Lectures on Large Stable Fields II
Generic types in stable groups

Dave Marker

Mathematics, Statistics, and Computer Science
University of Illinois at Chicago

February 1, 2021

www.math.uic.edu/∼marker/mc-S21

Where we are

Last time we used chain conditions in stable groups to prove that if K is an infinite stable field then the additive group $(K, +, \dots)$ is connected, i.e., has no definable finite index subgroups.

Our goal is to show the same for $(K^\times, \cdot, \dots)$.

Towards this end we will develop the theory of generic types in stable groups and use them to deduce the connectivity of the multiplicative group from the connectivity of the additive group.

Generic sets

Assume $(G, \cdot, \dots)$ is a stable group.

Definition

Let $A \subseteq G$ be definable. We say that A is *generic* if for some n there are $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that

$$G = a_1 A b_1 \cup \cdots \cup a_n A b_n.$$

We say that A is *left generic* if we can find $a_1, \dots, a_n$ such that

$$G = a_1 A \cup \cdots \cup a_n A.$$

We define *right generic* similarly.
We say that a type is generic if contains only generic formulas.

We will ultimately show that these three notions are the same.

Exercise

If A is generic so is $A^{-1} = \{g^{-1} : g \in A\}$

$$G = \bigcup_{i=1}^{n} b_i A c_i$$

Let $x \in A$. Then for some $a \in A$ and some i $x^-1 = b_i a c_i$. So $x = c_i^{-1} a^{-1} b_i^{-1}$ and

$$G = \bigcup c_i^{-1} A^{-1} b_i^{-1}.$$

Our first goal will be to show that generic types exist.

Lemma

Let $A \subseteq G$ be definable. If A is not left generic, then $G \setminus A$ is right generic.

Suppose not. Then for any $a_1, \ldots, a_n \in G$ there is

$$x \notin \bigcup_{i=1}^{n} (G \setminus A) a_i^{-1},$$

i.e., $x a_i \ \in A$ for $i = 1, \ldots, n$.
Similarly, for any $b_1, \ldots, b_n$, we can find

$$y \notin \bigcup_{i=1}^{n} b_i^{-1} A$$

i.e.,$b_i y \notin A$ for $i = 1, \ldots, n$.

1) For any $a_1, \ldots, a_n$ there is x such that $xa_i \in A$ for all i
2) For any $b_1, \ldots, b_n$ there is y such that $b_i y \notin A$ for all i

We build $c_1, \ldots, c_n, \ldots$ and $d_1, \ldots, d_n, \ldots$ as follows
Let c_1 be arbitrary.

Given $c_1, \ldots, c_n$ choose d_n using 2) so that $c_i d_n \notin A$ for $i = 1, \ldots, n$

Given $d_1, \ldots, d_n$ choose c_{n+1} using 1) such that $c_{n+1} d_i \in A$ for $i = 1, \ldots, n$

But then $c_i d_j \in A$ if and only if $i > j$ and the formula $xy \in A$ has the order property.

Corollary

If $A, B \subseteq G$ are definable and $A \cup B$ is generic, then at least one of A or B is generic.

Suppose

$$G = \bigcup_{i=1}^{n} a_i(A \cup B)b_i = \left(\bigcup a_i A b_i\right) \cup \left(\bigcup a_i B b_i\right).$$

By the previous lemma, either $\bigcup a_i A b_i$ is left generic or $\bigcup a_i B b_i$ is right generic. Suppose $\bigcup a_i A b_i$ is left generic. Then there are $c_1, \ldots, c_m$ such that

$$G = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} c_j a_i A b_i$$

and A is generic.
The other case is similar.

Existence of Generics

Corollary

There is a generic type.

Let $\Gamma = \{A : G \setminus A \text{ is not generic}\}$.

If $A_1, \ldots, A_n \in \Gamma$, then, by the previous corollary $\bigcup (G \setminus A_i)$ is not generic. Thus $\bigcap A_i \in \Gamma$.

Thus Γ is consistent. Let p be any complete type extending Γ. Then p contains no non-generic formulas.

