^{p^2} \supset \ldots$ gives rise to infinite forking chains.

Alternatively, suppose we have a model of size κ where K/K^p has cardinality κ. Looking at cosets we can easily construct $\kappa^{\aleph_0}$ types over K.

What are the stable fields?

Theorem (Macintyre '71)

Every infinite ω-stable field is algebraically closed.

Theorem (Cherlin–Shelah '80)

Every infinite superstable field is algebraically closed.

Conjecture: Every infinite stable field is separably closed.

Open Question: Is $\mathbb{C}(t)$ stable?

Large Fields

Theorem (Pop '96)

The following are equivalent for any field K
i) Every irreducible curve C defined over K with a smooth point in $C(K)$ has infinitely many points in $C(K)$.
ii) If V is an irreducible variety defined over K with a smooth point in $V(K)$, then $V(K)$ is Zariski dense in V.
iii) K is existentially closed in $K((t))$.

If these conditions hold we say K is *large*.

Large fields: separably closed, real closed, $\mathbb{Q}_p$, henselian valued fields, pseudofinite fields, PAC fields, PRC fields,...
Basically any theory of fields where we have some decent model theory.

Non-large fields: number fields, function fields (i.e., $\mathbb{C}(t), \ldots$).

Generalizations–What are the simple fields?

Examples: stable fields, psedudofinite fields,

Theorem (Chatzidakis–Pillay)

Any bounded PAC field is simple.

Question: Is every simple field bounded PAC?

Generalizations–What are the NIP fields?

Examples: stable, real closed fields, $\mathbb{Q}_p$, henselian valued fields with NIP characteristic 0 residue fields (i.e., $\mathbb{C}((t))$)

Shelah's Conjecture i) Any infinite NIP field is either real closed, algebraically closed or admits a nontrivial henselian valuation.

ii) Any infinite NIP field is either real closed, separably closed or admits a definable henselian valuation.

Henselianity Conjecture If (K, v) is an NIP valued field, then v is henselian.

Theorem (Johnson)

These hold for fields of finite dp-rank

And beyond? NTP_2 fields?

- $\prod_D \mathbb{Q}_p$, D an ultrafilter on primes
- bounded pseudo-real closed fields

Stable Groups

Let K be an infinite ω-stable field.

The starting point of Macintyre's proof that K is algebraically closed.

1) $(K, +, \dots)$ is connected, i.e., has no definable subgroup of finite index.
2) $(K^\times, \cdot, \dots)$ is connected.

1) uses DCC on definable subgroups in ω-stable theories

2) uses the fact that an ω-stable group is connected if and only if there is a unique type of maximal Morley rank so connectivity of the additive group implies connectivity of the multiplicative group.

Goal: develop enough about chain conditions and generic types in stable groups to prove 1) and 2) for stable fields.

Chain Conditions in Stable Groups–Baldwin-Saxl

Theorem

i) If $(G, \cdot, \ldots)$ *is* ω*-stable, there is no sequence of definable subgroups* $G \supset G_0 \supset G_1 \supset \ldots$.

If $H \subset G$ is a definable subgroup then either $RM(H) < RM(G)$ or $RM(H) = RM(G)$ and $\deg(H) < \deg(G)$.

Theorem

ii) If $(G,\cdot,\ldots)$ is superstable, there is no infinite descending chain $G \supset G_0 \supset G_1 \supset \ldots$ where G_{i+1} is an infinite index subgroup of G_i.

As in the earlier argument we can build an infinite forking chain or construct too many types if we have an infinite descending family of definable infinite index subgroups.

Theorem

iii) If $(G, \cdot, \ldots)$ is stable, there is no formula $\phi(v, \overline{w})$ and $\overline{a}_0, \overline{a}_1, \ldots$ such that $G \supset G_0 \supset G_1 \supset \ldots$ where $G_i = \{g \in G : \phi(g, \overline{a}_i)\}$. Indeed there is M_ϕ such that any such descending chain has length at most M_ϕ.

