Lectures on Large Stable Fields V
Large Fields

Dave Marker

Mathematics, Statistics, and Computer Science
University of Illinois at Chicago

February 22, 2021

www.math.uic.edu/~marker/mc-S21

Large fields

Definition

A field K is *large* if $C(K)$ is infinite whenever C is an irreducible curve defined over K with a smooth point in $C(K)$.

Large fields play an important role in Field Arithmetic particularly around inverse Galois theory problems.

Theorem (Pop)

If K is a large field and G is finite group, then there is $L \supset K(t)$ a Galois extension with Galois group G and $L \cap K^{\mathrm{alg}} = K$.

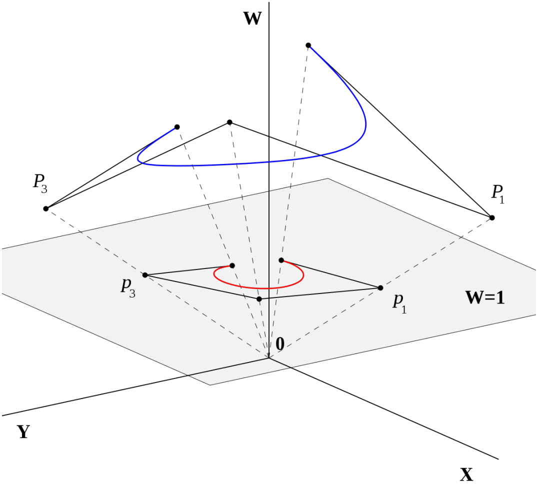
Reduction to plane curves

Let $C \subset \mathbb{A}^3$ be a curve defined over K not contained in a plane.

Let $\mathrm{p} \in \mathbb{A}^3 \setminus C$. For $\mathrm{a} \in C$, let l be the line between a and p and let (x, y) be the point where the plane $\mathbb{A}^2$ intersects l.
Let f be the map $\mathrm{a} \mapsto (x, y)$ and let $C' = f(C)$.

For $\mathrm{p} \in \mathbb{A}^3(K)$ suitably generic f is one-to-one except for finitely many points and $f(C(K)) = C'(K)$.

Moreover if a is a smooth point of C, then, for most $\mathrm{p} \in \mathbb{A}^3(K)$, $f(\mathrm{a})$ is a smooth point of C'.


W
P_3
P_1
p_3
p_1
W=1
0
Y
X

Smooth points

Let $V \subset \mathbb{A}^n$ be an irreducible variety defined by $f_1 = \cdots = f_m = 0$ and let $\mathrm{a} \in V$.

The *tangent space* to V at a is

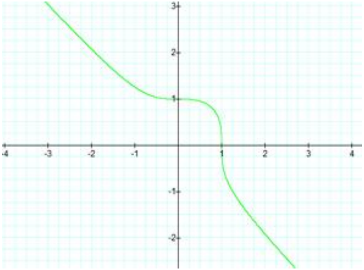
$$T_{\mathrm{a}} V = \left\{ \overline{u} \in \mathbb{A}^n : \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j}(\mathrm{a}) u_j = 0 \text{ for } i = 1, \ldots, m \right\}.$$

We say a is a *smooth point* if $\dim T_{\mathrm{a}}(V) = \dim V$.

For example, if V is the curve $f(X, Y) = 0$, $\dim T_{\mathrm{a}} V = 1$ unless $\frac{\partial f}{\partial X}(\mathrm{a}) = \frac{\partial f}{\partial Y}(\mathrm{a}) = 0$.

Nonlarge Fields

Consider the Fermat elliptic curve E given by $X^3 + Y^3 = 1$.



This is a smooth curve and the only points in $E(\mathbb{Q})$ are $(0,1)$ and $(1,0)$ Thus $\mathbb{Q}$ is not large.

More generally, if K is any number field or a function field (say., $K = k(t)$), then K is not large.

Real closed fields are large

First work in $\mathbb{R}$.

Suppose C is a curve given by $f(x,y)=0$, $f(a,b)=0$ and $\frac{\partial f}{\partial Y}(a,b)\neq 0$.

By the Implicit Function Theorem there is U a neighborhood of a and a real analytic function $g: U \to \mathbb{R}$ such that $g(a)=b$ and $f(x,g(x))=0$ for all $x \in U$.

Thus $\mathbb{R}$ is large.

We really do need a smooth point for this to work.
For example, consider the absolutely irreducible curve C given by $X^2 = Y^4 + Y^2$.
$(0,0)$ is the unique point in $C(\mathbb{R})$ but it's a singular point.

Lemma

The class of large fields is an elementary class.
In particular, if K is large and $K \equiv L$, then L is large.