More generally, if A is generic there is a generic type p with $A \in p$.

Goals: Useful Properties of Generics

- $\{\overline{b} : \phi(x, \overline{b}) \text{ is generic}\}$ is definable
- If a is generic over A and $a \mathop{\smile\!\!\!\!\!\!\;|}_A b$, then ab is generic over A.
- The set of generic types is bounded–there are at most $2^{|T|}$ generic types.
- p is generic if and only if $\mathrm{Stab}(p) = G^0$

Exercise

Suppose p is a generic type and A is a definable set. Then A is generic if and only if some translate cAd is in p.

($\Rightarrow$) If A is generic and $G = \bigcup_{i=1}^{n} b_i A c_i$, then some $b_i A c_i \in p$.

($\Leftarrow$) If p is generic and $cAd \in p$, then cAd is generic. Then

$$G = \bigcup_{i=1}^{n} a_i (cAd) b_i = \bigcup_{i=1}^{n} (a_i c) A (d b_i).$$

Definability of genericity

Lemma

For every formula $\phi(x, \overline{y})$ the set $\{\overline{b} : \phi(x, \overline{b})$ is generic$\}$ is definable.

Let p be a generic type.

Then $\phi(x, \overline{b})$ is generic if and only if only if there are c and d such that $\phi(cxd, \overline{b}) \in p$.

Let $\psi(x; \overline{y}, u, v)$ be the formula $\phi(uxv, \overline{y})$. By definability of types, there is a formula $d_p\psi(\overline{y}, u, v)$ such that

$$\psi(x; \overline{b}, c, d) \in p \Leftrightarrow d_p\psi(\overline{b}, c, d)$$

Then

$$\exists u \exists v \; d_p\psi(\overline{b}, u, v) \Leftrightarrow \phi(x, \overline{b}) \text{ is generic.}$$

Corollary

For every formula $\phi(x,\overline{y})$ there is an N such that $\phi(x,\overline{b})$ defines a generic set if and only if G is covered by N two-sided translates of $\phi(x,\overline{b})$

Let $\psi_n(\overline{y})$ be the formula

$$\exists u_1 \ldots \exists u_n \exists v_1 \ldots \exists v_n \forall x \bigvee_{i=1}^{n} \phi(u_i x v_i, \overline{y})$$

$\psi_n(\overline{y})$ asserts that G is covered by n two-sided translates of $\phi(x,\overline{y})$.

Then $\phi(x,\overline{b})$ is generic if and only if $\bigvee \psi_n(\overline{b})$.
But the set of $\overline{b}$ such that $\phi(x,\overline{b})$ is generic is definable. The result then follows by compactness.

Consider

$$\Gamma(\overline{y}) = \{\phi(x,\overline{y}) \text{ is generic }\} \cup \{\neg\psi_1(\overline{y}), \neg\psi_2(\overline{y}), \ldots\}.$$

Lemma

If p is a generic type and q is a nonforking extension then q is generic.

I will prove this for p is generic over a model $\mathbb{M}$ and leave the general case as an exercise.

If q is not generic, there is $\phi(x, \overline{b}) \in q$ non generic.

Let $\Psi_\phi(\overline{y})$ be the formula defining $\{\overline{c} : \phi(x, \overline{c})\}$ is generic and let $\theta(x, \overline{y})$ be the $\phi(x, \overline{y}) \wedge \Psi_\phi(\overline{y})$.
Then $\theta(x, \overline{b}) \in q$.

By definability of types there is $d_p\theta(\overline{y})$ such that

$$d_p(\overline{c}) \Leftrightarrow \theta(x, \overline{c}) \in p$$

and the same definition works for the non-forking extension q.
But then $\exists \overline{y}\ d_p\theta(\overline{y}$ holds in the monster model and hence in $\mathbb{M}$.
If $\overline{c} \in \mathbb{M}$ and $d_p\theta(\overline{c})$, then $\phi(x, \overline{c}) \in p$ and is non-generic, a contradiction.

