Lectures on Large Stable Fields VI
The Étale Open Topology

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Model theoretic characterization of large fields

Theorem (Pop)

K is large if and only if $K \prec_1 K((t))$

($\Leftarrow$) Suppose C is a plane curve and say $(0,0)$ is a smooth point of $f(X,Y)=0$ where, say, $\frac{\partial f}{\partial Y}(0,0) \neq 0$ and $(a_1,b_1),\ldots,(a_n,b_n)$ are all of the points in $C(K)$.

There is a formal power series $g(t) \in K((t))$ such that $f(t,g(t))=0$ [$(t,g(t))$ is a *formal branch* of the curve C]. In particular,

$$K((t)) \models \exists x \exists y \ \bigwedge_{i=1}^{n} x \neq a_i \wedge f(x,y)=0$$

but then so does K.

($\Rightarrow$) First show $K \prec_1 K(t)^h$ the henselization of $K(t)$.

Let K^* be a $|K|^+$-saturated elementary extension of K.
It suffices to show that if $K \subseteq L \subseteq K(t)^h$ is finitely generated over K then there is an embedding $\sigma : L \to K^*$ fixing K.

We may assume $\mathrm{td}(L/K) = 1$.
Since L/K is finitely generated and transcendence degree 1, there is a smooth projective curve C defined over K such that L is the function field of C.

Moreover, $L = K(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ is a generic point of C. In particular, $C(L) \neq \emptyset$.

Let $[y_0, \ldots, y_n]$ be homogeneous coordinates for an L-point of K. Suppose $v(y_j)$ is minimal. Let $z_i = y_i/y_j$ for $i = 1, \ldots, n$.

Then $[z_0, \ldots, z_n]$ are homogeneous coordinates for the same point with coordinates in $\mathcal{O}$ and $z_j = 1$.

Taking residues $[\overline{z}_0, \ldots, \overline{z}_n]$ are homogeneous coordinates for a point in $C(K)$.
Since C is smooth this is a smooth point.

Since K is large $C(K)$ is Zariski dense in K.

Let Γ(v) be the type saying that v is a generic point of C.
Since $C(K)$ is Zariski dense, Γ is consistent and can be realized by a $\in K^*$.
Then $L \cong K$(a).

To finish the proof of ($\Rightarrow$) apply

Theorem (Artin's Approximation Theorem)

$K(t)^h \prec_1 K(\!(t)\!)$

A standard application of Artin's Theorem is that if $F\langle\langle t\rangle\rangle$are the convergent power series over F where, say, $F = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$, then $F\langle\langle t\rangle\rangle \cap F(t)^{\mathrm{alg}} \prec_1 F((t))$.

Systems of Topologies

We will survey some of the results from Johnson, Tran, Walsberg and Ye's paper *Étale open topology and the stable field conjecture*.

Definition

Let K be a field. A *system of topologies* $\mathcal{T}$ is a family of topologies $\mathcal{T}_V$ on $V(K)$ for each K-variety V such that:

1. if $f : V \to W$ is a morphism of varieties, then $f|V(K)$ is $\mathcal{T}$-continuous;
2. if $f : V \to W$ is an open immersion of varieties, then $f|V(K)$ is a $\mathcal{T}$-open immersion.
3. if $f : V \to W$ is a closed embedding of varieties, then $f|V(K)$ is a $\mathcal{T}$-closed embedding.

Lemma

If $\mathcal{T}$ and $\mathcal{T}^*$ are systems of topologies that agree on $\mathbb{A}^n(K)$ for all n, then $\mathcal{T}$ and $\mathcal{T}^*$ agree on all varieties.

All varieties are built by patching affine varieties.

If $V \subset \mathbb{A}^n$ is $\{x : f_1(x) = \cdots = f_m(x) = 0\}$, let $f : \mathbb{A}^n \to \mathbb{A}^m$ be $f(x) = (f_1(x), \ldots, f_m(x))$.

Then $V = f^{-1}(0)$ must be $\mathcal{T}$-closed in any system of topologies $\mathcal{T}$.

Thus any system of topologies must refine the Zariski topology.

Examples

1. K any field and $\mathcal{T}_V$ the Zariski topology for each K-variety V.
2. K any field, $\mathcal{T}_V$ the discrete topology.
3. K an ordered field, $\mathcal{T}_{\mathbb{A}^1}$ the order topology on K and $\mathcal{T}_{\mathbb{A}^n}$ the product topology on K^n.
4. K a valued field, $\mathcal{T}_{\mathbb{A}^1}$ the valuation topology on $\mathbb{A}^1$
   (i.e., if for all a and g $\{x : v(x-a) > g\}$ is an open ball)
   and $\mathcal{T}_{\mathbb{A}^n}$ the product topology on K^n

Review: étale morphisms

Recall: $f : V \to W$ is *étale* if for all $a \in V$ the map $Df_a : T_a(V) \to T_{f(a)}(W)$ is an isomorphism.

