Lectures on Large Stable Fields III
Generic types in stable groups continued

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Last Time

Definition

A definable $A \subseteq G$ is *generic* if and only if there are $b_1, \ldots, b_n$, $c_1, \ldots, c_n$ such that $G = b_1 A c_1 \cup \cdots \cup b_n A c_n$.
A type is generic if it contains only generic formulas.

- If A is generic there is a generic type p with $A \in p$.
- For any $\phi(x; \overline{y})$ the set $\{\overline{b} : \phi(x; \overline{b}) \text{ is generic}\}$ is definable.
- For any $\phi(x; \overline{y})$ there is a bound N such that if $\phi(x, \overline{b})$ is generic we can cover G with at most N two sided translates

Last Time

- A non-forking extension of a generic type is generic.
- A generic type does not fork over $\emptyset$.
- There are at most $2^{|T|}$ generic types
- For all ϕ the set $\{p|\phi : p \text{ generic}\}$ is finite.

Lemma

If a is generic over A and $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$, then ab is generic and $ab \mathop{\smile\!\!\!\!\!\!\,|}_A b$.

If a is generic and $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$, then $\mathrm{tp}(a/Ab)$ is generic (recall: nonforking extensions of generic types are generic).

If $X \in \mathrm{tp}(ab/Ab)$, then $Xb^{-1} \in \mathrm{tp}(a/Ab)$ is generic. Thus X is also generic and $\mathrm{tp}(ab/Ab)$ is generic.

Since generic types don't fork over $\emptyset$, $ab \mathop{\smile\!\!\!\!\!\!\,|}_A b$.

Corollary

Every $g \in G$ is the product of two generics.

Let $g \in G$ and a generic with $a \mathop{\smile\!\!\!\!\!\!|}\, g$. Then $a^{-1}g$ is generic and $g = a(a^{-1}g)$.

Corollary

Let $A \subset G$ be definable. If A is generic, then A is left generic.
Thus generic $\Leftrightarrow$ left generic (similarly $\Leftrightarrow$ right generic)

Suppose A is generic and definable over a model $\mathcal{M}$.
By compactness it suffices to show that all $p \in S(\mathcal{M})$ are in a left translate of A.

Let b realize p. Let $a \in A$ with $\mathrm{tp}(a/\mathcal{M}b)$ generic.
Let $c = ab^{-1}$.
Then $\mathrm{tp}(c/\mathcal{M}b)$ is generic, $c \perp_{\mathcal{M}} b$ and $c \in Ab^{-1}$.
By symmetry $b \perp_{\mathcal{M}} c$.

So we have $c \in Ab^{-1}$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_{\mathcal{M}} c$.

Let $q = \mathrm{tp}(b/\mathcal{M})$. Let $\phi(v; w)$ be $w \in Av^{-1}$.
Using definability of types there is a formula $d_q\phi(w)$ such that

$$\phi(v; d) \in q \Leftrightarrow d_q\phi(d).$$

This definition also works for the non-forking extension $\mathrm{tp}(b/M, c)$.
But then $d_q\phi(c)$ so $\exists w\ d_q\phi(w)$ holds in the monster and hence in $\mathcal{M}$.
Thus there is $d \in \mathcal{M}$ with $d \in Ab^{-1}$ and $b \in d^{-1}A$.

Stabilizers

There is a natural action of G on the space of types.

If $p \in S_1(G)$, a realizes p and $g \in G$, let gp be the type of ga. Then

$$\phi(v) \in gp \Leftrightarrow \phi(gv) \in p.$$

If p is a type, then $\mathrm{Stab}(p) = \{g \in G : gp = p\}$ is the *stabilizer* of p.

Stabilizers of ϕ-types

We have to be careful thinking of G acting on ϕ-types as $\phi(gv;\overline{a})$ is not strictly speaking an instance of ϕ.

For any $\phi(v;\overline{y})$ we can consider $\psi(v;u,\overline{y}) = \phi(uv,\overline{y})$ Then

$$\psi(gv;b,\overline{a}) \Leftrightarrow \psi(v,bg,\overline{a})$$

So G acts on ψ-types.
Moreover, for any p, $p|\psi \Rightarrow p|\phi$. (For these lectures we will call such a ψ *robust*)

Define

$$\mathrm{Stab}_\psi(p) = \{g \in G : gp|\psi = p|\phi\}.$$

Then

$$\mathrm{Stab}(p) = \bigcap\{\mathrm{Stab}_\psi(p) : \psi \text{ robust}\}.$$

Suppose $\psi(v; u, \overline{y}) = \phi(uv, \overline{y})$

Lemma

$\mathrm{Stab}_\psi(p)$ is definable. Thus $\mathrm{Stab}(p)$ is $\bigwedge$-definable.

Then

$$g \in \mathrm{Stab}_\psi(p) \Leftrightarrow \forall c \forall \overline{b} \ \left(\psi(v; c, \overline{b}) \in p \Leftrightarrow \psi(v; cg, \overline{b}) \in p\right).$$

By definability of types there is $d_p\psi(u, \overline{y})$ such that

$$\psi(v, c, \overline{b}) \in p \Leftrightarrow d_p\psi(c, \overline{b}).$$

Then

$$\mathrm{Stab}_\psi(p) = \{g : \forall u \forall \overline{y} \ \left(d_p\psi(u, \overline{y}) \leftrightarrow d_p\psi(ug, \overline{y})\right)\}$$

Review–Connected Components

G^0 the *connected component* of G is the intersection of all definable finite index subgroups of G.

G^0_ϕ is the intersection of all conjugates of finite index subgroups of G defined using ϕ.