Corollary

Let p be a generic type of G. Then p does not fork over $\emptyset$.

Before proving this we need to review a few facts about stability and Shelah's local ∞-rank which we will denote R^∞.

Shelah's local ∞-rank R^∞

Suppose Δ is a collection of formulas $\phi(v_1, \dots, v_n, \overline{y})$, and let $\Phi(\overline{v})$ be a partial type.

We define $\mathrm{R}^\infty_\Delta(\Phi)$ inductively as follows

- $\mathrm{R}^\infty_\Delta(\Phi) \geq 0$ if $\Phi(\overline{v})$ is consistent;
- $\mathrm{R}^\infty_\Delta(\Phi) \geq \alpha + 1$ if and only if there are $\phi_1(\overline{v}, \overline{b}_1), \dots, \phi_n(\overline{v}, \overline{b}_n), \dots$, $\phi_i \in \Delta$, each consistent with $\Phi(\overline{v})$ but pairwise inconsistent with $\Phi(\overline{v})$ such that $\mathrm{R}^\infty_\Delta(\Phi(\overline{v}) \cup \{\phi_m(\overline{v})\}) \geq \alpha$;
- for α a limit ordinal, $\mathrm{R}^\infty_\Delta(\Phi(\overline{v})) \geq \alpha$ if it is greater than every $\beta < \alpha$.

Note that if Δ is all formulas this is exactly Morley rank.

Lemma

Let Δ be a finite set of stable formulas.

1. *There is an N such that $\mathrm{R}^{\infty}_{\Delta}(p) \leq N$ for all Δ-types p.*
2. *If $A \subset B$, $p \in S_{\Delta}(B)$, then p forks if and only if $\mathrm{R}^{\infty}_{\Delta}(p) < \mathrm{R}^{\infty}_{\Delta}(p|A)$.*

Corollary

Let p be a generic type of G. Then p does not fork over $\emptyset$.

Suppose $\phi(x, \overline{b}) \in p$ forks over $\emptyset$.

We can instead consider $\psi(x; u, v, \overline{y})$ given by $\phi(uxv, \overline{y})$. Then $\psi(x; 1, 1, \overline{b})$ is a forking formula.

Let p_ψ denote the ψ-type determined by p. It suffices to show that p_ψ doesn't fork over $\emptyset$.

Let $q \in S_\psi(G)$ be of maximal R_ψ^∞-rank.

Let $A \in p$ be defined using ψ.
Since p is generic, there are c and d such that $cAd \in q$.
Thus $\mathrm{R}_\psi^\infty(cAd) \geq \mathrm{R}_\psi^\infty(q)$.

But it's easy to see that $\mathrm{R}_\psi^\infty(A) = \mathrm{R}_\psi^\infty(cAd)$, using the fact that if $B_1, \ldots, B_n, \ldots$ are disjoint ψ-definable subsets of A, then $cB_1d, \ldots, cB_nd, \ldots$ are disjoint ψ-definable subsets of cAd.

Thus
$$\mathrm{R}_\psi^\infty(A) \geq \mathrm{R}_\psi^\infty(cAd) \geq \mathrm{R}_\psi^\infty(q).$$

Thus $\mathrm{R}_\psi^\infty(p|\psi)$ is maximal and $p|\psi$ does not fork over $\emptyset$.

Boundedly many generics

Corollary

There are a bounded number of generic types.

Since a generic type doesn't fork over $\emptyset$ there are only a bounded number (at most $2^{|T|}$) of generic types.
Every generic p is definable over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and there are at most $2^{|T|}$ defining schemes.

Finitely many generic ϕ-types

Corollary

$\{p|\phi : p$ generic$\}$ is finite.

For any I consider

$$\Gamma(v, y_i : i \in I) = \{\phi(x, \overline{y}_i) \text{ is generic} \wedge \neg\phi(v, \overline{y}_i) : i \in I\}.$$

If there are infinitely many ϕ-types of generics then Γ is consistent, but then there are unboundedly many ϕ-types of generics.