Think of étale maps as "covering maps".

- étale maps are dimension preserving, thus finite-to-one;
- étale maps are open
- We avoid inseparability, for example in characteristic $p > 0$, $f(x) = x^p$ is not étale because $Df_a(u) = 0$ for all a and all $u \in T_a(\mathbb{G}_m)$.

Étale maps avoid ramification, for example let $C = \{(x, y) : x = y^2\}$ and consider the projection $f(x, y) = x$. At any (a, b),

$$T_{(a,b)}(C) = \{(u, v) : u - 2bv = 0\} \text{ and } Df_{(a,b)}(u, v) = u$$

At $(0, 0)$, $Df_{(0,0)}(0, v) = 0$

The étale open topology

We say $U \subseteq \mathbb{A}^n(K)$ is an *étale image* if and only if there is an étale $f : V \to \mathbb{A}^n$ such that $U = f(V(K))$.

Lemma

The set of étale images in $\mathbb{A}^n(K)$ is closed under invertible affine transformations, finite intersections and finite union.

For unions–let $U_i = f(V_i(K))$ where $f_i : V_i \to \mathbb{A}^n$ is étale .
Let V be the disjoint union of V_1, V_2 and $f : V \to \mathbb{A}^n$ be $f(x) = f_i(x)$ for $x \in V_i$.
Then $f(V(K)) = U_1 \cup U_2$

The *etale open topology* $\mathcal{E}_K$ is the topological system generated by the étale images.

Restrictions

Suppose $K \subset L$ and $\mathcal{T}$ is a system of topologies for L.
We define a system of topologies $\mathcal{T}^*$ for K.

If V is a K-variety, we can view V as an L-variety. Then $V(L) \supseteq V(K)$

Let $\mathcal{T}^*_V$ be the subspace topology on $V(K)$, i.e., if $U \subset V(L)$ is open in $\mathcal{T}_V$, then $U \cap V(K)$ is open in $\mathcal{T}^*_V$.
Let $\mathcal{T}|K$ denote the restriction $\mathcal{T}^*$.

Lemma

$\mathcal{T}|K$ is a system of topologies for K.

Theorem

If L/K is algebraic, then $\mathcal{E}_K$ refines $\mathcal{E}_L|K$.

Separably Closed Fields

Proposition

If K is separably closed, then the étale open toplogy is the Zariski topology.

The étale topology refines the Zariski topology, so we need only show that every étale open image is Zariski open

Suppose $f : V \to \mathbb{A}^n$ is étale . Let U be the image of V. Then U is Zariski open.
We claim that $f(V(K)) = U(K)$. Let $\mathrm{a} \in U(K)$ and let $\mathrm{b} \in V$ with $f(\mathrm{b}) = \mathrm{a}$.

Since f is étale , $K(\mathrm{b})$ is a separable extension of K. Thus $\mathrm{b} \in K$ and $\mathrm{a} \in f(V(K))$.

The converse also holds.

Proposition

If K is not separably closed, then the étale open topology on $\mathbb{A}^1(K)$ is Hausdorff.

Corollary

K is separably closed if and only if the étale open topology is the Zariski topology

Suppose K is not separably closed.
Arguing as in previous lectures we can find L/K a finite Galois extension and $K \subseteq F \subset L$ where $L = F(\alpha)$ where either:

1. $\alpha^n \in F$ where $\mathrm{char}(F)$ does not divide n;
2. $\wp(\alpha) = \alpha^p - \alpha$ where $\mathrm{char}(F) = p > 0$.

$K \subseteq F \subset L$, $L = F(\alpha)$, $\alpha^n \in F$.

The map $f : F^\times \to F^\times$ by $f(x) = x^n$ is étale and non-surjective.
Thus $P = (F^\times)^n$ is proper étale open subset of F.

If $a \in F^\times \setminus P$, aP is étale open and $aP \cap P = \emptyset$.

Let $x, y \in F$. There is an affine transformation $g : F \to F$ with $f(x) = 0$ and $f(y) = a$.
Then $f^{-1}(P)$ and $f^{-1}(aP)$ are disjoint étale open sets separating x and y.
Thus $\mathcal{E}_F$ is Hausdorff.

But then $\mathcal{E}_F|K$ is Hausdorff–a subspace of a Hausdorff space is Hausdorff.
But $\mathcal{E}_K$ refines $\mathcal{E}_F|K$. Thus $\mathcal{E}_K$ is Hausdorff.

Large Fields

We argued last time that if K is large then every nonempty $U \in \mathcal{E}_K$ is infinite.

The converse also holds.

Proposition

If K is not large, then $\mathcal{E}_K$ is the discrete topology.