Let $\Phi_{d,n}$ assert that for all irreducible polynomials $F(X,Y)$ of degree at most d if there are x and y such that $F(x,y) = 0$ and $\frac{\partial F}{\partial X}(x,y) \neq 0$ or $\frac{\partial F}{\partial Y}(x,y) \neq 0$, then there at least n distinct solutions to $F(X,Y) = 0$.

Then $\{\Phi_{d,n} : d, n > 0\}$ axiomatizes the class of large fields.

Corollary

Any real closed field is large.

Henselian fields are large

Lemma

(K, v) is henselian $\Leftrightarrow$ for any $f \in \mathcal{O}[X]$ and $\alpha \in \mathcal{O}$ with $v(f(\alpha)) > 2v(f'(\alpha))$ there is $x \in \mathcal{O}$ with $f(x) = 0$.

Suppose (K, v) is henselian, $f(X, Y) \in K[X, Y]$ and (a, b) is a smooth point on the irreducible curve $f(X, Y) = 0$.

Without loss of generality $f(X, Y) \in \mathcal{O}[X, Y]$ (multiply by d with $v(d)$ suitably large).

By changing variables we may assume $a, b \in \mathcal{O}$.
Suppose $v(c) = \min(v(a), v(b)) < 0$. Let $a' = a/c$, $b' = b/c$ then (a', b') is a smooth point on the curve $f(cU, cV) = 0$.
Multiplying by c^{-N} for large enough N this curve is given by $g(U, V) = 0$ for some $g \in O[U, V]$.

Without loss of generality we have $f \in \mathcal{O}[X, Y]$ and $(a, b) \in \mathcal{O}$ with $f(a, b) = 0$ and, say, $\frac{\partial f}{\partial Y}(a, b) \neq 0$.

Let $v(\frac{\partial f}{\partial Y}(a, b)) = g < \infty$.
We can find U an open ball around a such that
$v(\frac{\partial f}{\partial Y}(x, b)) = g \wedge v(f(x, b)) > 2g$ for $x \in U$.

For $x \in U$ apply this version of Hensel's Lemma to $f(x, Y)$ at b.
Then for all $x \in U$ there is a y such that $f(x, y) = 0$.

PAC fields are large

Definition

K is pseudoalgebraically closed (PAC) if for $C(K) \neq \emptyset$ every absolutely irreducible irreducible curve C defined over K.

Recall: *absolutely irreducible* $\Leftrightarrow$ irreducible over K^{alg}.

Pseudofinite fields are PAC
Weil estimate: If C is an absolutely irreducible curve of genus g defined over $\mathbb{F}_q$ then $|C(\mathbb{F}_q) - q - 1| \leq g\sqrt{q}$

Lemma

If K is PAC and V is an absolutely irreducible variety defined over K then $V(K)$ is Zariski dense in V.

Lemma

If C is an irreducible curve defined over K and $C(K)$ contains a smooth point a, then C is absolutely irreducible.

.

Suppose not. Let $C_1, \ldots, C_n$ be the irreducible components of V with $a \in C_1$.
All C_i are defined over K^{sep}.
Since a is a smooth point, it is contained in a unique irreducible component of C.

Thus any automorphism of K^{sep} fixing K must fix C_1 the irreducible component containing a.
But then C_1 is defined over K, contradicting the irreducibility of K.

Corollary

Every PAC field is large.

Other large fields

We say K is *pseudo-real closed* (PRC) if for any absolutely irreducible curve C defined over K, if C has a point in every real closure of K, then $C(K) \neq \emptyset$.

Every PRC field is large.

Recall an algebraic extension $F/\mathbb{Q}$ is *totally real* if for all $\sigma : F \to \mathbb{C}$ we have $\sigma(F) \subset \mathbb{R}$.

$\mathbb{Q}(\sqrt{7})$ is totally real while $\mathbb{Q}(\sqrt[3]{7})$ is not.

Fact: $\mathbb{Q}^{\mathrm{tr}}$ the maximal totally real extension of $\mathbb{Q}$ is large.

Question: Let $\mathbb{Q}^{\mathrm{ab}}$ be the maximal abelian extension of $\mathbb{Q}$ (i.e., $\mathbb{Q}(U)$ where U is all roots of unity). Is $\mathbb{Q}^{\mathrm{ab}}$ large?

Points on higher dimensional varieties

Lemma

If K is large and V is an irreducible variety defined over K with a simple point in $V(K)$, then $V(K)$ is Zariski dense in V.

By induction on the dimension of V. By assumption this is true if $\dim(V) = 1$.
Let $a \in V$ be a smooth point and $U \subset V$ be Zariski open in V.

Let H be a generic hyperplane through a defined over K.
Let $W = H \cap V$.
Then W is irreducible of dimension $\dim(V)$-1 and a is a smooth point of W. [This is a variant of Bertini's Theorem.]
Moreover $W \cap U \neq \emptyset$.