Indeed this has nothing to do with the group structure. For any $\psi(v, \overline{w})$ NSOP (failure of Strict Order Property) implies we can not find $X_0 \supset X_1 \supset \ldots$ where X_i is defined by $\psi(v, \overline{a}_i)$

Theorem

Suppose $(G, \cdot, \ldots)$ is NIP. Let $\phi(v, \overline{w})$ be a formula. There is M such that if $(H_i : i \in I)$ is a **finite** *family of subgroups where $\phi(v, \overline{a}_i)$ defines H_i, then there are $i_1, \ldots, i_M$ such that*

$$\bigcap_{i \in I} H_i = H_{i_1} \cap \cdots \cap H_{i_M}.$$

Otherwise, for any n we can find $H_{i_1}, \ldots, H_{i_n}$ and $b_1, \ldots, b_n$ with

$$b_k \in \bigcap_{j \neq k} H_{i_j} \setminus \bigcap_{j=1}^{n} H_{i_j}.$$

For $X \subseteq \{1, \ldots, n\}$ let $c_X = \prod_{j \in X} b_j$.

$$c_X \in H_i \Leftrightarrow i \notin X$$

Corollary (Baldwin–Saxl)

Suppose $(G, \cdot, \dots)$ is stable. For every formula $\phi(x, \overline{y})$ there is a number M such that any properly descending chain of subgroups each of which is an arbitrary intersection of groups defined using ϕ has length at most M.

Suppose not. Then we can find $H_1, H_2, \dots$ defined by ϕ such that

$$H_1 \supset H_1 \cap H_2 \supset H_1 \cap H_2 \cap H_3 \supset \dots.$$

There is an N such that for all k there are $i_1, \dots, i_N \leq k$ such that

$$H_1 \cap \dots \cap H_k = H_{i_1} \cap \dots \cap H_{i_N}.$$

Let $\psi(x, \overline{y}_1, \dots, \overline{y}_N)$ be the formula

$$\bigwedge_{i=1}^{N} \phi(x, \overline{y}_i)$$

and apply the earlier result.

Note: An NIP field can have uniformly definable infinite chains of subgroups. Consider any valued field and $G_\gamma = \{x : v(x) > \gamma\}$.

Centralizers

If $A \subset G$, then $C(A) = \{g \in G : ga = ag \text{ for all } a \in A\}$.

Corollary

If $(G, \cdot, \ldots)$ is stable, then there is an M such that for any $A \subset G$ there is $A_0 \subseteq A$ with $|A_0| \leq M$ such that $C(A) = C(A_0)$.

$$C(A) = \bigcap_{a \in A} C(\{a\}) = C(\{a_1\}) \cap \ldots C(\{a_M\})$$

Note $C(A)$ is definable even though A may not be.

We say a group $(G, \cdot, \dots)$ is *connected* if there is no proper definable subgroup of finite index.

Corollary

Suppose $(K, +, \cdot, \dots)$ is stable. Then the additive group $(K, +, \dots)$ is connected.

Suppose H is a definable subgroup of $(K, +)$ of finite index. If $a \in K^\times$, then aH is also a finite index subgroup. Thus

$$I = \bigcap_{a \in K^\times} aH = a_1 H \cap \cdots \cap a_n H$$

is of finite index. But then I is a non-trivial ideal.

Connected Components

For $(G, \cdot, \dots)$ be stable let G^0 be the intersection of all definable finite index subgroups of G.

For each $\phi(v, \overline{w})$ let G^0_ϕ be the intersection of all conjugates of finite index subgroups defined using ϕ.

$G^0_\phi = \bigcap G^0_{\phi,n}$ where $G^0_{\phi,n}$ is the intersection of all conjugates of subgroups defined using ϕ of index at most n.

G^0_ϕ is normal and $\bigwedge$-definable over $\emptyset$. By stability it is a finite intersection of conjugates of subgroups defined using ϕ and hence definable.

Thus G^0_ϕ is definable over $\emptyset$ and of finite index.

$G^0 = \bigcap G^0_\phi$ is normal $\bigwedge$-definable over $\emptyset$ and $[G : G^0] \leq 2^{|T|}$.