There is a smooth curve C such that $C(K)$ contains a simple point a but $C(K)$ is finite.

There is an open subset U of C with a $\in U$ and an étale projection $\pi : U \to \mathbb{A}^1$.

Then $P = f(C(K))$ is a finite étale open image.

Translating we may assume $0 \in P$.

We can find $b \neq 0$ such $bP \cap P = \{0\}$. Thus $\{0\} \in \mathcal{E}_K$ and, translating, $\{a\} \in \mathcal{E}_K$ for all $a \in K$.

Minimal fields

A field K is *minimal* if every definable subset of K is finite or co-finite. [Note: we are not asserting this for every $K \prec L$ so, a priori, minimal might be a weaker condition than strongly minimal.]

Podewski's Conjecture Every minimal field is algebraically closed.

Proposition (Koenigsmann)

Every large minimal field K is algebraically closed.

We must have $K^p = K$. Thus K is perfect.

If K is not algebraically closed, it is not separably closed.
Thus there are $U, V \in \mathcal{E}_K$ disjoint and non-empty.

Since K is large, U and V are infinite, a contradiction.

Real closed fields

Let K be real closed.
Let $V = \{(x, y, z) : x^2 + y^2 = 1 \wedge yz = 1\}$.

The map $(x, y, z) \to x$ is étale and the image is $(-1, 1)$.

Taking affine transformations every open interval $(a, b) \in \mathcal{E}$.

Thus $\mathcal{E}_K$ refines the order topology on K.

Lemma

Suppose V and W are K-varieties and $f : V \to W$ is an étale morphism defined over K. Then f is an open map in the real topology on K.

When $K = \mathbb{R}$, f is a covering map and hence open in the usual topology. By transfer it's true in K.

Proposition

On a real closed field K, $\mathcal{E}_K$ is the order topology.

Henselian fields

Theorem

If (K, v) is a non separably closed Henselian valued field, then $\mathcal{E}_K$ is the valuation topology.

Lemma (Halevi, Hasson, Jahnke)

The valuation topology refines $\mathcal{E}_K$

To prove the theorem we need the converse that every open set in the valuation topology is open in $\mathcal{E}_K$.

Example: $\mathbb{Q}_p$ $(p \neq 2)$
Recall $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \exists y \; y^2 = px^2 + 1\}$

Let $C = \{(x, y, z) : y^2 = px^2 + 1 \wedge yz = 1\}$ and $f : C \to \mathbb{A}^1$ be $(x, y, z) \mapsto x$.
Then f is étale and $f(C(\mathbb{Q}_p)) = \mathbb{Z}_p$.

The general result uses:

Lemma (Prestel–Ziegler)

If K is henselian and $f(X) \in K[X]$ is separable with no zero in K then $f(K)$ is bounded away from 0 in the valuation topology.

For example in $\mathbb{Q}_5$ consider $f(X) = X^2 - 2$
Then $v(f(x)) \leq 0$ for all $x \in \mathbb{Q}_5$

Proof of Prestel–Ziegler

$f(X) \in K[X]$ separable, wlog f is monic.
There are $g, h \in K[X]$ such that $1 = fg + f'h$.

Work in $K \prec K^*$ very saturated with value group G.
There is $G_0 \subset G$ convex such that all coefficients of f, f', g, h have values in G_0.
Let $\Gamma = G/G_0$ and consider the coarsening $w = v/G_0$ on K^* with maximal ideal $\mathfrak{m}$ and henselian valuation ring $\mathcal{O}$. Then $f, f', g, h \in \mathcal{O}[X]$.

Suppose for contradiction that $f(a) \in \mathfrak{m}$.
If $w(a) < 0$, then $w(f(a)) < 0$ since f is monic. Thus $a \in \mathcal{O}$.

But $1 = f(a)g(a) + f'(a)h(a)$. Thus $f'(a) \notin \mathfrak{m}$.
By henselianity, we can find $b \in \mathcal{O}$ such that $f(b) = 0$, a contradiction.

Proposition

$\mathcal{E}_K$ refines the valuation topology.

Suppose $f(X) \in K[X]$ and $f(K)$ is bounded away from 0.
There is $U_0 \subset \mathbb{A}^1$ Zariski open with f étale on U_0.

Let $U = \{1/f(x) : x \in U_0\}$.
Then U is open in $\mathcal{E}_K$ and bounded in the valuation topology.
Translating we may assume that $0 \in U$.

Let W be open in the valuation topology. We want W open in $\mathcal{E}_K$.

For $b \in W$ there is $a_b \in K$ such that $a_b U \subset W - b$.

Then $W = \bigcup_{b \in W}(a_b U + b)$ is open in $\mathcal{E}_K$.

Thus $\mathcal{E}_K$ refines the valuation topology.