By induction there is a K-point in $W \cap U$. Thus $V(K)$ is Zariski dense in V.

Finite extensions of large fields

Lemma

If K is large and L/K is an algebraic extension, then L is large.

It suffices to prove this when L/K is finite. Suppose L/K has degree d. Let C be a smooth L-irreducible curve defined over L with $C(L)$ nonempty.

We look at the usual interpretation of L in K.
We identify L with K^d where addition is the usual addition on K^d and multiplication is a polynomial map from K^{2d} onto K^d.
The curve C is interpreted as $V \subset K^{2d}$ a d-dimensional variety defined over K.

Fact: V is irreducible and smooth [*Weil restriction of scalars.*]

Since $C(L) \neq \emptyset$, $V(K) \neq \emptyset$, hence $V(K)$ and $C(L)$ are infinite.

A model theoretic characterization of large fields

Theorem (Pop)

K is large if and only if $K \prec_1 K((t))$

($\Leftarrow$) Suppose C is a plane curve and say $(0,0)$ is a smooth point of $f(X,Y)=0$ where $\frac{\partial f}{\partial Y}(0,0) \neq 0$.

There is a formal power series $g(t) \in K((t))$ such that $f(t,g(t))=0$ [$(t,g(t))$ is a *formal branch* of the curve C]. In particular,

$$K((t)) \models \exists x \exists y \; x \neq 0 \wedge f(x,y) = 0$$

Iterating this we can get infinitely many points on C.

($\Rightarrow$) First show $K \prec_1 K(t)^h$ the henselization of $K(t)$.

Let K^* be a $|K|^+$-saturated elementary extension of K.
It suffices to show that if $K \subseteq L \subseteq K(t)^h$ is finitely generated over K then there is an embedding $\sigma : L \to K^*$ fixing K.

Since L/K is finitely generated and transcendence degree 1, there is a smooth projective curve C defined over K such that L is the function field of K.

Moreover, $L = K(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ is a generic point of C. So $C(L) \neq \emptyset$.

Let $[y_0, \ldots, y_n]$ be homogeneous coordinates for an L-point of K. Suppose $v(y_j)$ is minimal. Let $z_i = y_i/y_j$ for $i = 1, \ldots, n$.

Then $[z_0, \ldots, z_n]$ are homogeneous coordinates for the same point with coordinates in $\mathcal{O}$ and $z_i = 1$.

Taking residues $[\overline{z}_0, \ldots, \overline{z}_n]$ are homogeneous coordinates for a point in $C(K)$.
Since C is smooth this is a smooth point.

Since K is large $C(K)$ is Zariski dense in K.

Let Γ(v) be the type saying that v is a generic point of C.
Since $C(K)$ is Zariski dense Γ is consistent and can be realized by a $\in K^*$.
But $L \cong K$(a).

To finish the proof of $(\Rightarrow)$ apply

Theorem (Artin's Approximation Theorem)

$K(t)^h \prec_1 K(\!(t)\!)$

A standard application of Artin's Theorem is that if $F\langle\langle t\rangle\rangle$are the convergent power series over F where, say, $F = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$, then $F\langle\langle t\rangle\rangle \cap F(t)^{\mathrm{alg}} \prec_1 F((t))$.

étale maps

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be smooth varieties. Suppose $f : X \to Y$ is a morphism given by $f(\mathrm{x}) = (f_1(\mathrm{x}), \ldots, f_m(\mathrm{x}))$.

For $\mathrm{a} \in X$ we get a linear map $Df_{\mathrm{a}} : T_{\mathrm{a}}X \to T_{f(\mathrm{a})}Y$ given by

$$Df_{\mathrm{a}}(u_1, \ldots, u_n) = \left(\sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i}(\mathrm{a}) u_i, \cdots \sum_{i=1}^{n} \frac{\partial f_1}{\partial X_i}(\mathrm{a}) u_i \right)$$

Roughly, we say f is *étale* if Df_{a} is an isomorphism for each $\mathrm{a} \in X$

Intuition: Think of étale maps as covering maps.

In particular, étale maps are finite-to-one.

Example: Consider $f : \mathbb{A}^1 \to \mathbb{A}^1$ by $x \mapsto x^n$, where char(K) does not divide n.

If $a \neq 0$, $D_a f(u) = na^{n-1}u$ is an isomorphism between $T_a\mathbb{A}^1$ and $T_{a^n}\mathbb{A}^1$.

But at 0, $D_a(f)(u) = 0$ for all u so we don't have an isomophism
Thus f is not étale .

BUT if we consider $f(x) = x^n$ on the multiplicative group $\mathbb{G}_m$ then f is étale .

If char$(K) = p > 0$, the Artin–Schrier map $\wp(x) = x^p - x$ is étale as $\wp'(x) = -1$.

