Compactness Theorem Exercises

1) We say that a class of $\mathcal{L}$-structures $\mathcal{K}$ is an *elementary class* if there is an $\mathcal{L}$-theory T such that

$$\mathcal{M} \in \mathcal{K} \Leftrightarrow \mathcal{M} \models T.$$

We also say that T *axiomatizes* $\mathcal{K}$. Decide if the following classes are elementary. Show the class is elementary by giving an axiomatization or prove that it is not–probably by using the Compactness Theorem.

a) Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol.

a1) Let $\mathcal{K}$ be the class of all equivalence relations.

a2) Let $\mathcal{K}$ be the class of all equivalence relations where each class has size 2.

a3 Let $\mathcal{K}$ be the class of equivalence relations where each class is finite.

a4) Let $\mathcal{K}$ be the class of equivalence relations with infinitely many infinite classes.

b) Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a *graph* if $E^{\mathcal{M}}$ is symmetric.

b1) Let $\mathcal{K}$ be the class of connected graphs.

b2) Let $\mathcal{K}$ be the class of acyclic graphs.

b3) Let $\mathcal{K}$ be the class of bipartite graphs. [Recall that a graph is bipartite if we can partition the edges into two sets A and B such that there every edge has one vertex in A and one vertex in B. Hint: a graph is bipartite if and only if there are no cycles of odd length.]

c) Let $\mathcal{L} = \{\cdot, e\}$. For G a group let G^n be the set of n^{th}-powers.

c1) Let $\mathcal{K}$ be the class of divisible groups (i.e., groups where $G^n = G$ for all n.

c2) Let $\mathcal{K}$ be the class of groups G where $\bigcap_{n=1}^{\infty} G^n = \{e\}$.

c3) Let $\mathcal{K}$ be the class of torsion free groups.

c4) Let $\mathcal{K}$ be the class of torsion groups (i.e., groups where every element has finite order).

c5) Let $\mathcal{K}$ be the class of free groups.

2) Let $\mathcal{L} = \{+, \cdot, <, 0, 1\}$. We say that an ordered field F is *archimedian* for any $x, y > 0$ there are natural numbers m and n such that $x < my$ and $y < nx$. Prove that there is a nonarchimedian ordered field elementarily equivalent to the field of real numbers.