- G^0_ϕ is a finite intersection of conjugates of finite index subgroups defined using ϕ
- G^0_ϕ is a normal subgroup of finite index defined over $\emptyset$.
- $G^0 = \bigcap G^0_\phi$ is $\bigwedge$-definable over $\emptyset$, normal and of bounded index. $[G : G^0] \leq 2^{|T|}$.

Lemma

$\mathrm{Stab}(p) \subseteq G^0$

We will show that $\mathrm{Stab}(p) \subseteq G^0_\phi$ for all ϕ.

Suppose $p \in bG^0_\phi$. Let a realize p. Then $b^{-1}a \in G^0_\phi$.

If $g \in \mathrm{Stab}(p)$, then $b^{-1}ga \in G^0_\phi$.
Say $\alpha, \beta \in G^0_\phi$ such that $b^{-1}a = \alpha$ and $b^{-1}ga = \beta$.

$$b^{-1}g(b\alpha) = \beta \text{ and } b^{-1}gb = \beta\alpha^{-1} \in G^0_\phi.$$

But G^0_ϕ is normal. Thus $g \in G^0_\phi$.

Lemma

If p is generic, then $\mathrm{Stab}(p) \supseteq G^0$.

Suppose ψ is robust.

If p is generic and $g \in G$, then gp is generic.

We know that $\{p|\psi : p \text{ generic}\}$ is finite.
Thus $\mathrm{Stab}_\psi(p)$ is of finite index in G.
Hence $\mathrm{Stab}_\psi(p) \supseteq G^0$ and $\mathrm{Stab}(p) \supseteq G^0$.

Corollary

If p is generic, then $\mathrm{Stab}(p) = G^0$

We will soon show the converse.

We work over G sufficiently saturated so that all possible cosets of G^0 are represented.

If p is generic, then p is in some coset bG^0. Taking translations, we get generic types in all cosets of G^0.

Lemma

There is a unique generic type in G^0.

Suppose p and q are generic types in G^0. Let a and b realize p and q with $a \downarrow_G b$.
G^0 is the stabilizer of q and its non-forking extension $\widehat{q}$ to G, a (as q is generic). Thus ab realizes q

We could redo our whole theory of stabilizers looking at right actions of G on $S_1(G)$.
That would tell us that b is in the right-stabilizer of the non-forking extension of p to G, b. Thus ab realizes p and $p = q$.

Lemma

If $\mathrm{Stab}(p) = G^0$, *then* p *is generic.*

Let q be the unique generic type in G^0.
Let a realize p and b realize q with $a \mathop{\smile\!\!\!\!\!\mid}_G b$.

But $b \in G^0$, thus $b \in \mathrm{Stab}(\widehat{p})$ where $\widehat{p}$ is the non-forking extension of p to G, b.
Hence ba realizes p.

But $b \mathop{\smile\!\!\!\!\!\mid}_G a$ and b generic, implies ba is generic.
Thus p is generic.

Fundamental Theorem of Stable Groups

Theorem

- *G^0 is of bounded index.*
- *There is a unique generic type in every coset of G^0.*
- *G/G^0 is a profinite group acting transitively on the generics.*
- *p is generic if and only if $\mathrm{Stab}(p) = G^0$*

To see G/G^0 is profinite, note that

$$G/G^0 = \varprojlim \{G/H : H \text{ definable}, [G : H] < \infty\}.$$

Reminder–Connectedness of additive group

Lemma

If $(K, +, \cdot, \ldots)$ is an infinite stable field, then the additive group $(K, +, \ldots)$ is connected.

Suppose H is definable of finite index. For all $a \in K^\times$, aH is of finite index.

By Baldwin–Saxl,

$$I = \bigcap_{a \in K^\times} aH = a_1 H \cap \cdots \cap a_n H$$

for some $a_1, \ldots, a_n$.
I is of finite index so $I \neq \{0\}$. But I is an ideal.

Connectedness of $K^\times$

Corollary

Let K be an infinite stable field. Then the multiplicative group $(K^\times, \cdot, \dots)$ is connected

Since $(K, +, \dots)$ is connected there is a unique generic type p.

$x \mapsto bx$ is a group automorphism of $(K, +)$ for all $b \in K^\times$.
Thus bp is generic for all $b \in K^\times$

Working in $(K^\times, \cdot, \dots)$. This says that $\mathrm{Stab}(p) = K^\times$.
But then $(K^\times)^0 = K^\times$. Thus $K^\times$ is connected and p is the unique generic type of $K^\times$.

Superstable Groups

Lemma (Surjectivity Principle)

Let $(G, \cdot, \ldots)$ be a connected superstable group. If $\sigma : G \to G$ is a definable finite-to-one group homomorphism, then σ is surjective.

If $g \in G$, then $\sigma(g)$ and g are inter-algebraic.
Thus G and $\sigma(G)$ have the same U-rank.

If $[G : \sigma(G)]$ is infinite, then $U(G) > U(\sigma(G))$. [We can fork by naming a coset of $\sigma(G)$.]
Thus $[G : \sigma(G)]$ is finite and, since G is connected we must have $\sigma(G) = G$.

This fails in $\mathrm{SCF}_{p,e}$ for $e > 0$ since $x \mapsto x^p$ is not surjective.

Corollary

Suppose $(K, +, \cdot, \dots)$ is an infinite superstable field.

1. *The map $x \mapsto x^n$ is surjective for all n.*
2. *If $\mathrm{char}(K) = p > 0$, then map $x \mapsto x^p - x$ is surjective.*

$x \mapsto x^n$ is a homomorphism of the multiplicative group.
In characteristic p, $x \mapsto x^p - x$ is a homomorphism of the additive group.

Corollary

An infinite superstable field is perfect (i.e, every algebraic extension is separable).

In characteristic p we just need $x \mapsto x^p$ is surjective.