Properties of étale morphisms

- If $f : X \to Y$ and $g : Y \to Z$ are étale then so is $g \circ f$.
- If $f : \mathbb{A}^1 \to \mathbb{A}^1$ by $f(x) = \alpha x + \beta$ where $\alpha \neq 0$, then f is étale . $[D_\alpha f(u) = \alpha u]$
- (base change) If $f : X \to Z$ and $g : Y \to Z$ are étale and

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

  is the fibre product and $h : V \to Z$ by $h(x, y) = f(x) = g(y)$, then h is étale .

Étale open topology

Johnson, Tran, Walsberg and Ye introduced a topology on a field K as follows:

Let $\mathcal{E}_K = \{U \subseteq K :$ there is a smooth variety X defined over K and an étale $f : X \to \mathbb{A}^1$ defined over K such that $U = f(X(K))\}$.
We call $U \in \mathcal{E}_K$ an *étale image*.

Lemma

$\mathcal{E}_K$ is closed under finite intersections and affine transformations $x \mapsto ax + b$.

If $f : X \to \mathbb{A}^1$ and $g : Y \to \mathbb{A}^1$ are étale and $h : X \times_{\mathbb{A}^1} Y \to \mathbb{A}^1$ is the fibre product, then

$$h(X \times_{\mathbb{A}^1} Y) = f(X) \cap g(Y).$$

Thus $\mathcal{E}_K$ is a basis for a topology on K.

$\mathcal{E}_K$ for large fields K

Lemma

If K is a large field and $U \in \mathcal{E}_K$ is nonempty, then U is infinite.

There is X smooth and $f : X \to \mathbb{A}^1$ both defined over K such that $f(X(K)) = U$.

Since U is nonempty, $X(K)$ is nonempty, but every point on $X(K)$ is a smooth point.
But then $X(K)$ is infinite and, since f is finite-to-one, $f(X(K))$ is infinite.

Johnson, Tran, Walsberg and Ye prove that, conversely, for non-large fields the topology generated by $\mathcal{E}_K$ is discrete.

Large stable fields

Theorem (Johnson, Tran, Walsberg and Ye)

Every large stable field K is separably closed.

Suppose K is a large stable field and L/K is a finite Galois extension. We can find $K \subseteq F \subset L$ such that L/F is Galois of prime degree p. Note: F is stable and large.

Thus it suffices to show that for a large stable field K

- if $\operatorname{char}(K)$ does not divide n, then $x \mapsto x^n$ is surjective.
- If $\operatorname{char}(K) = p > 0$, then the Artin–Schreier map $\wp(x) = x^p - x$ is surjective.

Suppose char(K) does not divide n and $x \mapsto x^n$ is not surjective.
Then $(K^\times)^n$ is not generic in $K^\times$.

Let $P = (K^\times)^n - 1$.
Then $0 \in P$ and P is not generic.

Since P is not generic, $K^\times \setminus P$ is generic and there are $a_1, \ldots, a_n$ such that $K^\times = a_1(K^\times \setminus P) \cup \cdots \cup a_n(K^\times \setminus P)$.
Thus $\bigcap a_i P = \{0\}$.

Since $x \mapsto x^n$ is étale on $K^\times$, $(K^\times)^n \in \mathcal{E}_K$ and, since $\mathcal{E}_K$ is closed under affine transformations, $a_i P \in \mathcal{E}_K$ for $i = 1, \ldots, m$.

But then $\bigcap a_i P \in \mathcal{E}_K$ and it must be infinite, a contradiction.

Next suppose $\wp(x) = x^p - x$ is not surjective.

Then $\wp(K)$ is nongeneric.
Thus there are $a_1, \ldots, a_m$ such that $K^\times = \bigcup a_i(K^\times \setminus \wp(K))$.

But then $\bigcap a_i \wp(K) = \{0\}$

Note that $\wp$ is étale as $\wp'(x) = -1$ for all x.
Arguing as above we that see $\bigcap a_i \wp(K) = \{0\}$.

Virtually large stable fields

Definition

K is *virtually large* if there is a finite algebraic L/K such that L is large.

Fact: [Srinivasan] There are virtually large K that are not large.

Corollary

Every virtually large stable field is separably closed.

If K is stable, L/K is finite and L is large, then L is separably closed.

But if L/K is finite, non-trivial and L is separably closed, then Artin–Schreier theory tells us K is real closed and unstable.

To be continued....

I am going to take a two week break.

I will be back Monday March 15, 4:00 CST and will continue talking about some aspects of the Johnson, Tran, Walsberg, Ye paper.

If you are not on any of the UIC seminar mailing lists please send me an e-mail and I'll send a reminder a few days ahead of time. (*marker@uic.edu*)

Thanks for attending.